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On Relaxed Elastic Lines of Second Kind on a Curved Hypersurface in the n-Dimensional Euclidean Space

Ayhan Sarıoğlugil[1], Ayhan Tutar[1], Hellmuth Stachel[2]

[1] *Ondokuz Mayıs University, Science and Arts Faculty*
Department of Mathematics, 55139, Samsun, Türkiye
emails: {ayhans,atutar}@omu.edu.tr

[2] *Vienna University of Technology, Institute of Discrete Mathematics and Geometry*
Wiedner Hauptstrasse 8-10/104, A-1040 Wien, Austria
email: stachel@dmg.tuwien.ac.at

Abstract. In the present paper, the relaxed elastic lines of second kind on an oriented surface in the Euclidean n-space are defined and the Euler-Lagrange equations are derived. Furthermore, an example is presented. Special emphasis is laid on the particular case when these curves are at the same time geodesic.

Key Words: Relaxed elastic line, Geodesic, Intrinsic equation, Euler-Lagrange equation

MSC 2010: 53A07, 49Q20, 53C22

1. Introduction

The calculus of variations is playing an increasingly important role in the fields analysis, physics, geometry and engineering. So, many mathematicians have been interested in the calculus of variations, especially in elasticity theory, [1, 6, 7, 8, 9, 10, 12, 13].

For any curve with arc-length s, $0 \leq s \leq L$, and curvature k_1 the associated energy is given by

$$K = \int_0^L k_1^2 ds \tag{1.1}$$

The integral K is called the *total squared curvature*. For an *elastic line* of length L this integral is minimal.

The analysis of the total squared curvature has a long history going back to the infancy of the subject: the Calculus of Variations. Many of the qualitative properties of the corresponding variational solutions have been known ever since. The same cannot be said for the precise quantitative properties, especially when the length is not fixed. James Bernoulli

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posed a problem related to the elastic curves already in 1691 (possibly to quite his brother John who solved the problem of the catenary and then taunted James for wasting time trying to prove that the parabola provided the solution). Three years passed without a response to the challenge and James subsequently published his own solution. HUYGENS criticized James for omitting several elastic curves in his analysis. In 1742 Daniel BERNOULLI wrote a letter to EULER proposing the total squared curvature as the correct quantity to minimize. EULER finished his book on the Calculus of Variations in 1744 and applied his theory to the elastic curves using Daniel's suggestion. This work appeared in the first appendix *'De Curvis Elasticis'* to his book, and the objections raised by HUYGENS were addressed. Note that this early work predates the rich theory of elliptic functions developed by JACOBI and others one century later. In recent years the modified total squared curvature has emerged as a useful quantity in the study of geodesics, i.e., the straightest paths in a surface. LANGER and SINGER initiated the research in a series of papers dealing with closed elastic curves in spaces of constant curvature (flat, spherical and hyperbolic spaces), [7].

In Eq. (1.1), L is the length and $k_1(s)$ the curvature of the curve parametrized by its arc-length s. If no boundary conditions are imposed at $s = L$ and if no external forces act at any s, the elastic line is *relaxed*, [6]. The trajectory of a relaxed elastic line in space and on a plane must be straight because the position-independent quantity K takes its minimum value of zero when the squared curvature vanishes for all s. The trajectory of a relaxed elastic line constrained to lie on a general surface is, however, dependent on the intrinsic curvature of the surface, which in general bounds the possible values of K away from zero.

HILBERT and COHN-VOSSEN [4] stated incorrectly that a relaxed elastic line with specified position and tangent at $s = 0$ always has the trajectory of a geodesic. MANNING [9] obtained the intrinsic equations and NICKERSON-MANNING [10] have proved that the conclusion of HILBERT and COHN-VOSSEN [4] that a relaxed line always follows a geodesic is incorrect. Furthermore, NICKERSON-MANNING [9] has got the intrinsic equations of a relaxed line on an oriented surface. On the other hand, for understanding the mechanics of nuclesomal DNA, the Euler-Lagrange equations of a relaxed line on a restricted surface have been investigated by MANNING in [9].

Elastic lines of second kind on an oriented surface in E^3 have been studied by Z. ÜNAN and M. YILMAZ in [13]. In the underlying paper, the relaxed elastic lines of second kind on a curved oriented hypersurface M in the Euclidean space E^n are investigated.

2. Preliminaries

To begin with, we recall the fundamentals of the differential geometry of curves and hypersurfaces in the Euclidean n-space E^n with its inner product $\langle\ ,\ \rangle$. The Euclidean metric is given by

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2 = \sum_{i=1}^{n} dx_i^2$$

where $(x_1, x_2, \ldots, x_n)$ are cartesian coordinates in E^n.

Definition 1. Let $\alpha : I \subset \mathbb{R} \to E^n$ be a unit speed curve, i.e., parametrized by its arc-lengths, and let $(V_1, V_2, \ldots, V_n)$ be the Frenet frame field of α. Then, for each i, $1 \leq i < n$, the function

$$k_i \colon I \subset \mathbb{R} \to \mathbb{R} \tag{2.1}$$

defined for $s \in I$ by

$$k_i(s) = \langle \dot{V}_i(s), V_{i+1}(s) \rangle \tag{2.2}$$

is called the i^{th} *curvature function* of the curve α, and $k_i(s)$ is called the i^{th} *curvature* of α at $\alpha(s)$, [3].

Theorem 1. [Frenet Formulas] *If* $\alpha\colon I \subset \mathbb{R} \to E^n$ *is a unit speed curve then*

$$\dot{V}_i = -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s), \quad 1 \le i \le n, \tag{2.3}$$

where $k_0 = k_n = 0$, [3].

Thus, it is possible to write the Frenet formulas in matrix form as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \vdots \\ \dot{V}_{n-2} \\ \dot{V}_{n-1} \\ \dot{V}_n \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & \cdots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & k_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & -k_{n-2} & 0 & k_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -k_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{n-2} \\ V_{n-1} \\ V_n \end{bmatrix} \tag{2.4}$$

where k_1 and k_2 are are usually called *(ordinary) curvature* and *torsion*, respectively, [3].

