Equicevian Points and Cubics of a Triangle

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Abstract. A point $P$ in the plane of a given triangle $ABC$ is said to be equicevian if the cevians $AA_P$, $BB_P$, and $CC_P$ through $P$ are of equal length. In this note, we see that the set $\Omega$ of equicevian points can be obtained via three cubic curves, and we give a complete description of $\Omega$ including also the imaginary solutions. There exist up to ten equicevian points, among them the four focal points of the Steiner circumellipse. Besides, we present properties of the so-called equicevian cubics which in the irreducible case are strophoids, i.e., rational and circular cubics with orthogonal tangents at their node.

Key words: Equicevian points, equicevian cubics, strophoid, Steiner’s circum-ellipse, focal points, focal curves, pedal curves, Marden’s Theorem, Euclidean construction

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1. Introduction

It is well known that if any two of the four traditional centers (i.e., the centroid, the incenter, the circumcenter, and the orthocenter) coincide for a triangle, then the triangle must be equilateral (see, for example, [16]). The list can be extended to include other centers, such as the Fermat-Torricelli, the Gergonne, and Nagel points, as done in [12], and one would naturally feel that the list can be enlarged to include all reasonable or natural centers.

This paper is a result of our feeling that the celebrated Steiner-Lehmus Theorem should be placed in the context of coincidences of centers. The theorem states that if the internal angle bisectors of two angles in a triangle $ABC$ are equal, then the corresponding sides are equal. A poor-man’s version, which may turn out to be as strong, would state that if the three
internal angle bisectors are equal, then the triangle is equilateral. Stating it in this manner, one is tempted to view this as saying that if the incenter and the “equicevian center” of a triangle coincide, then it is equilateral, where the equicevian center would be the point the cevians through which are of equal length.

This raises the question whether there exists, for an arbitrary triangle, a point through which the cevians are equal, and whether such a point, if it exists, is unique. It is the main aim of this paper to address this problem, which turned out to be rather complicated. We shall see that a triangle can have as many as two real and two conjugate complex equicevian points (not counting vertices and points on the sidelines), and that a triangle can have two real equicevian points that are interior. There are triangles with no equicevian point at all. It will be proved that there is an intimate connection with the Steiner circumellipse and with Marden’s Theorem.

Related or not to the Steiner-Lehmus Theorem, the problem of existence and uniqueness of equicevian points is quite natural and interesting in its own. In fact, this problem has appeared in various forms at several places, and it would probably have appeared much more frequently had it been reasonably tractable.

2. The equicevian cubics

Definition 1. Let $ABC$ be a triangle. We call a point $P$ in the plane of $ABC$ $A$-equicevian if the lengths $|BB_P|$ and $|CC_P|$ of the cevians through $P$ are equal, and we define $B$-equicevian and $C$-equicevian points similarly. A point $P$ that is $A$- and $B$-equicevian (and hence also $C$-equicevian) is called equicevian. Thus, $P$ is equicevian if $|AA_P| = |BB_P| = |CC_P|$. If an equicevian point lies on one of the sidelines of $ABC$, we call it an improper and otherwise a proper equicevian point.

To study the set $\Omega_A$ of $A$-equicevian points of $ABC$, we place $ABC$ in the cartesian plane in such a way that

\[B = (-1, 0), \quad C = (1, 0), \quad A = (u, v), \quad v > 0,\]

and we take $P = (X, Y)$ to be an arbitrary point (see Figure 1). For simplicity, we assume that

\[|AB| \geq |AC|, \text{ i.e., } u \geq 0.\]

The equations of $AB$, $AC$, $AP$, $BP$, and $CP$ are given by

\[
AB: \quad y = \frac{(x+1)v}{u+1}, \quad AC: \quad y = \frac{(x-1)v}{u-1},
\]

\[
AP: \quad y = \frac{(Y-v)(x-u)}{X-u} + v, \quad BP: \quad y = \frac{(x+1)Y}{X+1}, \quad CP: \quad y = \frac{(x-1)Y}{X-1}.
\]

Thus the intersection $C_P$ of $AB$ and $CP$, the intersection $B_P$ of $AC$ and $BP$, and the intersection $A_P$ of $BC$ and $AP$ are

\[
C_P = \left(\frac{-vX + v - uY - Y}{vX - v - uY - Y}, \frac{-2vY}{vX - v - uY - Y}\right),
\]

\[
B_P = \left(\frac{vX + v + uY - Y}{vX + v - uY + Y}, \frac{2vY}{vX + v - uY + Y}\right),
\]

\[
A_P = \left(\frac{uY - vX}{Y - v}, 0\right).
\]
\[ \text{This implies} \]
\[ |AA_P|^2 = \frac{v^2[(X-u)^2 + (Y-v)^2]}{(Y-v)^2}, \quad |BB_P|^2 = \frac{4v^2[(X+1)^2 + Y^2]}{[v(X+1) - (u+1)Y]^2}, \]
\[ |CC_P|^2 = \frac{4v^2[(X-1)^2 + Y^2]}{[v(X-1) - (u-1)Y]^2}, \]
and therefore \((BB_P)^2 - (CC_P)^2 = 16v^2YH_A\), where
\[ H_A(X,Y) = (vX - uY)(X^2 + Y^2) + uv(X^2 - Y^2) - (u^2 - v^2 + 1)XY - (vX + uY) - uv. \]

Hence the \(A\)-equicevian points lie on the line \(BC\) or on the cubic which satisfies the equation \(H_A(X,Y) = 0\). We call this cubic the \(A\)-equicevian cubic associated with \(ABC\).

Similarly, the \(B\)-equicevian points lie on the line \(AC\) or on the \(B\)-equicevian cubic \(H_B(X,Y) = 0\), where
\[ H_B(X,Y) = ((u+3)Y - vX)(X^2 + Y^2) + (2u + 1)vX^2 - (2u + 5)vY^2 - (2u^2 - 2v^2 + 6u + 8)XY + (u^3 + 3u^2 + (u + 1)v^2 + 4u + 4)Y - (u^2 + v^2 + 2u - 4)vX + (u^2 + v^2 - 4)v. \]

The \(C\)-equicevian points lie on the line \(AB\) or on the \(C\)-equicevian cubic \(H_C(X,Y) = 0\), where
\[ H_C(X,Y) = ((u - 3)Y - vX)(X^2 + Y^2) + (2u - 1)vX^2 - (2u - 5)vY^2 - (2u^2 - 2v^2 - 6u + 8)XY + (u^3 - 3u^2 + (u - 1)v^2 + 4u - 4)Y - (u^2 + v^2 - 2u - 4)vX - (u^2 + v^2 - 4)v. \]

We summarize:

**Theorem 1.** All \(A\)-equicevian points lie on the line \(BC\) or on the \(A\)-equicevian cubic \(C_A\) which is the zero-set of the polynomial \(H_A(X,Y)\) defined in Eq. (3). Similarly, \(B\)-equicevian points lie on \(AC\) or on the \(B\)-equicevian cubic \(C_B\): \(H_B(X,Y) = 0\) given by (4), and \(C\)-equicevian points lie on \(AB\) or on the \(C\)-equicevian cubic \(C_C\): \(H_C(X,Y) = 0\) given by (5).

Figures 7, 8, 6, 9, and 11 show the cubics \(C_A, C_B, \) and \(C_C\) corresponding to the respective values \((u, v) = (0.2000, 1.7636), (0.620, 1.569), (0.00, 1.56 \text{ or } 1.85), \) and \((0.75, 1.70)\).

