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Characterizations of Ruled Surfaces in $\mathbb{R}^3$ and of Hyperquadrics in $\mathbb{R}^{n+1}$ via Relative Geometric Invariants

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Dedicated to Georg Stamou on the occasion of his 70^{th} birthday

Abstract. We consider hypersurfaces in the real Euclidean space $\mathbb{R}^{n+1}$ $(n \geq 2)$ which are relatively normalized. We give necessary and sufficient conditions a) for a surface of negative Gaussian curvature in $\mathbb{R}^3$ to be ruled, b) for a hypersurface of positive Gaussian curvature in $\mathbb{R}^{n+1}$ to be a hyperquadric and c) for a relative normalization to be constantly proportional to the equiaffine normalization.

Key Words: Ruled surfaces, ovaloids, hyperquadrics, equiaffine normalization, Pick-invariant

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1. Preliminaries

In this section we fix our notation and state some of the most important notions and formulae concerning the relative differential geometry of hypersurfaces in the real Euclidean space $\mathbb{R}^{n+1}$ $(n \geq 2)$. Our presentation is mainly based on the texts [3] and [5]. For a more detailed exposition of the subject the reader might read [4].

In the Euclidean space $\mathbb{R}^{n+1}$ let $\Phi = (M, \overline{x})$ be a C^r-hypersurface defined by an n-dimensional oriented and connected C^r-manifold M $(r \geq 3)$ and by a C^r-immersion $\overline{x}\colon\ M \to \mathbb{R}^{n+1}$, whose Gaussian curvature K_I never vanishes on M. A C^s-mapping $\overline{y}:\ M \to \mathbb{R}^{n+1}$ $(r > s \geq 1)$ is called a C^s*-relative normalization*, if

$$\operatorname{rank}\left(\{\overline{x}_{/1}, \overline{x}_{/2}, \ldots, \overline{x}_{/n}, \overline{y}\}\right) = n + 1, \tag{1a}$$
$$\operatorname{rank}\left(\{\overline{x}_{/1}, \overline{x}_{/2}, \ldots, \overline{x}_{/n}, \overline{y}_{/i}\}\right) = n, \quad \forall\ i = 1, 2, \ldots, n, \tag{1b}$$

for all $(u^1, u^2, \ldots, u^n) \in M$, where

$$f_{/i} := \frac{\partial f}{\partial u^i}, \quad f_{/ij} := \frac{\partial^2 f}{\partial u^i \partial u^j}, \quad \text{etc.}$$

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denote partial derivatives of a function (or a vector-valued function) f. We will also say that the pair $(\Phi, \overline{y})$ is a *relatively normalized hypersurface* of $\mathbb{R}^{n+1}$.

The *covector* $\overline{X}$ of the tangent vector space is defined by

$$\langle \overline{X}, \overline{x}_{/i} \rangle = 0 \quad \text{and} \quad \langle \overline{X}, \overline{y} \rangle = 1 \quad (i = 1, 2, \ldots, n), \tag{2}$$

where $\langle\ ,\ \rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$.

The quadratic differential form

$$G = G_{ij}\, du^i\, du^j, \quad \text{where} \quad G_{ij} := \langle \overline{X}, \overline{x}_{/ij} \rangle,$$

is definite or indefinite, depending on whether the Gaussian curvature K_I of Φ is positive or negative, and is called the *relative metric* of Φ. From now on we use G_{ij} as the fundamental tensor for "raising and lowering the indices" in the sense of classical tensor notation.

Let $\overline{\xi}: \ M \to \mathbb{R}^{n+1}$ be the Euclidean normalization of Φ. By virtue of (1) the *support function* of the relative normalization $\overline{y}$, which is defined by

$$q := \langle \overline{\xi}, \overline{y} \rangle : \ M \to \mathbb{R}, \quad q \in C^s(M),$$

never vanishes on M. In the sequel we choose $\overline{\xi}$ and $\overline{X}$ to have the same orientation. Then q is positive everywhere on M.

