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On a 3D Extension of the Simson–Wallace Theorem

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Abstract. The following 3D extension of the Simson–Wallace theorem is proved by a method which differs from that used in the past (Theorem 1): Let K, L, M, N be orthogonal projections of a point P to the faces BCD, ACD, ABD, and ABC of a tetrahedron $ABCD$. Then, all points P with the property that the tetrahedron $KLMN$ has a constant volume belong to a cubic surface (1). Next, the main theorem (Theorem 2) is proved which states that also the converse of Theorem 1 holds. Furthermore, we verify Theorem 2 for a regular tetrahedron by descriptive geometry methods using dynamic geometry software. To do this we take advantage of the fact that this cubic surface can be represented by a parametric system of conics which lie in mutually parallel planes (Theorem 3).

Key Words: Simson–Wallace loci, cubic surface, Cayley cubic, Monge projection

MSC 2000: 51N20, 51N05, 51N35

1. Introduction

The well-known Simson–Wallace theorem belongs to the most beautiful theorems of plane geometry. This theorem states (cf. [4]):

The orthogonal projections of a point P onto the sides of a triangle ABC are collinear if and only if P is a point of the circumcircle of ABC.

There exist several generalizations [6, 8, 10, 11, 12, 14] of this theorem. A generalization ascribed to Gergonne [2] is as follows (see Figure 1):

If P is a point of a circle which is concentric with the circumcircle of a triangle ABC then the orthogonal projections of P onto the sides of ABC form a triangle with constant area.

When the given area is zero, we get the classical Simson–Wallace theorem.

One could think that a generalization of the Simpson–Wallace theorem onto a tetrahedron in the three-dimensional space leads, by analogy with the planar case, to a sphere. But this is not the case. In [10, 11, 12] the 3D-extension of the Simpson–Wallace theorem onto a tetrahedron $ABCD$ is given:

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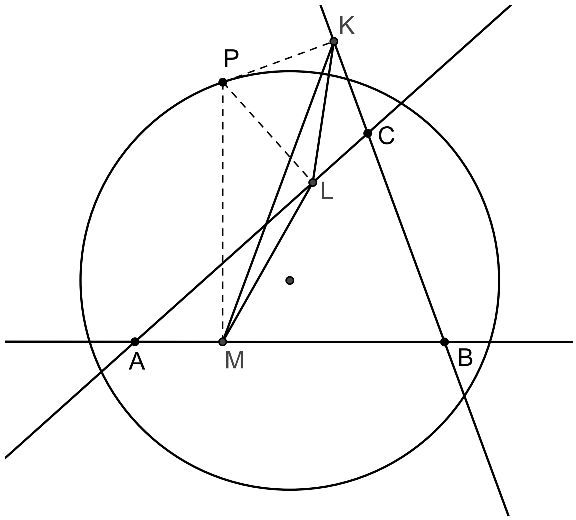

Figure 1: If P moves along the circle the triangle KLM has constant area

Theorem 1. *Let K, L, M, N be orthogonal projections of an arbitrary point P to the faces BCD, ACD, ABD, ABC of a tetrahedron $ABCD$. Then the point P such that the tetrahedron $KLMN$ has a constant volume s belongs to the cubic surface (1).*

First we give the proof of Theorem 1, so that we can realize the method we use throughout the paper. This proof differs from that given [11, 12], where the software CoCoA [1] was used. To avoid successive elimination, here the software Epsilon [15] is applied. All computational methods we use in this paper are methods of the theory of automated geometry theorem proving [2, 3, 7, 13, 15].

In Theorem 2 we show that also the converse to Theorem 1 holds. This ensures that the locus of the point P is the "whole" surface (1). The author tried to prove it a few years ago without success [12]. It turns out that the use of the software Epsilon which is based on the Wu–Ritt method [15, 16] can help in solving this problem.

Further some properties of the cubic surface (1) are investigated. If the volume s equals zero then we obtain a special cubic surface called the Cayley cubic. In Theorem 3 it is shown that if we take a regular tetrahedron and $s = 0$ then its associate Cayley cubic can be represented by a set of conics which are parallel to two opposite edges of the tetrahedron.

In the last section we verify Theorem 2 by descriptive geometry methods together with the dynamic geometry system GeoGebra [5].

