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Asymptotic Behaviour of the Maximum Curvature of Lamé Curves

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Abstract. The curve $\left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p = 1$ for $a, b, p > 0$ in the xy-plane is called a Lamé curve. It is also known as a superellipse and is one of the symbols of Scandinavian design. For fixed a and b, the above curve expands as p increases and shrinks as p decreases. The curve converges to a rectangle as $p \to \infty$, while it converges to a cross shape as $p \to 0^+$. In general, if $p > 2$, Lamé curves have shapes which lie between ellipses and rectangles. From the viewpoint of application, one of the fundamental problems is to detect the "optimal" value of the exponent p which creates the "most refined" shape. With this in mind, we closely examine how the curvature of Lamé curves depends on p. In particular, we derive an explicit expression of the asymptote of the maximum curvature, which is the main result of this paper.

Key Words: Lamé curve, superellipse, curvature, maximum curvature

MSC 2010: 53A04

1. Introduction

In this paper, we investigate the curve

$$\left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p = 1 \quad (a, b, p > 0), \tag{1}$$

in the xy-plane, where a, b and p are constants. This curve was analyzed for the first time in 1818 by the French mathematician Gabriel LAMÉ [7]. Therefore it is called a *Lamé curve*. On the other hand, in the middle of the 20th century, the Danish scientist and artist Piet HEIN suggested a Lamé curve, which he called a *superellipse*, for the roundabout at *Sergels torg* (Sergel's Square) in central Stockholm [2, 3, 4, 5, 6]. Since then, Lamé curves (superellipses) have become one of the symbols of Scandinavian design.

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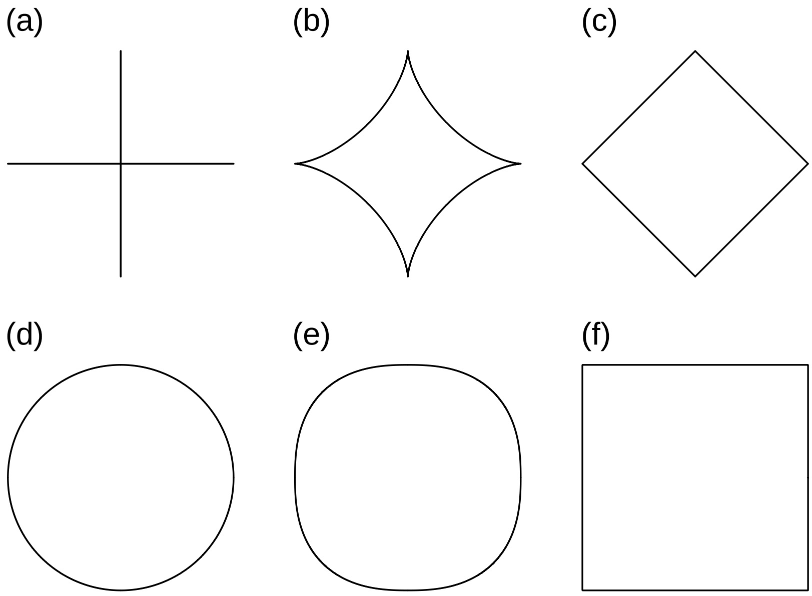

Figure 1: Examples of Lamé curves. In all cases holds $a = b$. The parameter p is chosen as follows: (a) $p = \frac{1}{20}$, (b) $p = \frac{2}{3}$, (c) $p = 1$, (d) $p = 2$, (e) $p = \frac{5}{2}$, (f) $p = 1000$.

As Figure 1 suggests, a Lamé curve expands as p increases and shrinks as p decreases. The curve converges, with respect to the Hausdorff distance, to a rectangle and a cross shape as $p \to \infty$ and $p \to 0^+$, respectively. We note that circles, ellipses, squares, rectangles, rhombuses and cross shapes are all special cases of Lamé curves [1].

We are especially interested in the case $p \geq 2$. Therefore, in what follows, let $p \geq 2$. In this case, Lamé curves have shapes which lie between ellipses and rectangles. In the context of the application to design engineering, one of the most challenging problems is to identify the "optimal" value of p for each given a and b. Of course, the criterion of optimality depends on the aim of the design, and thus the optimal exponent is not uniquely determined.

So far, several values have been proposed as a candidate for the best exponent. For example, in [5], GRIDGEMAN writes as follows (pp. 34–35):

> *"It is sometimes said that a superellipse is a nice compromise between an ellipse and the framing rectangle, and this suggests that the superellipse with exponent* $n = 3.1620380$ *has a special attraction, for it divides the area between the ellipse and the rectangle into two equal parts."*

(Note that n here corresponds to p in our notation.)

As for the architectural design of the roundabout at *Sergels torg*, Piet HEIN proposed a Lamé curve with $p = \frac{5}{2}$. It is said that with the help of a computer, he found that the exponent $p = \frac{5}{2}$ was most pleasing to the eye [3, 4]. From the practical point of view, if the curvature of the roundabout is too large, it hinders safe and smooth traffic flow. Therefore, it is quite reasonable to choose p so that the maximum value (denoted by $\kappa_{\max}(p)$) of the curvature becomes as small as possible[1]. For this reason, we examine in this paper the details of $\kappa_{\max}(p)$.

The structure of this paper is as follows: At the beginning of Section 2, we quickly review the derivation of the formula for the curvature of Lamé curves. Then, based on this formula,

[1]We choose the orientation of the curve in such a way that the curvature becomes nonnegative.

we closely investigate the behaviour of the maximum curvature $\kappa_{\max}(p)$. Our main result is Theorem 3, which gives an explicit expression of the asymptote of $\kappa_{\max}(p)$ as $p \to \infty$. In Section 3, we give three types of lower bounds for $\kappa_{\max}(p)$. At this point, we have not yet obtained an analytical estimation of the "optimal" exponent $p^* = \arg\min_p \kappa_{\max}(p)$. Hence, we conclude this paper with a comment on future studies, together with a numerical result.

2. Asymptotic behaviour of the maximum curvature

As noted in the previous section, let $p \geq 2$ in (1). Moreover, without loss of generality, we can assume $a \geq b$. Taking account of the symmetry of the curve, it suffices to treat the case where $x \geq 0$ and $y \geq 0$. In this case, the curve (1) is represented by the parametric equation

$$\begin{cases} x(t) = a\cos^{\frac{2}{p}} t \\ y(t) = b\sin^{\frac{2}{p}} t \end{cases} \qquad \left(0 \leq t \leq \frac{\pi}{2}\right). \tag{2}$$

Then, the curvature of the curve (2) is given as follows.