Let M be a hypersurface in E^n with the parameter representation

$$\begin{array}{rcl} X\colon \quad U \subset E^{n-1} & \to & E^n \\ (u_1, u_2, \ldots, u_{n-1}) & \mapsto & X(u_1, u_2, \ldots, u_{n-1}). \end{array} \tag{2.5}$$

The tangent space is defined as

$$T_M(P) = \mathrm{Sp}\left\{X_{u_1}, X_{u_2}, \ldots, X_{u_{n-1}}\right\}, \tag{2.6}$$

where the X_{u_i}, $1 \le i \le n-1$, are the partial derivatives of X. For the sake of brevity, we set $X_i = X_{u_i}$. We exclude singularities by the assumption $\dim T_M(P) = n-1$ for all points $P \in M$. Then the unit normal vector of M is

$$\xi = (-1)^{n-1} \frac{X_1 \wedge X_2 \wedge \cdots \wedge X_{n-1}}{\|X_1 \wedge X_2 \wedge \cdots \wedge X_{n-1}\|} \tag{2.7}$$

where $\wedge$ denotes the exterior product of the vectors $X_1, X_2, \ldots, X_{n-1}$.

Definition 2. Let M be a hypersurface in E^n. The linear map

$$\begin{array}{rcl} S\colon \ T_M(P) & \to & T_M(P) \\ v & \mapsto & S(v) = D_v\,\xi \end{array} \tag{2.8}$$

is called the *shape operator* or *Weingarten map* of M at $P \in M$.

Theorem 2. *Let* M *be a hypersurface in* E^n*. Then the map* S *is self-adjoint.*

Definition 3. Let M be a hypersurface in E^n. The map

$$\begin{aligned} I^q\colon\ T_M(P)\times T_M(P) &\to C^\infty(M,\mathbb{R}), \quad 1\le q\le n,\\ (v,w) &\mapsto I^q(v,w)=\langle S^{q-1}(v),\, w\rangle \end{aligned} \tag{2.9}$$

is called the q^{th} *fundamental form* of M at $P\in M$. In particular, for $q=1$ the map

$$\begin{aligned} I\colon\ T_M(P)\times T_M(P) &\to C^\infty(M,\mathbb{R})\\ (v,w) &\mapsto I(v,w)=\langle v,\, w\rangle \end{aligned} \tag{2.10}$$

is called *first fundamental form* of M at $P\in M$. For $q=2$ the map

$$\begin{aligned} I^2\colon\ T_M(P)\times T_M(P) &\to C^\infty(M,\mathbb{R})\\ (v,w) &\mapsto I^2(v,w)=\langle S(v),\, w\rangle \end{aligned} \tag{2.11}$$

is called *second fundamental form* of M at $P\in M$ and denoted by $I^2=II$. Finally, the map

$$\begin{aligned} I^3\colon\ T_M(P)\times T_M(P) &\to C^\infty(M,\mathbb{R})\\ (v,w) &\mapsto I^3(v,w)=\langle S^2(v),\, W\rangle \end{aligned} \tag{2.12}$$

is the *third fundamental form* of M at $P\in M$ and denoted by $I^3=III$.

Using (2.9), the first, the second and the third fundamental form can be rewritten respectively as follows:

$$I(v,w)=\sum_{i,j=1}^{n-1} g_{ij}\dot u_i\dot u_j=\sum_{i,j=1}^{n-1}\langle X_i,\, X_j\rangle\dot u_i\dot u_j\,, \tag{2.13}$$

$$II(v,w)=\sum_{i,j=1}^{n-1} b_{ij}\dot u_i\dot u_j=-\sum_{i,j=1}^{n-1}\langle X_{ij},\, \xi\rangle\dot u_i\dot u_j\,, \tag{2.14}$$

$$III(v,w)=\sum_{i,j=1}^{n-1} n_{ij}\dot u_i\dot u_j=\sum_{i,j=1}^{n-1}\langle \xi_i,\, \xi_j\rangle\dot u_i\dot u_j\,. \tag{2.15}$$

Definition 4. Let M be a hypersurface in E^n and α be a unit speed curve on M. The function

$$\begin{aligned} k_n\colon\ M &\to \mathbb{R}\\ P &\mapsto k_n(P)=\langle S(V),\, V\rangle = II(V,V) \end{aligned} \tag{2.16}$$

is called *normal* (or asymptotic) *curvature function*, and $k_n(P)$ is called the *normal* (or asymptotic) *curvature* of the curve α at $\alpha(s)$, [3]. If the normal curvature is zero then the curve α is called an *asymptotic curve*.

Definition 5. Let M be a hypersurface in E^n and α be a unit speed curve on M with the Darboux frame field $(E_1,E_2,\dots,E_{n-1},\xi)$. Then, for each i, $1\le i<n-1$, the function

$$\begin{aligned} k_{ig}\colon\ I\subset\mathbb{R} &\to \mathbb{R}\\ s &\mapsto k_{ig}(s)=\langle \dot E_i(s),\, E_{i+1}(s)\rangle \end{aligned} \tag{2.17}$$

is called the i^{th} *geodesic curvature function* of α, and $k_i(s)$ is called the i^{th} *geodesic curvature* of α at $\alpha(s)$, [2].

Theorem 3. *Let M be a hypersurface in E^n and α be a curve on M. Then the derivatives of the natural (Darboux) frame field $\{V_1 = E_1, E_2, \ldots, E_{n-1}, \xi\}$ (see Figure 1) are*

$$\begin{cases} \dot{E}_i = -k_{(i-1)g}E_{i-1} + k_{ig}E_{i+1} + II_{1i}\,\xi\,, \\ \dot{\xi} = -II_{11}E_1 - II_{12}E_2 - \cdots - II_{1(n-1)}E_{n-1}\,, \end{cases} \quad 1 \le i \le n-1, \tag{2.18}$$

where ξ is the unit normal vector of M and k_{ig} is the i^{th} geodesic curvature function of α, [2]*. Here, $k_{0g} = k_{(n-1)g} = 0$ and $II_{1j} = II(E_1, E_j)$.*

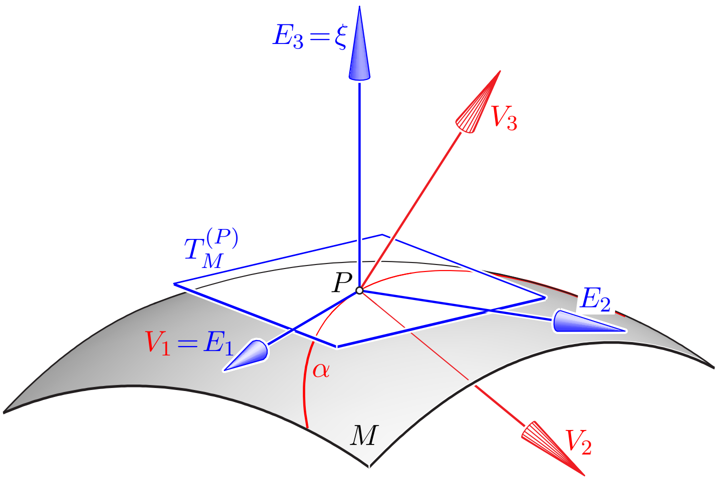

Figure 1: The Frenet frame $(V_1, V_2, \ldots, V_n)$ and the Darboux frame $(E_1, E_2, \ldots, E_{n-1})$ of the curve α on M at the point $P = \alpha(s)$.