2. The main theorem

First, we recall the basic ideas of the proof of Theorem 1. The novelty of this proof is the use of the software Epsilon with characteristic sets whereas in the previous proof [11, 12] we used the software CoCoA and successive elimination of variables. Secondly, we prove the main theorem — Theorem 2 — which states that also the converse statement holds.

Proof of Theorem 1: Let K, L, M, N be orthogonal projections of P onto the faces BCD, ACD, ABD and ABC of a tetrahedron $ABCD$.
Choose a rectangular system of coordinates so that $A = (0, 0, 0)$, $B = (a, 0, 0)$, $C = (b, c, 0)$, $D = (d, e, f)$, $K = (k_1, k_2, k_3)$, $L = (l_1, l_2, l_3)$, $M = (m_1, m_2, m_3)$, $N = (n_1, n_2, n_3)$, $P =$

(p, q, r), and let s be the oriented volume of $KLMN$. Suppose that $a \neq 0$, $c \neq 0$ and $f \neq 0$ since otherwise $ABCD$ is planar. Then

$$PK \perp BCD \iff \begin{cases} h_1: \ (b-a)(p-k_1) + c(q-k_2) = 0, \\ h_2: \ (d-a)(p-k_1) + e(q-k_2) + f(r-k_3) = 0, \end{cases}$$

$$K \in BCD \iff h_3: \ -acf - aek_3 + afk_2 + ack_3 + cfk_1 + bek_3 - cdk_3 - bfk_2 = 0,$$

$$PL \perp ACD \iff \begin{cases} h_4: \ b(p-l_1) + c(q-l_2) = 0, \\ h_5: \ d(p-l_1) + e(q-l_2) + f(r-l_3) = 0, \end{cases}$$

$$L \in ACD \iff h_6: \ cfl_1 + bel_3 - cdl_3 - bfl_2 = 0,$$

$$PM \perp ABD \iff \begin{cases} h_7: \ p - m_1 = 0, \\ h_8: \ d(p-m_1) + e(q-m_2) + f(r-m_3) = 0, \end{cases}$$

$$M \in ABD \iff h_9: \ em_3 - fm_2 = 0,$$

$$PN \perp ABC \iff \begin{cases} h_{10}: \ p - n_1 = 0, \\ h_{11}: \ b(p-n_1) + c(q-n_2) = 0, \end{cases}$$

$$N \in ABC \iff h_{12}: \ n_3 = 0.$$

The conclusion h_{13} is of the form

$$\text{Volume of } KLMN = s \iff h_{13}: \begin{vmatrix} k_1 & k_2 & k_3 & 1 \\ l_1 & l_2 & l_3 & 1 \\ m_1 & m_2 & m_3 & 1 \\ n_1 & n_2 & n_3 & 1 \end{vmatrix} - 6s = 0.$$

We eliminate the dependent variables k_1, k_2, k_3, l_1, l_2, l_3, m_1, m_2, m_3, n_1, n_2, n_3 in the system of algebraic equations $h_1 = 0$, $h_2 = 0, \ldots, h_{13} = 0$ with variable ordering $p \prec q \prec r \prec a \prec b \prec \cdots \prec f \prec s \prec k_1 \prec k_2 \prec \cdots \prec n_3$. The use of the *Wu–Ritt method* with characteristic sets in the Maple package Epsilon [15, 16] gives

```
with(epsilon);
CharSet({h1,h2,h3,h4,h5,h6,h7,h8,h9,h10,h11,h12,h13},[p,q,r,a,b,c,d,e,f,s,
k[1],k[2],k[3],l[1],l[2],l[3],m[1],m[2],m[3],n[1],n[2],n[3]]);
```

the equation

$$F := ac^2f^3G + sQ = 0, \tag{1}$$

where

$G = c^2f^2p^2q + cf(e^2+f^2-ce)p^2r + cf^2(a-2b)pq^2 + cf^2(a-2d)pr^2 + 2cef(b-d)pqr + b(b-a)f^2q^3 + f(be(a-b)+cd(d-a)+cf^2)q^2r + f^2(b^2-ab+c^2-2ce)qr^2 + (be(a-b)+cd(d-a)+ce(e-c))fr^3 - ac^2f^2pq + acf(ce-e^2-f^2)pr + abcf^2q^2 + (a(c^2d-2bce+be^2)-(cd-be)^2+f^2(ab-b^2-c^2))fqr + (ce^2(ab+ad-2bd)+c^2de(d-a)+be^3(b-a)+f^2(a(cd-be)+e(b^2+c^2)))r^2$

and

$Q = 6(e^2+f^2)((cd-be)^2+f^2(b^2+c^2))((c(a-d)-e(a-b))^2+f^2((a-b)^2+c^2))$.