It is not surprising that our search for equicevian points of a triangle results in an algebraic problem. Therefore it makes sense to extend Definition 1 to non-real points by calling (real or imaginary) points \(P\) equicevian if \(|AA_P|^2 = |BB_P|^2 = |CC_P|^2\) — in the sense of Eq. (2).
3. Properties of the equicevian cubics

3.1. Particular $A$-equicevian points

For analyzing the $A$-equicevian cubic $C_A$ it is useful to adopt the projective point of view. For this purpose we introduce homogeneous coordinates

$$(X_0, X_1, X_2)$$

with $X = X_1/X_0$ and $Y = X_2/X_0$,

and we denote the triple of unknowns by $X = (X_0, X_1, X_2)$. Homogenization of $H_A(X, Y)$ gives the polynomial

$$P(X) = (vX_1 - uX_2)(X_1^2 + X_2^2) + uvX_0(X_1^2 - X_2^2)$$
$$- (u^2 - v^2 + 1)X_0X_1X_2 - X_0^2(vX_1 + uX_2) - uvX_0^3$$

(6)

with the zero set $C_A^*$, which is the projective extension of $C_A$. For the gradient $\text{grad} P(X) = (P_{X_0}(X), P_{X_1}(X), P_{X_2}(X))$ of $P(X)$ we obtain

$$P_{X_0}(X) = uv(X_1^2 - X_2^2) + (v^2 - u^2 - 1)X_1X_2 - 2X_0(vX_1 + uX_2) - 3uvX_0^2,$$
$$P_{X_1}(X) = v(X_1^2 + X_2^2) + 2X_1(vX_1 - uX_2) + 2uvX_0X_1 - (u^2 - v^2 + 1)X_0X_2 - vX_0^2,$$
$$P_{X_2}(X) = -u(X_1^2 + X_2^2) + 2X_2(vX_1 - uX_2) - 2uvX_0X_2 - (u^2 - v^2 + 1)X_0X_1 - uX_0^2.$$  

(7)

We recall that at any regular point $p \in \mathbb{P} = (p_0, p_1, p_2) \in \mathbb{P}$ of the projective algebraic curve $P(X) = 0$ the tangent line satisfies the equation

$$P_{X_0}(p)X_0 + P_{X_1}(p)X_1 + P_{X_2}(p)X_2 = 0.$$  

Singular points $s \in \mathbb{P}$ are characterized by $\text{grad} P(s) = 0$.

From now on we denote by $\overline{A}$, $\overline{B}$, and $\overline{C}$ the respective images of $A$, $B$, and $C$ under reflections in the midpoints of the opposite sides. $\overline{ABC}$ is called the anticomplementary triangle of $ABC$.

**Theorem 2.** The $A$-equicevian cubic $C_A$ passes through $B$ and $C$ and has the point $\overline{A} = (-u, -v)$ as a node (see Figure 2).

For $u \neq 0$ the cubic $C_A$ is irreducible and therefore rational. Otherwise it splits into the axis $\overline{AA}$ of symmetry and the circumcircle of the triangle $\overline{ABC}$.

As a converse of the statement in Theorem 1, all points of the line $BC$ and the cubic $C_A$ other than $B$, $C$, and $\overline{A}$ are $A$-equicevian.

**Proof.** The vector $(1, -u, -v)$ is a common zero of all three derivatives $P_{X_0}, P_{X_1}$ and $P_{X_2}$; therefore point $\overline{A}$ is singular. We will see later that for $u \neq 0$ the cubic is irreducible; in this case $\overline{A}$ is the only singular point of the projective cubic $C_A^*$.

The line at infinity $X_0 = 0$ intersects $C_A^*$ at the absolute circle-points $(0, 1, \pm i) \in \mathbb{P}$ and at the real point $A_\infty = (0, u, v) \in \mathbb{P}$, which is located on the median of $A$. The tangent line at $A_\infty$ is the real asymptote of $C_A$. It satisfies the equation

$$t_{A_\infty}: -uvX_0 + v(u^2 + v^2)X_1 - u(u^2 + v^2)X_2 = 0.$$  

(8)

In order to obtain a rational parametrization of $C_A$, we return to cartesian coordinates. All lines through $\overline{A}$ — up to the parallel to the $y$-axis — can be set up in parameter form by

$$X(\tau) = \tau - u, \quad Y(\tau) = k\tau - v, \quad \tau \in \mathbb{R},$$  

(9)
Figure 2: The $A$-equicevian cubic $C_A$ is an oblique strophoid with node $\overline{A}$ and singular focal point $F_A$.

where $k$ denotes the tangent of the slope angle. Each single line intersects $C_A^*$ beside $\overline{A}$ in a unique remaining point. The corresponding parameter $\tau$ is

$$\tau = \frac{uvk^2 + (u^2 - v^2 - 1)k - uv}{(uk - v)(1 + k^2)}.$$  

(10)

This leads to the following rational representation of $C_A$ in terms of the parameter $k \in \mathbb{R}$:

$$X = \frac{u^2k^3 - 2uvk^2 + (v^2 + 1)k}{(v - uk)(1 + k^2)}, \quad Y = \frac{(u^2 - 1)k^2 - 2uvk + v^2}{(uk - v)(1 + k^2)}.$$  

(11)

A vanishing denominator characterizes points at infinity.

Whenever $C_A$ is reducible, it must contain a line through $\overline{A}$. For the slope $k$ of such a component the nominator and denominator in (10) must vanish simultaneously. However, there is no $k \in \mathbb{R}$ with this property. Only for $k \to \infty$, i.e.,

$$X = u, \quad Y = -\frac{u(v + 1)}{u},$$
we obtain in the case \( u = 0 \) an indeterminate point of intersection with \( \mathcal{C}_A \).

Conversely, for \( u = 0 \) the polynomial \( H_A(X, Y) \) from (3) factorizes into

\[
H_A(X, Y) = X [v(X^2 + Y^2) + (v^2 - 1)Y - v].
\]

The cubic \( \mathcal{C}_A \) splits into \( X = 0 \) and into the circle through \( B, C \) and \( \overline{A} \) (see Figure 6).

The points \( B \) and \( C \) are not \( A \)-equicevian since the cevians through \( B \) or \( C \), respectively, are not determined. Also \( \overline{A} \) is not \( A \)-equicevian because of \( |BB_P| = |CC_P| = \infty \). Hence

\[
\Omega_A = (\text{line } BC \cup \mathcal{C}_A) \setminus \{B, C, \overline{A}\}
\]

(12) is the set of \( A \)-equicevian points.

All properties listed in the following theorem follow either from the geometric definition \( |BB_P| = |CC_P| \) of \( \mathcal{C}_A \) or from the equation \( H_A = 0 \) given in (3).

**Theorem 3.** The remaining point of intersection between \( \mathcal{C}_A \) and the line \( BC \) is \( A_0 = (-u, 0) \), the pedal point of \( \overline{A} \).

The remaining intersection points of the cubic \( \mathcal{C}_A \) with the line \( AB \) need not be real; they are located on the circle with center \( C \) and diameter \( A\overline{B} \). Similarly, \( \mathcal{C}_A \) passes through the intersection points of sideline \( AC \) with the circle with center \( B \) and diameter \( A\overline{C} \).

\( \mathcal{C}_A \) intersects the bisector\(^1\) of \( BC \) at point \( A_1 = (0, -v) \); the remaining two intersection points are conjugate complex with coordinates \((0, \pm i)\).

**Remark 1.** The intersection of the \( A \)-equicevian cubic with the internal angle bisector of \( A \) is investigated by V. NICULA and C. POHOAT˘A in [20], by S. ABU-SAYMEH and M. HAJJA in [1], and by V. OXMAN in [21]. The intersection with the median through \( A \) is investigated by S. ABU-SAYMEH and M. HAJJA in [2]. The intersection with the altitude through \( A \) is investigated by S. ABU-SAYMEH and M. HAJJA in [3]. Partial results on equicevian points for isosceles triangles have appeared in the MONTHLY problem in [9]. Results pertaining to the lengths of the cevians through an equicevian point have appeared as a MONTHLY problem in [13] and as a MATHEMATICAL OLYMPIAD problem in [4]. In the MONTHLY problem in [22] the existence of two equicevian points is proved.

Also, partial results pertaining to what one may call equi-semi-cevian points of a triangle were obtained for isosceles triangles. These are the points \( P \) for which the cevians \( AA_P, BB_P, \) and \( CC_P \) have the property that \( |PA_P| = |PB_P| = |PC_P| \). According to M. FOX in his Feedback [10] on J.A. SCOTT’s note [25], this problem appears as early as the nineteenth century in [11], where it was realized that the problem is intractable even if one restricts attention to the case when the triangle is isosceles.

### 3.2. The equicevian cubics are strophoids

**Definition 2.** An irreducible cubic is called **circular** if it passes through the absolute circle-points \((0, 1, \pm i)\) in \( \mathbb{R} \). A circular cubic is called **strophoid** if it has a node with orthogonal tangents. A strophoid without any axis of symmetry is called **oblique** (see, e.g., [15, p. 515] or [26, pp. 37–39]).

\(^1\)The bisector of two points is the perpendicular bisector of the line segment joining them.
For any circular cubic, the tangent lines at the absolute circle-points meet at a point which is called singular focal point of the cubic.

