Journal for Geometry and Graphics
Volume 18 (2014), No. 1, 97–104.

Orthogonality Relations for Tetrahedra in Elliptic and Hyperbolic Space

Josef Schadlbauer, Hans-Peter Schröcker

Unit Geometry and CAD, University Innsbruck
Technikerstraße 13, 6020 Innsbruck, Austria
email: josef.schadlbauer@uibk.ac.at, hans-peter.schroecker@uibk.ac.at

Abstract. Two tetrahedra in Euclidean space are called orthologic if the four lines through the vertices of the first and perpendicular to the corresponding face planes of the second are concurrent. More generally, two tetrahedra are called skew-orthologic if these four lines lie in a regulus (they are skew generators of a hyperboloid). These relations are symmetric, that is, also the lines through vertices of the second and perpendicular to the corresponding face planes of the first tetrahedron are concurrent or lie in a regulus. Moreover, two tetrahedra are orthologic if and only if non-corresponding edges are orthogonal (they are "anti-orthogonal"). It is common to all results mentioned so far that they can be formulated exclusively in terms of incidence and orthogonality relations. This suggests to investigate them in non-Euclidean Cayley-Klein geometries where incidence is given by the underlying projective structure and orthogonality is replaced by polarity with respect to a quadric. We will prove symmetry of the defining condition of orthologic and skew-orthologic tetrahedra in these spaces. The theorems of orthogonal tetrahedra and orthogonal pairs of tetrahedra remain true and the notions of "anti-orthogonal" and "orthologic" still coincide in non-Euclidean geometry.

Key Words: orthologic tetrahedra, skew-orthologic tetrahedra, Cayley-Klein geometry, absolute polarity

MSC 2010: 51M10, 51N15

1. Introduction

Orthologic triangles and tetrahedra were introduced by J. Steiner in 1827 [10]. Since this time orthologic pairs attained a lot attention in elementary Euclidean geometry. A tetrahedron is a quadruple of four non-coplanar points (a_0, a_1, a_2, a_3) in Euclidean or projective three-space. We denote it as $\mathbf{A} = a_0a_1a_2a_3$. The two tetrahedra $\mathbf{A}$ and $\mathbf{B} = b_0b_1b_2b_3$ are called *orthologic*, if for all pairwise different $i, j, k, l \in \{0, 1, 2, 3\}$ the lines L_i through the vertices a_i

ISSN 1433-8157/\$ 2.50

of the first and perpendicular to the face planes $b_j \vee b_k \vee b_l$ of the second tetrahedron intersect in one point. This property is a symmetric relation between the two tetrahedra: Also the lines N_i through b_i and perpendicular to $a_j \vee a_k \vee a_l$ intersect.

Remark. Often, a tetrahedron is not viewed as a quadruple but as a *set* of four points. In this case, two tetrahedra are called orthologic if there exists an ordering of the vertices such that the corresponding quadruples are orthologic in our sense.

Some proofs of fundamental statements concerning orthologic tetrahedra can be found in [6, 7, 9]. There, the following characterization of orthologic tetrahedra is given:

Proposition 1. *The two tetrahedra* $\mathbf{A} = a_0a_1a_2a_3$ *and* $\mathbf{B} = b_0b_1b_2b_3$ *are orthologic if and only if non-corresponding edges*

$$a_i \vee a_j, \quad b_k \vee b_l \quad (i, j, k, l \text{ pairwise different})$$

are orthogonal.

L. Gerber investigated orthologic and skew-orthologic simplices in n-dimensional Euclidean space in [1, 2]. In case of $n = 3$, *skew-orthology* means that the four lines L_0, L_1, L_2, L_3 lie in a regulus (a family of skew generators of a hyperboloid). Gerber proved that this property is symmetric as well, that is, the lines N_0, N_1, N_2, N_3 also lie in a regulus.

It is common to all results so far that they can be stated exclusively in terms of incidence and orthogonality relations. This suggests to study them in the more general setting of Cayley-Klein geometry, where incidence is given by the underlying projective structure and orthogonality means polarity with respect to an absolute quadric. In order to avoid too many case distinctions, we confine ourselves to Cayley-Klein geometries with respect to a regular absolute quadric, that is, to elliptic geometry, hyperbolic geometry, and the geometry with a ruled absolute quadric which is unnamed according to [5, p. 178] and is called "hyperbolic of index two" in [3, pp. 136–137]. It turns out that the mentioned results remain true in this non-Euclidean setting. There, they admit transparent proofs and their Euclidean counterparts can be obtained as limiting cases.

