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Classification of Manifolds Resulting as Minkowski Operation Products of Basic Geometric Point Sets

Daniela Velichová

Institute of Mathematics and Physics, Faculty of Mechanical Engineering
Slovak University of Technology, Nám. slobody 17, 812 31 Bratislava, Slovakia
email: daniela.velichova@stuba.sk

Abstract. This paper brings a basic classification of possible results of the point set operations 'Minkowski sum' and 'Minkowski product' performed on elementary geometric figures as point subsets in the Euclidean space $\mathbf{E}^n$, which can be represented analytically by the relevant vector maps. The intrinsic geometric properties of the resulting manifolds are presented with illustrations of various appealing results in $\mathbf{E}^3$, and 3-dimensional orthographic views of manifolds from spaces $\mathbf{E}^4$ and $\mathbf{E}^6$.

Key Words: Minkowski point set operation, vector sum, exterior wedge product of vectors, Minkowski point set combination, Minkowski operator.

MSC 2010: 51N20, 53A05

1. Introduction

In accordance with the original definition of the Minkowski sum of two point sets based on their position with respect to a fixed reference point introduced by Hermann Minkowski in 1920, both Minkowski point set operations can be defined as operations on vectors determined by their Cartesian coordinates. The *Minkowski sum* of point sets is defined as vector sum and the *Minkowski product* of point sets as exterior (wedge) product of position vectors of all points in the respective sets. The vector maps of the operands in the two Minkowski point set operations determine the form of the vector maps of the resulting manifolds. Therefore, they can be represented analytically and their intrinsic geometric properties can be derived by means of differential characteristics of the two operand manifolds. Different properties of the background vector operations vector sum and wedge product of vectors influence the intrinsic characteristics of resulting manifolds, too.

The position of the operand sets with respect to the reference point at the origin of the Cartesian coordinate system is one of the key characteristics of the classification. Let us

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suppose first that none of the operand manifolds determined by the vector map defined on an interval in the real numbers contains the reference point O. Such situations will be considered as special singularities and treated separately.

We assume Cartesian coordinates with origin O in $\mathbf{E}^n$ and denote the non-zero position vector of points $p \neq O$ as $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $p = (p_1, p_2, \ldots, p_n)$. Manifolds represented by their vector maps are considered as infinite sets of points whose position vectors are determined as respective values of the vector functions which define the manifolds. Therefore, the operations Minkowski sum and Minkowski product applied on two manifolds can be performed as operations on their vector maps.

Starting from the Minkowski sum $\oplus$ and the Minkowski product $\otimes$ of two points, the resulting points are determined straightforwardly. The sum of points $a, b \in \mathbf{E}^n$ is again a point in $\mathbf{E}^n$, whereas their product is a point in the space $\mathbf{E}^d$, for $d = n\,(n-1)/2$.

$$a = (a_1, a_2, \ldots, a_n)\,, \quad b = (b_1, b_2, \ldots, b_n)$$

$$a \oplus b = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

$$\begin{aligned} a \otimes b &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n) \wedge (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + \cdots + b_n\mathbf{e}_n) \\ &= (a_1b_2 - a_2b_1)\,\mathbf{e}_{1\wedge 2} + \cdots + (a_{n-1}b_n - a_nb_{n-1})\,\mathbf{e}_{n-1\wedge n} \end{aligned}$$

The bivectors $\mathbf{e}_{1\wedge 2}$, $\mathbf{e}_{1\wedge 3}$, $\ldots$, $\mathbf{e}_{1\wedge n}$, $\mathbf{e}_{2\wedge 3}$, $\mathbf{e}_{2\wedge 4}$, $\ldots$, $\mathbf{e}_{2\wedge n}$, $\ldots$, $\mathbf{e}_{n-1\wedge n}$ form an ortho-normal basis of the space $\wedge^2\left(\mathbf{E}^d\right)$. For dimension $n = 3$ the wedge product can be considered as equivalent to the vector product. The Euclidean space $\mathbf{E}^3$ with the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is isomorphic to the space $\wedge^2(\mathbf{E}^3)\}$ with the basis $\{\mathbf{e}_{1\wedge 2}, \mathbf{e}_{1\wedge 3}, \mathbf{e}_{2\wedge 3}\}$, because there exists a regular linear transformation $\phi:\ \wedge^2(\mathbf{E}^3) \to \mathbf{E}^3$ mappig one basis to the other one. The matrix of this transformation is

$$M_\phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For an arbitrary vector $\mathbf{u}$ holds $\phi(\mathbf{u}) = \mathbf{u} \cdot M_\phi$, therefore

$$\phi(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_3, \quad \phi(\mathbf{e}_1 \wedge \mathbf{e}_3) = -\mathbf{e}_2, \quad \phi(\mathbf{e}_2 \wedge \mathbf{e}_3) = \mathbf{e}_1.$$

