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A Generalization of Ivory's Theorem

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Abstract. In this paper we use the definition of the energy of a curve on a surface (from [4] or [7]) and show that in a Liouville net over $\mathbb{C}$ the energy integrals along the two diagonals of any curvilinear quadrangle of net curves are equal. In particular cases also the lengths of the two diagonals are equal. For Liouville nets in $\mathbb{C}$ we prove a theorem about the energy of certain approximating polygons for the diagonals, in which the well-known planar version of Ivory's Theorem is included as a special case. This new theorem can therefore be seen as a generalization of Ivory's Theorem in the plane. In addition, we prove that this theorem is valid on the holomorphic Liouville curves constructed in [1].

Key Words: Ivory's Theorem, Energy/Action of a curve, holomorphic Liouville curve

MSC 2010: 53A05, 53A07

1. Introduction

The Theorem of Ivory in the Euclidean plane is about a general quadrangle formed by confocal quadrics. It states that the two straight line diagonals in each such quadrangle have equal lengths. In Figure 1 (kindly provided by H. Stachel) you can see a net of confocal ellipses and hyperbolas with common foci F_1 and F_2 and a quadrangle $ABCD$, where the diagonals AC and BD have the same length.

James Ivory proved his theorem in 1809 for the three-dimensional Euclidean case by straightforward calculation. It is now known (see [6]) that this theorem is valid in the n-dimensional Euclidean space, in a Minkowski (pseudo-Euclidean) plane and in n-dimensional hyperbolic spaces.

In this article we first introduce the notion of length and energy for a curve and a polygon. After that, we repeat the construction of a holomorphic conformal map φ from $\mathbb{C}$ to $\mathbb{C}$ from article [1], that generates all plane Liouville nets up to translations. Then we formulate and prove a theorem concerning the lengths and energies of the diagonals and their polygonal subdivisions in a plane Liouville net rectangle which contains Ivory's Theorem as a special case.[1] Then this theorem is extended to the holomorphic Liouville curves of article [1].

[1]Already W. Blaschke presented in his book [3] confocal conics as images of a rectangular grid under particular conformal mappings.

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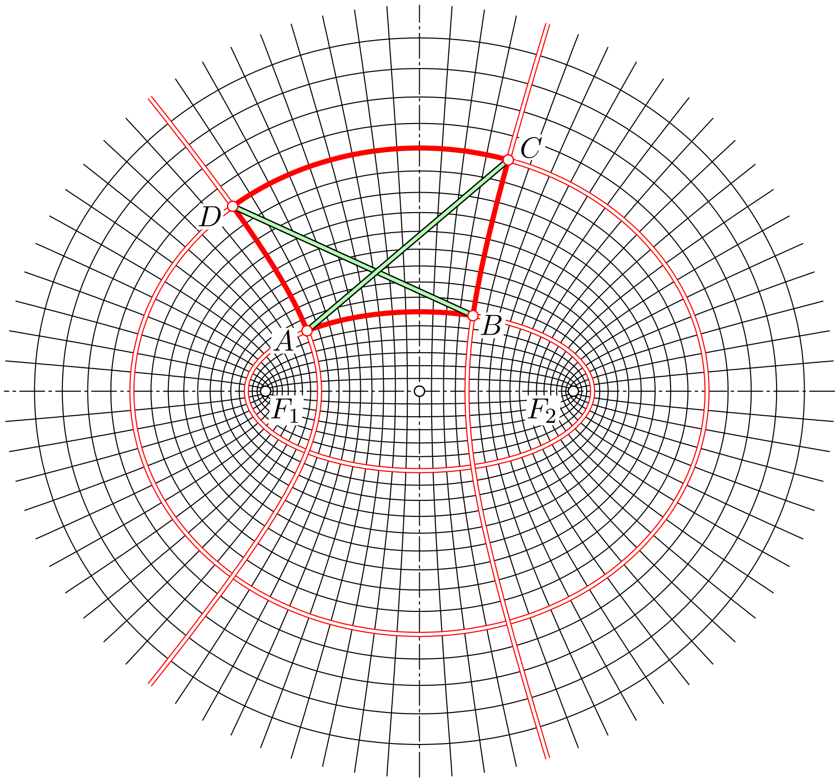

Figure 1: Ivory's Theorem in the Euclidean plane

My statements in the Theorems 5.1 and 6.2 dealing with polygons could be called 'discrete versions' of the energy-statements — in the sense of 'Discrete Differential Geometry'. Some particular comments, e.g., in the Theorems 6.1, 4.2, 4.4, and in Remarks 6.1, are not essential for the presented generalizations of Ivory's Theorem but only related to statements of [1].

2. Length and energy of a curve in a manifold

The length $L(p)$ and the energy $E(p)$ of a curve $p\colon\ [0,1] \to M$ in a Riemannian manifold M are given by the following expressions (see [4, Chapter 9, p. 194]) and they are related by the Schwarz inequality (with equality iff $\|\dot{p}\|$ is constant, which means that the parameter t is proportional to arc length):

$$L(p) = \int_0^1 \|\dot{p}\| dt = \int_0^1 \sqrt{\langle \dot{p}, \dot{p} \rangle}\, dt\,,$$
$$E(p) = \int_0^1 \|\dot{p}\|^2 dt = \int_0^1 \langle \dot{p}, \dot{p} \rangle\, dt\,,$$
$$L(p)^2 \leq E(p),$$

where $\dot{p}$ is the tangent vector of the curve and $\langle\ ,\ \rangle$ is the scalar product in the tangent space TM. For polygons approximating the curve p after a subdivision

$$0 = \frac{0}{m} < \frac{1}{m} < \cdots < \frac{k-1}{m} < \frac{k}{m} < \cdots < \frac{m-1}{m} < \frac{m}{m} = 1$$

of the interval of definition, the length $L(p,m)$ and the energy $E(p,m)$ are defined as follows:

$$L(p,m)=\sum_{k=1}^{m}\sqrt{\left\langle p\left(\frac{k}{m}\right)-p\left(\frac{k-1}{m}\right),\ p\left(\frac{k}{m}\right)-p\left(\frac{k-1}{m}\right)\right\rangle},$$
$$E(p,m)=\sum_{k=1}^{m}\left\langle p\left(\frac{k}{m}\right)-p\left(\frac{k-1}{m}\right),\ p\left(\frac{k}{m}\right)-p\left(\frac{k-1}{m}\right)\right\rangle.$$

