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Null Polarities as Generators of the Projective Group

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Abstract. It is well-known that the group of regular projective transformations of $\mathbb{P}^3(\mathbb{R})$ is isomorphic to the group of projective automorphisms of Klein's quadric $M_2^4 \subset \mathbb{P}^5(\mathbb{R})$. We introduce the Clifford algebra $\mathcal{C}\ell_{(3,3)}$ constructed over the quadratic space $\mathbb{R}^{(3,3)}$ and describe how points on Klein's quadric are embedded as null vectors, i.e., grade-1 elements squaring to zero. Furthermore, we discuss how geometric entities from Klein's model can be transferred to this homogeneous Clifford algebra model. Automorphic collineations of Klein's quadric can be described by the action of the so called sandwich operator applied to vectors $\mathfrak{v} \in \bigwedge^1 V$. Vectors correspond to null polarities in $\mathbb{P}^3(\mathbb{R})$. We introduce a factorization algorithm. With the help of this algorithm we are able to factorize an arbitrary versor $\mathfrak{g} \in \mathcal{C}\ell_{(3,3)}$ into a set of non-commuting vectors $\mathfrak{v}_i \in \bigwedge^1 V$, $i = 1, \ldots, k$, $1 \leq k \leq 6$ corresponding to null polarities with $\mathfrak{g} = \mathfrak{v}_1 \ldots \mathfrak{v}_k$. Thus, we present a new method to factorize every collineation in $\mathbb{P}^5(\mathbb{R})$ that is induced by a projective transformation acting on $\mathbb{P}^3(\mathbb{R})$ into a set of at most six involutoric automorphic collineations of Klein's quadric corresponding to null polarities respectively skew-symmetric 4×4 matrices. Moreover, we give an outlook for Lie's sphere geometry, i.e., the homogeneous Clifford algebra model constructed with the quadratic form corresponding to Lie's quadric $L_1^{n+1} \subset \mathbb{P}^{n+2}(\mathbb{R})$.

Key Words: Clifford algebra, line geometry, Klein's quadric, null polarity, factorization

MSC 2010: 51N15, 11E88, 51M30, 51B25

1. Introduction

Klein's quadric denoted by $M_2^4 \subset \mathbb{P}^5(\mathbb{R})$ and subspace intersections of Klein's quadric are well-known (cf. [12] or [14]). The group of regular projective transformations of $\mathbb{P}^3(\mathbb{R})$ is isomorphic to the group of automorphic collineations of Klein's quadric. Moreover, Cayley-Klein geometries can be represented by Clifford algebras, where the group of Cayley-Klein

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isometries is given by the Pin group of the corresponding Clifford algebra (see [6]). Therefore, we recall fundamental basics of Clifford algebras and recall the homogeneous Clifford algebra model corresponding to Klein's quadric. The relationsship between projective transformations and elements of the Pin group is given. The full construction can be found in [8].

2. Geometric algebra

General introductions to Clifford algebras can be found for example in [4] and [11].

Definition 1. Let V be a real valued vector space of dimension n. Furthermore, let $b : V \mapsto \mathbb{R}$ be a quadratic form on V. The pair (V, b) is called *quadratic space*.

We denote the matrix corresponding to b by B_{ij} with $i \leq j$, $j \leq n$. Therefore $b(x_i, x_j) = \mathrm{B}_{ij}$ for some basis vectors x_i and x_j.

Definition 2. The *Clifford algebra* is defined by the relations

$$x_i x_j + x_j x_i = 2\mathrm{B}_{ij}, \quad 1 \leq i, j \leq n. \tag{1}$$

Usually, the algebra is denoted by $\mathcal{C}\ell(V, b)$. By Silvester's law of inertia we can always find a basis $\{e_1, \ldots, e_n\}$ of V such that e_i^2 is either $1, -1$ or 0.

Definition 3. The number of basis vectors that square to $(1, -1, 0)$ is called *signature* (p, q, r). If $r \neq 0$ we call the geometric algebra *degenerated*. We denote a Clifford algebra by $\mathcal{C}\ell_{(p,q,r)}$.

Remark 1. A quadratic real space with signature $(p, q, 0)$ is abbreviated by $\mathbb{R}^{(p,q)}$.

With the new basis $\{e_1, \ldots, e_n\}$ the relations (1) become

$$e_i e_j + e_j e_i = 0, \quad i \neq j.$$

In the remainder of this paper we shall abbreviate the product of basis elements with lists

$$e_{12\ldots k} := e_1 e_2 \ldots e_k \text{ with } 0 \leq k \leq n.$$

The 2^n monomials $e_{i_1} e_{i_2} \ldots e_{i_k}$, $0 \leq k \leq n$, form the *standard basis* of the Clifford algebra. Furthermore, a Clifford algebra is the direct sum $\bigoplus_{i=0}^{n} \bigwedge^i V$ of all exterior products $\bigwedge^i V$ of any grade $0 \leq i \leq n$ where $e_{k_1} \ldots e_{k_i}$, $k_1 < \cdots < k_i$ form a basis of $\bigwedge^i V$. Therefore, a Clifford algebra is a *graded* algebra, and its dimension is calculated by

$$\dim \mathcal{C}\ell_{(p,q,r)} = \sum_{i=0}^{n} \dim \bigwedge^i V = \sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Moreover, the Clifford algebra $\mathcal{C}\ell_{(p,q,r)}$ can be decomposed in an *even* and an *odd part*

$$\mathcal{C}\ell_{(p,q,r)} = \mathcal{C}\ell^+_{(p,q,r)} \oplus \mathcal{C}\ell^-_{(p,q,r)} = \bigoplus_{\substack{i=0 \\ i \text{ even}}}^{n} \bigwedge^i V \oplus \bigoplus_{\substack{i=0 \\ i \text{ odd}}}^{n} \bigwedge^i V.$$

The even part $\mathcal{C}\ell^+_{(p,q,r)}$ is a subalgebra, because the product of two even graded monomials must be even graded since the generators cancel only in pairs. Elements contained in $\bigwedge^i V$ are called *i-vectors*, and the $\mathbb{R}$-linear combination of i-vectors $i \leq n$ is called a *multi-vector*. A multi-vector $\mathfrak{A}$ is called homogeneous if $[\mathfrak{A}]_i = \mathfrak{A}$, $i \leq n$ where $[\,\cdot\,]_m$ for $m \in \mathbb{N}$ denotes the grade-m part of the multi-vector $\mathfrak{A}$. The product of invertible vectors is called a *versor* and the product of vectors where at least one vector is not invertible is called a *null versor*.

Definition 4. The *center* of a ring $\mathcal{R}$ is the set of all elements that commute with all other elements

$$\mathcal{C}(\mathcal{R}) := \{c \in \mathcal{R} \mid cx = xc \text{ for all } x \in \mathcal{R}\}.$$

We are interested in the center of a Clifford algebra (see [5, p. 95]). The center of a Clifford algebra $\mathcal{C}\ell_{(p,q,r)}$ is

(1) $\mathcal{C}(\mathcal{C}\ell_{(p,q,r)}) = \{\alpha e_0 + \beta e_{12\ldots n} \mid \alpha, \beta \in \mathbb{R}\}$ if n is odd,

(2) $\mathcal{C}(\mathcal{C}\ell_{(p,q,r)}) = \{\alpha e_0 \mid \alpha \in \mathbb{R}\}$ if n is even.