Suppose the strophoid has an axis \( s \) of symmetry. Then the reflection in \( s \), which maps the strophoid onto itself, must keep the node and the focal point fixed as well as the real ideal point together with its tangent, the asymptote. Since the asymptote \( t_{A_\infty} \) with the equation (8) cannot contain any finite point of the curve, the axis \( s \) must be orthogonal to the asymptote and pass through the node and the focal point.

**Theorem 4.** In the irreducible case, i.e., for \( u \neq 0 \), the \( A \)-equicevian cubic \( C_A \) has the following properties.

1. \( C_A \) is an oblique strophoid. The two tangent lines \( t_1, t_2 \) at the node \( A \) bisect the angle \( \angle BAC \) (Figure 2).

2. The singular focal point \( F_A \) of \( C_A \) lies on the cubic.\(^2\) The line which connects \( F_A \) with the node \( A \) is a symmedian of the anticomplimentary triangle \( AB'C \), i.e., it is symmetric to the median \( AA \) with respect to (‘w.r.t.’ in brief) the angle bisector of \( \angle BAC \). On this symmedian the point \( F_A \) is the pedal point w.r.t. the circumcenter \( U_A \) of the triangle \( BAC \). Also the line \( A_0A_1 \) (Theorem 3) passes through the focal point \( F_A \).

3. The strophoid \( C_A \) is the pedal curve of a parabola \( P_A \) w.r.t. the point \( A \) (Figure 3). The directrix of \( P_A \) coincides with the median \( AA \). The focal point \( F_{pA} \) of \( P_A \) lies on the symmedian \( AFA \) and is the mirror image of \( A \) under reflection in the line \( F_AU_A \). The tangent of the parabola \( P_A \) at its vertex passes through \( F_A \).

**Proof.** By virtue of (7), the gradients of the polynomial \( P(X) \) in (6) at the absolute circle-points are

\[
g_1 = (2uv + (v^2 - u^2 - 1)i, 2(v - iu), 2i(-ui + v)),
g_2 = (2uv - (v^2 - u^2 - 1)i, 2(v + iu), -2i(ui + v)).
\]

The point \( F_A \) of intersection of the two conjugate complex tangent lines at the absolute circle-points can be computed by the vector product \( f = g_1 \times g_2 \). We obtain the homogeneous coordinates

\[
F = f \mathbb{R} = (-2(u^2 + v^2), u(u^2 + v^2 + 1), v(u^2 + v^2 - 1)) \mathbb{R}.
\]

It can be verified by straightforward computation that \( P(f) = 0 \). This means that the singular focal point \( F_A \) of \( C_A \) is located on the cubic.

The \( y \)-coordinate of \( F_A \) is negative if and only if \( v(u^2 + v^2 - 1) \) is positive, which is equivalent to \( \angle A < \pi/2 \). After reflection of \( F_A \) in the midpoint of \( BC \) it satisfies the equation (8) of the asymptote \( t_{A_\infty} \) (Figure 2).

The slopes \( k_1 \) and \( k_2 \) of the two tangent lines \( t_1, t_2 \) at \( A \) satisfy \( \tau = 0 \), i.e., by (10)

\[
uvw^2 + (u^2 - v^2 - 1)k - uv = 0.
\]

We note that under \( u \neq 0 \) these two lines are orthogonal because of \( k_1k_2 = -1 \).

Two lines through \( A \) with slopes \( k \) and \( k' \) are in harmonic position w.r.t. the tangent lines \( t_1, t_2 \) at the node \( A \) if and only if they are corresponding under the involution with \( t_1 \) and \( t_2 \) as fixed lines. This involution \( k \mapsto k' \) can be expressed as

\[
2uvwk' + (u^2 - v^2 - 1)(k + k') - 2uv = 0.
\]

\(^2\)It can be proved that a circular cubic with a node is a strophoid if and only if its singular focal point lies on the cubic. We will learn in the sequel that exactly in this case the absolute circle-points are associated points (see Definition 3 and Theorem 6, 3).
This is the inhomogeneous version of a symmetric bilinear form, and it has the property that for \( k' = k \) it becomes proportional to (14), which means that the fixed lines of this involution coincide with \( t_1 \) and \( t_2 \).

The slopes \( k = \frac{v}{u + 1} \) of \( \overline{AC} \) and \( k' = \frac{v}{u - 1} \) of \( \overline{AB} \) satisfy Eq. (15), since

\[
\frac{2uv^3}{u^2 - 1} + \frac{2uv(u^2 - v^2 - 1)}{u^2 - 1} - 2uv = 0.
\]

Therefore the tangent lines \( t_1, t_2 \) at the node \( \overline{A} \) bisect the angle \( \angle B \overline{A}C \).

The line \( \overline{AF}_{A} \) with the equation

\[
2uvX_0 + v(u^2 + v^2 + 1)X_1 - u(u^2 + v^2 - 1)X_2 = 0
\]

has the slope \( k = \frac{v(u^2 + v^2 + 1)}{u(u^2 + v^2 - 1)} \). Together with the slope \( k' = \frac{v}{u} \) of the median \( \overline{AA} \) the slope \( k \) satisfies Eq. (15) since

\[
\frac{2uv^3(u^2 + v^2 + 1)}{u^2(u^2 + v^2 - 1)} + \frac{v(u^2 - v^2 - 1)}{u(u^2 + v^2 - 1)} 2(u^2 + v^2) - 2uv = 0 .
\]

Therefore, \( t_1 \) and \( t_2 \) bisect also the angle between the median \( \overline{AA} \) and the line \( \overline{AF}_{A} \), which characterizes the latter as a symmedian of the triangle \( \overline{ABC} \). By the way, we never have \( kk' = -1 \); the line \( \overline{AF}_{A} \) is never orthogonal to the asymptote \( t_{A_{\infty}} \). Hence the strophoid \( C_{A} \) is oblique as stated in item 2.

It can be verified that the line orthogonal to \( \overline{AF}_{A} \) and passing through \( F_{A} \) contains the point

\[
U_{A} = \left(0, \frac{1 - u^2 - v^2}{2v} \right)
\]

which is the circumcenter of the triangle \( \angle B \overline{A}C \) (see Figures 2 and 3).

In order to prove item 3, we first compute the equation of the line \( t \) which passes through an arbitrary point \( T \in C_{A} \) and is orthogonal to \( \overline{AT} \) (Figure 3). Suppose point \( T \) is defined by the parameter \( k \) in the rational parametrization (11). Then we obtain

\[
-\frac{1}{k} \left[ X + \frac{u^2k^3 - 2uvk^2 + (v^2 + 1)k}{(uk - v)(1 + k^2)} \right] = Y - \frac{(u^2 - 1)k^2 - 2uvk + v^2}{(uk - v)(1 + k^2)} ,
\]

which after division by \( (1 + k^2) \) reduces to

\[
t: (uk - v)X + k(uk - v)Y + k = 0.
\]

The homogeneous line coordinates of \( t \),

\[
(U_0, U_1, U_2) = (k, (uk - v), (uk - v)k) ,
\]

satisfy the homogeneous quadratic equation

\[
U_0(vU_1 - uU_2) + U_1U_2 = 0 .
\]
Figure 3: The strophoid $C_A$ is the pedal curve of the parabola $P_A$ w.r.t. $A$, which is a point of the parabola’s directrix $AA$. The points $Q$ and $Q'$ are associated on $C_A$.

This is the tangential equation of a parabola, since the polynomial on the left hand side is irreducible and contains $(U_0, U_1, U_2) = (1, 0, 0)$ as a zero. After inversion of the symmetric coefficient matrix we obtain the (point) equation of the parabola

$$P_A: X_0^2 + 2X_0(uX_1 - vX_2) + u^2X_1^2 + 2uvX_1X_2 + v^2X_2^2 = 0.$$ 

Its point of contact with the line at infinity is $(0, v, -u)R$. Hence the axis of $P_A$ is orthogonal to the median $AA$. On the other hand, the axis as the polar of $A_\infty$ satisfies

$$(u^2 - v^2) + u(u^2 + v^2)X + v(u^2 + v^2)Y = 0.$$ 

The line $BC$ and the bisector of $B$ and $C$ are mutually orthogonal tangents of $P_A$. Therefore the midpoint of $BC$ is a point of the directrix, and the directrix coincides with the median $AA$. The finite tangent drawn from $A_\infty$ to $P_A$ contacts at the vertex and has the equation

$$-uv + (u^2 + v^2)(vX + uY) = 0.$$ 

By virtue of (13), this tangent passes through $F_A$.