For the vector product of vectors $\mathbf{u}$ and $\mathbf{v}$ holds

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\,\mathbf{e}_1 + (u_3v_1 - u_1v_3)\,\mathbf{e}_2 + (u_1v_2 - u_2v_1)\,\mathbf{e}_3$$

and

$$\begin{aligned} &\phi(\mathbf{u} \wedge \mathbf{v}) = \phi\left((u_1v_2 - u_2v_1)(\mathbf{e}_1 \wedge \mathbf{e}_2) + (u_1v_3 - u_3v_1)(\mathbf{e}_1 \wedge \mathbf{e}_3) + (u_2v_3 - u_3v_2)(\mathbf{e}_2 \wedge \mathbf{e}_3)\right) \\ &= (u_1v_2 - u_2v_1)\,\phi(\mathbf{e}_1 \wedge \mathbf{e}_2) + (u_1v_3 - u_3v_1)\,\phi(\mathbf{e}_1 \wedge \mathbf{e}_3) + (u_2v_3 - u_3v_2)\,\phi(\mathbf{e}_2 \wedge \mathbf{e}_3) = \mathbf{u} \times \mathbf{v}. \end{aligned}$$

The inverse ϕ^{-1} of the linear transformation ϕ yields $\phi^{-1}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \wedge \mathbf{v}$, while for its matrix holds $M_{\phi^{-1}} = M_\phi$.

The wedge product and the vector product of vectors in $\mathbf{E}^3$ are equivalent, and the Minkowski product of two points in $\mathbf{E}^3$ can therefore be considered as a point in $\mathbf{E}^3$. The wedge product of two vectors in $\mathbf{E}^4$ is a vector in $\mathbf{E}^6$, so the Minkowski product of two points in $\mathbf{E}^4$ is considered as a point in $\mathbf{E}^6$.

Let a differentiable manifold M be determined in the space $\mathbf{E}^n$ by its vector map

$$\mathbf{r}(u_i) = (x_1(u_i),\, x_2(u_i),\, \ldots,\, x_n(u_i)), \quad u_i \in \Omega,$$

where the coordinate functions $x_1(u_i)$, $x_2(u_i)$, $\ldots$, $x_n(u_i)$ are defined and at least once differentiable on the region $\Omega \subset \mathbb{R}^i$, $i \leq n$. The Minkowski sum of an arbitrary point $p = (p_1, p_2, \ldots, p_n)$ and the manifold M is a manifold $M' \subset \mathbf{E}^n$, which is the manifold M translated by the position vector $\mathbf{p}$. The Minkowski product of the manifold M and the point p is a manifold $M^* \subset \mathbf{E}^d$, determined by the vector map

$$\begin{aligned} p \otimes M &= (p_1\mathbf{e}_1 + \cdots + p_n\mathbf{e}_n) \wedge (x_1(u_i), x_2(u_i), \ldots, x_n(u_i)) \\ &= (p_1 x_2(u_i) - p_2 x_1(u_i),\, \ldots,\, p_{n-1} x_n(u_i) - p_n x_{n-1}(u_i)) \,. \end{aligned}$$

Examples from $\mathbf{E}^3$ are presented in Figure 1, where the Minkowski products of a point and a curve, a point and a planar region and a point and a surface patch are presented.

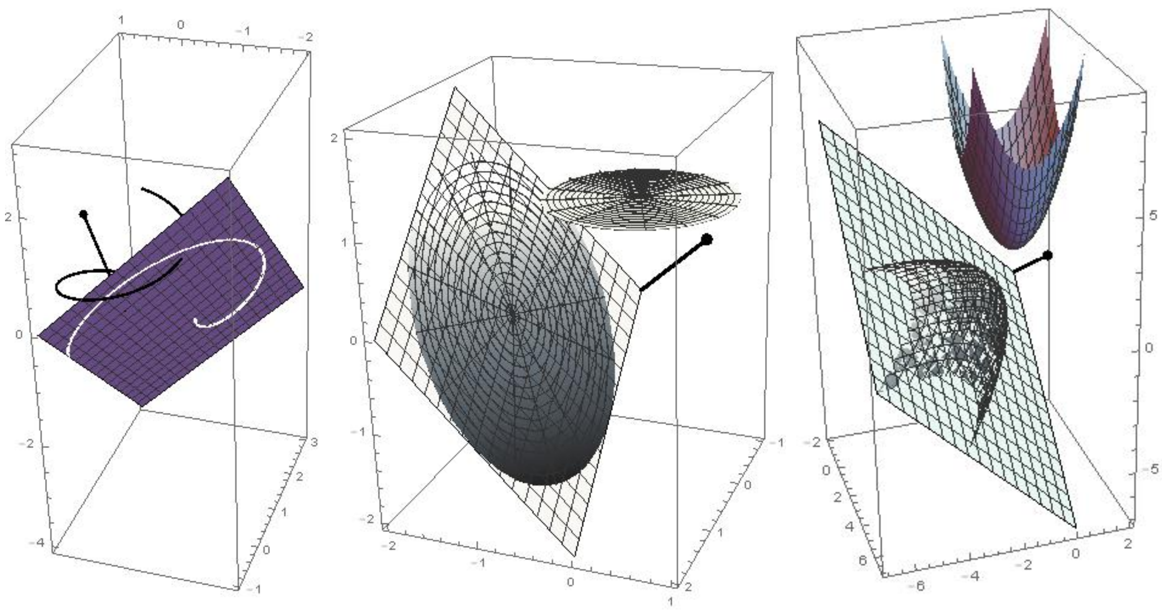

Figure 1: The Minkowski product of a point and various manifolds.