For the even part the center is

(3) $\mathcal{C}(\mathcal{C}\ell^+_{(p,q,r)}) = \{\alpha e_0 \mid \alpha \in \mathbb{R}\}$ if n is odd,

(4) $\mathcal{C}(\mathcal{C}\ell^+_{(p,q,r)}) = \{\alpha e_0 + \beta e_{12\ldots n} \mid \alpha, \beta \in \mathbb{R}\}$ if n is even.

2.1. Clifford algebra automorphisms

For our purposes two automorphisms that exist on each Clifford algebra are interesting.

A. The *conjugation* is an *anti-involution* denoted by an asterisk (see [13]). Its effect on generators is given by $e_i^* = -e_i$. There is no effect on scalars. Extending the conjugation by using linearity yields

$$(e_{i_1} e_{i_2} \ldots e_{i_k})^* = (-1)^k e_{i_k} \ldots e_{i_2} e_{i_1}$$

with $0 \le i_1 < i_2 < \cdots < i_k \le n$. The geometric product of a vector $\mathfrak{v} = \sum_{i=1}^{n} x_i e_i \in \bigwedge^1 V$ with its conjugate results in

$$\mathfrak{v}\mathfrak{v}^* = -x_1^2 - x_2^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2 = -b(v, v),$$

where $v = (x_1, \ldots x_n)^{\mathrm{T}} \in \mathbb{R}^n$.

Definition 5. The *inverse element* of a versor $\mathfrak{v} \in \mathcal{C}\ell_{(p,q,r)}$ is defined by $\mathfrak{v}^{-1} := \frac{\mathfrak{v}^*}{N(\mathfrak{v})}$ with $N(\mathfrak{v}) := \mathfrak{v}\mathfrak{v}^*$.

The map $N: \mathcal{C}\ell_{(p,q,r)} \to \mathcal{C}\ell_{(p,q,r)}$ is called the *norm* of the Clifford algebra. For general multi-vectors $\mathfrak{M} \in \mathcal{C}\ell_{(p,q,r)}$ inverse elements exist and are defined through the relation $\mathfrak{M}\mathfrak{M}^{-1} = \mathfrak{M}^{-1}\mathfrak{M} = 1$, but the determination is more difficult and can be found in [3]. Note that in general not every element is invertible.

B. The other automorphism we are dealing with is the *main involution*. It is denoted by α and defined by

$$\alpha(e_{i_1} e_{i_2} \ldots e_{i_k}) = (-1)^k e_{i_1} e_{i_2} \ldots e_{i_k}$$

for $0 \le i_1 < i_2 < \cdots < i_k \le n$. The main involution has no effect on the even subalgebra and it commutes with the conjugation, i.e., $\alpha(\mathfrak{M}^*) = \alpha(\mathfrak{M})^*$ for arbitrary $\mathfrak{M} \in \mathcal{C}\ell_{(p,q,r)}$.

2.2. Clifford algebra products

For vectors $\mathfrak{a}, \mathfrak{b} \in \bigwedge^1 V$ we can write the *inner product* in terms of the geometric product

$$\mathfrak{a} \cdot \mathfrak{b} := \frac{1}{2}(\mathfrak{a}\mathfrak{b} + \mathfrak{b}\mathfrak{a}). \tag{2}$$

A generalization of the inner product to homogeneous multi-vectors can be found in [7]. For $\mathfrak{A} \in \bigwedge^k V,\ \mathfrak{B} \in \bigwedge^l V$ the generalized inner product is defined by

$$\mathfrak{A} \cdot \mathfrak{B} := [\mathfrak{A}\mathfrak{B}]_{|k-l|}\,.$$

There is another product on vectors, i.e., the *outer (or exterior) product*

$$\mathfrak{a} \wedge \mathfrak{b} := \frac{1}{2}(\mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a}). \tag{3}$$

This product can also be generalized to homogeneous multi-vectors (see again [7]). For $\mathfrak{A} \in \bigwedge^k V,\ \mathfrak{B} \in \bigwedge^l V$ the generalized outer product is defined by

$$\mathfrak{A} \wedge \mathfrak{B} := [\mathfrak{A}\mathfrak{B}]_{|k+l|}\,.$$

From Eqs. (2) and (3) follows that for vectors the geometric product can be written as the sum of the inner and the outer product

$$\mathfrak{a}\mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b} + \mathfrak{a} \wedge \mathfrak{b}.$$

More general, this can be defined for multivectors with the commutator and the anti-commutator product (see [10]). For treating geometric entities within this algebra context the definition of a k-blade, the *inner product null space*, and its dual the *outer product null space* is needed.

Definition 6. A *k-blade* is the k-fold exterior product of vectors $\mathfrak{v} \in \bigwedge^1 V$. Therefore, a k-blade can be written as

$$\mathfrak{A} = \mathfrak{a}_1 \wedge \mathfrak{a}_2 \wedge \cdots \wedge \mathfrak{a}_k.$$

If the k-blade squares to zero, it is called a *null k-blade*. The *inner product null space* (IPNS) of a blade $\mathfrak{A} \in \bigwedge^k V$ (cf. [10]) is defined by

$$\mathbb{NI}(\mathfrak{A}) := \left\{\mathfrak{v} \in \bigwedge^1 V \mid \mathfrak{v} \cdot \mathfrak{A} = 0\right\}.$$

Moreover, the *outer product null space* (OPNS) of a blade $\mathfrak{A} \in \bigwedge^k V$ is defined by

$$\mathbb{NO}(\mathfrak{A}) := \left\{\mathfrak{v} \in \bigwedge^1 V \mid \mathfrak{v} \wedge \mathfrak{A} = 0\right\}.$$

Remark 2. The same set can be described with inner product or outer product null spaces. In the non-degenrate case the change between both representations is achieved with the *pseudoscalar* $\mathfrak{J} = e_1 \dots e_n$

$$\mathbb{NO}(\mathfrak{A}) = \mathbb{NI}(\mathfrak{A}\mathfrak{J}).$$

Lateron we work in a homogeneous Clifford algebra model, and therefore, it does not matter from which side we multiply with the pseudoscalar, since the inner product respectively outer product null space is not affected. Moreover, multiplication with the pseudoscalar corresponds to the application of the polarity corresponding to the measure quadric. In the degenerate case the *Poincaré-identity* has to be used (compare to [6]).

2.3. Pin and Spin groups

With respect to the geometric product the units, i.e., the invertible elements of a Clifford algebra denoted by $\mathcal{C}\ell^{\times}_{(p,q,r)}$ form a group.