3. The Wirtinger derivatives

Consider a map $\varphi\colon\ \mathbb{R}^2\to\mathbb{C}$ with $\varphi(x,y)=u(x,y)+iv(x,y)$, where $x,y,u,v\in\mathbb{R}$ and the partial derivatives u_x,u_y,v_x,v_y exist. Now we extend the domain of definition of φ from $\mathbb{R}^2$ to $\mathbb{C}^2$ and make a transformation, passing from (x,y) to $(z,\overline{z})$, where $z=x+iy$ and $\overline{z}=x-iy$. We denote this function still with $\varphi\colon\ \mathbb{C}^2\to\mathbb{C}$ and $\varphi(z,\overline{z})=u(x,y)+iv(x,y)$, where $x=\operatorname{re}z=\dfrac{z+\overline{z}}{2}$ and $y=\operatorname{im}z=\dfrac{z-\overline{z}}{2i}$. For the partial derivatives we have:

$$\frac{\partial}{\partial x}=\frac{\partial z}{\partial x}\frac{\partial}{\partial z}+\frac{\partial\overline{z}}{\partial x}\frac{\partial}{\partial\overline{z}}=\frac{\partial}{\partial z}+\frac{\partial}{\partial\overline{z}},$$
$$\frac{\partial}{\partial y}=\frac{\partial z}{\partial y}\frac{\partial}{\partial z}+\frac{\partial\overline{z}}{\partial y}\frac{\partial}{\partial\overline{z}}=i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial\overline{z}}\right).$$

From here we get the usual expressions for the Wirtinger derivatives:

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\qquad \frac{\partial}{\partial\overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

The map $\varphi\colon\ \mathbb{C}^2\to\mathbb{C}$ with $\varphi(z,\overline{z})=\varphi(x+iy,x-iy)=u(x,y)+iv(x,y)$ is conformal if and only if it is holomorphic. If the map φ is holomorphic, then u and v have first partial derivatives with respect to x and y and satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\quad\text{and}\quad\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

or, equivalently, the Wirtinger derivative of φ with respect to the complex conjugate of z is zero:

$$\varphi_{\overline{z}}=\frac{\partial\varphi}{\partial\overline{z}}=\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}+i\frac{\partial\varphi}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}\right)=0$$

which means that φ is functionally independent from the complex conjugate of z.

4. Construction of a plane Liouville map

We repeat here the construction of a holomorphic Liouville map $\varphi\colon\ \mathbb{C}\to\mathbb{C}$ from article [1], which is the one-dimensional case in article [1].

We start with the Liouville line element of φ, with $\psi:=\varphi_z$:

$$ds^2=(f(x)+g(y))\,(dx^2+dy^2)=\psi(z)\,\overline{\psi(z)}\,dz\,\overline{dz}\tag{4.1}$$

where $\psi(z)=\psi(x+iy)$ is a holomorphic function of the complex variable $z=x+iy$ with complex conjugate $\overline{z}=x-iy$, real part x and imaginary part y.

The differential is $dz = dx + i\,dy$. Then we have $dz\,\overline{dz} = dx^2 + dy^2$ and we set

$$\psi(z)\,\overline{\psi(z)} := f(x) + g(y). \tag{4.2}$$

This is equivalent to

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}\left(\psi(z)\,\overline{\psi(z)}\right) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\psi(z)\,\overline{\psi(z)}\right) = 0\,. \tag{4.3}$$

From the theory of complex differentiation (see Wirtinger derivatives) we get

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = i\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \overline{z}^2}\right). \tag{4.4}$$

Using this formula we can rewrite (4.3) as

$$\left(\psi(z)\,\overline{\psi(z)}\right)_{zz} = \left(\psi(z)\,\overline{\psi(z)}\right)_{\overline{z}\,\overline{z}}\,. \tag{4.5}$$

Now, by differentiating twice with respect to z and $\overline{z}$ we obtain

$$\begin{aligned}(\psi(z)\,\overline{\psi(z)})_{zz} &= 2\psi_z\overline{\psi}_z + \psi\,\overline{\psi}_{zz} + \psi_{zz}\overline{\psi} = \psi_{zz}\overline{\psi},\\ (\psi(z)\,\overline{\psi(z)})_{\overline{z}\,\overline{z}} &= 2\psi_{\overline{z}}\,\overline{\psi}_{\overline{z}} + \psi\,\overline{\psi}_{\overline{z}\,\overline{z}} + \psi_{\overline{z}\overline{z}}\,\overline{\psi} = \psi\,\overline{\psi}_{\overline{z}\overline{z}} = \psi\,\overline{\psi_{zz}}.\end{aligned}$$

Here we used the identity $\overline{\psi}_{\overline{z}} = \overline{\psi_z}$ and the fact that a holomorphic function ψ satisfies the Cauchy-Riemann equations, which can be written as $\psi_{\overline{z}} = \overline{\overline{\psi}_z} = \overline{\psi}_z = 0$. Therefore formula (4.5) is equivalent to

$$\begin{aligned}\psi_{zz}\overline{\psi} &= \psi\,\overline{\psi_{zz}}\\ \psi^{-1}\,\psi_{zz} &= (\overline{\psi_{zz}})(\overline{\psi}^{-1})\\ \psi^{-1}\,\psi_{zz} &= \overline{\psi^{-1}\,\psi_{zz}} =: a\,.\end{aligned} \tag{4.6}$$

Since the left hand side of (4.6) is holomorphic and the right hand side is anti-holomorphic, the function $\psi^{-1}\,\psi_{zz}$ must be a real constant $a \in \mathbb{R}$. With the notation $a = -k^2$ we get from (4.6) the one dimensional complex Helmholtz differential equation

$$\psi_{zz} + k^2\psi = 0\,. \tag{4.7}$$

The Helmholtz differential equation is a special case of the Schrödinger equation (time independent Schrödinger equation) and for $a = k = 0$ we get the Laplace equation $\psi_{zz} = 0$ as a special case of the Helmholtz equation.