In Section 2 some notational remarks and a very brief introduction to non-Euclidean geometry will be given. In Section 3 we state some basic theorems and in Section 4 we prove the main results concerning orthologic tetrahedra in elliptic and hyperbolic space.

2. Preliminaries

By $\mathbf{A} = a_0a_1a_2a_3$ we denote a tetrahedron with four non co-planar vertices a_0, a_1, a_2, a_3 in real projective three-space P^3. Lets take four pairwise different indices $i, j, k, l \in \{0, 1, 2, 3\}$. The line spanned by two points a_i and a_j is $A_{ij} = a_i \vee a_j$ and the plane spanned by a_i, a_j, and a_k is $\alpha_l = a_i \vee a_j \vee a_k$. The edges $a_i \vee a_j$ and $a_k \vee a_l$ are called *opposite*. Consider a second tetrahedron $\mathbf{B} = b_0b_1b_2b_3$ in P^3. The vertices a_i and b_i are called *corresponding*, the edges $A_{ij} = a_i \vee a_j$ and $B_{kl} = b_k \vee b_l$ are called *non-corresponding*, and the vertex a_i and the face $\beta_i = b_j \vee b_k \vee b_l$ are called *corresponding*.

Now we impose a non-Euclidean geometric structure on P^3 by choosing a regular absolute quadric $F \subset P^3$. Orthogonality in this Cayley-Klein geometry is defined as polarity with respect to F. In case of hyperbolic geometry, F is oval, in case of elliptic geometry, it has no real points but a real polar system. If F is singular, additional case distinctions would be necessary. The Euclidean case appears as a limiting case of elliptic or hyperbolic geometry [5, p. 190].

We denote the absolute polarity in F by π. It maps a point a to its polar plane α with respect to F. We use the same symbol for denoting the associated maps for planes and lines. That is $\pi(\alpha) = a$ if and only if $\pi(a) = \alpha$ and $K = \pi(L)$ if and only if every plane through K is the polar image of a point on L.

Via this polarity a more general concept of orthogonality can be introduced [11]:

Definition 1. Two lines L_0 and L_1 are called *orthogonal,* if and only if L_1 intersects the polar line $\pi(L_0)$ of L_0. Of course, in this case also L_0 intersects the polar line $\pi(L_1)$ of L_1.

Note that in this definition L_0 and L_1 are not necessarily intersecting. Similarly we can define orthogonality between a plane and a line.

Definition 2. A line L is perpendicular to the plane α, if and only if L is incident with the pole $\pi(\alpha)$ of α.

We are now in a position to formulate theorems on incidence and orthogonality relations between tetrahedra in the non-Euclidean geometry defined by the absolute quadric F.

3. Basic theorems

In this section we prove some basic orthogonality-related properties of tetrahedra or pairs of tetrahedra in elliptic or hyperbolic space. The following lemma will be very useful later.

Lemma 1. *The lines L_0, L_1, L_2 and L_3 are skew generators of a hyperboloid if and only if their polar images $K_i = \pi(L_i)$, $i \in \{0, 1, 2, 3\}$, are also skew generators of a hyperboloid.*

Lemma 1 is well-known and many simple proofs are conceivable, for example: Skew generators on a hyperboloid are characterized by an infinite number of transversal lines. If T intersects L_0, L_1, L_2, and L_3, its polar image $\pi(T)$ necessarily intersects K_0, K_1, K_2, and K_3. Thus, the latter lines are also skew generators on a hyperboloid.

Lemma 1 states that the polar image of a regulus is a regulus again. Moreover, when all L_i intersect in one point, their polar images K_i lie in a plane.

It is well-known that the altitudes of a tetrahedron lie in a regulus [1, 4] or are intersecting. This is also true in non-Euclidean spaces, as the following theorem shows.

Theorem 1. *Given a tetrahedron $\mathbf{A} = a_0a_1a_2a_3$ in elliptic or hyperbolic space, the four altitudes*

$$H_i = a_i \vee \pi(\alpha_i), \quad i \in \{0, 1, 2, 3\}$$

lie in a regulus or are intersecting.