Theorem 1. *The curvature $\kappa(p,t)$ of (2) at $(x(t), y(t))$, $\left(0 \leq t \leq \frac{\pi}{2}\right)$, is given by*

$$\kappa(p,t) = \frac{ab(p-1)\sin^{\xi} t \cos^{\xi} t}{\left(a^2 \sin^{\xi+2} t + b^2 \cos^{\xi+2} t\right)^{\frac{3}{2}}}, \quad \textit{where} \quad \xi = \frac{2(p-2)}{p}. \tag{3}$$

Remark 1. As noted before, p is constant. In what follows, however, we examine how the curvature of (2) depends on p. Hence, we denote it by $\kappa(p,t)$ rather than $\kappa(t)$. For $p = 2$ and $t = 0, \frac{\pi}{2}$, (3) contains 0^0 . It is interpreted as

$$\kappa(2,t) = \frac{ab}{\left(a^2 \sin^2 t + b^2 \cos^2 t\right)^{\frac{3}{2}}} \qquad \left(0 \leq t \leq \frac{\pi}{2}\right), \tag{4}$$

and therefore,

$$\kappa(2,0) = \frac{a}{b^2}, \quad \kappa\left(2, \frac{\pi}{2}\right) = \frac{b}{a^2}.$$

Proof of Theorem 1. Although this is a known fact, we include the proof for self-containedness, because this theorem plays a fundamental role in this paper. Put

$$f(x,y) = \left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p - 1.$$

For $x \geq 0$, $y \geq 0$, it can be rewritten as

$$f(x,y) = \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^p - 1.$$

Hence,

$$\begin{cases} f_x(x,y) = \dfrac{p}{a}\cos^{\frac{\xi}{2}+1} t, & f_{xx}(x,y) = \dfrac{p(p-1)}{a^2}\cos^{\xi} t, & f_{xy}(x,y) = 0, \\ f_y(x,y) = \dfrac{p}{b}\sin^{\frac{\xi}{2}+1} t, & f_{yy}(x,y) = \dfrac{p(p-1)}{b^2}\sin^{\xi} t, & f_{yx}(x,y) = 0. \end{cases} \tag{5}$$

Here, we note that (5) holds for $0 \le t \le \frac{\pi}{2}$ including $t = 0, \frac{\pi}{2}$. Substituting these relations into the formula

$$\kappa = \frac{\left|-f_{xx}f_y^2 - f_{yy}f_x^2 + f_{xy}f_xf_y + f_{yx}f_yf_x\right|}{\left(f_x^2 + f_y^2\right)^{\frac{3}{2}}},$$

we have the proof. □

For the sake of convenience, we rewrite $\kappa(p,t)$ in terms of $z = \tan^2 t$.

Corollary 1. *The curvature $\kappa(p,t)$ is rewritten as*

$$\kappa(p,t) = \begin{cases} ab(p-1)\left[\dfrac{(1+z)^{u+v}}{a^2z^u + b^2z^v}\right]^{\frac{3}{2}} & \text{for } p \ge 2,\ 0 < t < \dfrac{\pi}{2}, \\ 0 & \text{for } p > 2,\ t = 0, \dfrac{\pi}{2}, \\ \dfrac{a}{b^2} & \text{for } p = 2,\ t = 0, \\ \dfrac{b}{a^2} & \text{for } p = 2,\ t = \dfrac{\pi}{2}, \end{cases} \tag{6}$$

where

$$z = \tan^2 t, \quad u = \frac{2}{3}\left(2 - \frac{1}{p}\right), \quad v = \frac{2}{3}\left(\frac{2}{p} - 1\right). \tag{7}$$

In particular,

$$\kappa(2,t) = \frac{b}{a^2}\left(1 + \frac{a^2 - b^2}{a^2z + b^2}\right)^{\frac{3}{2}} \quad \left(0 \le t < \frac{\pi}{2}\right). \tag{8}$$

Proof: Let $0 < t < \frac{\pi}{2}$. From Theorem 1 follows

$$\kappa(p,t) = \frac{ab(p-1)}{\left(a^2\sin^{\frac{\xi}{3}+2} t\cos^{-\frac{2\xi}{3}} t + b^2\cos^{\frac{\xi}{3}+2} t\sin^{-\frac{2\xi}{3}} t\right)^{\frac{3}{2}}}. \tag{9}$$

Here we note that

$$a^2\sin^{\frac{\xi}{3}+2} t\cos^{-\frac{2\xi}{3}} t + b^2\cos^{\frac{\xi}{3}+2} t\sin^{-\frac{2\xi}{3}} t = \frac{a^2z^u + b^2z^v}{(1+z)^{u+v}}. \tag{10}$$

Substituting (10) into (9), we complete the proof of (6) for $0 < t < \frac{\pi}{2}$. The rest is easily verified. □

Theorem 1 shows that $\kappa(p,t)$ is continuous on $\left[0, \frac{\pi}{2}\right]$ as a function of t. Therefore, we can define a function $\kappa_{\max}(p)$ as follows.

Definition 1. For each $p \ge 2$, we denote by $\kappa_{\max}(p)$ the maximum value of $\kappa(p,t)$, that is,

$$\kappa_{\max}(p) = \max_{0 \le t \le \frac{\pi}{2}} \kappa(p,t) \quad (p \ge 2).$$

We call $\kappa_{\max}(p)$ the *maximum curvature*. Moreover, we put $c(p,z) = \kappa(p,t)$, that is,

$$c(p,z) = ab(p-1)\left[\frac{(1+z)^{u+v}}{a^2z^u + b^2z^v}\right]^{\frac{3}{2}}. \tag{11}$$

To examine the behaviour of $\kappa_{\max}(p)$, we prove some lemmas.

Lemma 1. *Let $p > 2$ be fixed. Then, there exists a unique $\alpha(p) > 0$ such that*

$$\kappa_{\max}(p) = c\,(p, \alpha(p)) = \kappa\left(p, \arctan\sqrt{\alpha(p)}\right).$$

In other words, $\alpha(p) = \arg\max_z c(p, z)$. Moreover, $z = \alpha(p)$ is the unique solution to

$$k(p, z) = a^2 v z^u + b^2 u z^v - a^2 u z^{u-1} - b^2 v z^{v-1} = 0 \quad (z > 0). \tag{12}$$

Proof: To maximize $\kappa(p, t)$, we only have to maximize

$$h(p, z) = \frac{(1+z)^{u+v}}{a^2 z^u + b^2 z^v}. \tag{13}$$

Here,

$$\frac{\partial h}{\partial z}(p, z) = \frac{(1+z)^{u+v-1} k(p, z)}{(a^2 z^u + b^2 z^v)^2} \quad (z > 0). \tag{14}$$

Therefore, the sign of $\dfrac{\partial h}{\partial z}(p, z)$ coincides with that of $k(p, z)$. Now, we differentiate $k(p, z)$ to obtain

$$\frac{\partial k}{\partial z}(p, z) = a^2 uv z^{u-1} + b^2 uv z^{v-1} - a^2 u(u-1) z^{u-2} - b^2 v(v-1) z^{v-2}.$$