Because of (2) we have

$$\overline{X} = q^{-1}\overline{\xi}, \quad G_{ij} = q^{-1} h_{ij}, \quad G^{(ij)} = q\, h^{(ij)}, \tag{3}$$

where h_{ij} are the components of the second fundamental form II of Φ and $h^{(ij)}$ resp. $G^{(ij)}$ the inverses of the tensors h_{ij} and G_{ij}. Let ∇^G_i denote the covariant derivative corresponding to G. By

$$A_{jkl} := \langle \overline{X}, \ \nabla^G_l \nabla^G_k \overline{x}_{/j} \rangle$$

the (symmetric) *Darboux tensor* is defined. It gives occasion to define the *Tchebychev-vector* $\overline{T}$ of the relative normalization $\overline{y}$

$$\overline{T} := T^m \, \overline{x}_{/m}, \quad \text{where} \quad T^m := \frac{1}{n} A_i^{im},$$

and the *Pick-invariant*

$$J := \frac{1}{n(n-1)} A_{jkl} A^{jkl}.$$

We mention, that when the second fundamental form II is positive definite, so does G and in this case $J \geq 0$ holds on M (see, e.g., [2, p. 133]).

Denoting by H_I the Euclidean mean curvature of Φ, by ∇^{II} resp. $\triangle^{II}$ the first resp. the second Beltrami differential operator with respect to the fundamental form II of Φ and by S_{II} the scalar curvature of II, the Pick-invariant is computed by (see [3])

$$J = \frac{3(n+2)}{4n(n-1)}\, q\, \nabla^{II}\left(\ln q, \ \ln q - \ln |K_I|^{\frac{2}{n+2}}\right) + q\, \frac{1}{n\,(n-1)} P, \tag{4}$$

where P is the function [6, p. 231]

$$P = n(n-1)\,(S_{II} - H_I) + (2K_I)^{-2}\, \nabla^{II} K_I. \tag{5}$$

The *relative shape operator* has the coefficients B_i^j such that

$$\overline{y}_{/i} =: -B_i^j \, \overline{x}_{/j}.$$

The *mean relative curvature*, which is defined by

$$H := \frac{1}{n} \operatorname{tr}\left(B_i^j\right),$$

is computed by (see [3])

$$H = q\, H_I + \frac{q}{n}\Big[\triangle^{II}(\ln q) + \nabla^{II}\left(\ln q,\ ln\left(q\,|K_I|^{-\frac{1}{2}}\right)\right)\Big]. \tag{6}$$

The scalar curvature S of the relative metric G, which is defined formally and is the curvature of the Riemannian or pseudo-Riemannian manifold (Φ, G), the mean relative curvature H and the Pick-invariant J satisfy the *Theorema Egregium of relative differential geometry*, which states that

$$H + J - S = \frac{n}{n-1}\,\|\overline{T}\|_G, \tag{7}$$

where $\|\overline{T}\|_G := G_{ij}\, T^i\, T^j$ is the *relative norm* of the Tchebychev-vector $\overline{T}$.

2. The Tchebychev-function and some related formulae

We consider the function

$$\varphi := \left(\frac{q}{q_{\text{AFF}}}\right)^{\frac{n+2}{2n}}, \tag{8}$$

where

$$q_{\text{AFF}} := |K_I|^{\frac{1}{n+2}}$$

is the support function of the *equiaffine normalization* $\overline{y}_{\text{AFF}}$ and we call it the *Tchebychev-function* of the relative normalization $\overline{y}$. It is known, that for the components of the Tchebychev-vector holds [3, p. 199]

$$T^i = G^{(ij)}(\ln\varphi)_{/j}.$$

Hence, by (3c), we obtain

$$\overline{T} = \nabla^G(\ln\varphi, \overline{x}) = q\,\nabla^{II}(\ln\varphi, \overline{x})$$

and

$$\left\|\overline{T}\right\|_G = \nabla^G(\ln\varphi) = q\,\nabla^{II}(\ln\varphi). \tag{9}$$

We notice that the Tchebychev-vector vanishes identically iff the Tchebychev-function φ is constant, i.e., by (8), iff $q = c\, q_{\text{AFF}}$, $c \in \mathbb{R}^*$, which means that the relative normalization $\overline{y}$ and the equiaffine normalization $\overline{y}_{\text{AFF}}$ are constantly proportional.