The Minkowski product of point p and the manifold M is the image of M under a 'quasi-central projection' from the centre p to the plane passing through the origin O and perpendicular to the position vector $\mathbf{p}$ of the projection centre p. Let $p = (p_1, p_2, p_3)$ and the manifold M be determined by the vector map

$$\mathbf{r}(u,v) = (x_1(u,v),\; x_2(u,v),\; x_3(u,v))\,, \quad (u,v) \in \Omega \subset \mathbb{R}^2.$$

Then the Minkowski product of M and p is a manifold $M^* = p \wedge M$ determined by the relation

$$\mathbf{r}^*(u,v) = \mathbf{r}(u,v) \cdot T = \mathbf{r}(u,v) \cdot \begin{pmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{pmatrix},$$

where T is the matrix representing this linear transformation. The image M^* is a planar figure located in the plane $p_1 x + p_2 y + p_3 z = 0$.

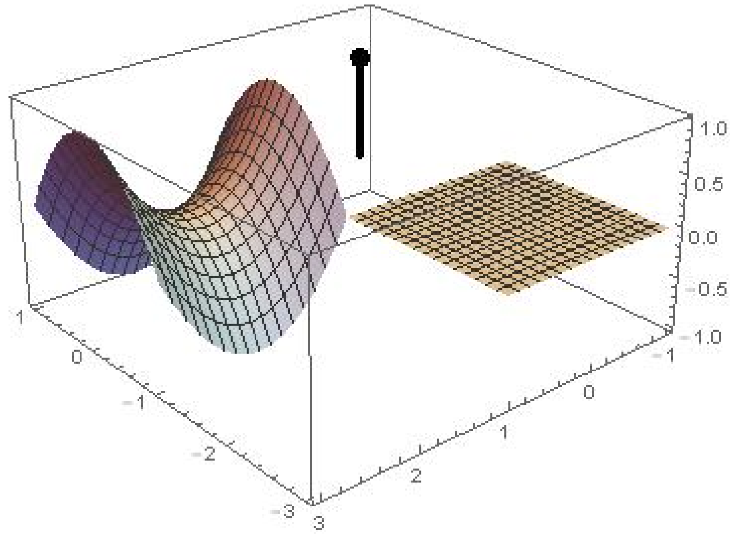

Figure 2: The Minkowski product of a point and a surface patch.

A special position of the point p on a particular coordinate axis yields a quasi-central projection to the perpendicular coordinate plane. Thus, three linear transformations are determined, composed from the orthographic projections to the respective coordinate plane xy, xz, and yz, the revolution in this plane through the angle $-\pi/2$ about the origin O, and the scaling by a non-zero coordinate p_3, p_2 or p_1 of the centre of projection. The respective matrices are

$$\begin{pmatrix} 0 & p_3 & 0 \\ -p_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -p_2 \\ 0 & 0 & 0 \\ p_2 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_1 \\ 0 & -p_1 & 0 \end{pmatrix}.$$

In Figure 2 an image of a hyperbolic paraboloid is presented under the quasi-central projection from the point $p = (0, 0, 1)$ onto the plane xy.

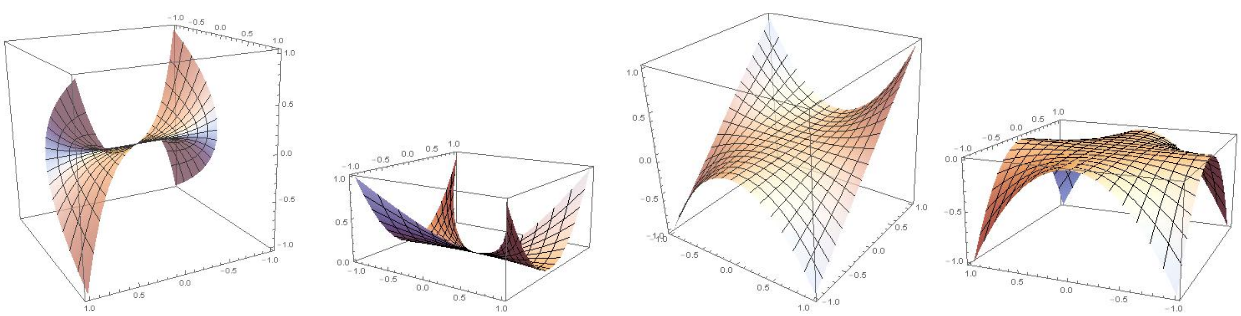

Figure 3: 3D views of a manifold from $\mathbf{E}^4$.

Let a 2-dimensional manifold in $\mathbf{E}^4$ be defined by the map

$$\mathbf{r}(u, v) = (u, v, u^2 v, u^2 v^2), \quad (u, v) \in \mathbb{R}^2.$$

3D orthographic views of this surface patch in the coordinate subspaces yzw, xzw, xyw and the view under the quasi-central projection from the point $p = (0, 0, 0, 1)$ to the coordinate subspace xyz (representing the revolved orthographic view to this coordinate space) are presented in Figure 3.