Noting (7) and the assumption $p > 2$, we know that

$$u > 0, \quad u - 1 > 0, \quad v < 0, \quad v - 1 < 0 \tag{15}$$

and therefore, $uv < 0$, $-u(u-1) < 0$, $-v(v-1) < 0$. Thus, we have

$$\frac{\partial k}{\partial z}(p, z) < 0. \tag{16}$$

In other words, $k(p, z)$ is monotonically decreasing as a function of z. Furthermore, noting (15) again, we observe that $\lim_{z \to 0^+} k(p, z) = \infty$ and $\lim_{z \to \infty} k(p, z) = -\infty$. From these results, we know that there exists a unique $\alpha(p) > 0$ such that

$$k(p, z) > 0 \quad (z < \alpha(p)), \quad k(p, \alpha(p)) = 0, \quad k(p, z) < 0 \quad (z > \alpha(p)).$$

Hence, from (14), we know that $h(p, z)$ has the maximum value at $z = \alpha(p)$. Moreover, from (7), $z = \alpha(p)$ implies $t = \arctan\sqrt{\alpha(p)}$. This completes the proof. □

For simplicity, we sometimes write α instead of $\alpha(p)$. In what follows, unless stated otherwise, we assume $a > b$.

Lemma 2. *Let $p > 2$. Then α satisfies*

$$\left(\frac{b}{a}\right)^{\eta} < \alpha < \left(\frac{b}{a}\right)^{\lambda} < 1, \quad \textit{where} \quad \eta = \frac{2p}{p-2}, \quad \lambda = \frac{p}{p-1}. \tag{17}$$

Proof: Noting $v < 0$, $u + v > 0$, $\eta > 0$, $\lambda > 0$ and $0 < b < a$, we observe that

$$k\left(p, \left(\frac{b}{a}\right)^{\eta}\right) = ab\left(\frac{b}{a}\right)^{-\frac{1}{3}} v\left[\left(\frac{b}{a}\right)^{\eta} - \left(\frac{b}{a}\right)^{-\eta}\right] > 0, \tag{18}$$

$$k\left(p, \left(\frac{b}{a}\right)^{\lambda}\right) = ab\left(\frac{b}{a}\right)^{-\frac{1}{3}\left(1-\frac{\lambda}{2}\right)} (u+v)\left[\left(\frac{b}{a}\right)^{\frac{\lambda}{2}} - \left(\frac{b}{a}\right)^{-\frac{\lambda}{2}}\right] < 0. \tag{19}$$

Since $k(p, z)$ is monotonically decreasing, (18) and (19) imply (17). □

Lemma 3. *The function $\alpha(p)$ has the following properties:*

(i) *$\alpha(p)$ is differentiable on $(2, \infty)$.*

(ii) *$\alpha(p)$ is monotonically increasing on $(2, \infty)$.*

(iii) $\lim_{p \to 2^+} \alpha(p) = 0$.

(iv) *$\alpha(p)$ is right-continuous at $p = 2$, where $\alpha(2) = \arg\max_z c(2, z)$.*

(v) $\lim_{p \to 2^+} \alpha(p)^v = 1$.

Proof: Let $p > 2$. We note that $k(p, \alpha) = 0$. Moreover, from (16), $\frac{\partial k}{\partial z}(p, \alpha) \neq 0$. Hence, $\alpha(p)$ is differentiable and

$$\alpha'(p) = -\frac{\partial k}{\partial p}(p, \alpha) \Big/ \frac{\partial k}{\partial z}(p, \alpha). \tag{20}$$

Further,

$$\frac{\partial k}{\partial p}(p, \alpha) = I + II,$$

where

$$I = a^2 v' \alpha^u + b^2 u' \alpha^v - a^2 u' \alpha^{u-1} - b^2 v' \alpha^{v-1},$$
$$II = \log\alpha \left(a^2 v u' \alpha^u + b^2 u v' \alpha^v - a^2 u u' \alpha^{u-1} - b^2 v v' \alpha^{v-1}\right).$$

Since

$$u > 0, \quad v < 0, \quad u' = \frac{2}{3p^2} > 0, \quad v' = -\frac{4}{3p^2} < 0, \quad \alpha < 1,$$

we know that

$$\log\alpha < 0, \quad a^2 v u' \alpha^u < 0, \quad b^2 u v' \alpha^v < 0, \quad -a^2 u u' \alpha^{u-1} < 0, \quad -b^2 v v' \alpha^{v-1} < 0.$$

This gives $II > 0$. Moreover, by (17),

$$I = \frac{2b^2\alpha^{v-1}}{3p^2}\left[(\alpha+2) - (2\alpha+1)\left(\frac{a}{b}\right)^2 \alpha^{\frac{2(p-1)}{p}}\right]$$
$$> \frac{2b^2\alpha^{v-1}}{3p^2}\left[(\alpha+2) - (2\alpha+1)\left(\frac{a}{b}\right)^2 \left(\frac{b}{a}\right)^{\frac{p}{p-1}\cdot\frac{2(p-1)}{p}}\right] = \frac{2b^2\alpha^{v-1}}{3p^2}(1-\alpha) > 0.$$

Therefore, $\frac{\partial k}{\partial p}(p, \alpha) > 0$. Noting (20) and $\frac{\partial k}{\partial z}(p, \alpha) < 0$, we obtain $\alpha'(p) > 0$. Thus, (i) and (ii) are proved.

We now prove (iii): For any $p > 2$, we have $k(p, \alpha) = 0$. Hence, $\lim_{p\to 2^+} k(p, \alpha) = 0$. On the other hand, since $\alpha(p)$ is monotonic and bounded, it converges as $p \to 2^+$. Assume that $\lim_{p\to 2^+} \alpha \neq 0$. Then, noting that $\lim_{p\to 2^+} u = 1$ and $\lim_{p\to 2^+} v = 0$, we obtain

$$\lim_{p\to 2^+} k(p, \alpha) = \lim_{p\to 2^+} \left(a^2 v\alpha^u + b^2 u\alpha^v - a^2 u\alpha^{u-1} - b^2 v\alpha^{v-1}\right) = b^2 - a^2 \neq 0.$$

This implies a contradiction. Therefore we have (iii). Moreover, it is easily verified from (8) that $\alpha(2) = 0$, which gives (iv).