Proof. We use a projective coordinate system, where

$$a_0 = [1, 0, 0, 0]^T, \quad a_1 = [0, 1, 0, 0]^T, \quad a_2 = [0, 0, 1, 0]^T, \quad a_3 = [0, 0, 0, 1]^T,$$

and the polarity π for points is given by the symmetric matrix $Q = (q_{ij})$, that is, $\pi(a) = Q \cdot a$. The Plücker coordinates [8, Section 2.1] of the four altitudes $H_i = a_i \vee \pi(\alpha_i)$ are

$$\begin{aligned} H_0 &= [q_{01}, q_{02}, q_{03}, 0, 0, 0]^T, & H_1 &= [q_{01}, 0, 0, 0, q_{13}, -q_{12}]^T, \\ H_2 &= [0, -q_{02}, 0, q_{23}, 0, -q_{12}]^T, & H_3 &= [0, 0, q_{03}, q_{23}, -q_{13}, 0]^T. \end{aligned}$$

The claim follows from $H_3 = H_0 - H_1 + H_2$. □

In Euclidean space, the four altitudes even lie on a hyperboloid of revolution [4]. This is not true in elliptic or hyperbolic space. (Here, hyperboloids of revolution are defined by the property that the pencil of quadrics spanned together with the absolute quadric F contains a double plane.)

Another generalization is the next theorem. It is known to be true in Euclidean space.

Theorem 2. *Let* $\mathbf{A} = a_0a_1a_2a_3$ *be a tetrahedron. If two pairs of opposite edges are orthogonal, then so is the third.*

Proof. We use the same projective coordinate system as in the proof of Theorem 1. Without loss of generality assume that

$$a_0 \vee a_1 \perp a_2 \vee a_3, \quad a_0 \vee a_2 \perp a_1 \vee a_3.$$

The Plücker coordinates of $A_{ij} = a_i \vee a_j$ are

$$A_{01} = [1, 0, 0, 0, 0, 0], \quad A_{02} = [0, 1, 0, 0, 0, 0], \quad A_{03} = [0, 0, 1, 0, 0, 0],$$

and the Plücker coordinates of $A^\star_{ij} = \pi(a_i) \cap \pi(a_j)$ are (only the relevant coordinates are shown at this point)

$$\begin{aligned} A^\star_{23} &= [*, *, *, q_{02}q_{13} - q_{03}q_{12}, *, *], \\ A^\star_{13} &= [*, *, *, *, q_{01}q_{23} - q_{03}q_{12}, *], \\ A^\star_{12} &= [*, *, *, *, *, q_{01}q_{23} - q_{02}q_{13}]. \end{aligned}$$

By assumption, A_{01} intersects $A^\star_{23}$ and A_{02} intersects $A^\star_{13}$. Thus, the Plücker product has to vanish, that is,

$$q_{02}q_{13} - q_{03}q_{12} = 0, \qquad q_{01}q_{23} - q_{03}q_{12} = 0.$$

Taking the difference of these two equations, we see that $q_{01}q_{23} - q_{02}q_{13} = 0$ which means that A_{03} intersects $A^\star_{12}$ or, equivalently, A_{03} and A_{12} are orthogonal ($a_0 \vee a_3 \perp a_1 \vee a_2$). □

4. Orthologic tetrahedra

So far we have only proved theorems concerning one tetrahedron. Now we take a closer look at pairs of tetrahedra. The next theorem is similar to Theorem 2, but relates two tetrahedra.

Theorem 3. *Let* $\mathbf{A} = a_0a_1a_2a_3$ *and* $\mathbf{B} = b_0b_1b_2b_3$ *be two tetrahedra. If five pairs of non-corresponding edges are orthogonal, then so is the sixth.*

Proof. Let $A_{ij} = a_i \vee a_j$ and $B_{ij} = b_i \vee b_j$ for pairwise different $i, j \in \{0, 1, 2, 3\}$. Without loss of generality we assume

$$A_{01} \perp B_{23}, \quad A_{02} \perp B_{13}, \quad A_{03} \perp B_{12}, \quad A_{12} \perp B_{03}, \quad A_{13} \perp B_{02}.$$