The respective second tangents through the absolute circle-points have the equation

$$\pm i + (\pm iu + v)X \mp i(\pm iu + v)Y = 0.$$ 

3In the reducible case $u = 0$ the lines $t$ which satisfy (17) form two pencils; the lines $t$ are parallel to the sideline $BC$ or they pass through the orthocenter $O$ of $ABC$. 
Their point of intersection is the parabola’s focal point $F_pA$ with the homogeneous coordinates \((u^2 + v^2), -u, v\) $\mathbb{R}$. Point $F_{pA}$ lies on the symmedian $\overline{AF_A}$; the second tangent drawn from $F_A$ to $P_A$ is the bisector of $F_{pA}$ and $\overline{A}$, which passes through the circumcenter $U_A$ of the triangle $ABC$.

Many properties of the equicevian cubic $C_A$ are related to a particular involutive point-pairing on the cubic and of projective origin. This will be addressed in the following two subsections.

3.3. Associated points

**Definition 3.** Tangents $t$ of the parabola $P_A$ intersect the cubic $C_A^*$ beside the pedal point $T$ w.r.t. $\overline{A}$ in two real or conjugate complex points $Q$ and $Q'$. We call them associated points of the cubic $C_A$.\(^4\)

In Figures 2 and 3 we find several pairs of associated points on $C_A^*$. The tangential equation of $P_A$ shows that the two tangents $t_1, t_2$ of $C_A$ at the node $\overline{A}$ are also tangent to $P_A$. Hence point $\overline{A}$ is self-associated. When the tangent $t$ of $P_A$ tends to infinity, the pedal point tends to $A_\infty$; therefore the absolute circle-points are associated, too. We summarize

**Theorem 5.** On the equicevian cubic $C_A^*$, the following pairs of points are associated: $(B, C)$, $(F_A, A_\infty)$, $(A_2, A_2')$ on the line $F_AU_A$, the conjugate complex points on the bisector of $BC$ (Theorem 3), the absolute circle-points, and $(\overline{A}, \overline{A})$.

**Theorem 6.** Pairs $(Q, Q')$ of associated points define on the $A$-equicevian cubic $C_A^*$ an involutive one-to-one correspondence with $\overline{A} \mapsto \overline{A}$ and the following properties:

1. The lines which connect $\overline{A}$ with a pair $(Q, Q')$ of associated points, $Q, Q' \neq \overline{A}$, are symmetric w.r.t. the bisectors $t_1, t_2$ of $\angle BAC$.

2. The midpoint of associated points $Q, Q'$ lies on the median $\overline{AA}$.

3. The tangents of $C_A$ at associated points meet each other at the point $T' \in C_A$, which is associated to the pedal point $T$ on the line $t = QQ'$ w.r.t. $\overline{A}$.

4. For any point $P \in C_A$, the lines $PQ$ and $PQ'$ are symmetric w.r.t. $P\overline{A}$.

5. Associated points share the lengths $|BB_Q| = |BB_{Q'}|$ and $|CC_Q| = |CC_Q'|$ of cevians.

**Proof.** item 1. Let point $Q' \in P_A$ with parameter $k'$ be one of the remaining intersection points of the line $t = QQ' \perp \overline{AT}$ with the cubic $C_A$. Then Eq. (16) implies

\[
k(u'k' - v)(1 + k'^2) - (uk - v)[u^2k'^3 - 2uvk'^2 + (v^2 + 1)k'] + k(uk - v) [(u^2 - 1)k'^2 - 2uvk' + v^2] = 0,
\]

which after some computations and division by $(k' - k)$ reduces to the quadratic equation in $k'$:

\[
[uk(u^2 - 1) - u^2v]k'^2 - 2uv(uk - v)k' - v(v^2 + 1 - uvk) = 0.
\]

The two zeros $k'_1$ and $k'_2$ with

\[
k'_1 + k'_2 = \frac{2uv(uk - v)}{uk(u^2 - 1) - u^2v}, \quad k'_1k'_2 = \frac{v(uk - v^2 - 1)}{uk(u^2 - 1) - u^2v}
\]

\(^4\)It can be proved that on the line $t$ the point of contact with the parabola $P_A$ and point $T$ separate the associated points $(Q, Q')$ harmonically.
satisfy the equation
\[ 2wv'k' + (u^2 - v^2 - 1)(k'_1 + k'_2) - 2uv = 0. \]
This means by (15) that the lines connecting \( A \) with a pair \((Q, Q')\) of associated points of \( C_A \) are symmetric w.r.t. the two node tangents \( t_1 \) and \( t_2 \).

We continue the proof of Theorem 6 after recalling some projective properties of cubics with a node.\(^5\)

### 3.4. Involutions on cubics with a node

Let \( C \) be a cubic with the node \( N \) (Figure 4).\(^6\) Each line through \( N \) intersects \( C \) beside \( N \) (multiplicity 2) in a single point. This defines a map of the pencil \( N \) of lines onto \( C \) which is one-to-one for lines which differ from the tangents \( t_1, t_2 \) at \( N \), while both tangents \( t_1 \) and \( t_2 \) are sent to \( N \). The following definition refers to this correspondance.

**Definition 4.** Let \( C \) be a cubic with the node \( N \). The involution in the pencil \( N \) with the two tangents \( t_1, t_2 \) at \( N \) as fixed lines induces an **involution \( \alpha \) of type 1** on \( C \). Any involution in the pencil \( N \) which interchanges \( t_1 \) and \( t_2 \) induces an **involution \( \beta \) of type 2** on \( C \).

Of course, involutions of both types keep the node \( N \) fixed. Note that these involutions do not preserve collinearity of triples of points on \( C \).

![Figure 4](image.png)

**Figure 4:** On the cubic \( C \) the involutions \( \alpha, \beta \) and \( \beta' = \alpha \circ \beta \) commute pairwise

**Lemma 7.** 1. For any involution \( \beta \) of type 2 on \( C \), all lines which connect corresponding points \( X, X\beta \in C, X \neq X\beta \), have a point \( Z \in C \setminus \{N\} \) in common, the ‘center’ of \( \beta \). Also the tangent lines at the fixed points \( Y, Y' \) of \( \beta \) pass through the center \( Z \) (see Figure 4).

2. Each involution \( \beta \) of type 2 commutes with the involution \( \alpha \) of type 1. Therefore \( \alpha \) maps each pair \((X, X\beta)\) of points corresponding under \( \beta \) again onto such a pair, and vice versa. In particular, the fixed points \( Y, Y' \) of \( \beta \) are corresponding under \( \alpha \).

---

\(^5\)There is a vast literature on nonsingular cubics, i.e., cubics without any singularity. In particular, the commutative group of points on the cubic is often addressed (e.g., [6]). Less attention is paid to cubics with a node.

\(^6\)The statements that follow are also valid for cubics with an isolated double point, i.e., in the case of conjugate complex node tangents.
3. Each involution $\beta$ of type 2 defines another involution $\beta' = \alpha \circ \beta = \beta \circ \alpha$ of type 2, and the involutions $\alpha$, $\beta$ and $\beta'$ commute pairwise. The centers $Z$ of $\beta$ and $Z'$ of $\beta'$ are corresponding under $\alpha$.

4. The lines connecting $Z'$ with corresponding points $X, X\beta \in C_A \setminus \{N, Z'\}$ constitute an involution in the pencil $Z'$. This involution keeps the line $Z'N$ fixed as well as the line through the fixed points of $\beta$.

5. For each point $Z' \in C \setminus \{N\}$, the lines connecting $Z'$ with corresponding points $(X, X')$ of $\alpha$ constitute an involution which keeps the line $Z'N$ fixed. For each quadrangle formed by pairs of points $(X, X')$ and $(Y, Y')$ corresponding under $\alpha$, the diagonal points $XY \cap X'Y'$ and $XY' \cap X'Y'$ lie on $C_\alpha$, and they are corresponding under $\alpha$, as well.

Proof. Two different involutions $\alpha, \beta$ in the pencil $N$ commute if and only if the fixed lines of one involution are corresponding under the other involution. Exactly in this case the composition $\beta' = \alpha \circ \beta$ is an involution, too.