From (iii), $\lim_{p\to 2^+} \left(b^2 u\alpha^v - a^2 u\alpha^{u-1} - b^2 v\alpha^{v-1}\right) = \lim_{p\to 2^+} k(p, \alpha) = 0$. So, there exists $p_0 > 2$ such that $b^2 u\alpha^v - a^2 u\alpha^{u-1} - b^2 v\alpha^{v-1} < 1 \quad (2 < p < p_0)$. Hence, if $2 < p < p_0$, then

$$-v\alpha^{v-1} < \frac{1 + a^2 u\alpha^{u-1}}{b^2} - u\alpha^v < \frac{1 + a^2 u}{b^2} < M,$$

where $M = \frac{1}{b^2}\left(1 + \frac{4a^2}{3}\right)$. This means

$$\left(-\frac{v}{M}\right)^{\frac{1}{1-v}} < \alpha < 1.$$

Thus,

$$|\log \alpha^v| = |v \log \alpha| < \left| v \log \left(-\frac{v}{M}\right)^{\frac{1}{1-v}} \right| = \frac{M}{1-v} \left| \left(-\frac{v}{M}\right) \log \left(-\frac{v}{M}\right) \right|.$$

Noting that $\lim_{p\to 2^+} \left(-\frac{v}{M}\right) = 0$, we have $\lim_{p\to 2^+} |\log \alpha^v| = 0$. This gives (v). □

Lemma 4. *Let $p > 2$. Then,*

$$\kappa'_{\max}(p) = \frac{\kappa_{\max}(p)}{p^2} \left[\frac{p^2}{p-1} - \log(1 + \alpha) - \frac{(a^2\alpha^u - 2b^2\alpha^v) \log \alpha}{a^2\alpha^u + b^2\alpha^v} \right].$$

Proof: Since $z = \alpha$ gives the maximum value of $c(p, z)$, we have $\frac{\partial c}{\partial z}(p, \alpha) = 0$. Therefore,

$$\kappa'_{\max}(p) = \frac{\partial c}{\partial p}(p, \alpha) + \frac{\partial c}{\partial z}(p, \alpha)\frac{d\alpha}{dp} = abh(p, \alpha)^{\frac{3}{2}} + ab(p-1)\frac{\partial}{\partial p}h(p, \alpha)^{\frac{3}{2}}. \tag{21}$$

Here, from (13),

$$\frac{\partial}{\partial p}h(p, \alpha)^{\frac{3}{2}} = -\frac{h(p, \alpha)^{\frac{3}{2}}}{p^2} \left[\log(1 + \alpha) + \frac{(a^2\alpha^u - 2b^2\alpha^v) \log \alpha}{a^2\alpha^u + b^2\alpha^v} \right]. \tag{22}$$

Substituting (22) into (21), we complete the proof. □

The following theorem shows how the maximum curvature $\kappa_{\max}(p)$ behaves at and near $p = 2$.

Theorem 2. *$\kappa_{\max}(p)$ has the following properties:*

(i) *$\kappa_{\max}(p)$ is right-continuous at $p = 2$. Thus, $\kappa_{\max}(p)$ is continuous on $[2, \infty)$.*

(ii) *The graph of $\kappa_{\max}(p)$ is tangent to the line $p = 2$ at $\left(2, \frac{a}{b^2}\right)$.*

Proof: From (11) and Lemma 3, we have

$$\lim_{p\to 2^+} \kappa_{\max}(p) = \lim_{p\to 2^+} c(p, \alpha) = ab\left(\frac{1}{b^2}\right)^{\frac{3}{2}} = \frac{a}{b^2}.$$

On the other hand, $\kappa_{\max}(2) = \kappa(2, 0) = \frac{a}{b^2}$. This gives (i).

Since $\kappa_{\max}(p)$ is continuous on $[2, \infty)$ and differentiable on $(2, \infty)$, the right derivative $(\kappa_{\max})'_+(2)$ is calculated as $(\kappa_{\max})'_+(2) = \lim_{p\to 2^+} \kappa'_{\max}(p) = -\infty$. Here we used Lemmas 3 and 4. Hence, we have (ii). □

The next aim is to examine the asymptotic behaviour of $\kappa_{\max}(p)$ as $p \to \infty$. Before stating the main theorem, we give an auxiliary lemma on $\alpha(p)$.

Lemma 5. *The function $\alpha(p)$ has the following properties:*

(i) *$\alpha(p)$ converges as $p \to \infty$ and the limit $\alpha_* = \lim_{p\to\infty} \alpha(p)$ satisfies*

$$0 < \left(\frac{b}{a}\right)^2 \le \alpha_* \le \frac{b}{a} < 1.$$

(ii) *$z = \alpha_*$ is the unique solution to*

$$a^2z^3 + 2a^2z^2 - 2b^2z - b^2 = 0 \quad (z > 0). \tag{23}$$

(iii) *α_* is given by*

$$\alpha_* = \frac{2\sqrt{2}}{3a}\sqrt{2a^2 + 3b^2}\cos\left(\frac{1}{3}\arccos\left(-\frac{a}{4\sqrt{2}}\frac{16a^2 + 9b^2}{(2a^2 + 3b^2)^{\frac{3}{2}}}\right)\right) - \frac{2}{3}.$$

Proof: (i) follows from (17) and Lemma 3 (ii).

Furthermore, since $k(p, \alpha) = 0$, we have $\lim_{p\to\infty} k(p, \alpha) = 0$. On the other hand, from (12),

$$\lim_{p\to\infty} k(p, \alpha) = -\frac{2}{3}\alpha_*^{-\frac{5}{3}}\left(a^2\alpha_*^3 + 2a^2\alpha_*^2 - 2b^2\alpha_* - b^2\right).$$

Therefore, we obtain $a^2\alpha_*^3 + 2a^2\alpha_*^2 - 2b^2\alpha_* - b^2 = 0$. Hence, α_* is a solution to (23). Now, we show that (23) has only one solution. To do this, put

$$G(z) = \frac{a^2z^3 + 2a^2z^2 - 2b^2z - b^2}{z} = a^2z^2 + 2a^2z - 2b^2 - \frac{b^2}{z} \quad (z > 0).$$

Then,

$$G'(z) = 2a^2z + 2a^2 + \frac{b^2}{z^2} > 0.$$

Therefore, $G(z)$ is monotonically increasing. This means that the equation $G(z) = 0$ (therefore (23)) has no more than one solution. Thus, we complete the proof of (ii).

Put $w = z + \frac{2}{3}$ in (23). Then the equation is rewritten as

$$w^3 = 2\left(\frac{2}{3} + \frac{b^2}{a^2}\right)w - \frac{1}{3}\left(\frac{16}{9} + \frac{b^2}{a^2}\right).$$

Here, we remind that one of the solutions to the cubic equation

$$w^3 = mw + n \quad (m, n \in \mathbb{R}) \tag{24}$$

is given by

$$w = 2\sqrt{\frac{m}{3}}\cos\left(\frac{1}{3}\arccos\frac{3n}{2m}\sqrt{\frac{3}{m}}\right),$$

provided that

$$\left|\frac{3n}{2m}\sqrt{\frac{3}{m}}\right| \leq 1. \tag{25}$$

With this in mind, put

$$m = 2\left(\frac{2}{3} + \frac{b^2}{a^2}\right), \quad n = -\frac{1}{3}\left(\frac{16}{9} + \frac{b^2}{a^2}\right).$$

Then,

$$\frac{3n}{2m}\sqrt{\frac{3}{m}} = -\frac{16 + 9\left(\frac{b}{a}\right)^2}{4\sqrt{2}\left[2 + 3\left(\frac{b}{a}\right)^2\right]^{\frac{3}{2}}}.$$