2. Minkowski sum and product of lines and planes

2.1. Two lines

We consider two lines in space $\mathbf{E}^3$ given by the parametric representations

$$\mathbf{p}(u) = p + u\mathbf{a}, \quad \mathbf{q}(v) = q + v\mathbf{b}, \quad u, v \in \mathbb{R}.$$

Their positions determine the form of their Minkowski sum

$$S = \mathbf{p}(u) \oplus \mathbf{q}(v) = p \oplus q + u\mathbf{a} + v\mathbf{b},$$

and their Minkowski product for $(u, v) \in \mathbb{R}^2$,

$$P = \mathbf{p}(u) \otimes \mathbf{q}(v) = p \wedge q + p \wedge u\mathbf{a} + q \wedge v\mathbf{b} + u\mathbf{a} \wedge v\mathbf{b}.$$

Two parallel lines with collinear direction vectors $\mathbf{a}$ and $\mathbf{b} = k\mathbf{a}$ determine as Minkowski sum a line S in the same direction passing through the point $p \oplus q$. When one line passes through the origin, the Minkowski sum coincides with the other line.

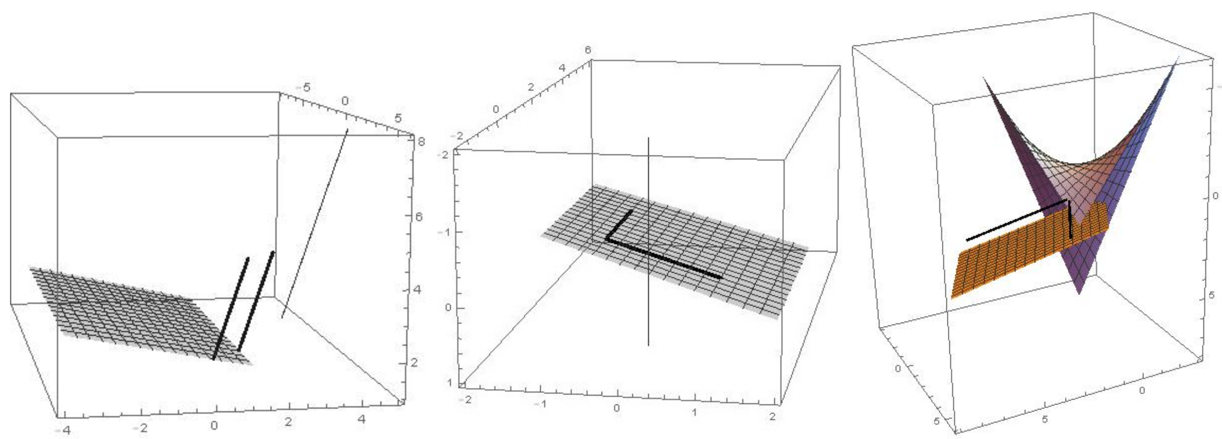

Figure 4: The Minkowski sum and the Minkowski product of parallel line segments (left) and of line segments intersecting at the origin(right and middle).

Singular forms of the Minkowski product of two parallel lines are the following. If the lines span a plane passing through the origin, their Minkowski product is a line passing through the origin perpendicularly to this plane, i.e., to both lines. The Minkowski product of a line with itself is a perpendicular line passing through the origin if the line does not contain the origin; otherwise it is just one point — the origin. Consider now two parallel lines that do not span a plane passing through the origin. Their Minkowski product is a plane passing through the point $p \otimes q$ and perpendicular to both lines (see Figure 4, left) with the general equation

$$a_1x + a_2y + a_3z + a_3(q_1p_2 - q_2p_1) + a_2(q_3p_1 - q_1p_3) + a_1(q_2p_1 - q_1p_2) = 0$$

where

$$\mathbf{a} = (a_1, a_2, a_3), \quad p = (p_1, p_2, p_3), \quad q = (q_1, q_2, q_3).$$

The Minkowski sum of intersecting lines with non-collinear direction vectors is a plane S passing through the point $p \oplus q$ and parallel to the plane formed by these lines. If the lines span a plane passing through origin, their Minkowski sum is the same plane (Figure 4, right).

If the two lines do not span a plane passing through origin they form a plane determined by their common point $r = (r_1, r_2, r_3)$ and their direction vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ with the equation

$$\begin{aligned}&(a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)y + (a_1b_2 - a_2b_1)z\\&-(a_2b_3 - a_3b_2)r_1 - (a_3b_1 - a_1b_3)r_2 - (a_1b_2 - a_2b_1)r_3 = 0.\end{aligned}$$

From this follows a condition which is satisfied by a plane not passing through the origin, namely

$$(a_2b_3 - a_3b_2)r_1 + (a_3b_1 - a_1b_3)r_2 + (a_1b_2 - a_2b_1)r_3 \neq 0.$$

The Minkowski product of such two lines is a quadratic surface with the equation

$$\begin{aligned}&a_1b_1x^2 + a_2b_2y^2 + a_3b_3z^2 + (a_1b_2 + a_2b_1)xy + (a_3b_1 + a_1b_3)xz + (a_2b_3 + a_3b_2)yz\\&+[r_1^2(b_2a_3 - a_2b_3) + r_1r_2(a_1b_3 - b_1a_3) + r_1r_3(b_1a_2 - a_1b_2)]x\\&+[r_2^2(a_1b_3 - b_1a_3) + r_1r_2(b_2a_3 - a_2b_3) + r_2r_3(b_1a_2 - a_1b_2)]y\\&+[r_3^2(b_1a_2 - a_1b_2) + r_1r_3(b_2a_3 - a_2b_3) + r_2r_3(a_1b_3 - b_1a_3)]z = 0\end{aligned}$$

It can be proved by standard methods that this is the general equation of a hyperbolic paraboloid (Figure 4, right).