- Let us solve the case $k = 0$ (Laplace equation) first. The solution is $\psi(z) = C_1 z + C_2$ where C_1, C_2 are complex integration constants. From this we get by integration $\varphi = \int \psi dz = C_1 z^2 + C_2 z + C_3$.
- The next case is $a = -k^2 > 0$, then $k = i\omega$ with $\omega \in \mathbb{R}$ and the Helmholtz equation reads in this case $\psi_{zz} - \omega^2\psi = 0$. The solution is $\psi(z) = C_1 e^{\omega z} + C_2 e^{-\omega z}$ and $\varphi = \int \psi dz = \frac{C_1}{\omega}e^{\omega z} - \frac{C_2}{\omega}e^{-\omega z} + C_3$.
- The last case is $a = -k^2 < 0$, then $k = \omega$ with $\omega \in \mathbb{R}$ and the Helmholtz equation reads in this case $\psi_{zz} + \omega^2\psi = 0$. The solution is $\psi(z) = C_1 e^{i\omega z} + C_2 e^{-i\omega z}$ and $\varphi = \int \psi dz = \frac{C_1}{i\omega}e^{i\omega z} - \frac{C_2}{i\omega}e^{-i\omega z} + C_3$.

We can ignore the translation of the solutions by the third integration constant C_3 and pass to the notation of [1].

Lemma 4.1. *The map φ maps a rectangular grid to a Liouville net if and only if φ is a solution of the Helmholtz equation $\varphi_{zzz} - a\varphi_z = 0$ (with complex constants α_s, β_s):*

1. *When $a > 0$: $\varphi(z) = \varphi_+(z) = \alpha_+ e^{\omega z} + \beta_+ e^{-\omega z}$.*
2. *When $a < 0$: $\varphi(z) = \varphi_-(z) = \alpha_- e^{i\omega z} + \beta_- e^{-i\omega z}$.*
3. *When $a = 0$: $\varphi(z) = \varphi_0(z) = \alpha_0 z + \beta_0 z^2$.*

Remarks 4.1. 1. The same solutions appear in [5, Theorem A] as the only solutions to a functional equation connected with Ivory's Theorem in the complex plane. This is not surprising, because the Liouville line element has the property that the parameter lines are geodesic ellipses and hyperbolas and the geodesic diagonals in each quadrangle formed by parameter lines satisfy Ivory's Theorem, i.e., they have the same geodesic length. This geometric characterization of the Liouville line element was proved by Dini, resp. by Zwirner according to [1].

2. Blaschke addresses in [2] mainly the 3D-case. Surprisingly, he doesn't speak of a Liouville net but of Stäckel's first fundamental form which is necessary and sufficient for Ivory's Theorem on Riemannian manifolds where geodesics serve as diagonals. It should however be noted that the Liouville line element is a special case of the Stäckel line element.

3. Some special solutions are:
   (a) $\varphi(z) = z$ — cartesian coordinates,
   (b) $\varphi(z) = z^2$ — parabolic coordinates,
   (c) $\varphi(z) = e^z$ — polar coordinates,
   (d) $\varphi(z) = \cosh(z)$ — elliptic coordinates.

In [1], the authors give conditions such that the metric admits a one-parameter family of isometries ($\theta \in \mathbb{R}$).

Theorem 4.2. *The metric admits a one-parameter family of isometries ($\theta \in \mathbb{R}$) in the following cases:*

1. *When $a > 0$: $\alpha_+\beta_+ = 0$. Example for the group of isometries: $z \mapsto z + i\theta$.*
2. *When $a < 0$: $\alpha_-\beta_- = 0$. Example for the group of isometries: $z \mapsto z + \theta$.*
3. *When $a = 0$: $\alpha_0\beta_0 = 0$. Example for the group of isometries: $z \mapsto e^{i\theta} z$.*

Remark 4.2. There is a beautiful geometric interpretation of these isometries: When we project the complex plane stereographically to the unit sphere centered at the origin from either the north or south pole, then we see that in each case the trajectories contain exactly one great circle, and these three great circles are pairwise orthogonal.

Proof. We prove the three cases:

1. $a > 0$: For the case $\alpha_+ = 0$ we set $c := \beta_+$ and for the case $\beta_+ = 0$ we set $c := \alpha_+$. Then both cases lead to $\varphi(z + i\theta) = \varphi_+(z + i\theta) = ce^{\pm\omega(z+i\theta)}$ and the metric is characterized by $\varphi_z\overline{\varphi_z} = \omega^2 c\bar{c}e^{\pm\omega(z+\bar{z})} = \omega^2 c\bar{c}e^{\pm 2\omega x}$, which is independent of θ.

2. $a < 0$: For the case $\alpha_- = 0$ we set $c := \beta_-$ and for the case $\beta_- = 0$ we set $c := \alpha_-$. Then both cases lead to $\varphi(z + \theta) = \varphi_-(z + \theta) = ce^{\pm i\omega(z+\theta)}$ and the metric is characterized by $\varphi_z\overline{\varphi_z} = \omega^2 c\bar{c}e^{\pm\omega(iz+\overline{iz})} = \omega^2 c\bar{c}e^{\mp 2\omega y}$, which is independent of θ.

3. $a=0$: For the case $\alpha_0=0$ we set $c:=\beta_0$, $m=2$ and for the case $\beta_0=0$ we set $c:=\alpha_0$, $m=1$. Then both cases lead to $\varphi(e^{i\theta}z)=\varphi_0(e^{i\theta}z)=c(e^{i\theta}z)^m$ and the metric is characterized by $\varphi_z\overline{\varphi_z}=m^2c\overline{c}(z\overline{z})^{m-1}=m^2c\overline{c}(x^2+y^2)^{m-1}$, which is independent of θ. □

We now calculate $\varphi^{(m)}=\partial^m\varphi/\partial z^m$:

Theorem 4.3. *For $\varphi:\mathbb{C}\to\mathbb{C}$ defined as $\varphi\in\{\varphi_-,\varphi_0,\varphi_+\}$ (see Lemma 4.1) we have $\varphi^{(m)}\in\left\{\varphi_-^{(m)},\varphi_0^{(m)},\varphi_+^{(m)}\right\}$ where*

$$\varphi_+^{(m)}(z)=\omega^m\left(\alpha_+e^{\omega z}+(-1)^m\beta_+e^{-\omega z}\right) \tag{4.8}$$
$$\varphi_-^{(m)}(z)=(i\omega)^m\left(\alpha_-e^{i\omega z}+(-1)^m\beta_-e^{-i\omega z}\right) \tag{4.9}$$
$$\varphi_0^{(m)}(z)=z^{1-m}\left(\frac{\alpha_0}{\Gamma(2-m)}+\frac{2\beta_0 z}{\Gamma(3-m)}\right) \tag{4.10}$$

Remark 4.3. The notation $\varphi^{(m)}$ used above represents for $m>0$ repeated (m-fold) differentiation of φ with respect to z, and for $m<0$ it represents repeated ($|m|$-fold) integration of φ with respect to z and with a vanishing integration constant.