From $V_1 = E_1$ follows $\dot{V}_1 = \dot{E}_1$, hence by Eqs. (2.3) and (2.18) $k_1V_2 = k_{1g}E_2 + II_{11}\xi$, and therefore

$$k_1^2 = k_{1g}^2 + II_{11}^2.$$

$\ddot{V}_1 = \ddot{E}_1$ implies for $n > 3$

$$\begin{aligned} \dot{k}_1V_2 + k_1k_2V_3 &= \left(\dot{k}_{1g} - II_{11}II_{12}\right)E_2 + (k_{1g}k_{2g} - II_{11}II_{13})\,E_3 - II_{11}II_{14}E_4 - \cdots \\ &\quad - II_{11}II_{1(n-1)}E_{n-1} + \left(k_{1g}II_{12} + \dot{II}_{11}\right)\xi\,. \end{aligned}$$

After computing

$$\|\dot{k}_1V_2 + k_1k_2V_3\|^2 = \dot{k}_1^2 + k_1^2k_2^2$$

we obtain

$$\begin{aligned} k_2^2 &= \frac{1}{k_1^2}\Big[-\dot{k}_1^2 + \dot{k}_{1g}^2 + \dot{II}_{11}^2 + II_{11}^2\sum_{i=2}^{n-1} II_{1i}^2 + k_{1g}^2\left(k_{2g}^2 + II_{12}^2\right) \\ &\quad -2II_{11}\left(k_{1g}k_{2g}II_{13} + II_{11}II_{12}\left(\frac{k_{1g}}{II_{11}}\right)^{\cdot}\right)\Big]. \end{aligned} \tag{2.19}$$

A curve α on M is called *geodesic* if $k_{1g} = 0$ for all s. In this case we have $k_1^2 = II_{11}^2$, and Eq. (2.19) reduces to

$$k_2^2 = \sum_{i=2}^{n-1} II_{1i}^2\,. \tag{2.20}$$

3. Elastic lines of second kind

Now we define elastic lines of second kind on a curved hypersurface M in E^n.

Definition 6. Let α be a C^k-curve parametrized by its arc-length s, $0 \le s \le L$, on an oriented curved surface in M. The integral

$$\Omega = \int_0^L k_2^2\, ds \tag{3.1}$$

with the torsion k_2 is called the *total squared second curvature*. The curve α is called a *relaxed elastic line of second kind* if it is an extremal for the variational problem of minimizing the total second squared curvature Ω within the family of all arcs of length L on M having the same initial point and initial direction.

When the geodesic curvature of α vanishes identically, we get after substituting (2.20) into (3.1)

$$\Omega_{II} = \int_0^L \Big[\sum_{i=2}^{n-1} II_{1i}^2\Big]\, ds\,. \tag{3.2}$$

The geodesic unit speed curve α on a curved hypersurface M in E^n is a relaxed elastic line of second kind if it is an extremal for the variational problem of minimizing the value of Ω_{II}.

Let $\overline{\alpha}$ be the curve that minimizes Ω_{II} among all curves of length L on a curved hypersurface M with stated boundary conditions at $s = 0$. For any other curve with stated boundary conditions we have

$$\begin{aligned}\Omega \;&=\; \int_0^L \frac{1}{k_1^2}\Big[-\dot k_1^2 + \dot k_{1g}^2 + \dot{II}_{11}^2 + II_{11}^2 \sum_{i=2}^{n-1} II_{1i}^2 + k_{1g}^2\left(k_{2g}^2 + II_{12}^2\right)\\ &\qquad\qquad -2II_{11}\Big(k_{1g}k_{2g}II_{13} + II_{11}II_{12}\Big(\frac{k_{1g}}{II_{11}}\Big)^{\cdot}\Big)\Big]ds\\ &\ge \overline{\Omega}\,,\end{aligned} \tag{3.3}$$

while if $\overline{\alpha}$ is a geodesic

$$\begin{aligned}\overline{\Omega} \;&=\; \int_0^L \overline{k}_2^2\, ds\\ &=\; \int_0^L \frac{1}{\overline{k}_1^2}\Big[-\dot{\overline{k}}_1^2 + \dot{\overline{k}}_{1g}^2 + \dot{\overline{II}}_{11}^2 + \overline{II}_{11}^2 \sum_{i=2}^{n-1} \overline{II}_{1i}^2 + \overline{k}_{1g}^2\left(\overline{k}_{2g}^2 + \overline{II}_{12}^2\right)\\ &\qquad\qquad -2\overline{II}_{11}\Big(\overline{k}_{1g}\,\overline{k}_{2g}\,\overline{II}_{13} + \overline{II}_{11}\overline{II}_{12}\Big(\frac{\overline{k}_{1g}}{\overline{II}_{11}}\Big)^{\cdot}\Big)\Big]ds\\ &=\; \int_0^L \Big[\sum_{i=2}^{n-1} \overline{II}_{1i}^2\Big] ds = \overline{\Omega}_{II}\,.\end{aligned} \tag{3.4}$$

In other words, if the curve that minimizes Ω_{II} is a geodesic, then this curve has least total second squared curvature.

To implement this rule for a given curved hypersurface, we first must find the curve that minimizes Ω_{II}. Then we must show that this curve is a solution of the differential equations of geodesic curves for the given curved hypersurface. If this is true then we have proved that the relaxed elastic line follows a geodesic trajectory.