As $a \neq 0$, $c \neq 0$ and $f \neq 0$ we see that (1) describes a cubic surface. □

Furthermore we can show the reverse statement as well, namely that for every point $P = (p, q, r)$ of the surface (1) the volume of the tetrahedron $KLMN$ equals s. We enter

```
with(epsilon);
Simson:=Theorem({h1,h2,h3,h4,h5,h6,h7,h8,h9,h10,h11,h12,F},{h13},[f,e,d,c,b,
a,p,q,r,s,k[1],k[2],k[3],l[1],l[2],l[3],m[1],m[2],m[3],n[1],n[2],n[3]]):
Prove(Simson);
```

and get the answer

The theorem is true under the following subsidiary conditions:

1) $c \neq 0$,
2) $f \neq 0$,
3) $e^2 + f^2 \neq 0$,
4) $(be - cd)^2 + f^2(b^2 + c^2) \neq 0$,
5) $(e(a-b) - c(a-d))^2 + f^2((a-b)^2 + c^2) \neq 0$.

All conditions 1) – 5) are fulfilled as we suppose that $c \neq 0$ and $f \neq 0$.

Remark 1. Note the variable ordering $f \prec e \prec d \prec c \prec b \prec a \prec p \prec q \prec r \prec s \prec k_1 \prec k_2 \prec \cdots \prec n_3$. By this ordering we get subsidiary conditions in such a form which directly depends on the non-zero values c and f. We can accept them without restricting the hypotheses of the theorem.

We can state the main theorem:

Theorem 2. *Let K, L, M, N be orthogonal projections of a point P to the faces BCD, ACD, ABD, ABC of a tetrahedron $ABCD$. Then the locus of P such that the tetrahedron $KLMN$ has a constant volume s is the surface (1).*

We call the surface F the *associate surface* of the tetrahedron $ABCD$ and the tetrahedron $ABCD$ the *associate tetrahedron* of the surface F.

3. Properties of the surface F

The surface (1) belongs to the family of algebraic surfaces of third degree called cubic surfaces, or briefly just *cubics* [7, 9]. We mention only those properties of (1) which we refer to in this article.

First suppose that $s = 0$. This surface has four singular points, the maximum number of singular points of a general cubic surface, which are placed at the vertices of the tetrahedron $ABCD$. A cubic surface with four singular points is called a *Cayley cubic*. (Do not mistake this surface for the ruled Cayley surface!) The Cayley cubic contains six lines AB, BC, CD, DA, AC and BD, the edges of $ABCD$, which are torsal lines of (1). Another three straight lines of the surface are intersections of tangent planes along opposite edges of $ABCD$. These three lines are coplanar [11, 12].

If $s \neq 0$ then the surface (1) does not have any singular point, and the vertices A, B, C, D of the associate tetrahedron do not lie on it. Note that the equation of this surface differs from the previous Cayley cubic only in a nonzero constant sQ.

In the following we demonstrate how the cubic surface (1) changes its shape with respect to different values of the volume s.

Figure 2: Cayley cubic (2) with $s = 0$. Four singular points are at vertices of the tetrahedron

Let the basic tetrahedron $ABCD$ be given by its vertices $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,1,0)$, and $D = (0,0,1)$. Substituting the values $a = 1$, $b = 0$, $c = 1$, $d = 0$, $e = 0$, and $f = 1$ into (1), we get the equation of the associate cubic surface

$$p^2q + pq^2 + p^2r + q^2r + pr^2 + qr^2 - pq - pr - qr + 18s = 0. \tag{2}$$

- For $s = 0$ we get the Cayley cubic with singular points at the vertices A, B, C, and D (Figure 2).
- For $s > 0$, for instance for $s = 1/1800$, we get the following cubic surface (Figure 3 left). Note that the cubic surface consists of two separated parts.
- Similarly the Cayley surface with negative s, for instance for $s = -1/1800$, is as follows (Figure 3 right).