Proof. Left as exercise for the interested reader. Hint: induction. □

Theorem 4.4. *The metric of $\varphi^{(m-1)}\in\left\{\varphi_-^{(m-1)},\varphi_0^{(m-1)},\varphi_+^{(m-1)}\right\}$ admits the same one-parameter group of isometries as φ under the same conditions.*

Proof. Left as exercise for the interested reader. Hint: Basically the same as the proof for Theorem 4.2, with adapted constants c and m. □

Theorem 4.5. *The line element of $\varphi^{(m-1)}\in\left\{\varphi_-^{(m-1)},\varphi_+^{(m-1)}\right\}$ is Liouville for all $m\in\mathbb{Z}$. The line element of $\varphi^{(m-1)}=\varphi_0^{(m-1)}$ is Liouville for all $1\le m\in\mathbb{Z}$ and additionally for $\beta_0=0$ and $m=0$.*

Proof. Left as exercise for the interested reader. □

5. Generalization of Ivory's Theorem in the complex one-dimensional case

The solutions $\varphi\in\{\varphi_-,\varphi_0,\varphi_+\}$ of the Helmholtz differential equation $\varphi_{zzz}=a\,\varphi_z$ represent all plane Liouville nets, up to translations. An example for a Liouville map φ from $\mathbb{C}$ to $\mathbb{C}$ can be seen in Figure 2. The Liouville maps are conformal (they preserve angles) because φ is holomorphic. The diagonals of the rectangles are mapped to isogonal trajectories of the parameter curves of the Liouville net.

The main theorem of this paper is the following:

Theorem 5.1 (A generalization of Ivory's Theorem in the plane)**.**
Let $ABCD$ be a rectangle in the complex plane with diagonals $d_1(t)=A+t(C-A)$ and $d_2(t)=B+t(D-B)$ for $t\in[0,1]$. Consider now the curves $p_1(t)=\varphi(d_1(t))$ and $p_2(t)=\varphi(d_2(t))$.
Then for $\varphi\in\{\varphi_-,\varphi_0,\varphi_+\}$ the two polygons formed by joining the points $p_1(\frac{k}{m})$ and $p_2(\frac{k}{m})$, respectively, where $0\le k\le m$, the energies are equal for each $m\in\mathbb{N}\setminus\{0\}$:

$$E(p_1,m)=E(p_2,m)\ \textit{and additionally}\ \ E(p_1)=E(p_2)$$

(see Figure 3). In the particular cases

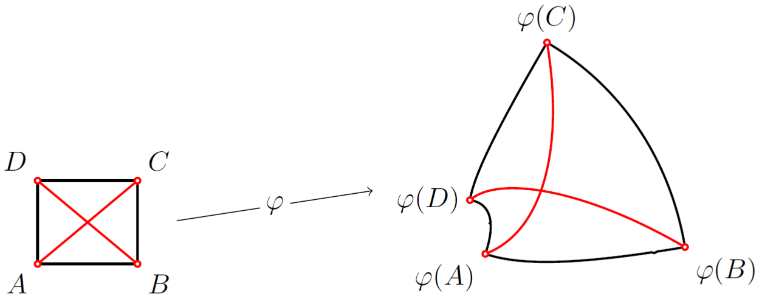

Figure 2: Mapping a plane rectangle to a plane Liouville net rectangle

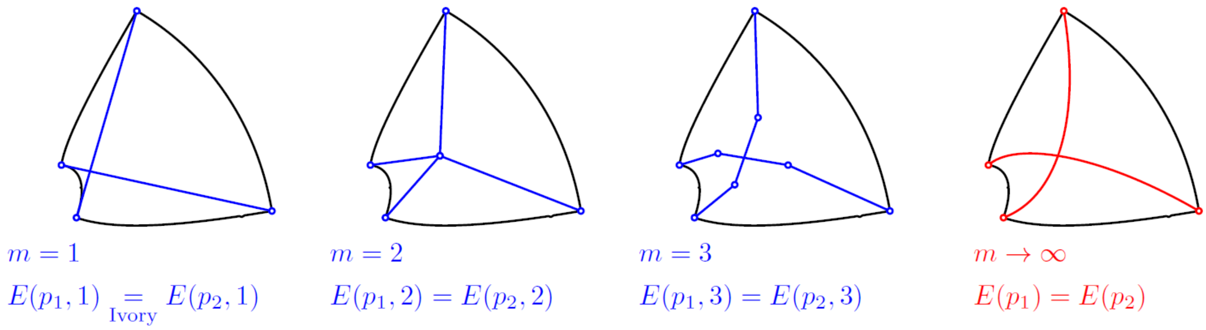

Figure 3: Subdivision of the diagonals in a plane Liouville net rectangle

1. *$\varphi = \varphi_-$ and $\alpha_- \beta_- = 0$,*
2. *$\varphi = \varphi_0$ and $\beta_0 = 0$, and*
3. *$\varphi = \varphi_+$ and $\alpha_+ \beta_+ = 0$*

also the lengths of the two polygons are equal, i.e.,

$$L(p_1, m) = L(p_2, m) \text{ and additionally } L(p_1) = L(p_2).$$

This theorem contains the plane version of Ivory's Theorem:

Corollary 5.2 (Ivory's Theorem)**.**
Take $m = 1$ in Theorem 5.1. Then for the lengths of the (straight line, geodesic) diagonals we have

$$L^2(p_1, 1) = E(p_1, 1) = E(p_2, 1) = L^2(p_2, 1).$$

Since the proof of the main theorem 5.1 is not short, we split this theorem into two parts 5.3 and 5.4 and prove them separately. Their proofs taken together are the proof of the main theorem.