4. Incomplete variational problem on a hypersurface

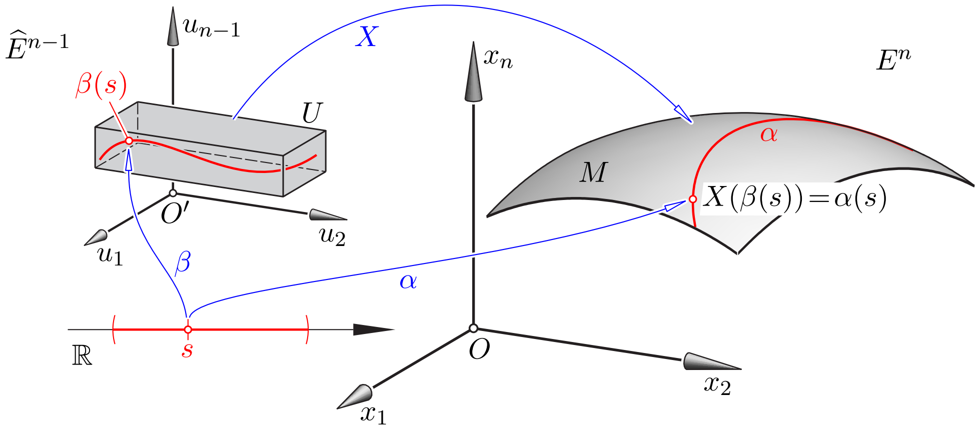

Figure 2: The curve $\alpha = X \circ \beta$ on the hypersurface M

Let M be a hypersurface in E^n and let α be a curve on M. By Figure 2 we have

$$\begin{aligned} \alpha = X \circ \beta \colon\ I &\to M \\ s &\mapsto \alpha(s) = X\left(u_1(s), u_2(s), \dots, u_{n-1}(s)\right) \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \beta \colon\ I &\to U \subset \widehat{E}^{n-1} \\ s &\mapsto \beta(s) = \left(u_1(s),\, u_2(s),\, \dots,\, u_{n-1}(s)\right). \end{aligned} \tag{4.2}$$

The tangent vector E_1 of α is

$$E_1 = \frac{d\alpha}{ds} = X_{u_1}\dot{u}_1 + X_{u_2}\dot{u}_2 + \cdots + X_{u_{n-1}}\dot{u}_{n-1} \tag{4.3}$$

where $\dot{u}_1 = \dfrac{du_i}{ds}$, $1 \le i \le n-1$. We write the side condition $I(E_1, E_1) = \langle E_1, E_1 \rangle = 1$ in the following as

$$g\left(u_1,\, u_2,\, \dots,\, u_{n-1},\, \dot{u}_1,\, \dot{u}_2,\, \dots,\, \dot{u}_{n-1}\right) = 1 \tag{4.4}$$

on the curved hypersurface M. Here

$$I(E_1,\, E_1) = \sum_{i,j=1}^{n-1} g_{ij}\dot{u}_i\dot{u}_j \tag{4.5}$$

where g_{ij} are the coefficients of the first fundamental form of M.

By (2.16) the normal curvature of M is written as

$$k_n = II\left(E_1,\, E_1\right) = \sum_{i,j=1}^{n-1} b_{ij}\dot{u}_i\dot{u}_j \tag{4.6}$$

where b_{ij} are the coefficients of the second fundamental form of M. The first geodesic curvature of the curve α on M is

$$k_{1g} = \sum_{i,j=1}^{n-1} g_{ij}\gamma_i\gamma_j \tag{4.7}$$

where

$$\gamma_i = \ddot{u}_i + \sum_{k,l=1}^{n-1} \Gamma^i_{kl}\dot{u}_k\dot{u}_l, \quad 1 \le i \le n-1. \tag{4.8}$$

The Γ^i_{kl}, $1 \le i,k,l \le n-1$, are *Christoffel symbols* of M. The equation of a geodesic curve, which is characterized by identically vanishing k_{1g}, is given by $\gamma_i = 0$, [9]. The Christoffel symbols Γ^r_{kl}, $1 \le k,l \le n-1$, can be written as

$$\Gamma^r_{kl} = \sum_{r=1}^{n-1} \langle X_{u_k u_l},\, X_{u_r}\rangle\, g^{ri} \tag{4.9}$$

where g^{ri} for $1 \le i,r \le n-1$ are the entries of the matrix $[g^{ri}] = [g_{ri}]^{-1}$.

On the other hand, we can write

$$II\,(E_1,\, E_2) = \frac{1}{W}\sum_{i,j=1}^{n} b_{ij}\gamma_i\dot{u}_j \tag{4.10}$$

where $W = II\,(E_1,\, E_1)$.

The problem now is to find functions $u_i(s)$, $1 \le i \le n-1$, that give stationary values to the integral

$$\Omega_{II} = \int_0^L \Big[\sum_{i=2}^{n-1} II^2_{1i}\Big]\, ds \tag{4.11}$$

subject to the side condition Eq. (4.4). We call this problem *incomplete* because it seeks to minimize Ω_{II} but not the total squared second curvature, [9].

Thus, the Euler equations for this incomplete problem are

$$\left\{\begin{aligned}
H_{u_1} - (H_{\dot{u}_1})^{\cdot} + (H_{\ddot{u}_1})^{\cdot\cdot} &= 0\\
H_{u_2} - (H_{\dot{u}_2})^{\cdot} + (H_{\ddot{u}_2})^{\cdot\cdot} &= 0\\
\vdots\qquad\qquad & \quad\vdots\\
H_{u_{k-1}} - \big(H_{\dot{u}_{k-1}}\big)^{\cdot} + \big(H_{\ddot{u}_{k-1}}\big)^{\cdot\cdot} &= 0\\
H_{u_k} - (H_{\dot{u}_k})^{\cdot} + (H_{\ddot{u}_k})^{\cdot\cdot} &= 0\\
H_{u_{k+1}} - \big(H_{\dot{u}_{k+1}}\big)^{\cdot} + \big(H_{\ddot{u}_{k+1}}\big)^{\cdot\cdot} &= 0\\
\vdots\qquad\qquad & \quad\vdots\\
H_{u_{n-2}} - \big(H_{\dot{u}_{n-2}}\big)^{\cdot} + \big(H_{\ddot{u}_{n-2}}\big)^{\cdot\cdot} &= 0\\
H_{u_{n-1}} - \big(H_{\dot{u}_{n-1}}\big)^{\cdot} + \big(H_{\ddot{u}_{n-1}}\big)^{\cdot\cdot} - \big(H_{\dddot{u}_{n-1}}\big)^{\cdot\cdot\cdot} &= 0
\end{aligned}\right. \tag{4.12}$$

where

$$H = \Big[\sum_{i=2}^{n-1} II^2_{1i}\Big] + \lambda(g-1), \tag{4.13}$$

and $\lambda = \lambda(s)$ is a Lagrange multiplier function.