Thus we obtain three types of cubic surfaces in accordance with the sign of the volume s (cf. the Gergonne generalization of the Simson–Wallace theorem in a plane, where we also get three types of loci (circles) [11]).

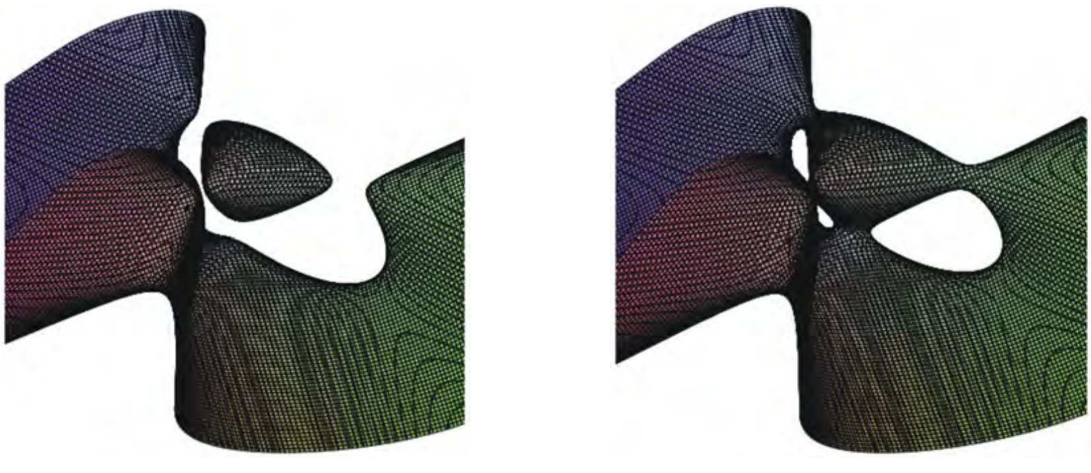

Figure 3: Cubic surfaces (2) with positive s (left), and negative s (right)

4. Regular tetrahedron and its associate surface

In this section we study a regular tetrahedron and its associate cubic surface. The resulting surfaces related to a tetrahedron have many algebraic, geometric and projective properties. E.g., in the case $s = 0$ the tetrahedron of singular points (Figure 2) and the straight lines on the surface are such properties. Obviously planes through such a line of F intersect F in conic sections. Aiming just at such projective geometric properties, it makes sense to look for a sort of *normal forms* of a Cayley cubic. In the following we connect such a normal form to the regular tetrahedron. For regular tetrahedra it turns out that the intersection of F with the ideal plane ω is completely reducible and $F \cap \omega$ consists of three real lines. We omit here the discussion of tetrahedra with respect to affine geometry and reducibility of $F \cap \omega$.

Setting $A = (0,0,0)$, $B = (2,0,0)$, $C = (1,\sqrt{3},0)$, and $D = (1,1/\sqrt{3},\sqrt{8/3})$, we get a regular tetrahedron. The cubic surface associated with this tetrahedron for a given s has the equation

$$3p^2q+\tfrac{3}{2}\sqrt{2}p^2r-q^3+\tfrac{3}{2}\sqrt{2}q^2r-\sqrt{2}r^3+2\sqrt{3}q^2+\tfrac{5}{2}\sqrt{3}r^2-6pq-3\sqrt{2}pr-\sqrt{6}qr+\tfrac{243}{16}\sqrt{6}s=0.$$

To avoid radicals and to obtain a simpler equation of the Cayley cubic associated with a regular tetrahedron, we use the mapping

$$(p,q,r,1) \;\to\; (p,q,r,1)\cdot M, \tag{3}$$

where

$$M = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{6}\sqrt{3} & -\frac{1}{6}\sqrt{3} & \frac{1}{3}\sqrt{3} & 0 \\ \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The similarity (3) maps the regular tetrahedron $ABCD$ with vertices $A = (0,0,0)$, $B = (2,0,0)$, $C = (1,\sqrt{3},0)$, and $D = (1,1/\sqrt{3},\sqrt{8/3})$ to the regular tetrahedron $A'B'C'D'$ with vertices $A' = (1,0,0)$, $B' = (0,1,0)$, $C' = (0,0,1)$, and $D' = (1,1,1)$ (Figure 4).