Definition 7. The *Clifford group* is defined by

$$\Gamma(\mathcal{C}\ell_{(p,q,r)}) := \left\{ \mathfrak{g} \in \mathcal{C}\ell^{\times}_{(p,q,r)} \mid \alpha(\mathfrak{g})\mathfrak{v}\mathfrak{g}^{-1} \in \bigwedge^1 V \text{ for all } \mathfrak{v} \in \bigwedge^1 V \right\}.$$

A proof that $\Gamma(\mathcal{C}\ell_{(p,q,r)})$ is indeed a group with respect to the geometric product can be found in [4]. We define two important subgroups of the Clifford group.

Definition 8. The *Pin group* is the subgroup of the Clifford group with $N(\mathfrak{g}) = \pm 1$.

$$\mathrm{Pin}_{(p,q,r)} := \left\{ \mathfrak{g} \in \mathcal{C}\ell_{(p,q,r)} \mid \mathfrak{g}\mathfrak{g}^* = \pm 1 \text{ and } \alpha(\mathfrak{g})\mathfrak{v}\mathfrak{g}^* \in \bigwedge^1 V \text{ for all } \mathfrak{v} \in \bigwedge^1 V \right\}.$$

Furthermore, we define the *Spin group* by $Pin_{(p,q,r)} \cap \mathcal{C}\ell^{+}_{(p,q,r)}$

$$\mathrm{Spin}_{(p,q,r)} := \left\{ \mathfrak{g} \in \mathcal{C}\ell^{+}_{(p,q,r)} \mid \mathfrak{g}\mathfrak{g}^* = \pm 1 \text{ and } \alpha(\mathfrak{g})\mathfrak{v}\mathfrak{g}^* \in \bigwedge^1 V \text{ for all } \mathfrak{v} \in \bigwedge^1 V \right\}.$$

Remark 3. For non-degenerated Clifford algebras the Pin group is a double cover of the orthogonal group of the quadratic space (V, b). Moreover, the Spin group is a double cover of the special orthogonal group of (V, b) (see [4]).

3. The homogeneous Clifford algebra model corresponding to Klein's quadric

To construct a homogeneous Clifford algebra model, we use a vector space as model for the projective space which is the base space of the model. Thus, we use $\mathbb{R}^6$ as model space for $\mathbb{P}^5(\mathbb{R})$ together with the quadric form b given by the corresponding symmetric matrix

$$\mathrm{B} = \begin{pmatrix} \mathrm{O} & \mathrm{I} \\ \mathrm{I} & \mathrm{O} \end{pmatrix}$$

where O denotes the 3×3 zero matrix and I the 3×3 identity matrix. The $\bigwedge^1 V$ subspace is identified with the vector space, and therefore, with the projective space. The square of a general vector $\mathfrak{v} = \sum_{i=1}^{6} x_i e_i$ results in

$$\mathfrak{v}\mathfrak{v} = x_1 x_4 + x_2 x_5 + x_3 x_6.$$

This is the equation of Klein's quadric, and a vector corresponds to a point on Klein's quadric if it is a null vector, i.e., a vector squaring to zero. Since the Clifford algebra allows the description of subspaces with the use of blades of higher grade, we can express linear line manifolds within the homogeneous Clifford algebra model (see [8, 9]).

3.1. Linear line manifolds

Clifford algebras carry the subspace structure of Grassmann algebras together with the metric properties derived with the corresponding Cayley-Klein geometry. Therefore, we construct blades of grade k, $k \leq 5$, and examine the geometric inner product and geometric outer product null spaces.

Two-blades: The exterior product of two vectors corresponds to a two-blade that describes a line P_1^1 in $\mathbb{P}^5(\mathbb{R})$. We use the outer product null space to determine the set of points on P_1^1 contained in M_2^4, and therefore, the set of lines in $\mathbb{P}^3(\mathbb{R})$ corresponding to the two-blade. If the whole line P_1^1 is contained in M_2^4 the corresponding two-blade is a null two-blade. Thus, the outer product null space corresponds to a pencil of lines in $\mathbb{P}^3(\mathbb{R})$ or to two lines in $\mathbb{P}^3(\mathbb{R})$ that may be identical. The inner product null space of a two-blade corresponds to a three-space intersection with Klein's quadric, i.e., a linear line congruence in $\mathbb{P}^3(\mathbb{R})$.

Three-blades: Three-blades can be generated as the exterior product of three vectors and describe two-spaces in $\mathbb{P}^5(\mathbb{R})$. The outer product null space of a three-blade corresponds to the two-space intersection of the two-space P_1^2 described by the three-blade with M_2^4. If the three-blade is a null three-blade P_1^2 is completely contained in Klein's quadric and the outer product null space corresponds to a bundle or a field of lines in $\mathbb{P}^3(\mathbb{R})$. The outer product null space of a non-null three-blade corresponds to a conic section of M_2^4 that defines a regulus in $\mathbb{P}^3(\mathbb{R})$. In a dual way the inner product null space of a three-blade is the intersection of the image of P_1^2 under the polarity corresponding to the measure quadric with Klein's quadric.

Four-blades: Outer product null spaces of four-blades correspond to three-space intersections with Klein's quadric, and the corresponding set of lines in $\mathbb{P}^3(\mathbb{R})$ is a linear congruence of lines. The dual of a three-space in $\mathbb{P}^5(\mathbb{R})$ is a line. Thus, the inner product null space of a four-blade is the intersection of this line with M_2^4.

Five-blades: Five-blades describe four-spaces in Klein's model, and the intersection of a four-space with Klein's quadric results in a linear congruence of lines. The outer product null space of a five-blade corresponds to a linear congruence of lines. Moreover, the inner produt null space of a five-blade yields a vector.

3.2. Clifford group action

The null vectors correspond to points on Klein's quadric. Moreover, the sandwich action of an arbitrary versor on a null vector results in a null vector again. Thus Klein's quadric is mapped to itself. To examine the action of the whole Clifford group, we first investigate the action of non-null vectors on null vectors.

Remark 4. Since we are working in a homogeneous Clifford algebra model, the multiplication with a real factor does not change the geometric meaning of an algebra element. Therefore, we use the conjugate element instead of the inverse element. Hence, the *sandwich operator* that we use is given by

$$\alpha(\mathfrak{a})\mathfrak{v}\mathfrak{a}^* = \mathfrak{a}\mathfrak{a}^*(\alpha(\mathfrak{a})\mathfrak{v}\mathfrak{a}^{-1}).$$

This operator does not involve an inverse, and therefore, it can also be applied if the element $\mathfrak{a}$ is not invertible.