Theorem 5.3. *Let A and C be two points in the complex plane. Construct the rectangle $ABCD$ with the points $B = \mathrm{re}(C) + i\,\mathrm{im}(A)$ and $D = \mathrm{re}(A) + i\,\mathrm{im}(C)$. The diagonals of the rectangle are $d_1(t) = A + t(C - A)$ and $d_2(t) = B + t(D - B)$, $t \in [0, 1]$.*
Then for $\varphi \in \{\varphi_-, \varphi_0, \varphi_+\}$ the image curves $p_1(t) = \varphi(d_1(t))$ and $p_2(t) = \varphi(d_2(t))$ of these diagonals have equal energies, i.e., $E(p_1) = E(p_2)$.
In the particular cases

1. *$\varphi = \varphi_-$ and $\alpha_- \beta_- = 0$,*
2. *$\varphi = \varphi_0$ and $\beta_0 = 0$, and*

3. $\varphi = \varphi_+$ *and* $\alpha_+ \beta_+ = 0$
also the lengths of the two diagonals are equal, i.e., $L(p_1) = L(p_2)$.

Proof. For convenience, let $a_1 = \mathrm{re}(A)$, $a_2 = \mathrm{im}(A)$, $c_1 = \mathrm{re}(C)$ and $c_2 = \mathrm{im}(C)$ in what follows.

1. Let $\varphi = \varphi_-$ and $\alpha = \alpha_- \neq 0$, $\beta = \beta_- \neq 0$. Then we have

$$\varphi(z) = \alpha e^{i\omega z} + \beta e^{-i\omega z} = \gamma \cos(\omega z + \delta)$$

with $\gamma = 2\sqrt{\alpha}\sqrt{\beta}$ and $\delta = -i \ln\left(\frac{\sqrt{\alpha}}{\sqrt{\beta}}\right)$.
We see that $\omega \in \mathbb{R}$ is a scaling factor in the pre-image plane and δ is a translation in the pre-image plane. The factor γ defines a stretch-rotation in the image plane. Therefore, without loss of generality (w.l.o.g.) we can set $\alpha = \beta = \frac{1}{2}$ and $\omega = 1$ to get $\varphi(z) = \cos z$.
We begin by computing $\dot{p}_1(t) = \frac{\partial p_1(t)}{\partial t}$ and $\dot{p}_2(t) = \frac{\partial p_2(t)}{\partial t}$:

$$\begin{aligned} \dot{p}_1(t) &= -\sin(d_1(t))d_1'(t) = (A - C)\sin(d_1(t)), \\ \dot{p}_2(t) &= -\sin(d_2(t))d_2'(t) = (B - D)\sin(d_2(t)). \end{aligned}$$

Next, we compute the squared length $\|\dot{p}_k(t)\|^2 = \dot{p}_k(t)\,\overline{\dot{p}_k(t)}$ for $k = 1, 2$:

$$\begin{aligned} \|\dot{p}_1(t)\|^2 &= (A - C)\overline{(A - C)}\ \sin(d_1(t))\sin(\overline{d_1(t)}) \\ &= \left((a_1 - c_1)^2 + (a_2 - c_2)^2\right)\sin(d_1(t))\sin(\overline{d_1(t)}), \\ \|\dot{p}_2(t)\|^2 &= (B - D)\overline{(B - D)}\ \sin(d_2(t))\sin(\overline{d_2(t)}) \\ &= \left((a_1 - c_1)^2 + (a_2 - c_2)^2\right)\sin(d_2(t))\sin(\overline{d_2(t)}). \end{aligned}$$

We can now calculate the difference

$$\|\dot{p}_1(t)\|^2 - \|\dot{p}_2(t)\|^2 = \left((a_1 - c_1)^2 + (a_2 - c_2)^2\right)\sin\left(a_1 + c_1\right)\sin((a_1 - c_1)\left(1 - 2t\right))$$

After integration we obtain the desired result for the energies,

$$E(p_1) - E(p_2) = \int_0^1 \|\dot{p}_1(t)\|^2 - \|\dot{p}_2(t)\|^2 dt = 0.$$

Consider now the case $\alpha = 0$. W.l.o.g. set $\omega = \beta = 1$. Then we have

$$\begin{aligned} \|\dot{p}_1(t)\|^2 &= \left((a_1 - c_1)^2 + (a_2 - c_2)^2\right)e^{2(a_2 + t(c_2 - a_2))} = \|\dot{p}_2(t)\|^2, \\ \|\dot{p}_1(t)\| &= \sqrt{\left((a_1 - c_1)^2 + (a_2 - c_2)^2\right)e^{2(a_2 + t(c_2 - a_2))}} = \|\dot{p}_2(t)\|. \end{aligned}$$

For the lengths follows

$$L(p_1) - L(p_2) = \int_0^1 \|\dot{p}_1(t)\| - \|\dot{p}_2(t)\| dt = \int_0^1 0\ dt = 0.$$

The case $\beta = 0$ is similar.

2. Let $\varphi = \varphi_0$ and $\alpha = \alpha_0$, $\beta = \beta_0$. We begin by computing $\dot{p}_1(t) = \dfrac{\partial p_1(t)}{\partial t}$ and $\dot{p}_2(t) = \dfrac{\partial p_2(t)}{\partial t}$.

$$\begin{aligned}\dot{p}_1(t) &= (c_1 - a_1 + i\,(c_2 - a_2))\,(\alpha + 2\,(1-t)\,\beta(a_1 + ia_2) + 2t\beta\,(c_1 + ic_2))\,,\\ \dot{p}_2(t) &= (a_1 - c_1 + i(c_2 - a_2))\,(\alpha + 2t\beta(a_1 + ic_2) + 2i\beta\,(1-t)\,(a_2 - ic_1))\,.\end{aligned}$$

Next, we compute the squared length $\|\dot{p}_k(t)\|^2 = \dot{p}_k(t)\,\overline{\dot{p}_k(t)}$ for $k = 1, 2$.