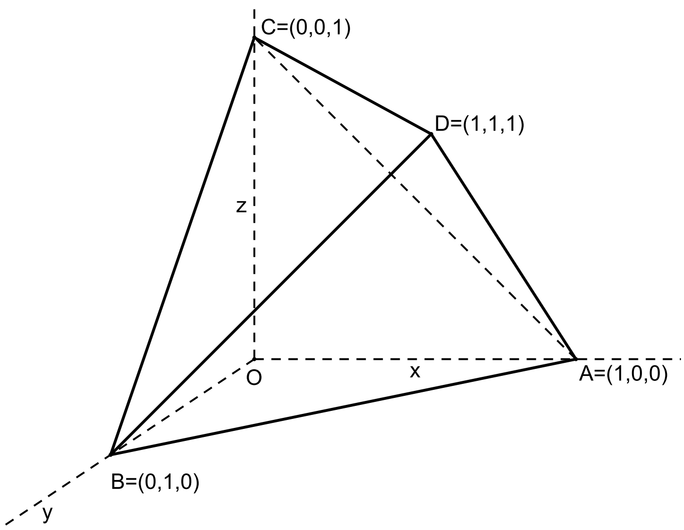

Figure 4: Regular tetrahedron in another coordinate system

The associate cubic surface for a given s transforms into

$$4pqr - (p+q+r-1)^2 - \frac{81}{4}\,s = 0. \tag{4}$$

Similarly, for a regular tetrahedron $ABCD$ with vertices $A = (a,0,0)$, $B = (0,a,0)$, $C = (0,0,a)$, $D = (a,a,a)$ and s we get by (3) the equation of an associate cubic surface in the form

$$4pqr - a(p+q+r-a)^2 - \frac{81}{4}\,s = 0. \tag{5}$$

Note that the constant by s in (5) does not depend on a.

Now we investigate properties of the surface (5) for $s = 0$,

$$4xyz - a(x+y+z-a)^2 = 0. \tag{6}$$

We changed the notation from p, q, r to common x, y, z which is now more comfortable. Suppose that $a > 0$.

Exploring sections of (6) with a pencil of mutually parallel planes $z = k$, where k is a real parameter, we get

$$4kxy - a(x+y+k-a)^2 = 0.$$

This yields for a fixed parameter k the conic

$$ax^2 + 2(a-2k)xy + ay^2 + 2a(k-a)x + 2a(k-a)y + a(k-a)^2 = 0 \tag{7}$$

with a canonical form

$$2kx^2 + 2(a-k)y^2 - ak(a-k) = 0. \tag{8}$$

Exploring (8) we get: If $k < 0$ or $k > a$ the conic is a hyperbola, if $k = 0$ or $k = a$ the conic is a double line, and finally if $0 < k < a$ the conic is an ellipse (Figure 5).

We can also study planar sections of (6) through a line which lies on the cubic surface. Recall that every plane intersects a cubic surface in a cubic curve.

If we want to get a conic section then a related cubic curve must be decomposed into a conic and a line. Thus the only conics on a cubic surface we get if the planar sections pass through a line which belongs to the cubic surface. As a cubic surface contains 27 lines (with multiplicities, some of them may be imaginary or ideal lines), then we get 27 systems of conics lying on the cubic.

In the case of a regular tetrahedron $ABCD$ the tangent planes of its associate Cayley cubic along a pair of opposite edges are mutually parallel and intersect at a line at infinity. This implies that planar sections of (6) which are parallel to two opposite edges of a regular $ABCD$ must be conics. Hence, for the surface (6), which is associated with a regular tetrahedron, we obtain the following theorem:

Theorem 3. *The level lines of the Cayley cubic (6) associated with a regular tetrahedron (Figure 4) are conics (7). Their top view image is a pencil of conics touching a square (Figure 5).*

In the next section we apply this theorem to a numerical verification of Theorem 2.