Sandwich action of vectors: Let $\mathfrak{a} = \sum_{i=1}^6 a_i e_i$ and $\mathfrak{v} = \sum_{i=1}^6 x_i e_i$ be two vectors with $\mathfrak{a}\mathfrak{a} \neq 0$ and $\mathfrak{v}\mathfrak{v} = 0$. The action of the sandwich operator $\alpha(\mathfrak{a})\mathfrak{v}\mathfrak{a}^*$ on the $\bigwedge^1 V$ subspace can

be expressed as product of a matrix with a vector. The matrix has the form:

$$\mathrm{M} = \begin{pmatrix} k_1 & a_1a_5 & a_1a_6 & a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_4 & k_2 & a_2a_6 & a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_4 & a_3a_5 & k_3 & a_3a_1 & a_3a_2 & a_3a_3 \\ a_4a_4 & a_4a_5 & a_4a_6 & k_4 & a_4a_2 & a_4a_3 \\ a_5a_4 & a_5a_5 & a_5a_6 & a_5a_1 & k_5 & a_5a_3 \\ a_6a_4 & a_6a_5 & a_6a_6 & a_6a_1 & a_6a_2 & k_6 \end{pmatrix} \tag{4}$$

with

$$\begin{aligned} k_1 &= -a_5a_2 - a_6a_3, & k_2 &= -a_6a_3 - a_4a_1, & k_3 &= -a_4a_1 - a_5a_2, \\ k_4 &= -a_5a_2 - a_6a_3, & k_5 &= -a_6a_3 - a_4a_1, & k_6 &= -a_4a_1 - a_5a_2. \end{aligned}$$

This involutoric automorphism of Klein's quadric corresponds to a null polarity acting on $\mathbb{P}^3(\mathbb{R})$ (see [8]). Furthermore, the Clifford group is generated by invertible vectors, and therefore, all elements of the Clifford group correspond to compositions of null polarities in $\mathbb{P}^3(\mathbb{R})$. Moreover, this means that every element of $\mathcal{C}\ell^+_{(3,3)}$ that is an even product of vectors corresponds to a collineation, and every element of $\mathcal{C}\ell^-_{(3,3)}$ that is an odd product of vectors corresponds to a correlation.

Collineations: A general element $\mathfrak{g} \in \mathcal{C}\ell^+_{(3,3)}$ corresponding to a collineation is given by

$$\begin{aligned} \mathfrak{g} = {} & g_1e_0 + g_2e_{12} + g_3e_{13} + g_4e_{14} + g_5e_{15} + g_6e_{16} + g_7e_{23} + g_8e_{24} + g_9e_{25} + g_{10}e_{26} + g_{11}e_{34} \\ & + g_{12}e_{35} + g_{13}e_{36} + g_{14}e_{45} + g_{15}e_{46} + g_{16}e_{56} + g_{17}e_{1234} + g_{18}e_{1235} + g_{19}e_{1236} + g_{20}e_{1245} \\ & + g_{21}e_{1246} + g_{22}e_{1256} + g_{23}e_{1345} + g_{24}e_{1346} + g_{25}e_{1356} + g_{26}e_{1456} + g_{27}e_{2345} + g_{28}e_{2346} \\ & + g_{29}e_{2356} + g_{30}e_{2456} + g_{31}e_{3456} + g_{32}e_{123456}. \end{aligned}$$

We derive constraints to this element by $\alpha(\mathfrak{g})\mathfrak{v}\mathfrak{g}^* \in \bigwedge^1 V$ for all $\mathfrak{v} \in \bigwedge^1 V$. If we try to compute the corresponding collineation we have to distinguish the type of the collineation, i.e., if it acts on points or on planes.

Action on points: First we assume $\mathfrak{g} \in \mathcal{C}\ell^+_{(3,3)}$ corresponds to a collineation acting on points. The automorphic collineation of Klein's quadric induced by $\mathfrak{g}$ can be transferred to a collineation that acts on $\mathbb{P}^3(\mathbb{R})$. Therefore, the element $g \in \mathcal{C}\ell^+_{(3,3)}$ is applied to a null three-blade corresponding to a bundle of lines under the constraint that $\mathfrak{g}$ is a versor (see [8]). This results in the matrix $(m_{i,j})$, $i, j = 0, \ldots, 3$, for the collineation with

$$\begin{aligned} m_{00} &= g_1 - g_{20} - g_{24} - g_{32} - g_{29} + g_9 + g_4 + g_{13}, \\ m_{11} &= g_{24} - g_9 + g_{20} - g_{13} - g_{32} + g_1 + g_4 - g_{29}, \\ m_{22} &= g_1 - g_{13} - g_{32} - g_4 + g_{29} + g_9 - g_{24} + g_{20}, \\ m_{33} &= g_{24} + g_{13} + g_{29} + g_1 - g_4 - g_9 - g_{20} - g_{32}, \end{aligned}$$

and

$$\begin{aligned} m_{01} &= 2(g_7 + g_{17}), & m_{02} &= 2(g_{18} - g_3), & m_{03} &= 2(g_{19} + g_2), \\ m_{10} &= -2(g_{26} + g_{16}), & m_{12} &= 2(g_5 + g_{25}), & m_{13} &= 2(g_6 - g_{22}), \\ m_{20} &= 2(g_{15} - g_{30}), & m_{21} &= 2(g_8 + g_{28}), & m_{23} &= 2(g_{21} + 2g_{10}), \\ m_{30} &= -2(g_{31} + g_{14}), & m_{31} &= 2(g_{11} - g_{27}), & m_{32} &= 2(g_{23} + g_{12}). \end{aligned}$$

The system of equations

$$(m_{i,j}) = (c_{i,j}), \quad i,j = 0,\ldots,3\,, \tag{5}$$

can be used to compute an element of $\mathcal{C}\ell^{+}_{(3,3)}$ corresponding to a collineation acting on points with matrix representation $\mathrm{C} = (c_{i,j})$, $i,j = 0,\ldots,3$. Therefore, constraint equations derived from $\alpha(\mathfrak{g})\mathfrak{v}\mathfrak{g}^* \in \bigwedge^1 V$ and $\mathfrak{g}\mathfrak{g}^* = \pm 1$ are used.

Action on planes: Moreover, the same automorphic collineation of Klein's quadric may correspond to a collineation in $\mathbb{P}^3(\mathbb{R})$ that maps planes to planes. This action can also be represented as matrix vector product. Therefore, the general element $\mathfrak{g} \in \mathcal{C}\ell^{+}_{(3,3)}$ is applied to a null three-blade corresponding to a field of lines. The action can be expressed as collineation with matrix $(m_{i,j})$, $i,j = 0,\ldots,3$,

$$\begin{aligned}
m_{00} &= g_{32}-g_{20}-g_{13}-g_{29}-g_{9}-g_{24}+g_{1}-g_{4},\\
m_{11} &= g_{1}+g_{9}+g_{20}+g_{24}+g_{13}-g_{29}+g_{32}-g_{4},\\
m_{22} &= g_{20}+g_{29}+g_{4}+g_{13}-g_{9}-g_{24}+g_{1}+g_{32},\\
m_{33} &= g_{9}+g_{24}+g_{29}-g_{13}+g_{1}+g_{4}-g_{20}+g_{32},
\end{aligned}$$

and

$$\begin{aligned}
m_{01} &= 2(g_{16}-g_{26}), & m_{02} &= -2(g_{15}+g_{30}), & m_{03} &= 2(g_{14}-g_{31}),\\
m_{10} &= 2(g_{17}-g_{7}), & m_{12} &= 2(g_{28}-g_{8}), & m_{13} &= -2(g_{27}+g_{11}),\\
m_{20} &= 2(g_{3}+g_{18}), & m_{21} &= 2(g_{25}-g_{5}), & m_{23} &= 2(g_{23}-g_{12}),\\
m_{30} &= 2(g_{19}-g_{2}), & m_{31} &= -2(g_{22}+g_{6}), & m_{32} &= 2(g_{21}-g_{10}).
\end{aligned}$$

Depending on the action of the collineation, i.e., if it acts on points or planes the elements of the Spin group can be transferred to their matrix representations and vice versa.