In the singular situation, when two intersecting lines span a plane passing through origin, their Minkowski product is a line passing through the origin and perpendicular to the plane spanned by the two lines (Figure 4, middle).

It is easy to prove that the Minkowski sum of two skew lines with linearly independent direction vectors $\mathbf{a}$ and $\mathbf{b}$ passing through points p and q, respectively, is a plane S determined by the two direction vectors and passing through the point $p \oplus q$. If one of the lines is passing through the origin, say $p \equiv O$, then the Minkowski sum (Figure 5, right) is a plane passing through the other line

$$S = \mathbf{p}(u) + \mathbf{q}(v) = q + u\mathbf{a} + v\mathbf{b}, \quad u, v \in \mathbb{R}.$$

The Minkowski product of two skew lines is a plane if and only if one of the lines is parallel to the plane passing through the origin and the other line. In case one of the skew lines is passing through the origin, $p \equiv O$, their Minkowski product

$$P = \mathbf{p}(u) \wedge \mathbf{q}(v) = q \wedge v\mathbf{b} + u\mathbf{a} \wedge v\mathbf{b} = (q + u\mathbf{a}) \wedge v\mathbf{b}, \quad (u, v) \in \mathbb{R}^2$$

is a plane through the origin and perpendicular to this line (Figure 5, right). In all other configurations it is a hyperbolic paraboloid (Figure 5, left).

2.2. Line and plane

Let a line and a plane be defined parametrically as

$$\mathbf{p}(t) = p + t\mathbf{a}, \quad t \in \mathbb{R}, \quad \text{and} \quad \mathbf{q}(u, v) = q + u\mathbf{b} + v\mathbf{c}, \quad u, v \in \mathbb{R}.$$

In case they are parallel, the line direction vector can be represented as a linear combination of the plane direction vectors, $\mathbf{a} = k\mathbf{b} + l\mathbf{c}$ for $k, l \in \mathbb{R}$, and their Minkowski sum for $t, u, v \in \mathbb{R}$ is represented as a plane

$$S = \mathbf{p}(t) + \mathbf{q}(u, v) = p \oplus q + (tk + u)\mathbf{b} + (tl + v)\mathbf{c}$$

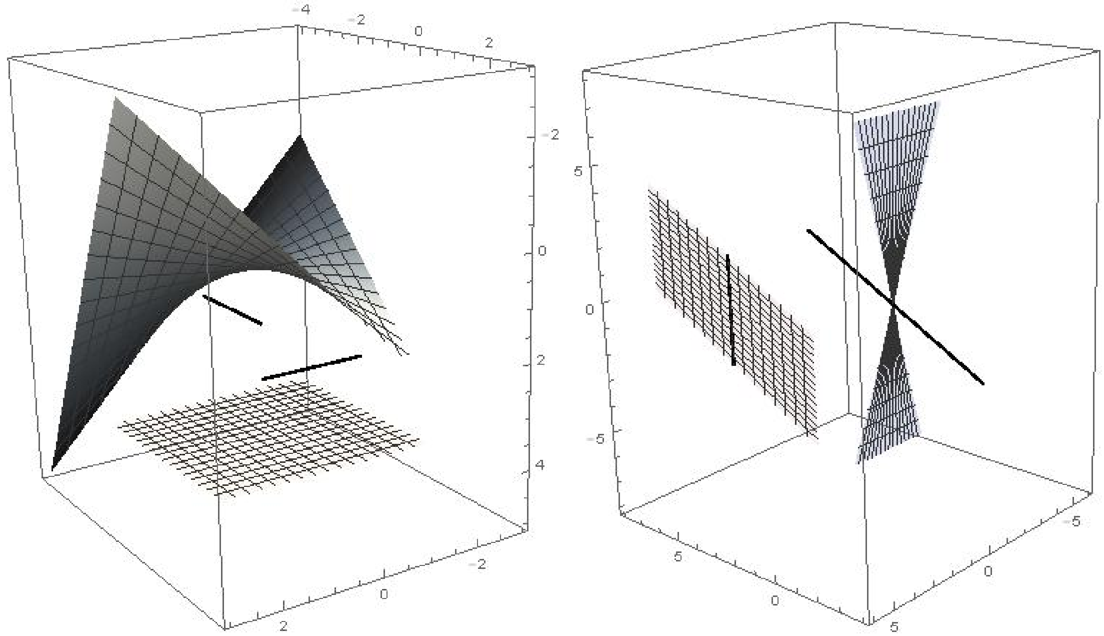

Figure 5: Minkowski sum and product of two skew line segments.

parallel to both, line and plane, and passing through the point $p \oplus q$, provided none of them contains the origin. If the line (plane) contains the origin, their Minkowski sum is the same plane (parallel plane passing through the line), as illustrated in Figure 6, left and middle.