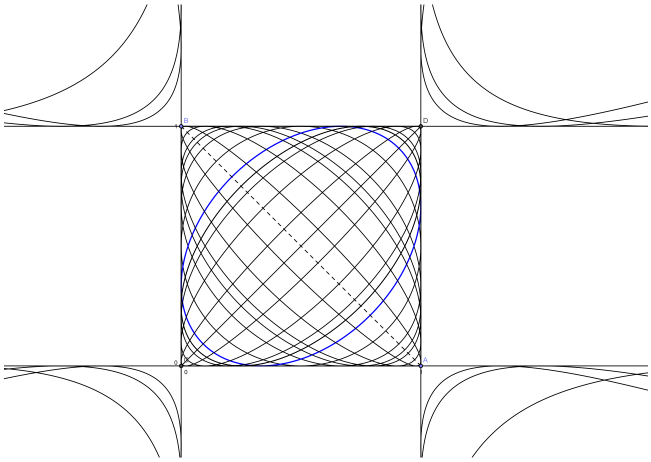

Figure 5: Horizontal view of planar sections of a Cayley cubic

5. Verification of Theorem 2 using descriptive geometry

We have seen that the 3D extension of Simson–Wallace theorem leads to a cubic surface and not to a sphere as one could expect. Whereas in the planar case of the Simson–Wallace theorem we can easily verify that the locus is the circumcircle of a triangle — either by ruler and compass or by a dynamic geometry software (DGS) —, in the 3D case the verification is problematic. For the presented normal form of a Cayley cubic we demonstrate Theorem 2 using descriptive geometry and the DGS GeoGebra [5]. We aim at the most interesting part of (6) — the tetrahedral part — or more precisely, the part which is described by the system (7) of conics for $0 \leq k \leq 1$ (Figure 6).

The verification of the Simson–Wallace theorem in the dynamic geometry system will be performed using the Monge orthogonal projection onto two mutually orthogonal planes. We prove that the feet K, L, M, N of perpendiculars from a point P of the Cayley cubic which is associated with a regular tetrahedron $ABCD$ onto the faces of $ABCD$ are coplanar. To do this it is sufficient to show that the straight lines KL and MN intersect at the point S (Figure 7).

We denote the horizontal projections with the index 1 whereas vertical and side projections

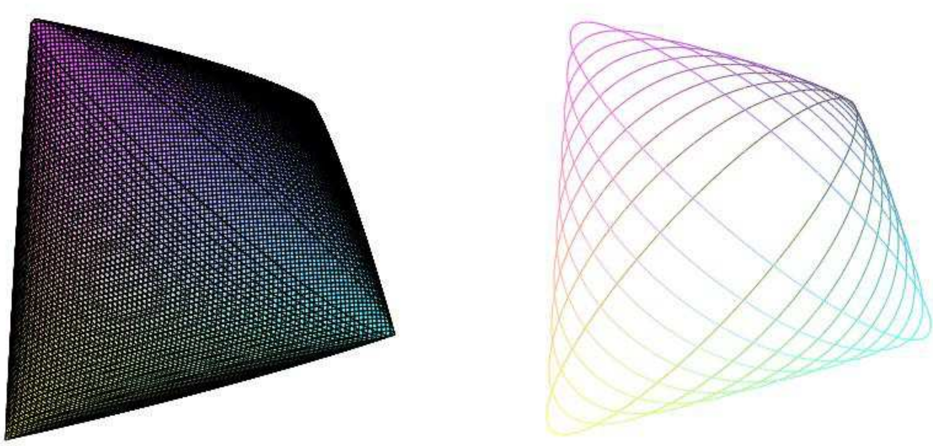

Figure 6: Cayley cubic associated with regular tetrahedron — tetrahedral part

get the indices 2 and 3, respectively.

We place the tetrahedron in a such position that the edge AB lies in the horizontal projection plane and is parallel to the vertical projection plane. The opposite edge CD is parallel to the horizontal plane and orthogonal to the vertical projection plane. To obtain a better view, we rotate the surface (6) through 45°. In the position displayed in the Figure 7 we have $A = (-\frac{1}{\sqrt{2}}, 0, 0)$, $B = (\frac{1}{\sqrt{2}}, 0, 0)$, $C = (0, -\frac{1}{\sqrt{2}}, 1)$ and $D = (0, \frac{1}{\sqrt{2}}, 1)$. The associate Cayley cubic has the equation

$$2x^2z - 2y^2z + 2y^2 + z^2 - z = 0. \tag{9}$$

By Theorem 3, this Cayley cubic can be represented by a set of conics

$$2kx^2 + 2(1-k)y^2 - k(1-k) = 0, \tag{10}$$

which lie in the planes $z = k$. Note that (10) follows from (8) for $a = 1$.