Correlations: General elements of the odd part $\mathcal{C}\ell^{-}_{(3,3)}$ have the form

$$\begin{aligned}
\mathfrak{h} = {} & h_1e_1+h_2e_2+h_3e_3+h_4e_4+h_5e_5+h_6e_6+h_7e_{123}+h_8e_{124}+h_9e_{125}+h_{10}e_{126}+h_{11}e_{134}\\
& +h_{12}e_{135}+h_{13}e_{136}+h_{14}e_{145}+h_{15}e_{146}+h_{16}e_{156}+h_{17}e_{234}+h_{18}e_{235}+h_{19}e_{236}+h_{20}e_{245}\\
& +h_{21}e_{246}+h_{22}e_{256}+h_{23}e_{345}+h_{24}e_{346}+h_{25}e_{356}+h_{26}e_{456}+h_{27}e_{12345}+h_{28}e_{12346}\\
& +h_{29}e_{12356}+h_{30}e_{12456}+h_{31}e_{13456}+h_{32}e_{23456}.
\end{aligned}$$

Their action on points or planes of the three-dimensional projective space $\mathbb{P}^3(\mathbb{R})$ can be described by the sandwich action on null three-blades corresponding to two-spaces contained entirely in M_2^4 that correspond to bundles of lines or fields of lines.

Action on planes: The action of a general element $\mathfrak{h} \in \mathcal{C}\ell^{-}_{(3,3)}$, when applied to a null three-blade corresponding to a two-space contained in Klein's quadric which is the image of a field of lines under the Klein map, can be expressed as 4×4 matrix $(m_{i,j})$, $i,j = 0,\ldots,3$, that determines a correlation with $m_{00}=-2h_7$, $m_{11}=-2h_{16}$, $m_{22}=2h_{21}$, $m_{33}=-2h_{23}$, and

$$\begin{aligned}
m_{01} &= h_9+h_{13}-h_{29}+h_1, & m_{02} &= h_2-h_8+h_{28}+h_{19}, & m_{03} &= h_3-h_{27}-h_{18}-h_{11},\\
m_{10} &= h_{29}+h_{13}+h_9-h_1, & m_{12} &= h_6-h_{22}+h_{15}+h_{30}, & m_{13} &= h_{31}-h_{25}-h_5-h_{14},\\
m_{20} &= h_{19}-h_8-h_2-h_{28}, & m_{21} &= h_{15}-h_{30}-h_6-h_{22}, & m_{23} &= h_4-h_{20}+h_{32}+h_{24},\\
m_{30} &= h_{27}-h_{11}-h_{18}-h_3, & m_{31} &= h_5-h_{31}-h_{25}-h_{14}, & m_{32} &= h_{24}-h_4-h_{32}-h_{20}.
\end{aligned}$$

Action on points: The same element $\mathfrak{h} \in \mathcal{C}\ell^-_{(3,3)}$ can be interpreted as correlation in $\mathbb{P}^3(\mathbb{R})$ that acts on points and maps them to planes. This action can be described by a product of a matrix $m_{(i,j)}$, $i,j = 0,\ldots,3$, with a vector. This matrix can be expressed with the coefficients of $\mathfrak{h} \in \mathcal{C}\ell^-_{(3,3)}$ and has the entries $m_{00}=2h_{26}$, $m_{11}=2h_{17}$, $m_{22}=-2h_{12}$, $m_{33}=2h_{10}$, and

$$\begin{aligned}
m_{01} &= h_{32}-h_4-h_{20}-h_{24}, & m_{02} &= h_{14}-h_{31}-h_{25}-h_5, & m_{03} &= h_{30}+h_{15}+h_{22}-h_6,\\
m_{10} &= h_4-h_{32}-h_{24}-h_{20}, & m_{12} &= h_{18}-h_{27}-h_3-h_{11}, & m_{13} &= h_2+h_8+h_{19}-h_{28},\\
m_{20} &= h_{31}+h_{14}-h_{25}+h_5, & m_{21} &= h_3-h_{11}+h_{27}+h_{18}, & m_{23} &= h_9-h_{13}-h_1-h_{29},\\
m_{30} &= h_{15}-h_{30}+h_6+h_{22}, & m_{31} &= h_8-h_2+h_{28}+h_{19}, & m_{32} &= h_1-h_{13}+h_{29}+h_9.
\end{aligned}$$

Depending on the action of the correlation, i.e., if it acts on points or planes, the elements of the Pin group can be transferred to their matrix representations and vice versa.

4. A factorization algorithm and its application

At this point we recall a theorem from [8].

Theorem 1. *Every regular collineation or correlation can be expressed as the product of six null polarities at the most.*

For a proof we refer to [8] or [9]. With the help of a modified version of a factorization algorithm introduced in [10, p. 107] we factorize arbitrary regular 4×4 matrices into the product of six skew symmetric matrices at the most. Thus, we present a new method to factorize regular projective transformations. Let $[\mathfrak{g}]_k$ denote the grade-k part of the versor $\mathfrak{g}$. For example, a versor $\mathfrak{g}$ with maximal grade four can be written as $\mathfrak{g} = [\mathfrak{g}]_0 + [\mathfrak{g}]_2 + [\mathfrak{g}]_4$. The part of maximal grade $[\mathfrak{g}]_{\max}$ is always a blade (see [10]). The non-null vector $\mathfrak{v} \in \bigwedge^1 V$ that is contained in $\mathbb{NO}([\mathfrak{g}]_{\max})$ satisfies

$$\mathfrak{v}\mathfrak{g}_{\max} = \mathfrak{v} \cdot \mathfrak{g}_{\max}. \tag{6}$$

Hence, $\mathfrak{g}' = \mathfrak{g}\mathfrak{v}^{-1}$ reduces the grade of the maximum blade of $\mathfrak{g}$ by one. With Remark 4 this can be simplified to $\mathfrak{g}' = \mathfrak{g}\mathfrak{v}^*$ for homogeneous Clifford algebra models. Moreover, for vectors $\mathfrak{v} \in \bigwedge^1 V$ we have $\mathfrak{v}^* = -\mathfrak{v}$, and therefore, we can use $\mathfrak{g}' = \mathfrak{g}\mathfrak{v}$ to reduce the degree of the initial versor $\mathfrak{g}$. The result of this product is again a versor. Repeated application of this process results in a set of vectors $\mathfrak{v}_1,\ldots,\mathfrak{v}_{\max}$ whose geometric product results in the versor $\mathfrak{g}$ except for a real factor. Note that this procedure may fail for null versors. We show this algorithm in an example