Denote the polar image of B_{ij} by $B^\star_{ij}$. Then

$$A_{01} \cap B^\star_{23} \neq \emptyset,\ A_{02} \cap B^\star_{13} \neq \emptyset,\ A_{03} \cap B^\star_{12} \neq \emptyset,\ A_{12} \cap B^\star_{03} \neq \emptyset,\ A_{13} \cap B^\star_{02} \neq \emptyset,$$

because of Definition 1. There exists a polarity χ in a regular quadric Q such that $\chi(A_{ij}) = B^\star_{kl}$. By Theorem 2, the tetrahedron is orthogonal with respect to the orthogonality given by the quadric Q. In particular, we have $A_{23} \cap B^\star_{01} \neq \emptyset$, or $A_{23} \perp B_{01}$. □

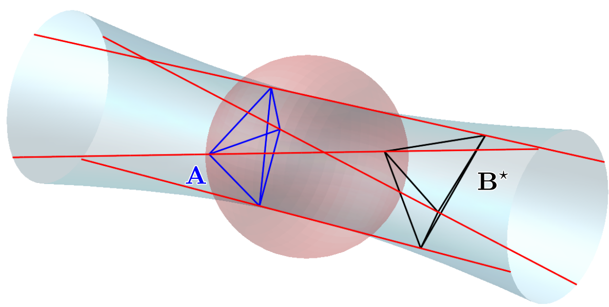

Figure 1: Ruled quadric (light-blue) with inscribed tetrahedra **A** (blue) and $\mathbf{B}^\star$ (black)

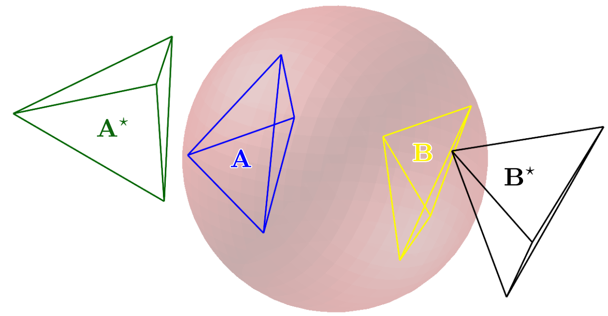

Figure 2: Tetrahedra with polar images: **A** (blue), $\mathbf{A}^\star$ (green), $\mathbf{B}^\star$ (black) and **B** (yellow)

For the upcoming proofs the following lemma is a useful statement.

Lemma 2. *Given are two tetrahedra* $\mathbf{A} = a_0a_1a_2a_3$ *and* $\mathbf{B} = b_0b_1b_2b_3$ *with respective face planes* α_i *and* β_i.

a) The connecting lines of corresponding vertices a_i *and* b_i *lie in a regulus if and only if the intersection lines of corresponding face planes* α_i *and* β_i *lie in a regulus.*

b) The connecting lines of corresponding vertices a_i *and* b_i *are concurrent if and only if the intersection lines of corresponding face planes* α_i *and* β_i *are co-planar.*

Proof. There exists a polarity χ in a regular quadric Q such that $\chi(a_i) = a_j \vee a_k \vee a_l$ and $\chi(b_i) = b_j \vee b_k \vee b_l$. This polarity interchanges the connecting lines of corresponding vertices with the intersection lines of corresponding face planes. This implies the second claim. The first claim follows from Lemma 1. □

Now we proceed with pairs of (skew-) orthologic tetrahedra.

Definition 3. Let $\mathbf{A} = a_0a_1a_2a_3$ and $\mathbf{B} = b_0b_1b_2b_3$ be two tetrahedra. The tetrahedron **A** is called *orthologic* with respect to **B**, if the lines through the vertex a_i of **A** and orthogonal to the corresponding face plane β_i of **B** are concurrent for $i \in \{0, 1, 2, 3\}$. If these four lines lie in a regulus, we say that **A** is *skew-orthologic* with respect to **B**.

As in the Euclidean case, the property of being (skew-) orthologic is symmetric.

Theorem 4. *Let* **A** *and* **B** *be two tetrahedra. Then* **A** *is (skew-) orthologic with respect to* **B** *if and only if* **B** *is (skew-) orthologic with respect to* **A**.