In order to reduce the computational cost, we set up the given projective cubic $C$ in normal form \[8\]

$$C: \ X_0(X_1^2 - X_2^2) + X_3^3 = 0.$$

Point $N = (1, 0, 0)\mathbb{R}$ is the node with the tangents $X_1 \pm X_2 = 0$. The cubic $C$ admits the rational parametrization

$$X(t) = (t^3, -t(t^2 - 1), (t^2 - 1))$$

with $X(1) = X(-1) = (1, 0, 0)$ as the node. The parameter $t$ serves as a projective coordinate in the pencil $N$. Hence $t \mapsto t' = 1/t$ induces the involution

$$\alpha: \ C \to C, \ (t^3, -t(t^2 - 1), (t^2 - 1))\mathbb{R} \mapsto (1, (t^2 - 1), -t(t^2 - 1))\mathbb{R}.$$

The mapping $t \mapsto \frac{at - c}{ct - a}, a^2 \neq c^2$, induces on $C$ an involution $\beta$ of second kind:

$$\beta: \ (t^3, t(1 - t^2), (t^2 - 1))\mathbb{R} \mapsto ((at - c)^3, (at - c)(c^2 - a^2)(t^2 - 1), (ct - a)(a^2 - c^2)(t^2 - 1))\mathbb{R}.$$

For each $t \in \mathbb{R}$ the line connecting $X$ with $X\beta$ has the line coordinate vector

$$X(t) \times X(t)\beta = ((c^2 - a^2)(t^2 - 1), (c^2 - a^2)t^2 + act - c^2, ct(at - c)).$$

All these lines pass through the point

$$Z = (-c^3, c(c^2 - a^2), a(c^2 - a^2))\mathbb{R},$$

which lies on $C$ and belongs to the parameter $t = -c/a$.

When $X$ converges towards a fixed point $Y$ of $\beta$, the line $XX\beta$ converges against the tangent of $C$ at the fixed point. Cubics with a node are of class 4 (see \[8\]). Therefore, conversely, through point $Z \in C$ at most two tangents of $C$ can be drawn which differ from the tangent at $Z$.

The $\alpha$-image $Z'$ corresponds to $t = -a/c$ and has the homogeneous coordinates

$$Z' = (a^3, a(c^2 - a^2), c(c^2 - a^2))\mathbb{R}.$$

For the lines connecting $Z'$ with points $X$ and $X\beta$ we obtain the homogeneous line coordinates

$$Z'X = ((c^2 - a^2)(t^2 - 1), a^2 - act + (c^2 - a^2)t^2, at(a - ct))\mathbb{R},$$

$$Z'X\beta = ((a^2 - c^2)(t^2 - 1), a^2t^2 - act + (c^2 - a^2), a(at - c))\mathbb{R}.$$
Now we can verify that there is a projective transformation mapping $Z'X$ onto $Z'X\beta$, since in matrix form

$$
\begin{pmatrix}
(a^2 - c^2)(t^2 - 1) \\
a^2t^2 - act + (c^2 - a^2) \\
a(at - c)
\end{pmatrix} = \begin{pmatrix}
a^2 - c^2 & 0 & 0 \\
2a^2 - c^2 & c^2 - a^2 & 0 \\
ac & 0 & c^2 - a^2
\end{pmatrix} \begin{pmatrix}
(c^2 - a^2)(t^2 - 1) \\
2a^2 - c^2 & c^2 - a^2 & 0 \\
a(t - ct)
\end{pmatrix}.
$$

This means for the pencil $Z'$, there is a projectivity $Z'X \mapsto Z'X\beta$ which by definition is involutory and fixes the lines through $N$ and through the fixed points of $\beta$.

Since $\beta$ and $\beta' = \alpha \circ \beta$ commute, the line connecting the fixed points $Y,Y'$ of $\beta$ (see Figure 4) passes through $Z'$, and vice versa. For any point $X \in C \setminus \{N\}$ with $X \neq X\beta, X\beta'$, the four points $X, X\beta, X\beta'$, and $X' = X\alpha$ form a quadrangle with $Z$ and $Z'$ as diagonal points. We note coinciding lines $Z'X\beta = Z'X'$ and $ZX\beta' = ZX'$.

Item 5 is just a consequence of the fact that for any pair $(X,Y)$, $X,Y \in C \setminus \{N\}$, there is an involution $\beta$ of the second kind with $X \mapsto Y$. But also for each given point $Z \in C \setminus \{N\}$ there is such an involution with the center $Z$.

By the way, the plane cubic $C$ with a node can also be seen as the image of a spatial cubic under central projection. This offers a second way to prove Lemma 7 (note the analysis of a “strophoidal” spatial cubic in [7, 318–332]).

**Remark 2.** If point $Z$ is chosen as the unit point $E$ of a commutative group which can be defined on $C \setminus \{N\}$ (see, e.g., [6]), then by Lemma 7, 1. and 2. the image $Z'$ under $\alpha$ is the only point other than $Z = E$ which satisfies $Z' \circ Z' = E$. The involution $\alpha$ sends point $X \in C \setminus \{N\}$ onto $X' = Z' \circ X$.

### 3.5. Properties of equicevian cubics ctd.

According to Theorem 6, 1., associated points $(X,X')$ of the $A$-equicevian cubic $C_A$ are corresponding under the involution $\alpha$ of type 1, as defined in Definition 4. Hence we can directly apply Lemma 7.

**Proof of Theorem 6, items 2–5.** Let $\beta$ be the involution on $C_A$ induced by pairs of orthogonal lines through $A$. Then the absolute circle-points are the two fixed points of $\beta$. By virtue of Lemma 7, 1., the center $Z$ of $\beta$ coincides with the singular focal point $F_A$, and point $A_\infty$ serves as point $Z'$. By Lemma 7, 4. and 5., for each finite pair $(X,X')$ of associated points the connecting lines with $A_\infty$ are symmetrical w.r.t. $A_\infty A = \overline{AA}$. Hence the midpoint of $(X,X')$ lies on the median $A\overline{AB}$ (Figure 3).

Let $t$ be the line connecting the associated points $Q,Q' \in C_A \setminus \{A\}$. There is an involution $\beta_t$ of second kind which fixes $Q$. By Lemma 7, 2., $\beta_t$ fixes $Q'$ as well, and by Lemma 7, 1., the tangent lines of $C_A$ at $Q$ and $Q'$ meet at the center of $\beta_t$. This center is associated to the pedal point of $t$ w.r.t. $A$. Note in Figure 2, e.g., the points $(B,C)$ with pedal point $A_0$ and the associated meeting point $A'_0$ of the tangents at $B$ and $C$, or the pair $(A_2, A_2')$ with pedal point $F_A$ and $F'_A = A_\infty$ (Theorem 5).

In order to prove Theorem 6, 4. and 5., we choose two pairs of associated points of $C_A$, the pair $(Q,Q')$ and the two absolute circle-points. By virtue of Lemma 7, 5., the connecting lines with any point $Z' \in C_A$ belong to an involution which fixes $Z'N$. The second fixed line must be orthogonal to $Z'N$ since also the isotropic lines are corresponding. Consequently, the other two corresponding lines, $Z'Q$ and $Z'Q'$, must be symmetric w.r.t. $Z'N$. Applied to the choice $Z' = C$ (see Figure 3), the lines $CQ$ and $CQ'$ are symmetric w.r.t. to $\overline{AB}$ and the
altitude through \( C \). The latter guarantees that the cevians \( CC_Q \) and \( CC_Q' \) have the same length. The same holds for vertex \( B \). \( \square \)

**Remark 3.** We refer to Theorem 6, 4.: Let \((Q, Q')\) be fixed. Then the strophoid \( C_A \) is the locus of points \( P \) such that the lines \( PQ \) and \( PQ' \) have a bisector passing through \( A \). If the three given points \( Q, Q' \) and \( A \) were collinear and \( A \) different from the midpoint of \( QQ' \), the requested locus of points \( P \) is the Apollonius circle together with the diameter line \( QQ' \), hence a reducible strophoid.

The following two properties of strophoids are related to circles. The first is often used to characterize strophoids as particular sets of points (see, e.g., [17, pp. 95, 155], [18, pp. 59–70], or [24]).

**Theorem 8.** 1. Let a point \( D \) vary on the median \( AA' \). Then the equicevian cubic \( C_A \) is the locus of intersection points of the line \( DF_A \) and the circle centered at \( D \) and passing through the node \( A' \) (Figure 2).

2. Each pair \((Q, Q')\) of associated points of \( C_A \) lies on a circle which is centered on the line \( F_A U_A \) and passes through \( A \) and \( F_A \) (Figure 3). The circumcircle of the triangle \( ABC \) with center \( U_A \) is a particular case; it intersects \( C_A \) at \( B \) and \( C \).