Example 1. Let $\mathrm{K} \in \mathrm{PGL}(\mathbb{P}^3(\mathbb{R}))$ be a collineation of $\mathbb{P}^3(\mathbb{R})$ that acts on points

$$K = \begin{pmatrix} 1 & 0 & 3 & 0\\ 1 & 1 & 0 & 1\\ 1 & 2 & 1 & 0\\ 1 & 1 & 2 & 1 \end{pmatrix}.$$

To get a versor $\mathfrak{g} \in \mathcal{C}\ell^+_{(3,3)}$ corresponding to this collineation we have to solve the system (5).

$$\begin{aligned}
2(g_7+g_{17}) &= 0, & 2(g_{18}-g_3) &= 3, & 2(g_{19}+g_2) &= 0, & -2(g_{26}+g_{16}) &= 1,\\
2(g_5+g_{25}) &= 0, & 2(g_6-g_{22}) &= 1, & 2(g_{15}-g_{30}) &= 1, & 2(g_8+g_{28}) &= 2,\\
2(g_{21}+2g_{10}) &= 0, & -2(g_{31}+g_{14}) &= 1, & 2(g_{11}-g_{27}) &= 1, & 2(g_{23}+g_{12}) &= 2,
\end{aligned}$$

$$\begin{aligned}
g_1-g_{20}-g_{24}-g_{32}-g_{29}+g_9+g_4+g_{13} &= 1,\\
g_{24}-g_9+g_{20}-g_{13}-g_{32}+g_1+g_4-g_{29} &= 1,\\
g_1-g_{13}-g_{32}-g_4+g_{29}+g_9-g_{24}+g_{20} &= 1,\\
g_{24}+g_{13}+g_{29}+g_1-g_4-g_9-g_{20}-g_{32} &= 1.
\end{aligned}$$

There are two possibilities to guarantee that the resulting versor is in the Spin group, i.e., $\mathfrak{g}\mathfrak{g}^* = 1$ or $\mathfrak{g}\mathfrak{g}^* = -1$. We compute both solutions and start with the constraint equations implied by $\alpha(\mathfrak{g})\mathfrak{v}\mathfrak{g}^* \in \bigwedge^1 V$ for all $\mathfrak{v} \in \bigwedge^1 V$ and $\mathfrak{g}\mathfrak{g}^* = 1$. The corresponding Spin group element has the form:

$$\begin{aligned}
\mathfrak{g}_+ = \frac{1}{8\sqrt{2}}\big(&7e_0+6e_{12}-6e_{13}+e_{14}-2e_{15}-6e_{23}+6e_{24}-e_{25}-2e_{26}+2e_{34}+6e_{35}-5e_{36}-4e_{45}\\
&+2e_{46}+6e_{1234}-4e_{56}+6e_{1235}-6e_{1236}-5e_{1245}+2e_{1246}-4e_{1256}+2e_{1345}-e_{1346}+2e_{1356}\\
&-2e_{2345}+2e_{2346}+e_{2356}-2e_{2456}-e_{123456}\big).
\end{aligned}$$

If we demand that $\mathfrak{g}\mathfrak{g}^* = -1$ the resulting Spin group element is computed as

$$\begin{aligned}
\mathfrak{g}_- = \frac{1}{8\sqrt{2}}\big(&e_0-6e_{12}-6e_{13}-e_{14}+2e_{15}+4e_{16}+6e_{23}+2e_{24}+e_{25}+2e_{26}+2e_{34}+2e_{35}+5e_{36}\\
&+2e_{46}-6e_{1234}+6e_{1235}+6e_{1236}+5e_{1245}-2e_{1246}+6e_{1345}+e_{1346}-2e_{1356}-4e_{1456}\\
&-2e_{2345}+6e_{2346}-e_{2356}-2e_{2456}-4e_{3456}-7e_{123456}\big).
\end{aligned}$$

Both elements $\mathfrak{g}_+$ and $\mathfrak{g}_-$ correspond to the same collineation whose entries can be computed with the coefficients of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ and $(m_{i,j})$, $i,j = 0,\ldots,3$.

Depending on the matrix of the projective transformation, i.e., if it describes a collineation or correlation that acts on points or planes, the corresponding versor can be computed with the use of the corresponding systems derived in the previous section.

Remark 5. The matrix

$$K = \begin{pmatrix} -1 & 0 & 3 & 0\\ 1 & 1 & 0 & 1\\ 1 & 2 & 1 & 0\\ 1 & 1 & 2 & 1 \end{pmatrix}$$

results in a versor with complex entries. Thus, we have to work over the complex numbers.

Moreover, it can be verified that $\mathfrak{g}_+ \mathrel{\widehat{=}} \mathfrak{J}\mathfrak{g}_-$, where '$\widehat{=}$' means 'equal up to a scalar factor'. Multiplication with the pseudoscalar does not change the action of the versor since

$$(\mathfrak{J}\mathfrak{g})\mathfrak{v}(\mathfrak{J}\mathfrak{g})^* = \alpha(\mathfrak{J})\alpha(\mathfrak{g})\mathfrak{v}\mathfrak{g}^*\mathfrak{J}^* = \mathfrak{J}\mathfrak{J}^*\alpha(\mathfrak{g})\mathfrak{v}\mathfrak{g}^* \mathrel{\widehat{=}} \alpha(\mathfrak{g})\mathfrak{v}\mathfrak{g}^*.$$

Remark 6. Multiplication with an element of the center $\mathcal{C}(\mathcal{C}\ell_{(3,3)}) = ae_0 + be_{123456}$, $a,b \in \mathbb{C}$, has no effect on the sandwich action of a versor. Therefore, we have a group isomorphism

$$\mathrm{PGL}(4,\mathbb{C})/\mathcal{C}(\mathrm{PGL}(4,\mathbb{C})) \;\to\; \mathrm{Pin}\,(\mathcal{C}\ell_{(3,3)})/\mathcal{C}(\mathcal{C}\ell_{(3,3)}).$$

Now we factorize the element $\mathfrak{g}_+$ into vectors that correspond to null polarities. Due to the homogeneous setting we start with the numerator of $\mathfrak{g}_+$ of Example 1 denoted by $\mathfrak{g}$ $(= 8\sqrt{2}\mathfrak{g}_+)$.