The statement of Theorem 4 is visualized in Figs. 1–3. We chose the absolute quadric F as a (Euclidean) sphere. Thus, the Cayley-Klein geometry is hyperbolic. In order to construct a pair of skew-orthologic tetrahedra **A** and **B**, we choose four lines L_0, L_1, L_2, L_3 in a regulus (red lines in Figure 1). The lines are chosen such that they intersect F in two points. Hence, it is possible to pick a point $a_i \in L_i$ in the interior of F and a point $\beta_i^\star$ in the exterior of F ($i \in \{0, 1, 2, 3\}$). Thus, we obtain two tetrahedra $\mathbf{A} = a_0a_1a_2a_3$ and $\mathbf{B}^\star = \beta_0^\star\beta_1^\star\beta_2^\star\beta_3^\star$. The polar image of a_i is the plane $a_i^\star$, the polar image of $\beta_i^\star$ is the plane β_i. These planes determine two further tetrahedra $\mathbf{A}^\star$ and **B** which are depicted in green and yellow in Figure 2. Theorem 4 states that the connecting lines of corresponding vertices of these two tetrahedra lie on a regulus (Figure 3).

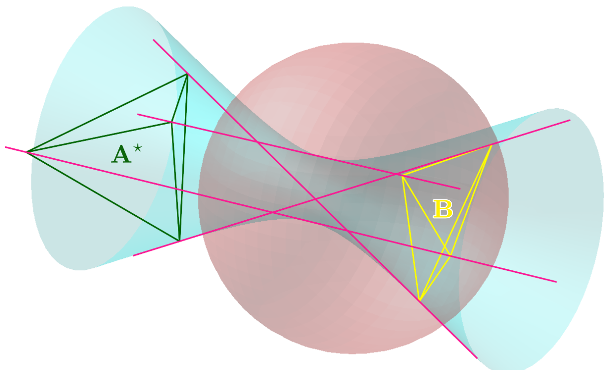

Figure 3: Ruled quadric (cyan) with inscribed tetrahedra $\mathbf{A}^\star$ and $\mathbf{B}$

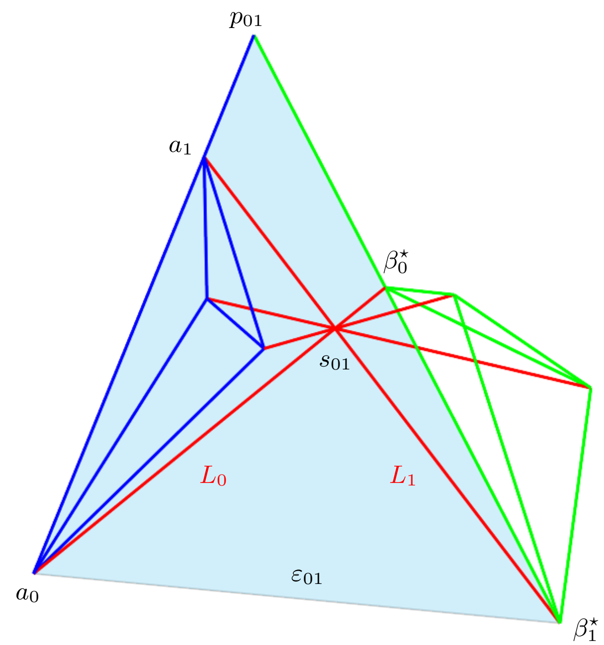

Figure 4: Sketch for the proof of Theorem 5

Proof of Theorem 4. We begin with the skew-orthologic case. Denote by L_i the line through a_i and orthogonal to β_i and by $L_i^\star = \pi(L_i)$ its polar image. By Lemma 1, the lines $L_i^\star$ lie in a regulus. If the indices $i, j, k, l \in \{0, 1, 2, 3\}$ are pairwise different, the line $L_i^\star$ lies in the plane $\beta_i = b_j \vee b_k \vee b_l$. Moreover, the line $L_i^\star$ intersects $\pi(\alpha_j) \vee \pi(\alpha_k)$, $\pi(\alpha_j) \vee \pi(\alpha_l)$, and $\pi(\alpha_k) \vee \pi(\alpha_l)$, that is, it lies in the plane $\pi(\alpha_j) \vee \pi(\alpha_k) \vee \pi(\alpha_l)$. In other words, $L_i^\star$ is the line of intersection of corresponding face planes of the two tetrahedra $\mathbf{B}$ and $\pi(\alpha_0)\pi(\alpha_1)\pi(\alpha_2)\pi(\alpha_3)$. We have to show that the connecting lines $b_i \vee \pi(\alpha_i)$ lie in a regulus. This follows from Lemma 2.a with a_i replaced by $\pi(\alpha_i)$.