To show that (25) is satisfied, we define $l(x)$ by

$$l(x) = \frac{9x + 16}{4\sqrt{2}\,(3x + 2)^{\frac{3}{2}}} \quad (x \geq 0).$$

Then, $l(x) > 0$, $l(0) = 1$ and $l'(x) < 0$. Therefore, $0 < l(x) < 1 \ \ (x > 0)$, which means

$$-1 < \frac{3n}{2m}\sqrt{\frac{3}{m}} < 0. \tag{26}$$

This gives (25). Thus,

$$\begin{aligned} w_* &= 2\sqrt{\frac{m}{3}}\cos\left(\frac{1}{3}\arccos\frac{3n}{2m}\sqrt{\frac{3}{m}}\right) \\ &= \frac{2\sqrt{2}}{3a}\sqrt{2a^2 + 3b^2}\cos\left(\frac{1}{3}\arccos\left(-\frac{a}{4\sqrt{2}}\frac{16a^2 + 9b^2}{(2a^2 + 3b^2)^{\frac{3}{2}}}\right)\right) \end{aligned}$$

is a solution to (24). Moreover, by (26),

$$\frac{\pi}{2} < \arccos\frac{3n}{2m}\sqrt{\frac{3}{m}} < \pi.$$

Hence, we know that

$$w_* > 2\sqrt{\frac{m}{3}} \cdot \frac{1}{2} = \sqrt{\frac{2}{3}\left(\frac{2}{3} + \frac{b^2}{a^2}\right)} > \frac{2}{3}.$$

Therefore, $w_* - \frac{2}{3}$ is the unique positive solution to (23), which means (iii). □

We are now in a position to state the main result of this paper.

Theorem 3. *The asymptote of $\kappa_{\max}(p)$ as $p \to \infty$ is calculated as*

$$\frac{ab\left(\alpha_*^{\frac{1}{2}} + \alpha_*^{-\frac{1}{2}}\right)}{\left(a^2\alpha_* + b^2\alpha_*^{-1}\right)^{\frac{3}{2}}}\left[p - 1 + \log(1+\alpha_*) + \frac{(a^2\alpha_* - 2b^2\alpha_*^{-1})\log\alpha_*}{a^2\alpha_* + b^2\alpha_*^{-1}}\right],$$

where α_* is given by Lemma 5 (iii).

Proof: Noting

$$\frac{\partial}{\partial z} h(p,\alpha)^{\frac{3}{2}} = \frac{1}{ab(p-1)}\frac{\partial c}{\partial z}(p,\alpha) = 0$$

and the relation (22), we obtain

$$\begin{aligned}
&\lim_{p\to\infty}\left[\kappa_{\max}(p) - \frac{ab(1+\alpha_*)}{\left(a^2\alpha_*^{\frac{4}{3}} + b^2\alpha_*^{-\frac{2}{3}}\right)^{\frac{3}{2}}}(p-1)\right] \\
&= \lim_{p\to\infty} ab\left\{h(p,\alpha)^{\frac{3}{2}} - \left[\frac{(1+\alpha_*)^{\frac{2}{3}}}{a^2\alpha_*^{\frac{4}{3}} + b^2\alpha_*^{-\frac{2}{3}}}\right]^{\frac{3}{2}}\right\}\Big/\left(\frac{1}{p-1}\right) \\
&= ab\lim_{p\to\infty}\frac{d}{dp}h(p,\alpha)^{\frac{3}{2}}\Big/\frac{d}{dp}\left(\frac{1}{p-1}\right) \\
&= -ab\lim_{p\to\infty}(p-1)^2\left[\frac{\partial}{\partial p}h(p,\alpha)^{\frac{3}{2}} + \frac{\partial}{\partial z}h(p,\alpha)^{\frac{3}{2}}\frac{d\alpha}{dp}\right] \\
&= ab\lim_{p\to\infty}\frac{(p-1)^2}{p^2}h(p,\alpha)^{\frac{3}{2}}\left[\log(1+\alpha) + \frac{(a^2\alpha^u - 2b^2\alpha^v)\log\alpha}{a^2\alpha^u + b^2\alpha^v}\right] \\
&= \frac{ab(1+\alpha_*)}{\left(a^2\alpha_*^{\frac{4}{3}} + b^2\alpha_*^{-\frac{2}{3}}\right)^{\frac{3}{2}}}\left[\log(1+\alpha_*) + \frac{\left(a^2\alpha_*^{\frac{4}{3}} - 2b^2\alpha_*^{-\frac{2}{3}}\right)\log\alpha_*}{a^2\alpha_*^{\frac{4}{3}} + b^2\alpha_*^{-\frac{2}{3}}}\right].
\end{aligned}$$

This gives the proof. □

In the case $a = b$, we can explicitly calculate $\kappa_{\max}(p)$.

Theorem 4. *Let $a = b$.*

(i) *$\alpha(2)$ is not uniquely determined. In fact, $c(2,z) = \frac{1}{a} = \kappa_{\max}(2)$ $(z \geq 0)$ (cf. Lemma 3 (iv)).*

(ii) *$\kappa_{\max}(p) = c(p,1) = \kappa\left(p, \frac{\pi}{4}\right) = \frac{p-1}{a}\,2^{\frac{2-p}{2p}}$ (cf. Lemma 1).*

(iii) *$\kappa_{\max}(p)$ is continuous on $[2,\infty)$. In particular, $\kappa_{\max}(p)$ is right-continuous at $p = 2$ (cf. Theorem 2 (i)).*