Example 2. To factorize $\mathfrak{g}$, we first observe that $[\mathfrak{g}]_{\max} = [\mathfrak{g}]_6 = -e_{123456}$. Multiplication with a scalar does not change the outer product null space of a k-blade, and therefore follows

$$\mathbb{NO}([\mathfrak{g}]_{\max}) = \mathbb{NO}(e_{123456}) = \{\mathfrak{v} \in \bigwedge^1 V\}.$$

We can now choose an arbitrary non-null vector $\mathfrak{v}_1 \in \mathbb{NO}([\mathfrak{g}]_{\max})$. Let $\mathfrak{v}_1 = e_1+e_4$ be the first factor. The versor $\mathfrak{g}_1 = \mathfrak{g}\mathfrak{v}_1$ has maximal grade five and the grade-5 part is given by

$$[\mathfrak{g}_1]_5 \mathrel{\widehat{=}} e_{23456} - 4e_{12345} + 4e_{12346} - 3e_{12456} + e_{13456}.$$

The outer product null space of $[\mathfrak{g}_1]_5$ is computed as

$$\mathbb{NO}([\mathfrak{g}_1]_5) = \{\mathfrak{a} \in \bigwedge^1 V \mid \mathfrak{a} \wedge [\mathfrak{g}_1]_5 = 0\}.$$

This results in the set of all vectors $\mathfrak{a} = \sum_{i=1}^{6} a_i e_i$ satisfying $a_1 - a_2 - 3a_3 + 4a_5 + 4a_6 = 0$. We choose the non-null vector $\mathfrak{v}_2 = 4e_2 + e_5 \in \mathbb{NO}([\mathfrak{g}_1]_5)$ and compute the grade-four part of $\mathfrak{g}_2 = \mathfrak{g}_1\mathfrak{v}_2$

$$\begin{aligned}[\mathfrak{g}_2]_4 \mathrel{\widehat{=}} {} & e_{2356} - 12e_{1234} + 5e_{2345} - 4e_{1236} + 4e_{1235} + e_{1245} + e_{1345} - 8e_{2346} \\ & + 4e_{2456} - e_{3456} + 8e_{1246} - 3e_{1256} - 4e_{1346} + e_{1356} + e_{1456}.\end{aligned}$$

Again we determine a vector contained in the outer product null space of $[\mathfrak{g}_2]_4$. This null space is three-dimensional and spanned by the vectors

$$\mathfrak{b}_1 = e_1 + 5e_2 + e_6, \quad \mathfrak{b}_2 = 4e_1 + 8e_2 + e_5, \quad \mathfrak{b}_3 = e_1 + e_2 + e_4, \quad \mathfrak{b}_4 = e_1 + 4e_2 - e_3.$$

We choose the non-null vector $\mathfrak{b}_3$ and set $\mathfrak{v}_3 = \mathfrak{b}_3$. In the next step we have to compute the outer product null space of $[\mathfrak{g}_3]_3$ with $\mathfrak{g}_3 = \mathfrak{g}_2\mathfrak{v}_3$ that is two-dimensional and spanned by

$$\mathfrak{b}_1 = e_1 + 4e_2 - 3e_3, \quad \mathfrak{b}_2 = 3e_1 + 7e_2 - e_4 + e_5, \quad \mathfrak{b}_3 = 4e_2 - e_4 + e_6\,.$$

To keep the factors simple, we set

$$\mathfrak{v}_4 = \mathfrak{b}_1 + \mathfrak{b}_3 = -e_1 + e_3 - e_4 + e_6.$$

The reduced versor $\mathfrak{g}_4 = \mathfrak{g}_3\mathfrak{v}_4$ has maximal grade two and the outer product null space of the grade-two part is spanned by the two vectors:

$$\mathfrak{b}_1 = 2e_1 + 3e_2 + e_3 - e_4 + e_5, \quad \mathfrak{b}_2 = 4e_2 - e_4 + e_6.$$

Note that the vector $\mathfrak{b}_2$ is a null vector, and therefore, we choose $\mathfrak{v}_5 = \mathfrak{b}_1$ as the next factor. The last factor $\mathfrak{v}_6$ is obtained by

$$\mathfrak{v}_6 = \mathfrak{g}_4\mathfrak{v}_5 = 8e_1 + 8e_2 + 4e_3 - 3e_4 + 4e_5 - e_6.$$

This results in the factorization

$$\mathfrak{v}_6\mathfrak{v}_5\mathfrak{v}_4\mathfrak{v}_3\mathfrak{v}_2\mathfrak{v}_1 \mathrel{\widehat{=}} \mathfrak{g}.$$

The next step is to carry over the factorization to projective transformations. Each vector corresponds to a null polarity. Therefore, we aim at a matrix product of the form

$$\mathrm{M}_6\mathrm{M}_5\mathrm{M}_4\mathrm{M}_3\mathrm{M}_2\mathrm{M}_1 \mathrel{\widehat{=}} \mathrm{K}.$$

Since we started with a collineation that maps points to points, we have to ensure that the vectors are transferred to projective mappings of the right type. For example, the product $\mathrm{M}_2\mathrm{M}_1$ must be the matrix corresponding to a collineation that maps points to points. Hence, M_1 has to correspond to a null polarity acting on points and M_2 to a null polarity acting on planes. Thus, the matrix product has to be understood as

$$\mathrm{M}_6\mathrm{M}_5^*\mathrm{M}_4\mathrm{M}_3^*\mathrm{M}_2\mathrm{M}_1^* \mathrel{\widehat{=}} \mathrm{K},$$

where M_i^*, $i = 1, 3, 5$, is a null polarity acting on points and M_i, $i = 2, 4, 6$, is a null polarity acting on planes. We compute the matrices M_i^* and M_i with the matrix representations computed in section 3.2. This results in the six null polarities:

$$\mathrm{M}_1^* = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{M}_2 = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \\ -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathrm{M}_3^* = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix},$$

$$\mathrm{M}_4 = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{M}_5^* = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & -1 & 3 \\ 1 & 1 & 0 & -2 \\ 0 & -3 & 2 & 0 \end{pmatrix}, \quad \mathrm{M}_6 = \begin{pmatrix} 0 & 8 & 8 & 4 \\ -8 & 0 & -1 & -4 \\ -8 & 1 & 0 & -3 \\ -4 & 4 & 3 & 0 \end{pmatrix}.$$

With these six skew symmetric matrices it can be verified that

$$\mathrm{M}_6\mathrm{M}_5^*\mathrm{M}_4\mathrm{M}_3^*\mathrm{M}_2\mathrm{M}_1^* = -\frac{1}{4}\,\mathrm{K}.$$

Thus, we have found one possible factorization of the initial projective mapping into six null polarities.

It is worth to check the maximal grade of a versor that is the solution of the other systems that were derived in Section 3.2 if a factorization of a 4×4 matrix is searched. The maximal grade of the versor determines the number of skew-symmetric matrices that are necessary to factorize the given matrix and for correlations the maximal possible grade is five.

5. The homogeneous Clifford algebra model of Lie's sphere geometry

The same construction that we described for Klein's quadric can be applied to any quadric. As a second example that shall demonstrate the power of this calculus, we examine Lie sphere geometry in a Clifford algebra context. Lie sphere geometry is the geometry of oriented spheres. Especially, for the three-dimensional case the set of oriented spheres can be mapped to a hyperquadric L_1^4 in five-dimensional projective space $\mathbb{P}^5(\mathbb{R})$. The construction goes back to S. Lie and was treated again by W. Blaschke (cf. [1]). A modern treatment of this topic can be found in [2]. Moreover, the Lie construction can be achieved for arbitrary dimension.