In the orthologic case we can argue similarly but have to show that co-planarity of the lines $L_i^\star$ implies concurrency of the lines $b_i \vee \pi(\alpha_i)$ ($i \in \{0, 1, 2, 3\}$). This is Lemma 2.b. □

With this theorem we can take a closer look at orthogonal and anti-orthogonal tetrahedra. Their definition is again independent of the underlying space.

Definition 4. Let $\mathbf{A}$ and $\mathbf{B}$ be two tetrahedra. $\mathbf{A}$ and $\mathbf{B}$ are called *orthogonal*, if all corresponding edges are orthogonal:

$$a_i \vee a_j \perp b_i \vee b_j; \quad i \neq j.$$

$\mathbf{A}$ and $\mathbf{B}$ are called *anti-orthogonal*, if all non-corresponding edges are orthogonal:

$$a_i \vee a_j \perp b_k \vee b_l; \quad i, j, k, l \text{ pairwise different.}$$

In Euclidean geometry, anti-orthogonality and orthology are equivalent. [9] is an accessible reference with a simple proof. This result can be generalized to non-Euclidean spaces:

Theorem 5. *Two tetrahedra* $\mathbf{A} = a_0a_1a_2a_3$ *and* $\mathbf{B} = b_0b_1b_2b_3$ *are orthologic if and only if they are anti-orthogonal.*

Proof. Denote the face planes of $\mathbf{B}$ by $\beta_i = b_j \vee b_k \vee b_l$ ($i, j, k, l \in \{0,1,2,3\}$ pairwise different) and the vertices of the polar tetrahedron $\mathbf{B}^\star$ by $\beta_i^\star = \pi(\beta_i)$. Assume at first that the lines $L_i = a_i \vee \beta_i^\star$ (red lines in Figure 4) intersect in the point s, that is, $\mathbf{A}$ and $\mathbf{B}$ are orthologic. The lines $a_i \vee a_j$ lie in the plane spanned by L_i and L_j in which also $\beta_i^\star \vee \beta_j^\star = \pi(b_k) \vee \pi(b_l)$ lies. This means that the lines $a_i \vee a_j$ and $\beta_i^\star \vee \beta_j^\star$ intersect in a point p_{ij} — the tetrahedra $\mathbf{A}$ and $\mathbf{B}$ are anti-orthogonal.

Conversely, if $a_i \vee a_j$ intersects $\beta_i^\star \vee \beta_j^\star$ for all pairwise different $i, j \in \{0,1,2,3\}$, the lines $L_i = a_i \vee \beta_i^\star$ and $L_j = a_j \vee \beta_j^\star$ lie in the plane ε_{ij} spanned by a_i, a_j, $\beta_i^\star$, $\beta_j^\star$, and intersect in points $s_{ij} = L_i \cap L_j$. Without loss of generality take $i = 0$ and $j = 1$, then L_0 and L_1 intersect in s_{01} and span the plane ε_{01}. By assumption, L_2 intersects both, L_0 and L_1. There are two possibilities: $L_2 \subset \varepsilon_{01}$ or $s_{01} \in L_2$. In the first case, L_3 lies in ε_{01} as well, because it has to intersect L_0, L_1, and L_2. This leads to a degenerate tetrahedron $\mathbf{B}$ and is invalid. In the second case, L_3 necessarily contains s_{01} and the two tetrahedra $\mathbf{A}$ and $\mathbf{B}$ are orthologic. □