The Minkowski product of a line and a plane

$$P = \mathbf{p}(t) \wedge \mathbf{q}(u,v) = p \otimes q + p \wedge (u\mathbf{b} + v\mathbf{c}) - q \wedge t(k\mathbf{b} + l\mathbf{c})$$

is a hyperbolic paraboloid, if the plane is not passing through origin. In such case, the Minkowski product of the line parallel to the plane is a plane perpendicular to the line (Figure 6, right).

A line and a non-parallel plane determine the Minkowski sum

$$S = \mathbf{p}(t) + \mathbf{q}(u,v) = p \oplus q + t\mathbf{a} + u\mathbf{b} + v\mathbf{c}$$

forming the whole space $\mathbf{E}^3$ for $t, u, v \in \mathbb{R}$.

Their Minkowski product

$$P = \mathbf{p}(t) \wedge \mathbf{q}(u,v) = p \otimes q + (p + t\mathbf{a}) \wedge (u\mathbf{b} + v\mathbf{c}) + t\mathbf{a} \wedge q$$

is the same space $\mathbf{E}^3$, if the line does not pass through the origin. If just the plane passes through the origin, the Minkowski product is formed as a one-parametric system of planes with a common line passing through the origin and perpendicular to the line (Figure 7, left). If just the line is passing through the origin, then their Minkowski product is a plane through the origin and perpendicular to this line

$$P = t\mathbf{a} \wedge (q + u\mathbf{b} + v\mathbf{c}), \quad t, u, v \in \mathbb{R}$$

(Figure 7, middle). If the line and the plane share the origin as their common point, the resulting plane intersects the given plane in the line through the origin and perpendicular to the given line (Figure 7, right).

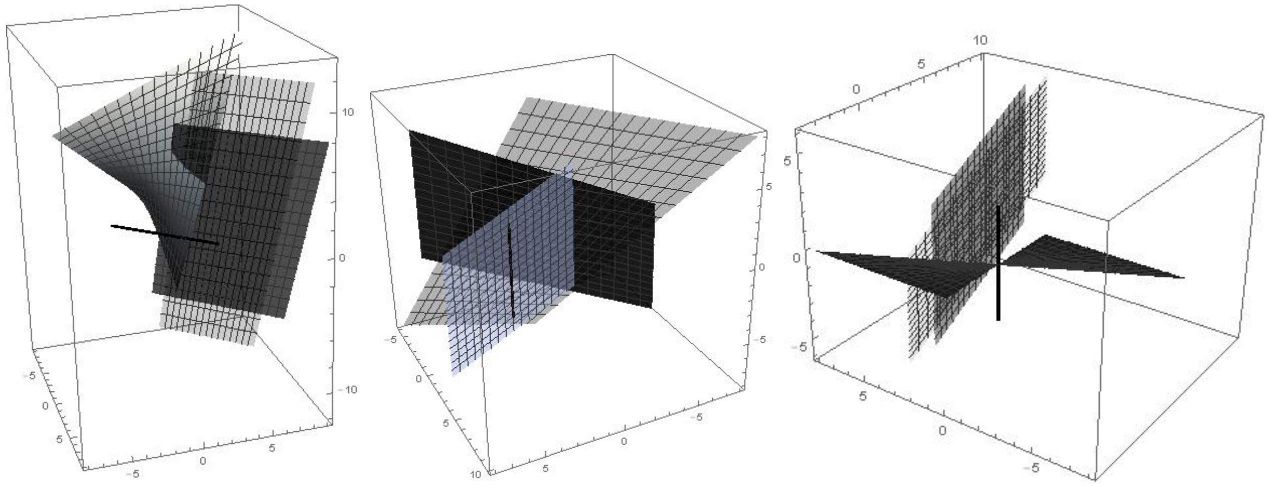

Figure 6: Minkowski sum and product of a line and a parallel plane.

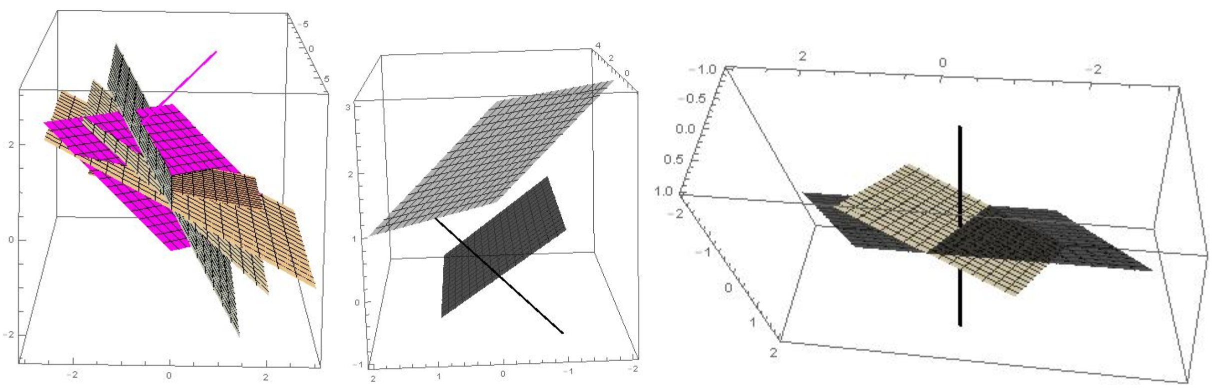

Figure 7: The Minkowski product of a plane and an intersecting line.