3. The two pencils of circles mentioned before share a circle (dotted in Figure 3) which passes through the points \( A_2 \) and \( A'_2 \) on the bisectors of the two tangent lines \( t_1, t_2 \) at the node \( A \) (see also Figure 2).

**Proof.** We recall the involution \( \beta \) induced by pairs of orthogonal lines through \( A \) (note page 145): The circle with diameter \( XX_0 \beta \) must pass through \( A \) because of the right angle there. Since by virtue of Lemma 7, 4. the lines \( A_\infty X \) and \( A_\infty X_\beta \) are harmonic w.r.t. \( A_\infty A \) and the line at infinity, the center of this circle lies on the median \( AA' \). Due to Lemma 7, 1., the line \( XX_0 \beta \) passes through \( F_A \). In Figure 2, e.g., the pedal point \( A_0 \) of \( BC \) and the point \( A_1 \) on the bisector of \( B \) and \( C \) are corresponding in \( \beta \). The two intersection points \( A_2, A'_2 \) on the line \( F_A U_A \), which is perpendicular to \( A_\beta F_A \), are not only corresponding under \( \beta \), but also associated, and therefore fixed points of the involution \( \beta' = \alpha \circ \beta \) with the center \( A_\infty \).

In order to prove item 2, we start with the associated points \((Q, Q')\) on line \( t \) with pedal point \( T \) (Figure 3); the midpoint of \( QQ' \) lies on the median \( AA' \). Beside \( t \), also the line through \( Q \) orthogonal to \( \overline{AQ} \) and the line through \( Q' \) orthogonal to \( \overline{AQ'} \) are tangent to the parabola \( P_A \). These three tangent lines define a triangle (shaded in Figure 3) which according to a well-known theorem passes through the parabola’s focal point \( F_{pA} \). On the other hand, due to the right angles at \( Q \) and \( Q' \), this circle passes through \( A \), too. Therefore, its center lies on the bisector \( F_A U_A \) of \( F_{pA} \) and \( A \), but also on the bisector of \( Q \) and \( Q' \), which intersects the median \( AA' \) at the midpoint of \( QQ' \) (Theorem 6, 2.). \( \square \)

3.6. Equicevian cubics are focal curves

**Theorem 9.** Pairs \((Q, Q')\) of associated points on the \( A \)-equicevian cubic \( C_A \setminus \{B, C, A\} \) are the focal points of conics \( N \) which pass through \( B \) and \( C \) with the respective tangents \( B A \) and \( CA \) (Figure 5).

---

7 By the way, the second part of Lemma 7, 5. implies that the bisector of \( QQ' \) intersects \( C_A \) in two conjugate complex points. They are the finite intersection points of the isotropic lines through \( Q \) and those through \( Q' \).
Proof. For each conic \( N \) of the said set, the median \( AA \) is a diameter. We can specify any conic in this set by choosing its center \( M \) on the median \( AA \). Any choice different from \( A \) and from the midpoint of \( BC \) guarantees a regular conic.

The median \( AA \) is the directrix of the negative pedal curve \( P_A \). We show first that the axes of the conic \( N \) coincide with the tangents \( 3, 3' \) drawn from \( M \) to the parabola \( P_A \).

For this purpose we recall a theorem which is attributed to G. Desargues: Given any quadrilateral, there is a range of conics tangent to the four sidelines. Let \( P \) be a point which is not placed on any of these lines. Then the tangents drawn from \( P \) onto the conics of this range are pairs of an involution. Also the lines connecting \( P \) with opposite vertices of the given quadrilateral belong to this involution.

We pick out four tangents \( t_1, \ldots, t_4 \) of the parabola \( P_A \) (note the shaded quadrangle in Figure 5): the line \( t_1 \) at infinity, the sideline \( t_2 = BC \), the altitude \( t_3 \) at \( C \), which intersects the median \( AA \) at \( R \), and the second tangent \( t_4 \) drawn from \( R \) to \( P_A \). This tangent \( t_4 \) is orthogonal to \( t_3 \) and therefore parallel to \( AB \).

Point \( C = t_2 \cap t_3 \) and the ideal point \( t_1 \cap t_4 \) of \( AB \) are opposite vertices as well as \( R = t_3 \cap t_4 \) together with the ideal point \( t_1 \cap t_2 \) of \( BC \). By virtue of Desargues’ theorem, in the pencil \( M \) the lines 1 = \( MC \) and 1’ parallel to \( AB \) make a pair of the said involution. The same holds for 2 = \( MR = AA \) and 2’ parallel to \( BC \). The pairs (1, 1’) and (2, 2’) are at the same time conjugate diameters of the conic \( N \), which contacts the line \( CA \) at \( C \) and has \( A \) as the pole of \( BC \). The tangents 3 and 3’ of \( P_A \) form the orthogonal pair in the said involution; therefore they are the axes of \( N \).

On line 3, the two associated points \( Q, Q' \) of \( P_A \) are symmetric w.r.t. \( M \) (Theorem 6, 2.),
and their connections with $C$ are symmetric w.r.t. the altitude $t_3$ of $ABC$ (Theorem 6, 4.). This defines the pair $(Q, Q')$ uniquely.

It is well known that at each point $C$ of a conic the tangent and the normal bisect the angle between the connections with the focal points, which of course are also symmetric w.r.t. the center. Consequently, the associated points $Q, Q'$ on line 3 are identical with the focal points of $N$.

We know already from Footnote 7 (page 146) that exactly on one of these two orthogonal tangents $3, 3'$ of $P_A$ the points of intersection with the cubic are real. However, all four focal points are points of the $A$-equicevian cubic $C_A$.

Conversely, each pair $(Q, Q')$ of associated points other than $(\overline{A}, A)$ and $(B, C)$ has a midpoint $M \in \overline{AA}$, where a conic $N$ of the requested kind can be centered (note [15, p 515, footnote 235]). The particular pair $(F_A, A_{\infty})$ defines a parabola. Hence $F_A$ is the focal point of the parabola, which is tangent to $B\overline{A}$ and $C\overline{A}$ at $B$ and $C$, respectively. \hfill \Box

4. The equicevian points of a triangle

Points $P$ which are $A$-equicevian and $B$-equicevian, satisfy the property

$$|AA_P|^2 = |BB_P|^2 = |CC_P|^2.$$ 

Hence they are $C$-equicevian, too.

In order to find all equicevian points for a given triangle $ABC$, we have to intersect the sets $\Omega_A$ and $\Omega_B$ where according to Theorem 1 and by (12)

$$\Omega_A = (BC \cup C_A) \setminus \{B, C, \overline{A}\} \quad \text{and analogously} \quad \Omega_B = (AC \cup C_B) \setminus \{A, C, \overline{B}\}.$$ 

According to Theorem 3, on each sideline of the triangle there are at most two real equicevian points $P$. Those on $BC$ are characterized by the conditions $|AP| = |BC|$ and $P \neq B, C$ (note Figure 7).

![Figure 6: At isosceles triangles, two improper real equicevian points are either on the axis of symmetry or symmetric to it. At equilateral triangles the two become coincident.](image)
On at least two sidelines the equicevian points must be real, because if the altitude $h_A$ of $A$ is longer than $|BC|$, we obtain

$$h_B \leq |BC| < h_A \leq |AC|$$

and analogously $h_C < |AB|$.

Suppose one of the equicevian points $P$ on the sideline $BC$ coincides with the vertex $B$ or $C$. Then the triangle must be isosceles. Figure 6 shows the case $u = 0$ with $|AB| = |AC|$: By virtue of Theorem 2, the cubic $C_A$ splits into the axis of symmetry $AA$ and the circumcircle of $ABC$; the second node of $C_A$ is the orthocenter $O$ of $ABC$. Exactly for equilateral triangles the intersection $\Omega_A \cap \Omega_B \cap \Omega_C$ consists of a single point, the center of $ABC$, which is a node for all three cubics.

4.1. Remaining equicevian points

Proper equicevian points are among the intersection points between the cubics $C_A$ and $C_B$, where $C_B$ — the zero-set of the polynomial $H_B(X,Y)$ in (4) — has properties analogous to those of $C_A$ listed in Theorems 2, 3, 4, 6, and 8. These two cubics share the absolute circle-points, the vertex $C$ and the real or conjugate complex points of intersection of the line $AB$.
and the circle with center $C$ and radius $|AB|$ (Figures 7 and 8). Point $C$ and the absolute circle-points are intersection points of multiplicity 1 for the following reasons:

- The tangents of $C_A$ and $C_B$ at $C$ are the respective mirror images of $BC$ and $AC$ under reflection in the altitude of $C$ (note Figure 2). Therefore they must be different; they include an angle which is congruent to $\angle ACB$.
- The tangents of $C_A$ and $C_B$ at the absolute circle-points have the respective singular focal points $F_A$ and $F_B$ as their only real points. A coincidence of $F_A$ and $F_B$ would mean by Theorems 9 and 5 that there is a parabola which contacts all three sides of $ABC$ at their respective midpoints $C$, $A$ and $B$; the parabola would have a symmetry center at the centroid $G$ of $ABC$.