Let a point P move along an ellipse of the surface (9). From the point P we construct projections of the feet K, L of perpendiculars to the faces BCD and ACD. We used the side projection to simplify the construction. Similarly, we construct projections of the feet M, N of perpendiculars of P to the faces ABC and ABD. Note that the vertical projection of the associate Cayley cubic (9) of $ABCD$ is a parabola $z = -2x^2 + 1$ and a line $z = 0$.

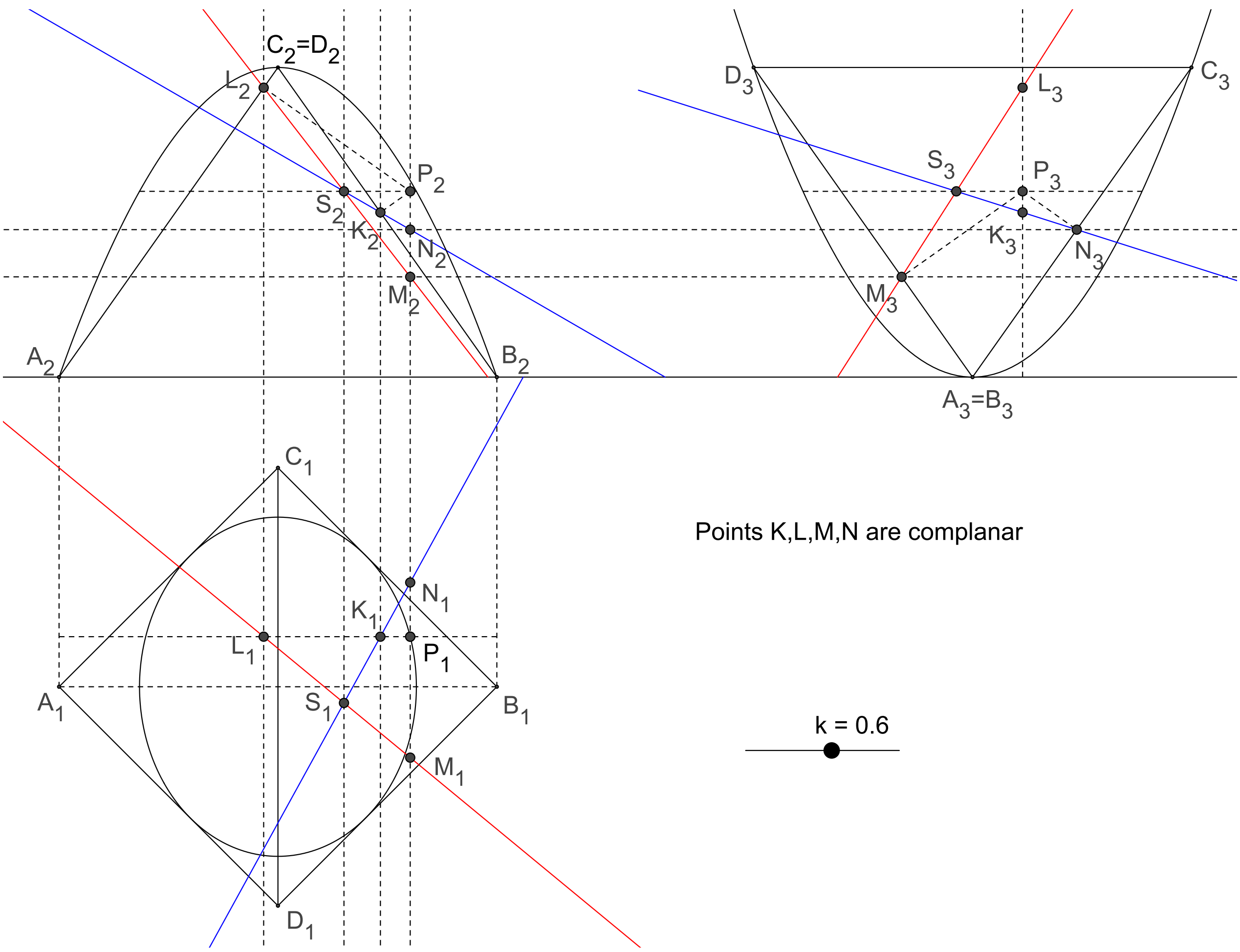

Figure 7: Points K, L, M, N are coplanar

The whole system is fully interactive. We can move both, an ellipse of a section using a slider with a parameter k of the ellipse (10), and a point P along the ellipse using a pointer. In this way we can cover the whole tetrahedral part. Finally we construct the intersection S_1 of the lines K_1N_1 and L_1M_1, and the intersection S_2 of lines K_2N_2 and L_2M_2. Now using the window "Relation between two objects" we get the answer that S_2 lies on the line through S_1 and which is orthogonal to x axis, i.e., the lines KN and LM are concurrent. This can be connected with the text *"Points K,L,M,N are coplanar"*. Thus moving the point P along the Cayley cubic, we see the text that the points K, L, M, N are coplanar. If we detach the point P from the cubic, i.e., from the ellipse, using the window "Detach point", the text disappears. In this way we are able to verify the validity of the theorem for infinitely many positions of the point P. The verification by DGS is now complete.

Note that this verification is based on a numerical description of geometry objects. From the mathematical point of view, this cannot be considered as a rigorous mathematical proof.

6. Conclusions

The Wallace–Simson theorem has been generalized several times in history. The 3D extension given in this paper is based on results of commutative algebra in the last third of the last century.

In this paper we proved that if K, L, M, N are orthogonal projections of a point P onto the faces of a tetrahedron $ABCD$ then the locus of the point P such that the tetrahedron $KLMN$ has a given volume s is the cubic surface (1).

The special case of the cubic surface (1) for $s = 0$ is a Cayley cubic. We have shown that the Cayley cubic which is associated to a regular tetrahedron can be represented by a set of conics as level lines. This enables a verification of Theorem 2 by the methods of descriptive geometry in connection with a dynamic geometry software.