2.3. Two planes

Let two intersecting planes in $\mathbf{E}^3$ be determined parametrically by the vector maps

$$\mathbf{p}(u,v) = p + u\mathbf{a} + v\mathbf{b}, \quad u,v \in \mathbb{R}, \quad \text{and} \quad q(s,t) = q + s\mathbf{a} + t\mathbf{c}, \quad s,t \in \mathbb{R}.$$

Their Minkowski sum

$$S = \mathbf{p}(u,v) + \mathbf{q}(s,t) = p \oplus q + (u+s)\mathbf{a} + v\mathbf{b} + t\mathbf{c}$$

and their Minkowski product

$$P = \mathbf{p}(u,v) \wedge \mathbf{q}(s,t) = p \otimes q + p \wedge (s\mathbf{a} + t\mathbf{c}) - q \wedge (u\mathbf{a} + v\mathbf{b}) + ut(\mathbf{a} \wedge \mathbf{c}) - sv(\mathbf{a} \wedge \mathbf{b}) + vt(\mathbf{b} \wedge \mathbf{c})$$

form the whole space $\mathbf{E}^3$. If both planes pass through the origin, their Minkowski product is a one-parametric system of hyperbolic paraboloids sharing the intersection line of the two planes, which passes through the origin (Figure 8, left).

If the planes are parallel,

$$\mathbf{p}(u,v) = p + u\mathbf{a} + v\mathbf{b}, \quad u,v \in \mathbb{R}, \quad \text{and} \quad \mathbf{q}(s,t) = q + s\mathbf{a} + t\mathbf{b}, \quad s,t \in \mathbb{R},$$

their Minkowski sum

$$S = \mathbf{p}(u,v) + \mathbf{q}(s,t) = p \oplus q + (u+s)\mathbf{a} + (v+t)\mathbf{b}$$

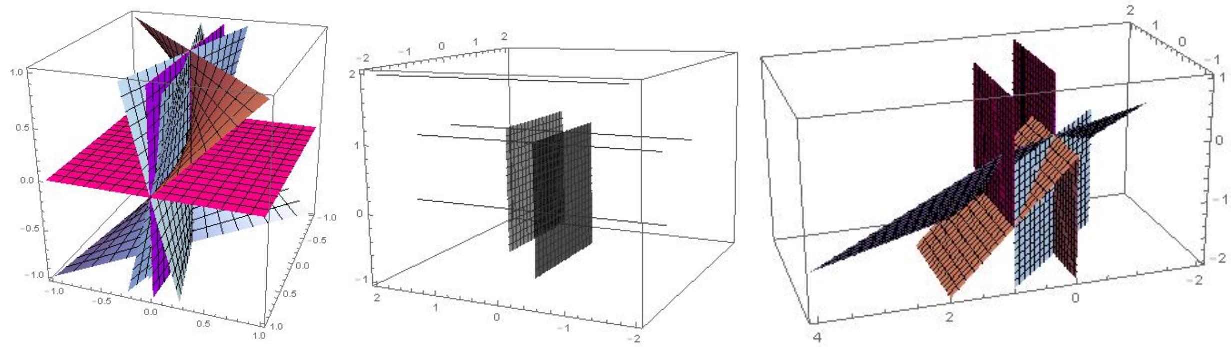

Figure 8: Minkowski product of two planes.

is a plane passing through the point $p \oplus q$ in direction of both parallel planes.

The Minkowski product of two parallel planes is the whole space $\mathbf{E}^3$ in case none of the planes is passing through the origin,

$$P = \mathbf{p}(u, v) \wedge \mathbf{q}(s, t) = p \otimes q + p \wedge (s\mathbf{a} + t\mathbf{b}) - (q \wedge (u\mathbf{a} + v\mathbf{b})) + (ut - vs)(\mathbf{a} \wedge \mathbf{b})),$$

formed by a system of planes (Figure 8, right). If one of the parallel planes passes through the origin, the space is formed by a system of lines perpendicular to both planes (Figure 8, middle).