A higher intersection multiplicity between $C_A$ and $C_B$ can only appear at the points on line $AB$ (note, e.g., the cubics $C_B$ and $C_C$ in Figure 8 at $A_0$). According to Bézout’s Theorem, there are at most four points of intersection remaining, hence four real or pairwise conjugate complex proper equicevian points. We call them remaining equicevian points. They all must be finite as the medians through $A$ and $B$ can never be parallel. Note that all remaining equicevian points must be common to all three cubics (see Figures 7 and 8).

---

8The lines $\overline{AF_A}$, $\overline{BF_B}$ and $\overline{CF_C}$ are symmedians of the triangle $\triangle ABC$; they meet at the symmedian point $\overline{S}$ (= Lemoine- or Grebepoint, $X_6$ in the list of triangle centers given in [14]).
Figure 9: The negative pedal curves $P_A$, $P_B$, and $P_C$ of the three equicevian cubics $C_A$, $C_B$, and $C_C$ share the tangents $m$ and $n$ passing through the centroid $G$. These tangents are the principal axes of the Steiner circumellipse $S$ of $ABC$. The real remaining equicevian points $E_1$ and $E_2$ are the focal points of $S$.

4.2. The main theorem

**Theorem 10.** For any triangle $ABC$, the remaining four equicevian points, i.e., the finite intersection points of the three equicevian cubics $C_A$, $C_B$, and $C_C$, are identical with the two real and two conjugate complex focal points of the Steiner circumellipse $S$ of the triangle $ABC$. For non-isosceles triangles the axes of $S$ are common tangents of the three negative pedal curves, the parabolas $P_A$, $P_B$ and $P_C$.

**Proof.** We recall Theorem 9 and choose the centroid $G$ as the center of a conic $S$ which contacts $BA$ at $B$ and $CA$ at $C$. Since $G$ lies also on the median $BB$ and its conjugate diameter w.r.t. $S$ is parallel to $AC$ (note the lines 2 and 2’ in Figure 5), the same conic $S$ will contact $AB$ at $A$. Hence $S$ will be the Steiner circumellipse of the triangle $ABC$ (Figure 9). The two real focal points of $S$ as well as its two conjugate complex focal points are associated points not only of $C_A$, but also of $C_B$ and $C_C$ and hence the remaining four equicevian points. For non-equilateral triangles, the axes $m$, $n$ of $S$ are unique. They contain pairs of associated points w.r.t. $C_A$, $C_B$, and $C_C$. According to Definition 3, the axes $m$, $n$ of the Steiner ellipse $S$ are common tangents of the three negative pedal curves $P_A$, $P_B$ and $P_C$ (Figure 9).

**Lemma 11.** Given a non-equilateral triangle, among the two axes $m$, $n$ of the Steiner ellipse $S$, the major axis does not meet the interior of the longest side.
Figure 10: The pairs \((1, 1'), (2, 2')\) and \((3, 3')\) are conjugate diameters of the Steiner circumellipse. In the displayed case \(|BC| \geq |AB| \geq |AC|\) the real remaining equicevian points are located in the shaded closed area.

Under \(|BC| \geq |AB| \geq |AC|\) segments of the median through \(B\) and the parallel to \(BC\) through the centroid belong to the boundary of the centrally symmetric closed area where the real focal points of \(S\) are located (see Figure 10).

**Proof.** It is well known that the major axis of an ellipse lies within the acute angle formed by any two conjugate diameters. There are three pairs \((1, 1'), (2, 2'), (3, 3')\) of conjugate diameter lines of the Steiner ellipse \(S\) with its center at the centroid \(G\) (see Figure 10): each median is conjugate to the parallel of the respective side. The six involved lines through \(G\) meet the sides either at the vertices or at the midpoints or they intersect the sides in the ratio \(1:2\).

We first focus on non-isosceles triangles and assume \(|BC| > |AB| > |AC|\). For each side, the acute angle between the corresponding pair of conjugate diameters contains the common vertex with the neighbor side which is longer. Therefore the common intersection of the three acute angles cannot contain any point of the longest side \(BC\) (note shaded in Figure 10).

In the case of an isosceles triangle with either \(|BC| = |AB| > |AC|\) or \(|BC| > |AB| = |AC|\) either 3, 3' or 1, 1' are already the respective axes of \(S\). For the same reason as before in the first case 3 is the major axis, in the second case 1' (compare Figure 6).

Since \(S\) is the Steiner inellipse of the triangle \(\overline{ABC}\), the real remaining equicevian points, which are symmetric w.r.t. \(G\), must lie in the interior of \(\overline{ABC}\) (Figure 10).

We express the main theorem also in coordinates:

**Theorem 12.** Based on the coordinates (1) of the triangle \(ABC\) (see Figure 1), the remaining equicevian points \(E_1, \ldots, E_4\) are as below:

\[
\begin{align*}
E_{1,2} &= \left(\left(1 \pm 2k\right)\frac{u}{3}, \left(1 \pm \frac{2i}{k}\right)\frac{v}{3}\right), \\
E_{3,4} &= \left(\frac{1}{3} \left(u \pm \frac{2i}{k}\right), \frac{1}{3} (v \pm 2iku)\right), \\
\text{where } k &= \frac{1}{u\sqrt{2}} \sqrt{u^2 - v^2 + 3 + \sqrt{(u^2 + v^2 + 3)^2 - 12v^2}}, \\
\text{for } u \neq 0 &\text{ and } v^2 \leq 3 : E_{1,2} = \frac{1}{3} \left(u \pm 2\sqrt{3-v^2}, v\right), E_{3,4} = \frac{1}{3} \left(u, v \pm 2i\sqrt{3-v^2}\right), \\
\text{and } v^2 &\geq 3 : E_{1,2} = \frac{1}{3} \left(u, v \pm 2\sqrt{v^2-3}\right), E_{3,4} = \frac{1}{3} \left(u \pm 2i\sqrt{v^2-3}, v\right).
\end{align*}
\]
Proof. Due to the following corollary, which is equivalent to Marden's theorem\(^9\), the computation of the real remaining equicevian points \(E_1, E_2\) is straightforward. The two imaginary solutions \(E_3, E_4\) are the finite points of intersection of the isotropic lines passing through \(E_1\) and \(E_2\).

**Corollary 13.** Let \(a, b, c \in \mathbb{C}\) be the complex coordinates of the vertices of the triangle \(ABC\) w.r.t. a Cartesian coordinate system with the origin at the centroid \(G\). Then the complex coordinates of the two real remaining equicevian points of the triangle \(ABC\) are

\[
e_{1,2} = \pm \sqrt{\frac{2}{3}(a^2 + b^2 + c^2)}, \quad \text{where } a + b + c = 0.
\]

Proof. We start with recalling a wellknown lemma: For any ellipse centered at the origin let \(p, q \in \mathbb{C}\) be the complex coordinates of the endpoints of two conjugate diameters. Then \(f_{1,2} = \pm \sqrt{p^2 + q^2}\) are the complex coordinates of the real focal points.

This can be verified in the following way: Since a rotation about the origin through an angle \(\psi\), i.e., the multiplication with \(e^{i\psi}\), acts on \(f_{1,2}\) in the same way as on \(p\) and \(q\), we can set

\[
p = r \cos \varphi + is \sin \varphi \quad \text{and} \quad q = -r \sin \varphi + is \cos \varphi,
\]

where \(r, s \in \mathbb{R}\) are the semiaxes of the given ellipse. Then \(p^2 + q^2 = r^2 - s^2\) which confirms the statement.

For the Steiner circumellipse we can set \(p = a\) and \(q = \frac{1}{\sqrt{3}}(c - b)\). This implies for the real focal points and equicevian points

\[
e_{1,2}^2 = a^2 + \frac{1}{3}(b^2 + c^2 - 2bc).
\]

We subtract from the right hand side the vanishing term \(\frac{1}{3}[a^2 - (b + c)^2]\) and obtain

\[
e_{1,2}^2 = a^2 + \frac{1}{3}(b^2 + c^2 - 2bc - a^2 + b^2 + 2bc + c^2) = \frac{2}{3}(a^2 + b^2 + c^2),
\]

which is the statement of Corollary 13.