5.1. Lie's quadric

A point model for the set of oriented hyperspheres, hyperplanes, and points (considered as spheres of radius 0) of $\mathbb{R}^n$ is given by the projective hyperquadric

$$L_1^{n+1}\colon\ -x_0^2 + x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 = 0.$$

For our purposes it is convenient that we restrict ourselves to the case of oriented spheres in three-dimensional Euclidean space. Nevertheless, we formulate the calculus for arbitrary dimensions. The quadric $L_1^{n+1} \subset \mathbb{P}^{n+2}(\mathbb{R})$ is of dimension $n+1$, degree 2, and is called Lie's quadric. The maximal dimension of subspaces contained by L_1^{n+1} is 1, and therefore, there are no two-spaces contained entirely in L_1^{n+1}. Oriented hyperspheres, hyperplanes, and points are represented in Lie coordinates as shown in Table 1. It is not difficult to recover the Euclidean representation from Lie coordinates. If $x_0 + x_1 = 0$ and if $x_{n+2} = 0$ we have the point at infinity. If $x_{n+2} \neq 0$ we bring the point to the form $(h, -h, N, 1)^{\mathrm{T}}\,\mathbb{R}$ by dividing by x_{n+2}. If $x_0 + x_1 \neq 0$ and if $x_{n+2} = 0$, we have a proper point. We obtain its normal form by dividing by $x_0 + x_1$. The last case is if $x_{n+2} \neq 0$. In this case we have an oriented sphere. Again we get its normal form through division by $x_0 + x_1$.

Table 1: Correspondence between Euclidean entities and Lie-coordinates.

Euclidean	**Lie**
points: $u \in \mathbb{R}^n$	$\left(\frac{1+u\cdot u}{2},\ \frac{1-u\cdot u}{2},\ u_1,\ \ldots,\ u_n,\ 0\right)^{\mathrm{T}}\mathbb{R}$
∞	$(1, -1, 0, \ldots, 0, 0)^{\mathrm{T}}\,\mathbb{R}$
sphere: center $p \in \mathbb{R}^n$, signed radius r	$\left(\frac{1+p\cdot p-r^2}{2},\ \frac{1-p\cdot p+r^2}{2},\ p_1,\ \ldots,\ p_n,\ r\right)^{\mathrm{T}}\mathbb{R}$
planes: $u \cdot N = h$, unit normal $N \in \mathbb{R}^n$	$(h, -h, N_1, \ldots, N_n, 1)^{\mathrm{T}}\,\mathbb{R}$

The fundamental invariant of Lie sphere geometry is the oriented contact of spheres. It is not difficult to show that two spheres are in oriented contact if, and only if, their Lie coordinates $s_1, s_2 \in L_1^{n+1}$ satisfy $\ell(s_1, s_2) = 0$, where $\ell(\,\cdot\,,\,\cdot\,)$ denotes the bilinear form corresponding to L_1^{n+1}.

Especially for $n = 3$ the lines on L_1^4 correspond to so called parabolic pencils of spheres. These pencils consist of all oriented spheres with one common point of contact. Furthermore, each parabolic pencil contains exactly one point, i.e., sphere of radius 0. If this point sphere is not ∞ the pencil contains exactly one oriented hyperplane Σ.

Remark 7. Conics on Lie's quadric correspond to Dupin cyclides, that are the envelopes of two one-parameter families of spheres.

The group of Lie transformations shows up as the group of projective automorphisms of L_1^{n+1}. This group is isomorphic to $\mathrm{O}(n+1,2)/\pm 1$ (see Cecil [2]). Since the Pin group of the Clifford algebra $\mathcal{C}\ell_{(n+1,2,0)}$ is a double cover of $\mathrm{O}(n+1,2)$ we can use this group to describe Lie transformations.

5.2. The homogeneous Clifford algebra model corresponding to Lie sphere geometry

In this section we discuss the Clifford algebra model for Lie sphere geometry in the three-dimensional case. Therefore, the projective space we are dealing with is a five-dimensional space $\mathbb{P}^5(\mathbb{R})$. The homogeneous Clifford algebra model is obtained with the six-dimensional real vector space $\mathbb{R}^6$ as a model for the projective image space together with the quadratic

form of Lie's quadric

$$\mathrm{Q} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

This algebra has signature $(p, q, r) = (4, 2, 0)$ and is of dimension $2^6 = 64$. Again, the advantage of the Clifford algebra lies in the common description of the application of Lie transformations. Arbitrary projective subspaces of $\mathbb{P}^5(\mathbb{R})$ are transformed by the sandwich operator. As an example we determine all Lie inversions that leave the point at infinity fixed, i.e., the subgroup of *Laguerre* transformations. The point at infinity has the form $\mathfrak{p} = e_1 - e_2$ (compare to Table 1). A general invertible vector is given by $\mathfrak{a} = \sum_{i=1}^{6} a_i e_i$ with $\mathfrak{a}\mathfrak{a} \neq 0$. The application of the sandwich operator to $\mathfrak{p}$ results in

$$\begin{aligned} \alpha(\mathfrak{a})\mathfrak{p}\mathfrak{a}^* = &-2(a_1+a_2)a_3e_3 - 2(a_1+a_2)a_4e_4 - (a_1^2+a_2^2+a_3^2+a_4^2+a_5^2-a_6^2+2a_2a_1)e_1 \\ &- (a_1^2+a_2^2-a_3^2-a_4^2-a_5^2+a_6^2+2a_2a_1)e_2 - 2(a_1+a_2)a_5e_5 - 2(a_1+a_2)a_6e_6. \end{aligned}$$

To guarantee that this entity represents the point at infinity, we first see that $a_1+a_2=0$. With this condition the coefficients of e_3, e_4, e_5, and e_6 vanish. Moreover, the sum of the coefficients e_1 and e_2 has to vanish. This results in

$$-2a_1^2 - 2a_2^2 - 4a_2a_1 = -2(a_1 + a_2)^2 = 0.$$

Therefore, the only condition to a Lie inversion that it represents a Laguerre transformation is given by $a_1 + a_2 = 0$, and the subgroup of Laguerre transformations is generated by all vectors with $a_1 + a_2 = 0$.

Remark 8. Analogue to Theorem 1 we can formulate a similar theorem for Lie sphere geometry in arbitrary dimensions. Since the projective model space for n-dimensional Lie sphere geometry has dimension $n + 2$, the vector space for the homogeneous Clifford algebra model has dimension $n + 3$. That means, the highest grade is equal to $n + 3$, and therefore, every group element can be written as the composition of $n+3$ vectors at the most. Especially for the case $n = 3$, we have similar results as for Klein's quadric. In this case six involutions are necessary to generate the whole group.

Let us reformulate this remark as a theorem.

Theorem 2. *Every Lie transformation in n-dimensional space is the composition of $n+3$ involutions, that correspond to the sandwich action of vectors.*

6. Conclusion

With the help of the homogeneous Clifford algebra model corresponding to line geometry we introduced null polarities as fundamental involutions. After we transferred linear line manifolds to this model, we derived the correspondence between regular projective transformations and versors. Every versor can be factorized into vectors that correspond to null polarities. Therefore, every 4×4 matrix corresponding to a projective transformation can be expressed as the product of six skew symmetric matrices at the most. Moreover, we presented a homogeneous Clifford algebra model corresponding to Lie sphere geometry.

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