The following example shows that there is no characterization of skew-orthologic tetrahedra in terms of simple orthogonality or anti-orthogonality conditions. The absolute quadric F and its induced polarity π is represented by the matrix

$$F = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It has real points, that is, the space is of hyperbolic type. To get a pair of skew-orthologic tetrahedra $\mathbf{A}$ and $\mathbf{B}$, the points a_i and the poles $\beta_i^\star = \pi(\beta_i)$ ($i \in \{0,1,2,3\}$) are taken on a one sheeted hyperboloid of revolution, with $a_i \vee \beta_i^\star$ belonging to one ruling of the hyperboloid (a detailed description of this construction was given right before the proof of Theorem 4). We choose

$$\begin{aligned} &a_0 = [1,1,1,1]^T, \quad a_1 = [1,-2,1,2]^T, \quad a_2 = [1,-1,-3,3]^T, \quad a_3 = [1,-3,-1,3]^T, \\ &\beta_0^\star = [1,1,2,2]^T, \quad \beta_1^\star = [1,-3,1,3]^T, \quad \beta_2^\star = [1,-1,-4,4]^T, \quad \beta_3^\star = [1,-7,-4,8]^T. \end{aligned}$$

With the absolute polarity π and the given points $\beta_i^\star$, the face planes of $\mathbf{B}$ can be calculated as $\beta_i = \pi(\beta_i^\star) = F \cdot \beta_i^\star$. The vertices b_i are the intersection of the three corresponding face planes β_j, β_k and β_l. For the orthogonal (resp. anti-orthogonal) tetrahedra the lines $A_{ij}^\star = \pi(a_i) \cap \pi(a_j)$ and $B_{ij} = b_i \vee b_j$ (resp. $B_{kl} = b_k \vee b_l$) have to intersect. The calculated Plücker coordinates are

$$\begin{aligned} A_{01}^\star &= [1,-4,3,3,0,-1]^T, & B_{01} &= [-8,-10,-12,3,0,-2]^T, \\ A_{02}^\star &= [3,-2,-1,1,2,-1]^T, & B_{02} &= [20,3,19,4,5,-5]^T, \\ A_{03}^\star &= [2,-3,1,2,1,-1]^T, & B_{03} &= [16,9,13,-2,5,-1]^T, \\ A_{12}^\star &= [9,4,7,-1,4,-1]^T, & B_{12} &= [12,-11,5,4,3,-3]^T, \\ A_{13}^\star &= [5,0,5,1,2,-1]^T, & B_{13} &= [8,-3,-1,1,3,-1]^T, \\ A_{23}^\star &= [-3,-3,-4,1,-1,0]^T, & B_{23} &= [4,-9,7,4,1,-1]^T. \end{aligned}$$

Corresponding lines intersect, if the Plücker product vanishes, which is not the case in this example. So in general skew-orthologic tetrahedra are neither orthogonal nor anti-orthogonal.

5. Conclusion

As we have seen, some of the Euclidean properties of orthologic tetrahedra can be generalized to non-Euclidean spaces. So far only incidence and orthogonality relations in non-Euclidean geometries were considered. In further research one can ask for additional metric properties in the sense of [1, 2].

References

[1] L. Gerber: *Associated and Perspective Simplexes*. Trans. Amer. Math. Soc. **201**, 43–55 (1975).

[2] L. Gerber: *Associated and Skew-Orthologic Simplexes*. Trans. Amer. Math. Soc. **231**/(1), 47–63 (1977).

[3] O. Giering: *Vorlesungen über höhere Geometrie*. Vieweg, Braunschweig, Wiesbaden 1982.

[4] H. Havlicek, G. Weiss: *Altitudes of a tetrahedron and traceless quadratic forms*. Amer. Math. Monthly **110**, 679–693 (2003).

[5] F. Klein: *Vorlesungen über nicht-euklidische Geometrie*. Springer-Verlag, Berlin 1968.

[6] J. Neuberg: *Mémoire sur le tétraèdre*. F. Hayez, Bruxelles 1884.

[7] J. Neuberg: *Über orthologische Tetraeder*. Monatsh. Math **18**/(1), 212–218 (1907).

[8] H. Pottmann, J. Wallner: *Computational Line Geometry*. Mathematics and Visualization, Springer-Verlag, Berlin 2001.

[9] H.-P. Schröcker: *Orthologic Tetrahedra with Intersecting Edges*. KoG **13**, 13–18 (2009).

[10] J. Steiner: *Vorgelegte Lehrsätze*. Crelle's Journal **II**, 287–292 (1827).

[11] E.A. Weiss: *Einführung in die Liniengeometrie und Kinematik*. B.G. Teubner, 1935.

Received August 2, 2012; final form March 24, 2014