The system is of order $(4n-3)$, of fourth order in each u_i, of first order in λ. There are $(4n-3)$ constants of integration in the general solution.

One of them is fixed by the following argument: Differentiate Eq. (4.4) once with respect to the arc-length s to make it of fourth order in the u_i's, so that the system of $(4n-3)$th order is transformed into normal form. When the resulting equation $\dddot{g} = 0$ is integrated, Eq. (4.4) dictates that the constants of integration are 0, 0 and 1.

The other $(4n-3)$ integration constants in the general solution are determined by the boundary conditions. Since the side condition (4.4) does not contain the parameter $\ddot{u}_i$, the boundary terms of the variation are not determined in the usual straightforward way. We nevertheless find the following $(4n-6)$ boundary terms if (4.4) can be solved for $\dot{u}_k$ as a function of $\ddot{u}_1, \ddot{u}_2, \ldots, \ddot{u}_{k-1}, \ddot{u}_{k+1}, \ldots, \ddot{u}_{n-1}$, and $u_1, u_2, \ldots, u_{n-1}$ to give $\dot{u}_k = V(u_1, u_2, \ldots, u_{n-1}, \dot{u}_1, \dot{u}_2, \ldots, \dot{u}_{k-1}, \dot{u}_{k+1}, \ldots, \dot{u}_{n-1})$, evaluated at $s = 0$ and L:

$$\left\{\begin{array}{r} \left[H_{\dot{u}_1} - (H_{\ddot{u}_1})^{\cdot} + H_{\ddot{u}_k} V_{u_1}\right] \delta u_1\Big|_0^L \\ \left[H_{\dot{u}_2} - (H_{\ddot{u}_2})^{\cdot} + H_{\ddot{u}_k} V_{u_2}\right] \delta u_2\Big|_0^L \\ \vdots \\ \left[H_{\dot{u}_{k-1}} - \left(H_{\ddot{u}_{k-1}}\right)^{\cdot} + H_{\ddot{u}_k} V_{u_{k-1}}\right] \delta u_{k-1}\Big|_0^L \\ \left[H_{\dot{u}_k} - (H_{\ddot{u}_k})^{\cdot} + H_{\ddot{u}_k} V_{u_k}\right] \delta u_k\Big|_0^L \\ \left[H_{\dot{u}_{k+1}} - \left(H_{\ddot{u}_{k+1}}\right)^{\cdot} + H_{\ddot{u}_k} V_{u_{k+1}}\right] \delta u_{k+1}\Big|_0^L \\ \vdots \\ \left[H_{\dot{u}_{n-2}} - \left(H_{\ddot{u}_{n-2}}\right)^{\cdot} + H_{\ddot{u}_k} V_{u_{n-2}}\right] \delta u_{n-2}\Big|_0^L \\ \left[H_{\dot{u}_{n-1}} - \left(H_{\ddot{u}_{n-1}}\right)^{\cdot} + H_{\ddot{u}_k} V_{u_{n-1}}\right] \delta u_{n-1}\Big|_0^L \\ \left[H_{\ddot{u}_1} + H_{\ddot{u}_k} V_{\dot{u}_1}\right] \delta \dot{u}_1\Big|_0^L \\ \left[H_{\ddot{u}_2} + H_{\ddot{u}_k} V_{\dot{u}_2}\right] \delta \dot{u}_2\Big|_0^L \\ \vdots \\ \left[H_{\ddot{u}_{k-1}} + H_{\ddot{u}_k} V_{\dot{u}_{k-1}}\right] \delta \dot{u}_{k-1}\Big|_0^L \\ \left[H_{\ddot{u}_{k+1}} + H_{\ddot{u}_k} V_{\dot{u}_{k+1}}\right] \delta \dot{u}_{k+1}\Big|_0^L \\ \vdots \\ \left[H_{\ddot{u}_{n-2}} + H_{\ddot{u}_k} V_{\dot{u}_{n-2}}\right] \delta \dot{u}_{n-2}\Big|_0^L \\ \left[H_{\ddot{u}_{n-1}} + H_{\ddot{u}_k} V_{\dot{u}_{n-1}}\right] \delta \dot{u}_{n-1}\Big|_0^L \end{array}\right. \tag{4.14}$$

Suppose that we place no other restrictions on the elastic line than the confinement to the hypersurface. Then the $(4n-6)$ integration constants are determined by setting equal zero each of the $(4n-6)$ factors multiplying δu_i and $\delta \dot{u}_i$ in the "V boundary terms". These conditions are completely natural [11].

If the natural conditions are allowed at $s = L$, but the initial position and direction of the relaxed elastic line of the second kind are specified, too, then we get a different set of equations to determine the $(4n-6)$ integration constants. The $(2n-3)$ boundary conditions are obtained by setting equal to zero each of the $(2n-3)$ integration constants multiplying

δu_k and $\delta \dot{u}_k$, with the integration constants evaluated at $s = L$. In addition, $(n-1)$ boundary conditions are specified by the values of u_k at $s = 0$ by $u_k(0) = 0$ for $1 \le k \le n-1$. Thus the initial tangent vector of a relaxed elastic line of the second kind is

$$V_1(0) = \sum_{i=1}^{n-1} \dot{u}_{i0}\, X_{u_i}\left(u_{10},\, u_{20},\, \ldots,\, u_{(n-1)0}\right). \tag{4.15}$$

5. The complete variational problem

Now, we will investigate the complete variational problem, which seeks to minimize the total squared torsion by given

$$\begin{aligned}\Omega \;=\; \int_0^L \frac{1}{k_1^2}\Big[-\dot{k}_1^2 + \dot{k}_{1g}^2 + \dot{I\!I}_{11}^2 + I\!I_{11}^2 \sum_{i=2}^{n-1} I\!I_{1i}^2 + k_{1g}^2\left(k_{2g}^2 + I\!I_{12}^2\right) \\ -2I\!I_{11}\Big(k_{1g}k_{2g}I\!I_{13} + I\!I_{11}I\!I_{12}\Big(\frac{k_{1g}}{I\!I_{11}}\Big)^{\cdot}\Big)\Big]\,ds\end{aligned} \tag{5.1}$$

The problem at hand is to find functions $u_i(s)$ and $\lambda(s)$ that give stationary values to the above integral, subject to the side condition (4.4).