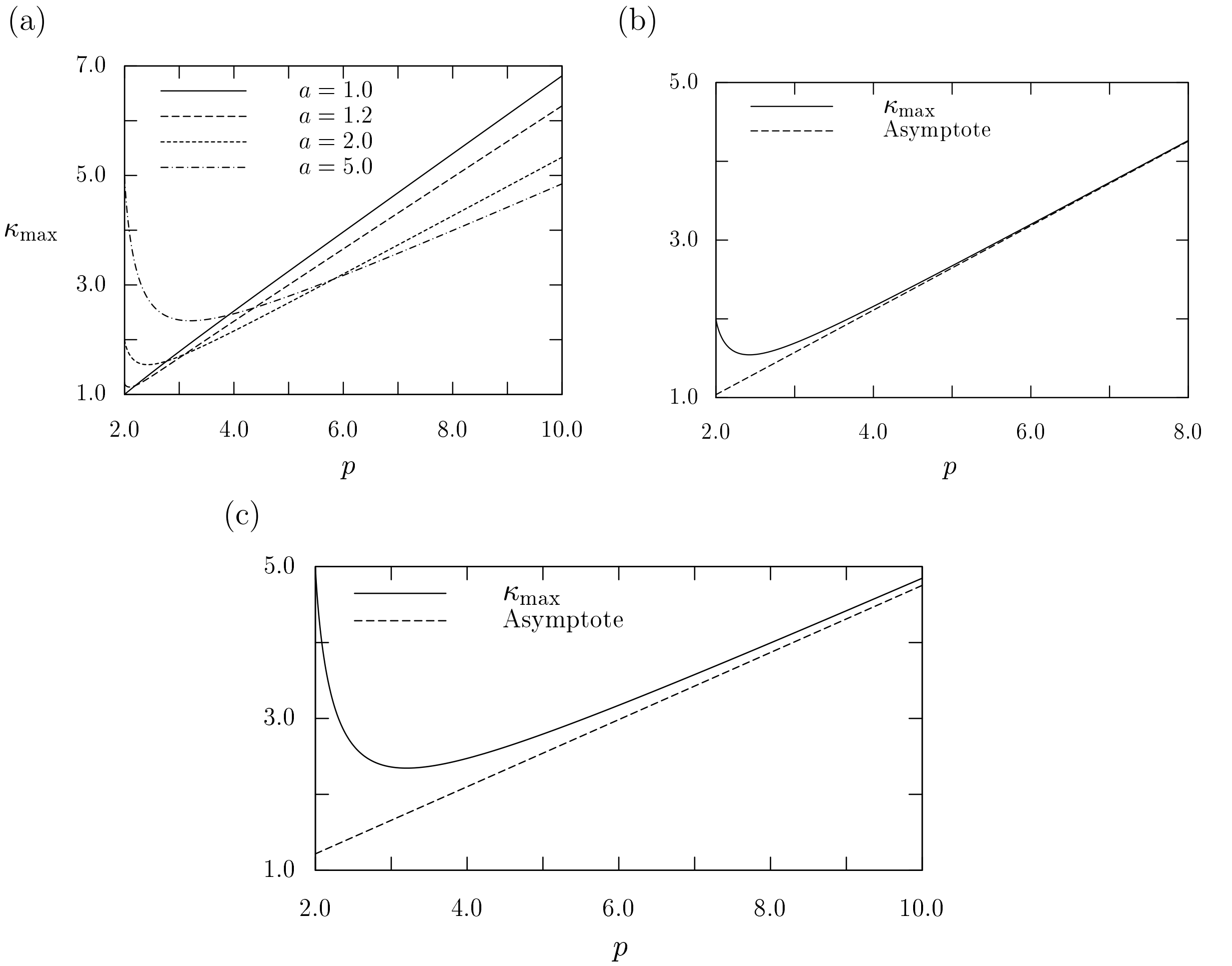

Figure 2: Graphs of $\kappa_{\max}(p)$. In all cases, $b = 1$. (a) $\kappa_{\max}(p)$ for $a = 1.0, 1.2, 2.0, 5.0$; (b) $\kappa_{\max}(p)$ and its asymptote for $a = 2.0$; (c) $\kappa_{\max}(p)$ and its asymptote for $a = 5.0$.

(iv) *The graph of* $\kappa_{\max}(p)$ *is tangent to the line* $\dfrac{4 - \log 2}{4a}(p-2) + \dfrac{1}{a}$ *at* $\left(2, \dfrac{1}{a}\right)$ *(cf. Theorem 2 (ii)).*

(v) *The asymptote of* $\kappa_{\max}(p)$ *as* $p \to \infty$ *is given by* $\dfrac{1}{\sqrt{2}a}(p - 1 + \log 2)$ *(cf. Theorem 3).*

Proof: (i) is obvious (cf. (4)). Let $p > 2$. Then, from (12), $k(p, 1) = (a^2 - b^2)(v - u) = 0$, which means $\alpha(p) = 1$ and $\kappa_{\max}(p) = c(p, 1)$. Thus, substituting $z = 1$ into (11), we have (ii) for $p > 2$. Moreover, (i) implies that (ii) also holds for $p = 2$. (iii) immediately follows from (ii). (iv) is easily verified from (ii). Finally, we can prove (v) in the same way as in Theorem 3. □

At the end of this section, we show in Figure 2 some examples of the graphs of $\kappa_{\max}(p)$ and their asymptotes. The values of $\kappa_{\max}(p)$ are obtained by numerical calculation while the asymptotes are given by Theorem 3.

3. Lower bounds for the maximum curvature

Figure 2 in Section 2 suggests that $\kappa_{\max}(p)$ is well approximated by its asymptote if p is sufficiently large. However, the asymptote does not provide enough information for smaller

p. Therefore, the purpose here is to derive efficient lower bounds for $\kappa_{\max}(p)$.

Throughout this section, let $p > 2$ and $a > b$. Taking (17) into account, we prove the following theorem.

Theorem 5. *$\kappa_{\max}(p)$ satisfies*

$$\kappa_{\max}(p) > \frac{p-1}{2\sqrt{2}a^{\lambda}b^{\lambda}}\left(a^{\lambda}+b^{\lambda}\right)^{2-\frac{1}{\lambda}}, \quad \text{where} \quad \lambda = \frac{p}{p-1}. \tag{LB1}$$

Proof: Note that

$$c\left(p, \left(\frac{b}{a}\right)^{\lambda}\right) = \frac{p-1}{2\sqrt{2}a^{\lambda}b^{\lambda}}\left(a^{\lambda}+b^{\lambda}\right)^{2-\frac{1}{\lambda}}.$$

Since only $z = \alpha$ gives the maximum value $\kappa_{\max}(p)$ of $c(p, z)$, we have the proof. □

In the same way as in Theorem 5, we obtain

$$\kappa_{\max}(p) > \frac{p-1}{(a^{\eta}+b^{\eta})^{\frac{1}{\eta}}}, \quad \text{where} \quad \eta = \frac{2p}{p-2}, \tag{27}$$

because

$$c\left(p, \left(\frac{b}{a}\right)^{\eta}\right) = \frac{p-1}{(a^{\eta}+b^{\eta})^{\frac{1}{\eta}}}.$$

However, the next proposition indicates that (27) does not improve LB1 in Theorem 5.

Proposition 1. *The following inequality holds.*

$$\frac{p-1}{2\sqrt{2}a^{\lambda}b^{\lambda}}\left(a^{\lambda}+b^{\lambda}\right)^{2-\frac{1}{\lambda}} > \frac{p-1}{(a^{\eta}+b^{\eta})^{\frac{1}{\eta}}}.$$

Proof:

$$\begin{aligned}
\frac{p-1}{2\sqrt{2}a^{\lambda}b^{\lambda}}\left(a^{\lambda}+b^{\lambda}\right)^{2-\frac{1}{\lambda}} &= \left(\frac{a^{\lambda}+b^{\lambda}}{2\sqrt{a^{\lambda}b^{\lambda}}}\right)^{2}\frac{\sqrt{2}(p-1)}{(a^{\lambda}+b^{\lambda})^{\frac{1}{\lambda}}} > \frac{\sqrt{2}(p-1)}{(a^{\lambda}+b^{\lambda})^{\frac{1}{\lambda}}} \\
&= \sqrt{2}\frac{(a^{\eta}+b^{\eta})^{\frac{1}{\eta}}}{(a^{\lambda}+b^{\lambda})^{\frac{1}{\lambda}}}\frac{p-1}{(a^{\eta}+b^{\eta})^{\frac{1}{\eta}}} = \sqrt{2}\frac{\left[1+\left(\frac{b}{a}\right)^{\eta}\right]^{\frac{1}{\eta}}}{\left[1+\left(\frac{b}{a}\right)^{\lambda}\right]^{\frac{1}{\lambda}}}\frac{p-1}{(a^{\eta}+b^{\eta})^{\frac{1}{\eta}}}.
\end{aligned}$$