$$\begin{aligned}\|\dot{p}_1(t)\|^2 = &\left((a_1 - c_1)^2 + (a_2 - c_2)^2\right)\left(\overline{\alpha} + 2\overline{\beta}\,((1-t)\,a_1 + i\,(t-1)\,a_2 + t\,(c_1 - ic_2))\right)\\ &\cdot(\alpha + 2\beta a_1\,(1-t) + 2i\beta a_2\,(1-t) + 2t\beta\,(c_1 + ic_2))\,,\\ \|\dot{p}_2(t)\|^2 = &\left((a_1 - c_1)^2 + (a_2 - c_2)^2\right)\left(\overline{\alpha} + 2\overline{\beta}\,(ta_1 + i\,((t-1)\,(a_2 + ic_1) - tc_2))\right)\\ &\cdot(\alpha + 2t\beta a_1 - 2i\beta\,((t-1)\,(a_2 - ic_1) - tc_2))\,.\end{aligned}$$

We can now calculate the difference

$$\begin{aligned}\|\dot{p}_1(t)\|^2 - \|\dot{p}_2(t)\|^2 = &2\,(1-2t)\,(a_1 - c_1)\left(\beta\overline{\alpha} + \overline{\beta}\,(\alpha + 2\beta\,(a_1 + c_1))\right)\\ &\cdot\left((a_1 - c_1)^2 + (a_2 - c_2)^2\right).\end{aligned}$$

After integration we arrive at the desired result for the energies,

$$E(p_1) - E(p_2) = \int_0^1 \|\dot{p}_1(t)\|^2 - \|\dot{p}_2(t)\|^2 dt = 0.$$

In the case $\beta = 0$ we have

$$\begin{aligned}\|\dot{p}_1(t)\|^2 &= \alpha\overline{\alpha}\left((a_1 - c_1)^2 + (a_2 - c_2)^2\right) = \|\dot{p}_2(t)\|^2,\\ \|\dot{p}_1(t)\| &= \sqrt{\alpha\overline{\alpha}\left((a_1 - c_1)^2 + (a_2 - c_2)^2\right)} = \|\dot{p}_2(t)\|.\end{aligned}$$

For the lengths follows

$$L(p_1) - L(p_2) = \int_0^1 \|\dot{p}_1(t)\| - \|\dot{p}_2(t)\| dt = \int_0^1 0\ dt = 0.$$

3. The case for φ_+ is similar to φ_- . The only difference is a $90°$ rotation in the pre-image plane.

The proof is now complete. □

Theorem 5.4. *Let $ABCD$ be a rectangle in the complex plane with the diagonals $d_1(t) = A + t(C - A)$ and $d_2(t) = B + t(D - B)$, $t \in [0, 1]$.*
Then, for $\varphi \in \{\varphi_-, \varphi_0, \varphi_+\}$ and $p_1(t) = \varphi(d_1(t))$ and $p_2(t) = \varphi(d_2(t))$ two polygons formed by joining the points $p_1(\frac{k}{m})$ and respectively $p_2(\frac{k}{m})$, where $0 \leq k \leq m$, the energies are equal for each natural number $m > 0$, i.e., $E(p_1, m) = E(p_2, m)$ (see Figure 3).
In the particular cases

1. *$\varphi = \varphi_-$ and $\alpha_-\,\beta_- = 0$,*
2. *$\varphi = \varphi_0$ and $\beta_0 = 0$, and*
3. *$\varphi = \varphi_+$ and $\alpha_+\,\beta_+ = 0$*

also the lengths of the two polygons are equal, i.e., $L(p_1, m) = L(p_2, m)$.

Proof. We set $p_1(t_k) = \varphi(d_1(t_k))$ and $p_2(t_k) = \varphi(d_2(t_k))$. Now we calculate $E(p_1, m) - E(p_2, m)$ and $L(p_1, m) - L(p_2, m)$ and replace t_k by $\frac{k}{m}$, where

$$E(p, m) = \sum_{k=1}^{m} \langle\, p\,(t_k) - p\,(t_{k-1})\,, p\,(t_k) - p\,(t_{k-1})\,\rangle\,,$$
$$L(p, m) = \sum_{k=1}^{m} \sqrt{\langle\, p\,(t_k) - p\,(t_{k-1})\,, p\,(t_k) - p\,(t_{k-1})\,\rangle}.$$

There are three cases $\varphi \in \{\varphi_-, \varphi_0, \varphi_+\}$ to distinguish.

1. The case $\varphi = \varphi_-$ is similar to the case φ_+ which will be proved below. The only difference is a $90°$ rotation in the pre-image plane.

2. Let $\varphi = \varphi_0$, $\alpha = \alpha_0$, $\beta = \beta_0$, and

$$C := -2(a_1 - c_1)((a_1 - c_1)^2 + (a_2 - c_2)^2)(\beta\overline{\alpha} + \overline{\beta}\alpha + 2\beta\overline{\beta}(a_1 + c_1)).$$

Then

$$E(p_1, m) - E(p_2, m) = C \sum_{k=1}^{m} (t_k - t_{k-1})^2 (t_k + t_{k-1} - 1),$$
$$L(p_1, m) - L(p_2, m) = \sqrt{C} \sum_{k=1}^{m} (\sqrt{\ldots} - \sqrt{\ldots}).$$

For $\beta = 0$ follows $C = 0$; therefore the lengths are equal.
For $t_k = \frac{k}{m}$ the difference $t_k - t_{k-1} = \frac{1}{m}$ is the same for all k. Therefore we can write using the constant $C_2 := C\frac{1}{m^2}$

$$E(p_1, m) - E(p_2, m) = C_2 \sum_{k=1}^{m} (t_k + t_{k-1} - 1).$$

We have to show that for $s_k := t_k + t_{k-1} - 1 = \dfrac{2k - 1 - m}{m}$

$$\sum_{k=1}^{m} s_k = 0$$

for all natural numbers $m > 0$. Since $s_k = -s_{m+1-k}$, we are done.