The Euler-Lagrange equations for the complete variational problem are

$$\left\{\begin{aligned} H_{u_1} - (H_{\dot{u}_1})^{\cdot} + (H_{\ddot{u}_1})^{\cdot\cdot} - \big(H_{\dddot{u}_1}\big)^{\cdot\cdot\cdot} &= 0 \\ H_{u_2} - (H_{\dot{u}_2})^{\cdot} + (H_{\ddot{u}_2})^{\cdot\cdot} - \big(H_{\dddot{u}_2}\big)^{\cdot\cdot\cdot} &= 0 \\ \vdots \qquad\qquad & \quad \vdots \\ H_{u_k} - (H_{\dot{u}_k})^{\cdot} + (H_{\ddot{u}_k})^{\cdot\cdot} - \big(H_{\dddot{u}_k}\big)^{\cdot\cdot\cdot} &= 0 \\ \vdots \qquad\qquad & \quad \vdots \\ H_{u_{n-2}} - \big(H_{\dot{u}_{n-2}}\big)^{\cdot} + \big(H_{\ddot{u}_{n-2}}\big)^{\cdot\cdot} - \big(H_{\dddot{u}_{n-2}}\big)^{\cdot\cdot\cdot} &= 0 \\ H_{u_{n-1}} - \big(H_{\dot{u}_{n-1}}\big)^{\cdot} + \big(H_{\ddot{u}_{n-1}}\big)^{\cdot\cdot} - \big(H_{\dddot{u}_{n-1}}\big)^{\cdot\cdot\cdot} &= 0 \end{aligned}\right. \tag{5.2}$$

where

$$\begin{aligned}H \;=\; \frac{1}{k_1^2}\Big[-\dot{k}_1^2 + \dot{k}_{1g}^2 + \dot{I\!I}_{11}^2 + I\!I_{11}^2 \sum_{i=2}^{n-1} I\!I_{1i}^2 + k_{1g}^2\left(k_{2g}^2 + I\!I_{12}^2\right) \\ -2I\!I_{11}\Big(k_{1g}k_{2g}I\!I_{13} + I\!I_{11}I\!I_{12}\Big(\frac{k_{1g}}{I\!I_{11}}\Big)^{\cdot}\Big)\Big] + \lambda(g-1).\end{aligned} \tag{5.3}$$

The system is of order $6n-5$, of sixth order in each u_i and of first order in λ. There are $(6n-5)$ constants of integration in the general solution. To get the system in normal form, (4.4) must be differentiated five times with respect to s. Reintegration of $g^{(5)} = 0$ gives the respective fixed values 0, 0, 0, 0, and 1 to the resulting five constants of integration. The other $(6n-5)$ constants in the general solution are determined by the boundary conditions. Since the side condition Eq. (4.4) does not contain the parameters $\ddot{u}_1,\, \ddot{u}_2,\, \ldots,\, \ddot{u}_{n-1}$, the boundary terms of the variation are not determined in the usual straightforward way. We nevertheless find the following boundary terms if Eq. (4.4) can be solved for u_k, $1 \le k \le n-1$, as a function of the parameters $u_1,\, u_2,\, \ldots,\, u_{n-1},\, \dot{u}_1,\, \dot{u}_2,\, \ldots,\, \dot{u}_{k-1},\, \dot{u}_{k+1},\, \ldots,\, \dot{u}_{n-1}$, in order to give $\dot{u}_k = V(u_1, u_2, \ldots, u_{n-1}, \dot{u}_1, \dot{u}_2, \ldots, \dot{u}_{k-1}, \dot{u}_{k+1}, \ldots, \dot{u}_{n-1})$, evaluated at $s = 0$ and L.

$$\left\{\begin{array}{r} \left[H_{\dot{u}_1} - (H_{\ddot{u}_1})^{\cdot} + (H_{\dddot{u}_1})^{\cdot\cdot} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{u_1}\right] \delta u_1|_0^L \\ \left[H_{\dot{u}_2} - (H_{\ddot{u}_2})^{\cdot} + (H_{\dddot{u}_2})^{\cdot\cdot} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{u_2}\right] \delta u_2|_0^L \\ \vdots \\ \left[H_{\dot{u}_{k-1}} - (H_{\ddot{u}_{k-1}})^{\cdot} + \left(H_{\dddot{u}_{k-1}}\right)^{\cdot\cdot} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{u_{k-1}}\right] \delta u_{k-1}|_0^L \\ \left[H_{\dot{u}_k} - (H_{\ddot{u}_k})^{\cdot} + (H_{\dddot{u}_k})^{\cdot\cdot} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{u_k}\right] \delta u_k|_0^L \\ \left[H_{\dot{u}_{k+1}} - (H_{\ddot{u}_{k+1}})^{\cdot} + \left(H_{\dddot{u}_{k+1}}\right)^{\cdot\cdot} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{u_{k+1}}\right] \delta u_{k+1}|_0^L \\ \vdots \\ \left[H_{\dot{u}_{n-1}} - (H_{\ddot{u}_{n-1}})^{\cdot} + \left(H_{\dddot{u}_{n-1}}\right)^{\cdot\cdot} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{u_{n-1}}\right] \delta u_{n-1}|_0^L \\ \left[H_{\ddot{u}_1} - (H_{\dddot{u}_1})^{\cdot} + H_{\dddot{u}_1} V_{\dot{u}_1} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{\dot{u}_1}\right] \delta \dot{u}_1|_0^L \\ \left[H_{\ddot{u}_2} - (H_{\dddot{u}_2})^{\cdot} + H_{\dddot{u}_2} V_{\dot{u}_2} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{\dot{u}_2}\right] \delta \dot{u}_2|_0^L \\ \vdots \\ \left[H_{\ddot{u}_{k-1}} - \left(H_{\dddot{u}_{k-1}}\right)^{\cdot} + H_{\dddot{u}_{k-1}} V_{\dot{u}_{k-1}} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{\dot{u}_{k-1}}\right] \delta \dot{u}_{k-1}|_0^L \\ \left[H_{\ddot{u}_{k+1}} - \left(H_{\dddot{u}_{k+1}}\right)^{\cdot} + H_{\dddot{u}_{k+1}} V_{\dot{u}_{k+1}} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{\dot{u}_{k+1}}\right] \delta \dot{u}_{k+1}|_0^L \\ \vdots \\ \left[H_{\ddot{u}_{n-2}} - \left(H_{\dddot{u}_{n-2}}\right)^{\cdot} + H_{\dddot{u}_{n-2}} V_{\dot{u}_{n-2}} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{\dot{u}_{n-2}}\right] \delta \dot{u}_{n-2}|_0^L \\ \left[H_{\ddot{u}_{n-1}} - \left(H_{\dddot{u}_{n-1}}\right)^{\cdot} + H_{\dddot{u}_{n-1}} V_{\dot{u}_{n-1}} + \left[H_{\ddot{u}_k} - (H_{\dddot{u}_k})^{\cdot} + H_{\dddot{u}_k} V_{\dot{u}_k}\right] V_{\dot{u}_{n-1}}\right] \delta \dot{u}_{n-1}|_0^L \end{array}\right. \tag{5.4}$$