From the relation (4) we obtain the Pick-invariant of the Euclidean normalization ($q = 1$)

$$J_{\text{EUK}} = \frac{1}{n(n-1)} P.$$

Hence by using (5) we find

$$J_{\text{EUK}} = S_{II} - H_I + \frac{(n+2)^2}{4n(n-1)}\, \nabla^{II} (\ln q_{\text{AFF}})\,. \tag{10}$$

From (8) and (10) we conclude that the relation (4) can be written as

$$\frac{J}{q} = \frac{3(n+2)}{4n(n-1)} \left[\frac{4n^2}{(n+2)^2}\, \nabla^{II} (\ln\varphi) - \nabla^{II} (\ln q_{\text{AFF}})\right] + J_{\text{EUK}}. \tag{11}$$

For the equiaffine ($\varphi = 1$) Pick-invariant J_{AFF} we deduce

$$\frac{J_{\text{AFF}}}{q_{\text{AFF}}} = \frac{-3(n+2)}{4n(n-1)}\, \nabla^{II} (\ln q_{\text{AFF}}) + J_{\text{EUK}}. \tag{12}$$

By subtracting (12) from (11) we obtain

$$\frac{J}{q} - \frac{J_{\text{AFF}}}{q_{\text{AFF}}} = \frac{3n}{(n-1)(n+2)}\, \nabla^{II} (\ln\varphi)\,. \tag{13}$$

Similarly, taking (6) and (8) into account, we find

$$\begin{aligned} \frac{H}{q} - H_I = & \frac{2}{n+2} \triangle^{II} (\ln\varphi) + \frac{4n}{(n+2)^2}\, \nabla^{II} (\ln\varphi) \\ & - \frac{n-2}{n+2}\, \nabla^{II} (\ln\varphi, \ln q_{\text{AFF}}) + \frac{1}{n} \triangle^{II} (\ln q_{\text{AFF}}) - \frac{1}{2}\, \nabla^{II} (\ln q_{\text{AFF}})\,. \end{aligned} \tag{14}$$

For the mean equiaffine curvature H_{AFF} we infer

$$\frac{H_{\text{AFF}}}{q_{\text{AFF}}} - H_I = \frac{1}{n} \triangle^{II} (\ln q_{\text{AFF}}) - \frac{1}{2}\, \nabla^{II} (\ln q_{\text{AFF}})\,. \tag{15}$$

By subtracting (15) from (14) we obtain

$$\frac{H}{q} - \frac{H_{\text{AFF}}}{q_{\text{AFF}}} = \frac{2}{n+2} \triangle^{II} (\ln\varphi) + \frac{4n}{(n+2)^2}\, \nabla^{II} (\ln\varphi) - \frac{n-2}{n+2}\, \nabla^{II} (\ln\varphi,\ \ln q_{\text{AFF}})\,. \tag{16}$$

The relations (7), (9), (13) and (16) may be combined into

$$\frac{S}{q} - \frac{J_{\text{AFF}} + H_{\text{AFF}}}{q_{\text{AFF}}} = \frac{2}{n+2} \triangle^{II} (\ln\varphi) - \frac{n(n-2)}{(n+2)^2} \nabla^{II} (\ln\varphi) - \frac{n-2}{n+2}\, \nabla^{II} (\ln\varphi,\ \ln q_{\text{AFF}})\,,$$

and with reference to

$$S_{\text{AFF}} = J_{\text{AFF}} + H_{\text{AFF}}, \tag{17}$$

where S_{AFF} denotes the inner equiaffine curvature, we conclude that

$$\frac{S}{q} - \frac{S_{\text{AFF}}}{q_{\text{AFF}}} = \frac{2}{n+2} \triangle^{II} (\ln\varphi) - \frac{n(n-2)}{(n+2)^2} \nabla^{II} (\ln\varphi) - \frac{n-2}{n+2}\, \nabla^{II} (\ln\varphi,\ \ln q_{\text{AFF}})\,. \tag{18}$$

3. Characterizations of ruled surfaces in $\mathbb{R}^3$ and of hyperquadrics in $\mathbb{R}^{n+1}$

Let now α be any real number. By using the relations (13) and (16)–(18) we obtain

$$\frac{\alpha(S-H)+J}{q} = (\alpha+1)\,\frac{J_{\text{AFF}}}{q_{\text{AFF}}} - \frac{n\,[\alpha(n-1)-3]}{(n-1)\,(n+2)}\nabla^{II}\ln\varphi.$$