Therefore, all we have to show is

$$\frac{\left[1+\left(\frac{b}{a}\right)^{\eta}\right]^{\frac{1}{\eta}}}{\left[1+\left(\frac{b}{a}\right)^{\lambda}\right]^{\frac{1}{\lambda}}} \geq \frac{1}{\sqrt{2}}.$$

To do this, we put

$$F(x) = \frac{(1+x^{\eta})^{\frac{1}{\eta}}}{(1+x^{\lambda})^{\frac{1}{\lambda}}} \quad (0 < x \leq 1).$$

Then, noting $\eta > \lambda$, we obtain

$$F'(x) = \frac{(1+x^{\eta})^{\frac{1}{\eta}-1}(x^{\eta-1}-x^{\lambda-1})}{(1+x^{\lambda})^{\frac{1}{\lambda}+1}} < 0.$$

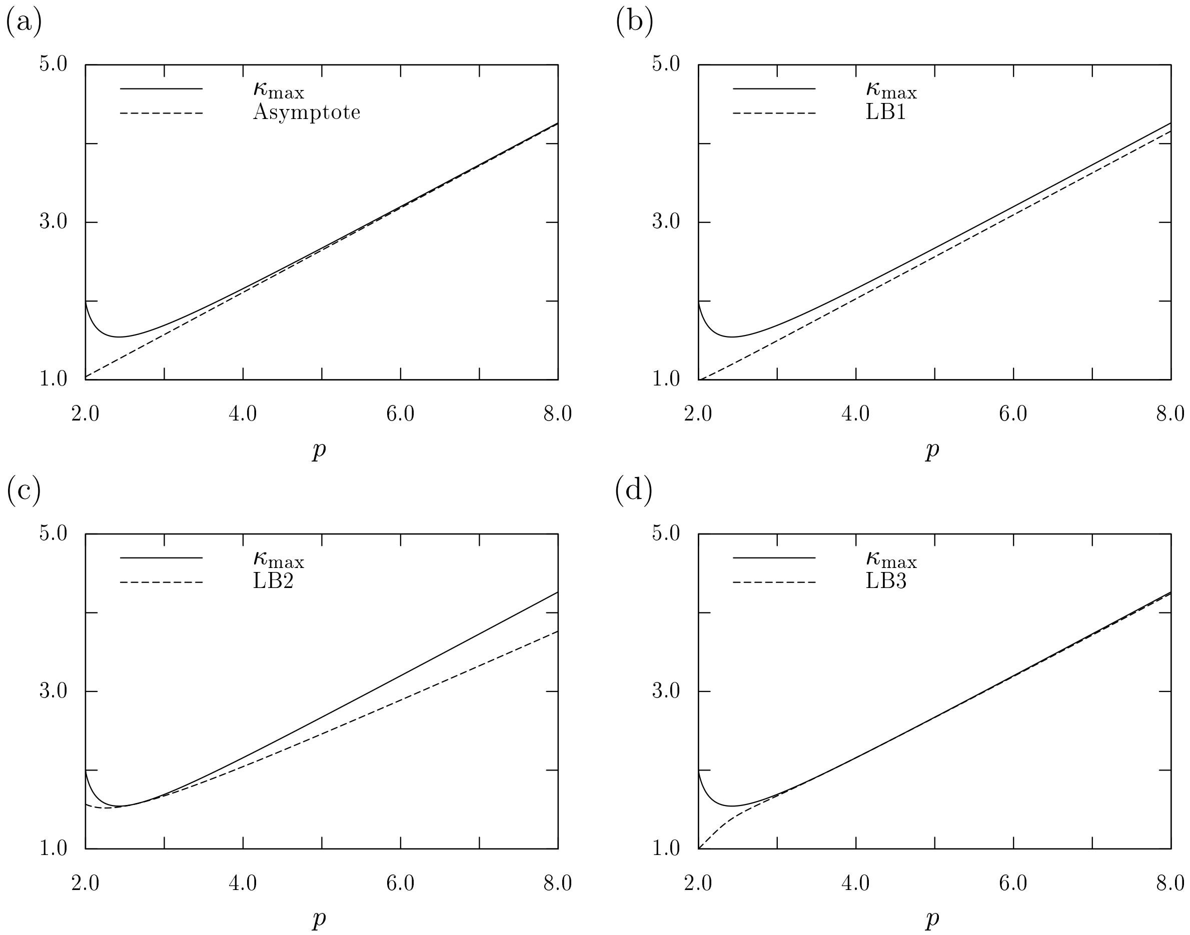

Figure 3: Asymptote and lower bounds of $\kappa_{\max}(p)$. In all cases, $a = 2$ and $b = 1$.
(a) Asymptote, (b) LB1, (c) LB2, (d) LB3.

Hence,

$$F(x) > F(1) = 2^{\frac{1}{\eta}-\frac{1}{\lambda}} = \frac{1}{\sqrt{2}} \quad (0 < x < 1).$$

This completes the proof. □

Empirically, it seems that the following lower bounds, which we call LB2 and LB3, give better estimations.

Theorem 6. *$\kappa_{\max}(p)$ satisfies*

(i) $$\kappa_{\max}(p) \geq \frac{(p-1)\left(a^{2\lambda}+b^{2\lambda}\right)^{2-\frac{1}{\lambda}}}{(ab)^{2\lambda-2}\left(a^2+b^2\right)^{\frac{3}{2}}}. \qquad \text{(LB2)}$$

(ii) $$\kappa_{\max}(p) \geq \frac{(p-1)\left(a^{\frac{\eta}{2}}+b^{\frac{\eta}{2}}\right)^{\frac{3}{2}-\frac{1}{\eta}}}{ab\left(a^{\frac{\eta}{2}-1}+b^{\frac{\eta}{2}-1}\right)^{\frac{3}{2}}}. \qquad \text{(LB3)}$$

Proof: It can be verified that

$$c\left(p, \left(\frac{b}{a}\right)^{2\lambda}\right) = \frac{(p-1)\left(a^{2\lambda}+b^{2\lambda}\right)^{2-\frac{1}{\lambda}}}{(ab)^{2\lambda-2}\left(a^2+b^2\right)^{\frac{3}{2}}}, \quad c\left(p, \left(\frac{b}{a}\right)^{\frac{\eta}{2}}\right) = \frac{(p-1)\left(a^{\frac{\eta}{2}}+b^{\frac{\eta}{2}}\right)^{\frac{3}{2}-\frac{1}{\eta}}}{ab\left(a^{\frac{\eta}{2}-1}+b^{\frac{\eta}{2}-1}\right)^{\frac{3}{2}}}.$$