On the other hand, if V_{u_k} for $1 \le k \le n-1$ is singular it can involve a division by zero. Then we must be use a different set of $(12n - 16)$ boundary terms, after solving Eq. (3.4) for $\dot{u}_k = V(u_1, u_2, \ldots, u_{n-1}, \dot{u}_1, \dot{u}_2, \ldots, \dot{u}_{k-2}, \dot{u}_{k-1}, \ldots, \dot{u}_{n-1})$.

Suppose that we place on the elastic line no other restriction than the confinement to the hypersurface. Then, $(3n - 5)$ integration constants are determined by setting equal to zero each of the $(3n - 5)$ factors of δ_{u_k} and $\delta_{\dot{u}_k}$, $1 \le k \le n-1$, in the V boundary terms. These boundary conditions are completely natural (free) (see for details [9] and [12]).

6. Relaxed elastic lines of second kind on a hypercylinder

A spherical hypercylinder with radius 1 in E^n is given by

$$C^{n-1} = \left\{(x_1, x_2, \ldots, x_n)\colon \sum_{i=1}^{n-1} x_i^2 = 1\right\} \tag{6.1}$$

It can be parameterized by

$$X = \left(\cos u_1, \sin u_1 \cos u_2, \sin u_1 \sin u_2 \cos u_3, \ldots, \left(\prod_{i=1}^{n-3} \sin u_i\right) \cos u_{n-2}, \left(\prod_{i=1}^{n-2} \sin u_i\right), u_{n-1}\right) \tag{6.2}$$

where the u_i's for $1 \le i \le n-2$ are polar angles, and x_n determines the axis.

The derivatives of X with respect to the parameter u_i, $1 \le i \le n-1$, are

$$\left\{\begin{aligned}
X_{u_1} &= \Big(-\sin u_1,\ \cos u_1 \cos u_2,\ \cos u_1 \sin u_2 \cos u_3,\ \dots,\ \cos u_1 \Big(\prod_{i=2}^{n-3} \sin u_i\Big) \cos u_{n-2},\\
&\quad \cos u_1 \prod_{i=2}^{n-2} \sin u_i\,,\ 0\Big)\\
X_{u_2} &= \Big(0,\ -\sin u_1 \sin u_2,\ \sin u_1 \cos u_2 \cos u_3,\ \dots,\ \sin u_1 \cos u_2 \Big(\prod_{i=3}^{n-3} \sin u_i\Big) \cos u_{n-2},\\
&\quad \sin u_1 \cos u_2 \prod_{i=3}^{n-2} \sin u_i\,,\ 0\Big)\\
X_{u_3} &= \Big(0,\ 0,\ -\sin u_1 \sin u_2 \sin u_3,\ \dots,\ \sin u_1 \sin u_2 \cos u_3 \Big(\prod_{i=4}^{n-3} \sin u_i\Big) \cos u_{n-2},\\
&\quad \sin u_1 \sin u_2 \cos u_3 \prod_{i=4}^{n-2} \sin u_i\,,\ 0\Big)\\
&\ \ \vdots \qquad\qquad\qquad \vdots\\
X_{u_{n-3}} &= \Big(0,\ 0,\ \dots,\ 0,\ -\Big(\prod_{i=1}^{n-4} \sin u_i\Big) \cos u_{n-3} \cos u_{n-2},\ \Big(\prod_{i=1}^{n-4} \sin u_i\Big) \cos u_{n-3} \sin u_{n-2},\ 0\Big)\\
X_{u_{n-2}} &= \Big(0,\ 0,\ \dots,\ -\Big(\prod_{i=1}^{n-3} \sin u_i\Big) \sin u_{n-2},\ \Big(\prod_{i=1}^{n-3} \sin u_i\Big) \cos u_{n-2},\ 0\Big)\\
X_{u_{n-1}} &= (0,\ 0,\ \dots,\ 0,\ 1)
\end{aligned}\right. \tag{6.3}$$

The vector system $\big(X_{u_1},\ X_{u_2},\ \dots,\ X_{u_{n-1}}\big)$ is an orthogonal basis of $T_{C^{n-1}}(P)$. After applying the method of Gram-Schmidt orthonormalization, we have

$$\left\{\begin{aligned}
E_1 &= \Big(-\sin u_1,\ \cos u_1 \cos u_2,\ \cos u_1 \sin u_2 \cos u_3,\ \dots,\ \cos u_1 \Big(\prod_{i=2}^{n-3} \sin u_i\Big) \cos u_{n-2},\\
&\quad \cos u_1 \prod_{i=2}^{n-2} \sin u_i\,,\ 0\Big)\\
E_2 &= \Big(0,\ -\sin u_2,\ \cos u_2 \cos u_3,\ \dots,\ \cos u_2 \Big(\prod_{i=3}^{n-3} \sin u_i\Big) \cos u_{n-2},\ \cos u_2 \prod_{i=3}^{n-2} \sin u_i\,,\ 0\Big)\\
E_3 &= \Big(0,\ 0,\ -\sin u_3,\ \dots,\ \cos u_3 \Big(\prod_{i=4}^{n-3} \sin u_i\Big) \cos u_{n-2},\ \cos u_3 \prod_{i=4}^{n-2} \sin u_i\,,\ 0\Big)\\
&\ \ \vdots \qquad\qquad\qquad \vdots\\
E_{n-3} &= \Big(0,\ 0,\ \dots,\ 0,\ -\cos u_{n-2},\ \sin u_{n-2},\ 0\Big)\\
E_{n-2} &= \Big(0,\ 0,\ \dots,\ -\sin u_{n-2},\ \cos u_{n-2},\ 0\Big)\\
E_{n-1} &= \Big(0,\ 0,\ 0,\ \dots,\ 0,\ 1\Big)
\end{aligned}\right. \tag{6.4}$$