Remark 4. Corollary 13 is also the basis for a new proof of Marden’s Theorem: The vertices \(\overline{A}, \overline{B}\) and \(\overline{C}\) with the complex coordinates \(-2a, -2b\) and \(-2c\), respectively, are the zeros of the polynomial \(p(z) = (z + 2a)(z + 2b)(z + 2c)\). Because of \(a + b + c = 0\) it reduces to

\[
p(z) = z^3 - 2(a^2 + b^2 + c^2)z - 8abc.
\]

As a consequence, the derivative \(p'(z) = 3z^2 - 2(a^2 + b^2 + c^2)\) has the two roots \(e_1\) and \(e_2\) as given above.

### 4.3. Ruler and compass construction

Having Theorem 10 in mind, it is not hard to find ruler-and-compass constructions for the vertices and the focal points of the Steiner circumellipse. Nevertheless, also a direct construction of the remaining equicevian points deserves publication, as on the other hand there are no apparent relations to constructions for ellipses — like, e.g., to the construction of the axes from conjugate diameters.

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\(^9\)This theorem states (see, e.g., \([19, 5]\)): If in the complex plane the roots of a third-degree polynomial \(p(z)\) are the vertices of a triangle \(\overline{ABC}\), then the roots of the first derivative \(p'(z)\) are the focal points of the Steiner inellipse of \(\overline{ABC}\).
Figure 11: Ruler and compass construction of the remaining real equicevian points $E_1$ and $E_2$. The displayed equicevian cubics $C_A, C_B, C_C$ are only used as an illustration.

**Theorem 14.** The remaining real equicevian points $E_1$ and $E_2$ of a given triangle $ABC$ can be constructed by ruler and compass in the following way (Figure 11):

1. Draw the anticomplementary triangle $\overline{ABC}$ and the medians $\overline{AA}, \overline{BB}$ and $\overline{CC}$ meeting at the centroid $G$ of both the triangles $ABC$ and $\overline{ABC}$.

2. Determine the circumcenter $U_A$ of the triangle $\overline{ABC}$ and reflect the median $\overline{AA}$ in one angle bisector at $A$ in order to obtain the symmedian $s_A$ of $\overline{ABC}$. The pedal point of $s_A$ w.r.t. $U_A$ is denoted by $F_A$ (= focal point of $C_A$).

3. Draw the circle with diameter $GA$ and intersect it with the diameter line through $F_A$. The connections of these points with $G$ give two orthogonal lines $m$ and $n$.

4. The line $F_AU_A$ intersects $m$ in point $M_A$ and $n$ in $N_A$. Draw circles with centers $M_A$ and $N_A$, respectively, which pass through $\overline{A}$. The remaining equicevian points are the intersection points of the first circle with $n$ and of the second circle with $m$. By Lemma 11, the real solutions are found on the line which does not meet the interior of the triangle’s longest side.

**Proof.** Concerning item 3, the circle with diameter $\overline{AG}$ must on the one hand pass through the pedal points of $m$ and $n$ w.r.t. $\overline{A}$. On the other hand, this is a particular circle of the set explained in Theorem 8,1. The circles in item 4 belong to the pencil of circles described in Theorem 8,2. \qed
4.4. Real proper equicevian points in the interior

**Theorem 15.** Each triangle $ABC$ has 0, 1 or 2 real proper and 0 or 2, ..., 6 real improper equicevian points.

1. Only equilateral triangles have no real improper equicevian point; the center is the unique proper equicevian point.
2. For non-equilateral triangles, the number of proper real equicevian points is less than two if and only if one vertex lies on the ellipse $E$ which has the opposite side as its minor axis and the ratio $2 : 1$ of semiaxes.
3. The number of proper equicevian points is zero if and only if the triangle is isosceles with side lengths $2 : \sqrt{3} : \sqrt{3}$.

**Proof.** Suppose a real equicevian point $E$ lies on the sideline $BC$. Since the focal points of $S$ differ from the vertices $ABC$, point $E$ must coincide with the pedal point $A_0 = (-u, 0)$ of $A$ (see Figures 2 and 8). However, $A_0$ is equivecian if and only if $|A_0A| = |BC|$, i.e., $4u^2 + v^2 = 4$. This means, vertex $A$ lies on the ellipse $E$ with $B$ and $C$ as endpoints of the minor axis and the ratio $2 : 1$ of semiaxes (Figure 8).

When both real remaining equicevian points lie on sidelines, then according to Figure 10 one of them belongs to the side’s third next to the longest side while the other one lies in the neighboring half but outside the first third. On the other hand, the two points must be symmetric with respect to the centroid. There remains as the only possibility that the major axis of $S$ is parallel to the longest side. The triangle must be isosceles, and if $BC$ is not the longest side then point $A_0$ must separate $BC$ in the ratio $1 : 2$. This gives $u = \frac{1}{3}$ and $v = 2\sqrt{1 - u^2} = \frac{4}{3}\sqrt{2}$.

The final theorem gives necessary and sufficient conditions for the existence of proper equicevian points in the interior of the triangle $ABC$. These criteria are expressed in terms of the constant $k$ which is used in Theorem 12 and can also be written as

$$k = \frac{1}{u\sqrt{2}} \sqrt{u^2 - v^2 + 3 + \sqrt{(u^2 - v^2 + 3)^2 + 4u^2v^2}}. \quad (20)$$

The following geometric interpretation of $k$ plays also a role in [22].

**Lemma 16.** Let $S$ and $S'$ denote the respective third vertices of the two regular triangles with the basis $BC$. Then for each triangle $ABC$ with $A$ not located on the line $SS'$ the constant $k$ equals the ratio between major and minor axis of the unique ellipse which has the focal points $S, S'$ and passes through $A$.

**Proof.** Let $a, b$ be the major and minor semiaxes of the ellipse with focal points $S, S'$, which w.r.t. the coordinate system used in (1) have the coordinates $(0, \pm \sqrt{3})$. In terms of $k = \frac{a}{b} > 1$ these semiaxes satisfy

$$a^2 = \frac{3k^2}{k^2 - 1} \quad \text{and} \quad b^2 = \frac{3}{k^2 - 1}.$$

The ellipse passes through $A$ if and only if

$$(k^2u^2 + v^2)(k^2 - 1) = 3k^2.$$

The only positive solution of this equation is $k$ as given in (20).
Theorem 17. Let $ABC$ be a non-regular triangle with coordinates (1) and $u,v > 0$.

1. The real equicevian point $E_1$ from Theorem 12 lies in the interior of the triangle $ABC$ if and only if the constant $k$ from (20) satisfies $u < \frac{1}{k+1}$.

2. Point $E_2$ from Theorem 12 lies in the interior of the triangle $ABC$ if and only if $k > 2$ and $u < \frac{1}{k-1}$.

3. Both real equicevian points $E_1,E_2$ from Theorem 12 are in the interior of $ABC$ if and only if $k > 2$ and $u < \frac{1}{k+1}$.

Proof. We embed the plane of the triangle $ABC$ as the plane $z = 0$ into the Euclidean 3-space. Then point $P$ is an interior point of the triangle $ABC$ with coordinates $A = (u,v,0)$, $B = (-1,0,0)$ and $C = (1,0,0)$ and $u,v > 0$ if and only if all three vector products $\overrightarrow{BC} \times \overrightarrow{BP}$, $\overrightarrow{CA} \times \overrightarrow{CP}$ and $\overrightarrow{AB} \times \overrightarrow{CP}$ have a positive $z$-coordinate. By virtue of Theorem 12, for $P = E_1$ and $P = E_2$ we obtain the following triple of $z$-coordinates:

$$\frac{2(k+1)v}{3k}, \quad \frac{2(k-1)v}{3k}(1 \mp (k+1)u), \quad \frac{2(k+1)v}{3k}(1 \pm ku + u).$$

For $E_1$ the upper signs are valid; therefore the first and third terms are always positive (also in agreement with Lemma 11). The condition stated in the theorem characterizes a positive middle term.

For $E_2$ the lower signs are valid, and in this case the stated condition is equivalent to the requirement that the first and third terms are positive.

Item 3 follows from $\frac{1}{k+1} < \frac{1}{k-1}$, which holds for all $k > 1$. \hfill $\square$

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References


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