For $\alpha = \dfrac{3}{n-1}$ we get

$$\frac{3(S-H)+(n-1)J}{q} = (n+2)\frac{J_{\text{AFF}}}{q_{\text{AFF}}}. \tag{19}$$

This result implies the following

Proposition 1. *Let $(\Phi, \overline{y})$ be a relatively normalized hypersurface of $\mathbb{R}^{n+1}$. Then the function*

$$\frac{3(S-H)+(n-1)J}{q}$$

is independent of the relative normalization and vanishes iff $J_{\text{AFF}} = 0$.

On account of the relations (7) and (19) we infer that

$$\|\overline{T}\|_G = \frac{(n-1)\,(n+2)}{3n}\left(J - \frac{q}{q_{\text{AFF}}}J_{\text{AFF}}\right) = \frac{n+2}{n}\left(H - S + \frac{q}{q_{\text{AFF}}}J_{\text{AFF}}\right). \tag{20}$$

From (20) follows immediately

$$J_{\text{AFF}} = 0 \iff 3n\left\|\overline{T}\right\|_G = (n-1)\,(n+2)\,J \iff n\left\|\overline{T}\right\|_G = (n+2)\,(H-S)\,. \tag{21}$$

We suppose that $n = 2$ and $K_I < 0$. It is well known (see [1, p. 125]), that the vanishing of J_{AFF} characterizes the ruled surfaces of $\mathbb{R}^3$ among the surfaces of negative Gaussian curvature. So, from the relations (19) and (21) we obtain the following characterizations for ruled surfaces in $\mathbb{R}^3$:

Proposition 2. *Let $\Phi \subset \mathbb{R}^3$ be a surface of negative Gaussian curvature. Then Φ is a ruled surface iff there exists a relative normalization of Φ, for which one of the following equivalent properties holds true:*

(*a*) $3(S-H)+J = 0$,

(*b*) $3\|\overline{T}\|_G = 2J$,

(*c*) $\|\overline{T}\|_G = 2(H-S)$.

Let now be $n \geq 2$ and $K_I > 0$. Moreover, without loss of generality, we assume that the second fundamental form II is positive definite. It is also well-known (see [5, p. 380]) that in this case the equiaffine Pick-invariant is non-negative and that it vanishes iff Φ is a hyperquadric. So, by using the relations (19) and (21), we can characterize the hyperquadrics of $\mathbb{R}^{n+1}$ among all hypersurfaces of positive Gaussian curvature as the following proposition states:

Proposition 3. *Let $\Phi \subset \mathbb{R}^{n+1}$ be a hypersurface of positive Gaussian curvature. Then Φ is a hyperquadric iff there exists a relative normalization of Φ, for which one of the following equivalent properties holds true:*

(*a*) $3(S-H)+(n-1)J = 0$,

(*b*) $3n\left\|\overline{T}\right\|_G = (n-1)(n+2)J$,

(*c*) $n\left\|\overline{T}\right\|_G = (n+2)(H-S)$.

4. The vanishing of the Pick-invariant and some integral formulae

Another consequence of relation (13) are the following two propositions:

Proposition 4. *Let $\Phi \subset \mathbb{R}^{n+1}$ be a hypersurface of positive Gaussian curvature. For the Pick-invariant of every relative normalization $\overline{y}$ the following relation is valid*

$$\frac{J}{q} - \frac{J_{\text{AFF}}}{q_{\text{AFF}}} \geq 0. \tag{22}$$

The equality holds iff the relative normalization $\overline{y}$ and the equiaffine normalization $\overline{y}_{\text{AFF}}$ are constantly proportional.