3. Let $\varphi = \varphi_+$, $\alpha = \alpha_+$, $\beta = \beta_+$ and

$$C := 4(\alpha\overline{\alpha}e^{\omega(a_1+c_1)} - \beta\overline{\beta}e^{-\omega(a_1+c_1)}).$$

Then

$$E(p_1, m) - E(p_2, m) = C \sum_{k=1}^{m} (\cos(\omega(a_2 - c_2)(t_k - t_{k-1})) - \cosh(\omega(a_1 - c_1)(t_k - t_{k-1})))$$
$$\cdot \sinh(\omega(a_1 - c_1)(t_k + t_{k-1} - 1)).$$

For $t_k = \frac{k}{m}$ the difference $t_k - t_{k-1} = \frac{1}{m}$ is the same for all k. Therefore we can write using the constant

$$C_2 := C\left(\cos\left(\frac{\omega(a_2-c_2)}{m}\right) - \cosh\left(\frac{\omega(a_1-c_1)}{m}\right)\right)$$

$$E(p_1,m) - E(p_2,m) = C_2 \sum_{k=1}^{m} \sinh(\omega(a_1-c_1)(t_k+t_{k-1}-1)).$$

We have to show that for $s_k := t_k + t_{k-1} - 1 = \frac{2k-1-m}{m}$

$$\sum_{k=1}^{m} \sinh\left(\omega(a_1-c_1)s_k\right) = 0$$

for all natural numbers m. Because of $s_k = -s_{m+1-k}$ and $\sinh(-t) = -\sinh(t)$ we are done with the energies.

For the lengths we have for $\alpha = 0$ with a constant C_3

$$\begin{aligned} L(p_1,m) - L(p_2,m) &= C_3 \sum_{k=1}^{m} \left(\sqrt{\cosh(q_k)+\sinh(q_k)} - \sqrt{\cosh(r_k)+\sinh(r_k)}\right) \\ &= C_3 \sum_{k=1}^{m} T_k \end{aligned}$$

with

$$\begin{aligned} q_k &= \omega\left(a_1\left(\frac{-2m+2k-1}{m}\right) + c_1\left(\frac{1-2k}{m}\right)\right), \\ r_k &= \omega\left(a_1\left(\frac{1-2k}{m}\right) + c_1\left(\frac{-2m+2k-1}{m}\right)\right). \end{aligned}$$

For all $1 \le k \le m$ we have $q_k = r_{m+1-k}$. Therefore the two terms

$$T_k = \sqrt{\cosh(q_k)+\sinh(q_k)} - \sqrt{\cosh(r_k)+\sinh(r_k)}$$

and

$$\begin{aligned} T_{m+1-k} &= \sqrt{\cosh(q_{m+1-k})+\sinh(q_{m+1-k})} - \sqrt{\cosh(r_{m+1-k})+\sinh(r_{m+1-k})} \\ &= \sqrt{\cosh(r_k)+\sinh(r_k)} - \sqrt{\cosh(q_k)+\sinh(q_k)} \end{aligned}$$

cancel each other, and the sum above vanishes.

The case $\beta = 0$ is similar, only a_1 and $-c_1$ get interchanged in the case $\alpha = 0$, and of course C has another value.

The proof is now complete. $\square$

6. Generalizing Ivory's Theorem on holomorphic Liouville curves

In the following theorems in this section we deal with the holomorphic Liouville curve $\varphi : \mathbb{C} \to \mathbb{C}^n$, $0 < n \in \mathbb{N}$, from [1, Theorem 3.1, p. 29],

$$\varphi = \varphi_- + \varphi_0 + \varphi_+ \tag{6.1}$$

where with the unitary basis $(u_1, \ldots, u_n)$ of $\mathbb{C}^n$

$$\varphi_+(z) = \sum_{p=1}^{r} \left(\alpha_p e^{z\omega_p} + \beta_p e^{-z\omega_p}\right) u_p,$$
$$\varphi_-(z) = \sum_{p=r+1}^{r+s} \left(\alpha_p e^{iz\omega_p} + \beta_p e^{-iz\omega_p}\right) u_p,$$
$$\varphi_0(z) = \sum_{p=r+s+1}^{n} z\left(\alpha_p + \beta_p z\right) u_p$$

is the solution of the differential equation

$$\varphi_{zzz} = A\,\varphi_z$$

with a hermitian matrix A. It is necessary to make a minor correction to [1, Proposition 3.3, p. 31] and to [1, Remark 3.5, p. 32], which deal with the isometries of the metric induced by the holomorphic Liouville curve $\varphi\colon \mathbb{C} \to \mathbb{C}^n$. Our proposals are as follows:

Theorem 6.1 (Replacement for Proposition 3.3 in [1], p. 31)**.**
Let $\varphi\colon \mathbb{C} \to \mathbb{C}^n$ be given by (6.1). Then the metric induced on $\mathbb{C}$ admits a family of isometries $\tau(t)$ indexed by the real variable t in one of the following cases:

1. *$r = n$, $s = 0$ and $\alpha_p\beta_p = 0$ for $1 \le p \le n$: Example: $\tau(t)\colon\ z \mapsto z + it$.*
2. *$r = 0$, $s = n$ and $\alpha_p\beta_p = 0$ for $1 \le p \le n$: Example: $\tau(t)\colon\ z \mapsto z + t$.*
3. *$r = 0$, $s = 0$ and $\alpha_p\beta_p = 0$ for $1 \le p \le n$: Example: $\tau(t)\colon z \mapsto e^{it}z$.*

Remark 6.1 (Replacement for Remark 3.5 in [1], p. 32)). With

$$g_t\Big(\sum_{p=1}^{n} \gamma_p u_p\Big) = \begin{cases} \sum_{p=1}^{n} \gamma_p e^{i\Theta_p(t)} u_p, & \text{if } A \text{ is non-singular,} \\ \sum_{p=1}^{n} \gamma_p e^{i\Theta_p(t)} u_p + (\alpha_n b(t))u_n, & \text{if } A \text{ is singular,} \end{cases} \tag{6.2}$$

where we put

$$\Theta_p(t) = \begin{cases} i\,\omega_p b(t), & \text{if } s = 0 \text{ and } \alpha_p = 0, \\ -i\,\omega_p b(t), & \text{if } s = 0 \text{ and } \beta_p = 0, \\ -\omega_p b(t), & \text{if } r = 0 \text{ and } \alpha_p = 0, \\ \omega_p b(t), & \text{if } r = 0 \text{ and } \beta_p = 0, \end{cases} \tag{6.3}$$

we have again three cases:

1. $r = n$, $s = 0$ and $\alpha_p\beta_p = 0$ for $1 \le p \le n:$ g_t as defined by (6.2) for $s = 0$.
2. $r = 0$, $s = n$ and $\alpha_p\beta_p = 0$ for $1 \le p \le n:$ g_t as defined by (6.2) for $r = 0$.
3. $r = 0$, $s = 0$ and $\alpha_p\beta_p = 0$ for $1 \le p \le n:$ $g_t\big(\sum_{p=1}^{n} \gamma_p u_p\big) = \sum_{p=1}^{n} \gamma_p e^{it} u_p$, or, equivalently, $g_t(v) = e^{it} v$ for all $v \in \mathbb{C}^n$.