From here we get

$$\xi = \left(\cos u_1,\ \sin u_1 \cos u_2,\ \dots,\ \Big(\prod_{i=1}^{n-3} \sin u_i\Big) \cos u_{n-2},\ \prod_{i=1}^{n-2} \sin u_i\,,\ 0\right). \tag{6.5}$$

By (6.3) we have

$$\begin{cases} g_{11} &= 1 \\ g_{22} &= \sin^2 u_1 \\ g_{33} &= \sin^2 u_1 \sin^2 u_2 \\ \vdots & \quad \vdots \\ g_{(n-3)(n-3)} &= \Big(\prod_{i=1}^{n-4} \sin u_i\Big) \cos^2 u_{n-3} \\ g_{(n-2)(n-2)} &= \prod_{i=1}^{n-3} \sin u_i \\ g_{(n-1)(n-1)} &= 1 \\ g_{ij} &= 0 \quad \text{for } i \neq j. \end{cases} \tag{6.6}$$

Substituting (6.3) into (4.4) we obtain

$$\dot u_1^2 + \sin^2 u_1 \dot u_2^2 + \sin^2 u_1 \sin^2 u_2 \dot u_3^2 + \cdots + \Big(\prod_{i=1}^{n-4} \sin^2 u_i\Big) \dot u_{n-3}^2 + \Big(\prod_{i=1}^{n-3} \sin^2 u_i\Big) \dot u_{n-2}^2 + \dot u_{n-1}^2 = 1 \tag{6.7}$$

Furthermore, the entries g^{ij} of the matrix $\big(g_{ij}\big)^{-1}$ are

$$g^{ij} = \begin{cases} 0 & \text{for } \ i \neq j \\ \dfrac{1}{g_{ij}} & \text{for } \ i = j, \ 1 \leq i \leq n-1. \end{cases} \tag{6.8}$$

On the other hand, the coefficients b_{ij} of the second fundamental form II of C^{n-1} are

$$b_{ij} = \begin{cases} 0 & \text{for } \ i \neq j \\ -g_{ij} & \text{for } \ i = j = 1, 2, \ldots, n-2 \\ 0 & \text{for } \ i = j = n-1. \end{cases} \tag{6.9}$$

From Eq. (4.6) we obtain

$$k_n^2 = [II(E_1, E_1)]^2 = \Big[\sum_{i,j=1}^{n-2} g_{ij} \dot u_i \dot u_j\Big]^2. \tag{6.10}$$

By Eq. (4.7) we have

$$k_{1g}^2 = \Big[\sum_{i,j=1}^{n-1} g_{ij} u_i u_j\Big]^2. \tag{6.11}$$

By straigthtforward computation we get

$$\Gamma_{ij}^k = 0\,. \tag{6.12}$$

Thus we have

$$\begin{aligned} \gamma_1 &= \ddot u_1 \\ \gamma_2 &= \ddot u_2 \\ \vdots \ & \quad \vdots \\ \gamma_{n-2} &= \ddot u_{n-2} \\ \gamma_{n-1} &= \ddot u_{n-1} \end{aligned} \tag{6.13}$$

On the other hand

$$\begin{aligned} II_{11} &= 1 \\ II(E_1,\,E_2) &= 0 \\ II(E_1,\,E_3) &= 0 \\ \vdots \quad & \quad \vdots \\ II(E_1,\,E_{n-3}) &= 0 \\ II(E_1,\,E_{n-2}) &= 0 \\ II(E_{n-1},\,E_{n-1}) &= 0 \end{aligned} \tag{6.14}$$

Furthermore, we have

$$k_{1g} = \langle \dot{E},\,E_2\rangle = 0 \quad\text{and}\quad k_{2g} = \langle \dot{E},\,E_3\rangle = 0. \tag{6.15}$$

Substituting (6.10) and (6.11) into (6.13) we get

$$H = \lambda(g-1).$$

Then, H satisfies (5.2).

From (6.12) and (6.15) we get

$$u_i = a_i\,s + b_i, \quad a_i, b_i \in \mathbb{R}, \quad 1 \le i \le n-1\,. \tag{6.16}$$

Thus we obtain the parametric representation of the relaxed line of second kind on C^{n-1} as

$$\alpha(s) = \begin{cases} \alpha_1(s) &= \cos(a_1 s + b_1) \\ \alpha_2(s) &= \sin(a_1 s + b_1)\cos(a_2 s + b_2) \\ \quad\vdots & \quad\quad\vdots \\ \alpha_{n-2}(s) &= \prod_{i=1}^{n-3} \sin(a_i s + b_i)\cos(a_{i+1} s + b_{i+1}) \\ \alpha_{n-1}(s) &= \prod_{i=1}^{n-2} \sin(a_i s + b_i) \\ \alpha_n(s) &= a_{n-1} s + b_{n-1} \end{cases} \tag{6.17}$$

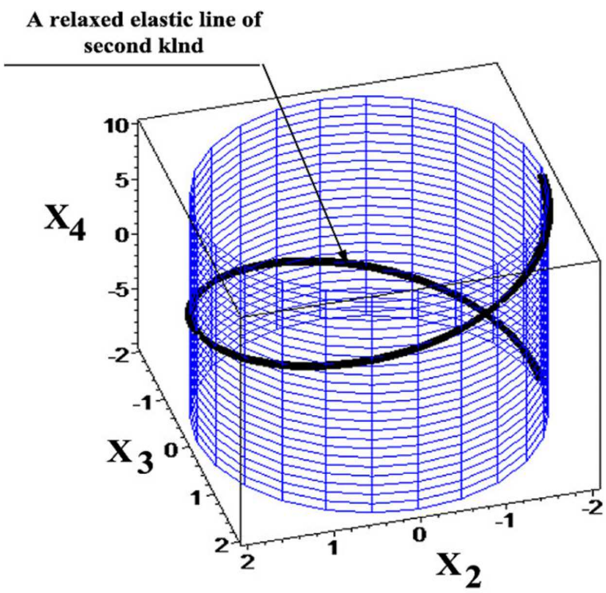

Figure 3: Projection of a relaxed line of second kind on a hypercylinder C^4

For example, we set $n = 4$: We immediately obtain a relaxed line of second kind of the spherical cylinder C^3. In Figure 3 the hypercylinder C^3 (with radius 2) is projected into the hyperplane $x_1 = 0$, [2].

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