Proof. Because of the assumption $K_I > 0$ we have $\nabla^{II}(\ln\varphi) \geq 0$. So the inequality follows from (13). Furthermore,

$$\frac{J}{q} - \frac{J_{\text{AFF}}}{q_{\text{AFF}}} = 0 \iff \nabla^{II}(\ln\varphi) = 0 \iff \varphi = \text{const.} \iff q = c\, q_{\text{AFF}},\ c \in \mathbb{R}^*,$$

which proves the assertion. □

Proposition 5. *Let $\Phi \subset \mathbb{R}^{n+1}$ be a hypersurface of positive Gaussian curvature. If there is a relative normalization $\overline{y}$, whose Pick-invariant vanishes identically, then Φ is a hyperquadric. Furthermore $\overline{y}$ is constantly proportional to the equiaffine normalization $\overline{y}_{\text{AFF}}$.*

Proof. Let $\overline{y}$ be a relative normalization of Φ with vanishing Pick-invariant. Then, from the relation (13) we obtain

$$-\frac{J_{\text{AFF}}}{q_{\text{AFF}}} = \frac{3n}{(n-1)(n+2)}\, \nabla^{II}(\ln\varphi)\,. \tag{23}$$

Because of $J_{\text{AFF}} \geq 0$ and $\nabla^{II} \ln\varphi \geq 0$, both members of (23) vanish. But $J_{\text{AFF}} \geq 0$ implies that Φ is a hyperquadric and $\nabla^{II} \ln\varphi = 0$ implies that the function φ is constant, which means that $q = c\, q_{\text{AFF}}, c \in \mathbb{R}^*$, and the proof is completed. □

We conclude the paper by considering closed surfaces of positive Gaussian curvature (ovaloids) in $\mathbb{R}^3$. For $n = 2$ relation (16) becomes

$$\frac{H}{q} - \frac{H_{\text{AFF}}}{q_{\text{AFF}}} = \frac{1}{2} \triangle^{II}(\ln\varphi) + \frac{1}{2} \nabla^{II}(\ln\varphi),$$

from which we have

Proposition 6. *Let $(\Phi, \overline{y})$ be a relatively normalized ovaloid in $\mathbb{R}^3$. Then*

$$\iint_M \left(\frac{H}{q} - \frac{H_{\text{AFF}}}{q_{\text{AFF}}}\right) \mathrm{d}O_{II} \geq 0,$$

where $\mathrm{d}O_{II}$ is the element of area of Φ with respect to the second fundamental form II of Φ. The equality is valid iff the relative normalization $\overline{y}$ is constantly proportional to the equiaffine normalization $\overline{y}_{\text{AFF}}$.

Furthermore, for $n = 2$, relation (18) becomes

$$\frac{S}{q} - \frac{S_{\text{AFF}}}{q_{\text{AFF}}} = \frac{1}{2} \triangle^{II}(\ln\varphi). \tag{24}$$

From this equation we easily deduce:

Proposition 7. *Let* $(\Phi, \overline{y})$ *be a relatively normalized ovaloid in* $\mathbb{R}^3$*. If the function*

$$\frac{S}{q} - \frac{S_{\text{AFF}}}{q_{\text{AFF}}}$$

does not change its sign on M*, then the relative normalization* $\overline{y}$ *and the equiaffine normalization* $\overline{y}_{\text{AFF}}$ *are constantly proportional.*

Finally, from the relations (10), (12), (15) and (17) for $n = 2$ we obtain

$$\frac{S_{\text{AFF}}}{q_{\text{AFF}}} - S_{II} = \frac{1}{2} \triangle^{II} (\ln q_{\text{AFF}}). \tag{25}$$

If we now integrate (24) and (25) over M we get

$$\iint_M \frac{S}{q} \,\mathrm{d}O_{II} = \iint_M \frac{S_{\text{AFF}}}{q_{\text{AFF}}} \,\mathrm{d}O_{II} = \iint_M S_{II} \,\mathrm{d}O_{II} = 2\pi\chi,$$

where $\chi = 2$ is the Euler characteristic of Φ. Hence we arrive at

Proposition 8. *Let* $(\Phi, \overline{y})$ *be a relatively normalized ovaloid in* $\mathbb{R}^3$*. Then the following integral formula is valid*

$$\iint_M \frac{S}{q} \,\mathrm{d}O_{II} = 4\pi.$$

Corollary 9. *For an ovaloid* $\Phi \subset \mathbb{R}^3$ *the following integral formula is valid*

$$\iint_M \frac{S_{\text{AFF}}}{q_{\text{AFF}}} \,\mathrm{d}O_{II} = 4\pi.$$

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