This gives the proof. □

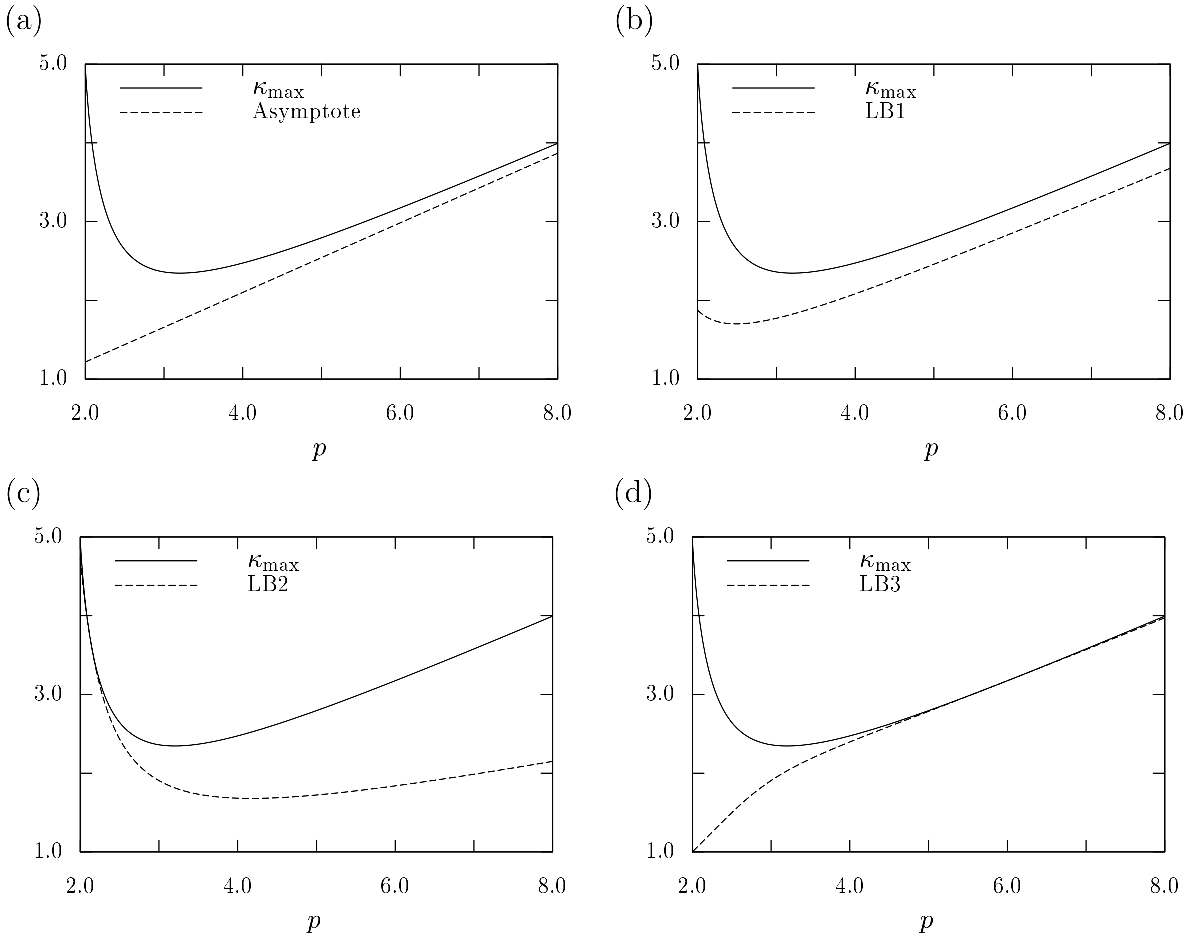

Figure 4: Asymptote and lower bounds of $\kappa_{\max}(p)$. In all cases, $a = 5$ and $b = 1$.

In Fig. 3 and 4, we show the lower bounds LB1, LB2 and LB3 given by Theorems 5 and 6. It seems that the combination of LB2 and LB3, that is,

$$\max\left\{\frac{(p-1)\left(a^{2\lambda}+b^{2\lambda}\right)^{2-\frac{1}{\lambda}}}{(ab)^{2\lambda-2}\left(a^2+b^2\right)^{\frac{3}{2}}}, \frac{(p-1)\left(a^{\frac{\eta}{2}}+b^{\frac{\eta}{2}}\right)^{\frac{3}{2}-\frac{1}{\eta}}}{ab\left(a^{\frac{\eta}{2}-1}+b^{\frac{\eta}{2}-1}\right)^{\frac{3}{2}}}\right\}$$

greatly improves the estimation, though it is still not satisfactory for some interval of p.

4. Conclusion

In this paper, we closely examined the maximum curvature $\kappa_{\max}(p)$ of Lamé curves given by $\left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p = 1$, $(a, b, p > 0)$. In particular, we derived an explicit expression of its asymptote (Theorem 3). In addition, we obtained several types of lower bounds (LB1, LB2, LB3) for $\kappa_{\max}(p)$. Numerical calculations indicate that the asymptote and the lower bounds give good approximations to $\kappa_{\max}(p)$ when p is relatively large. On the other hand, further study is needed to analytically assess the curvature for smaller p, which leads to the determination of the "optimal" exponent $p^* = \arg\min_{p\geq 2} \kappa_{\max}(p)$. As a reference, we show in Figure 5 approximate values of p^* obtained by numerical calculation.

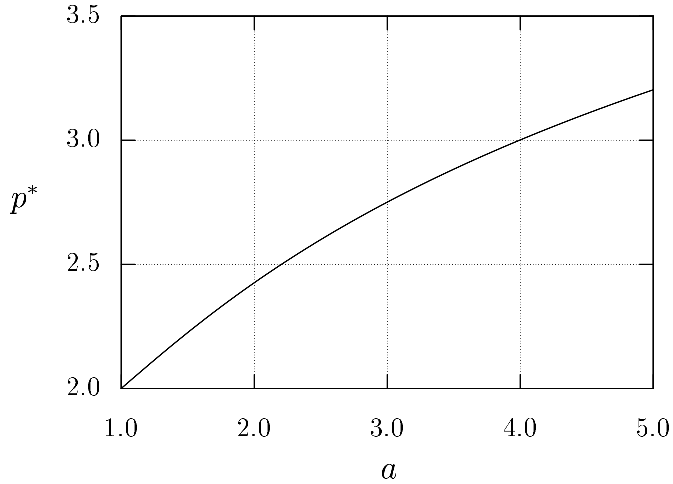

Figure 5: Approximate values of the optimal exponent p^* for each a. The other constant b is fixed as $b = 1$. These values are obtained by numerical calculation.

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