Acknowledgements

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References

[1] A. Capani, G. Niesi, L. Robbiano: *CoCoA, a System for Doing Computations in Commutative Algebra.* `http://cocoa.dima.unige.it`

[2] S.C. Chou: *Mechanical Geometry Theorem Proving.* D. Reidel Publishing Company, Dordrecht 1987.

[3] D. Cox, J. Little, D.O'Shea: *Ideals, Varieties, and Algorithms.* Second Edition, Springer, New York, Berlin, Heidelberg 1997.

[4] H.S.M. Coxeter, S.L. Greitzer: *Geometry revisited.* Toronto, New York 1967.

[5] GeoGebra [online]: `http://www.geogebra.at`

[6] O. Giering: *Affine and Projective Generalization of Wallace Lines.* J. Geometry Graphics **1**, 119–133 (1997).

[7] G.M. Gruel, G. Pfister, H. Schönemann: *Singular 2.0. A Computer Algebra System for Polynomial Computations.* Univ. Kaiserslautern 2001.

[8] M. Guzmán: *An Extension of the Wallace–Simson Theorem: Projecting in Arbitrary Directions*. Amer. Math. Monthly **106**, 574–580 (1999).

[9] B. Hunt: *The Geometry of Some Special Arithmetic Quotients*. Springer, New York 1996.

[10] E. Roanes-Lozano, E. Roanes-Macías: *Automatic Determination of Geometric Loci. 3D-Extension of Simson–Steiner Theorem*. In J.A. Campbell, E. Roanes-Lozano (eds.): *Lecture Notes in Artificial Intelligence* **1930**, Springer, New York, Heidelberg 2001, pp. 157–173.

[11] P. Pech: *Selected topics in geometry with classical vs. computer proving*. World Scientific Publishing, New Jersey, London, Singapore 2007.

[12] P. Pech: *On Simson–Wallace Theorem and Its Generalizations*. J. Geometry Graphics **9**, 141–156 (2005).

[13] T. Recio, H. Sterk, M.P. Vélez: *Project 1. Automatic Geometry Theorem Proving*. In A. Cohen, H. Cuipers, H. Sterk (eds.): *Some Tapas of Computer Algebra, Algorithms and Computations in Mathematics*, Vol. 4, Springer, New York, Heidelberg 1998, pp. 276–296.

[14] R. Riesinger: *On Wallace Loci from the Projective Point of View*. J. Geometry Graphics **8**, 201–213 (2004).

[15] D. Wang: *Elimination Practice. Software Tools and Applications*. Imperial College Press, London 2004.

[16] D. Wang: *Elimination Methods*. Springer, Wien, New York 2001.

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