A direct consequence of Theorem 5.1 is the following

Theorem 6.2 (A generalization of Ivory's Theorem for a Liouville holomorphic curve)**.**
Let $ABCD$ be a rectangle in the complex plane with diagonals $d_1(t) = A + t(C - A)$ and $d_2(t) = B + t(D - B)$, $t \in [0, 1]$. Then for $\varphi : \mathbb{C} \to \mathbb{C}^n$ as defined in (6.1) the images $p_1(t) = \varphi(d_1(t))$ and $p_2(t) = \varphi(d_2(t))$ of the diagonals have the following property:

For the two polygons formed by joining the points $p_1(\frac{k}{m})$ and, respectively, $p_2(\frac{k}{m})$ for $0 \le k \le m$ the energies are equal for each natural number $m > 0$ and each component with $1 \le p \le n$, i.e.,

$$E_p(p_1, m) = E_p(p_2, m) \quad \text{and also} \quad E_p(p_1) = E_p(p_2).$$

When in addition both of the following conditions are satisfied,

1. $\alpha_p \beta_p = 0$ *for* $1 \le p \le r + s$, *and*
2. $\beta_p = 0$ *for* $r + s + 1 \le p \le n$,

then also the lengths of the two polygons are equal, i.e.,

$$L_p(p_1, m) = L_p(p_2, m) \quad \text{and} \quad L_p(p_1) = L_p(p_2).$$

Proof: Apply the one-dimensional case on each component. □

By forming the sum of the components in Theorem 6.2 we obtain the following

Corollary 6.3. *Let $ABCD$ be a rectangle in the complex plane with diagonals $d_1(t) = A + t(C - A)$ and $d_2(t) = B + t(D - B)$, $t \in [0, 1]$. Consider now the curves $p_1(t) = \varphi(d_1(t))$ and $p_2(t) = \varphi(d_2(t))$ with $\varphi\colon \mathbb{C} \to \mathbb{C}^n$ as defined by (6.1).*

Then the two polygons formed by joining the points $p_1(\frac{k}{m})$ and, respectively, $p_2(\frac{k}{m})$ for $0 \le k \le m$ the energies are equal for each natural number $m > 0$, i.e.,

$$E(p_1, m) = \sum_{p=1}^{n} E_p(p_1, m) = \sum_{p=1}^{n} E_p(p_2, m) = E(p_2, m)$$

and

$$E(p_1) = \sum_{p=1}^{n} E_p(p_1) = \sum_{p=1}^{n} E_p(p_2) = E(p_2).$$

In addition, when the following two conditions are satisfied,

1. $\alpha_p \beta_p = 0$ *for* $1 \le p \le r + s$, *and*
2. $\beta_p = 0$ *for* $r + s + 1 \le p \le n$,

the sums of the lengths of the two polygons are equal, too, i.e.,

$$S(p_1, m) = \sum_{p=1}^{n} L_p(p_1, m) = \sum_{p=1}^{n} L_p(p_2, m) = S(p_2, m)$$

and

$$S(p_1) = \sum_{p=1}^{n} L_p(p_1) = \sum_{p=1}^{n} L_p(p_2) = S(p_2).$$

Theorem 6.4. *The two Theorems 6.2 and 6.3 also apply to the minimal Liouville surfaces of [1]. In fact they apply not only to the minimal Liouville surfaces, but more generally to all Liouville surfaces defined as the real part of φ. When the metric of φ admits isometric deformations then the theorems above also hold for the deformed surfaces.*

Proof: The real part is $\frac{1}{2}(\varphi + \overline{\varphi})$ and the derivative with respect to z is $\frac{1}{2}(\varphi_z)$. The line element is therefore $\frac{1}{4}(\varphi_z)^* \varphi_z \, \overline{dz} \, dz$ which is $\frac{1}{4}$ times the line element of φ. Multiplication of both sides by the same constant does not change the property of being equal in the theorems above. An isometric deformation does not change the metric, therefore the formulas in the proofs of the theorems above do not change. □

7. Conclusion

Among the theorems presented in this paper, the Theorems 5.1, 6.2 and 6.3 seem to be new.

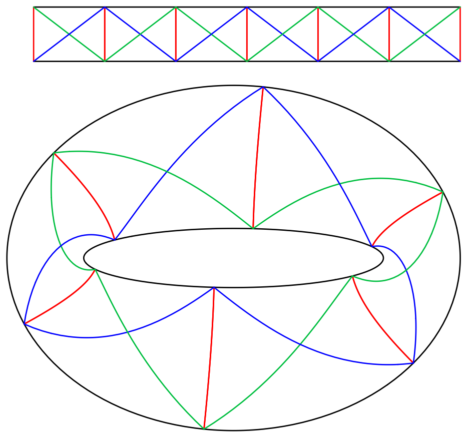

Figure 4: Elliptic billiard as conformal map of rectangular billiard

A plane rectangular billiard is mapped by $\varphi\colon\ \mathbb{C} \to \mathbb{C}$ conformally to a plane *Liouville billiard*. The trajectories in this Liouville billiard are not straight lines, but the wall reflects them correctly. The trajectories seem to bend in a field, where the foci play a special role (as attractors?). Figure 4 shows an example for periodic orbits (blue and green lines) which by our theorems or because of symmetry have the same total energy.

Due to the conformity of $\varphi\colon\ \mathbb{C} \to \mathbb{C}$ the diagonals of the rectangles are mapped onto isogonal trajectories of the parameter curves of the Liouville net. The same holds for the curved billiard in Figure 4.

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