3. Minkowski sum and product of two curves in 3D

Consider two curve segments determined parametrically as

$$\mathbf{k}(u) = (x_k(u),\, y_k(u),\, z_k(u)), \quad u \in K \subset \mathbb{R}, \quad \text{and} \quad \mathbf{l}(v) = (x_l(v),\, y_l(v),\, z_l(v)), \quad v \in L \subset \mathbb{R}.$$

The Minkowski sum of the two curves is a surface of translation, defined on the planar region $\Omega = K \times L \subset \mathbb{R}^2$. The patch

$$\mathbf{s}(u, v) = (x_k(u) + x_l(v),\ y_k(u) + y_l(v),\ z_k(u) + z_l(v))$$

can be generated by translation of one curve along the other. The differential characteristics of such a surface patch can be represented by means of derivatives of the curve vector maps, abbreviated as

$$\mathbf{k}'(u) = (x_k'(u),\, y_k'(u),\, z_k'(u)),\ u \in K \subset \mathbb{R}, \quad \text{and} \quad \mathbf{l}'(v) = (x_l'(v),\, y_l'(v),\, z_l'(v)),\ v \in L \subset \mathbb{R}.$$

The first and the second fundamental form of the resulting surface and their discriminants can be expressed by the following formulas:

$$\phi_1 = \|\mathbf{k}'\|^2\, du^2 + 2\mathbf{k}' \cdot \mathbf{l}'\, dudv + \|\mathbf{l}'\|^2\, dv^2, \quad D_1 = (\|\mathbf{k}'\| \cdot \|\mathbf{l}'\|)^2 - (\mathbf{k}' \cdot \mathbf{l}')^2 = \|\mathbf{k}'\| \times \|\mathbf{l}'\|^2,$$

$$\phi_2 = Ldu^2 + Ndv^2, \quad L = \frac{[\mathbf{k}', \mathbf{l}', \mathbf{k}'']}{\|\mathbf{k}'\| \times \|\mathbf{l}'\|}, \quad N = \frac{[\mathbf{k}', \mathbf{l}', \mathbf{l}'']}{\|\mathbf{k}'\| \times \|\mathbf{l}'\|},$$

$$D_2 = L \cdot N = \frac{D}{D_1}, \quad D = \begin{vmatrix} \|\mathbf{k}'\|^2 & \mathbf{k}' \cdot \mathbf{l}' & \mathbf{k}' \cdot \mathbf{l}'' \\ \mathbf{k}' \cdot \mathbf{l}' & \|\mathbf{l}'\|^2 & \mathbf{l}' \cdot \mathbf{l}'' \\ \mathbf{k}' \cdot \mathbf{k}'' & \mathbf{k}'' \cdot \mathbf{l}' & \mathbf{k}'' \cdot \mathbf{l}'' \end{vmatrix}.$$

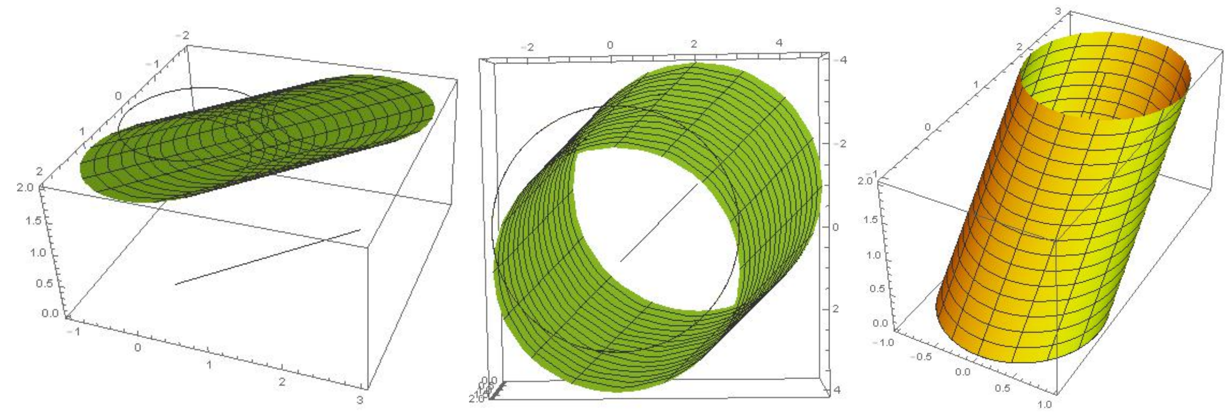

Figure 9: Minkowski sum of a circle and a line segment in various positions.

The Gauss curvature is given by formula $K = D \cdot D_1^{-2}$.

Examples of the Minkowski sum of a line segment and a circle are presented in Figure 9. The planar region presented on the left and in the middle can be generated as Minkowski sum of a line segment parallel to (or located in) the plane of the circle, the circular cylindrical surface patch on the right for a non-parallel line segment.

The Minkowski product of two curves is a surface of translation defined on the planar region $\Omega = K \times L \subset \mathbb{R}^2$,

$$\mathbf{p}(u,v) = \mathbf{k}(u) \wedge \mathbf{l}(v) = \begin{pmatrix} y_k(u)z_l(v) - z_k(u)y_l(v) \\ z_k(u)x_l(v) - x_k(u)z_l(v) \\ x_k(u)y_l(v) - y_k(u)x_l(v) \end{pmatrix}^T, \quad (u,v) \in \Omega,$$

generated by all such points in the space, whose position vector is the vector product of the position vector of one point of the curve k with the position vector of one point of the curve l. In case of elementary planar curves some wellknown surface patches can be generated, e.g., ruled surfaces — cylinders, transition surfaces or conoids, looped strips, torus, and others.

Using properties of the vector product, the Lagrange identity and the following abbreviations

$$\mathbf{k}' \cdot \mathbf{l} = \mathbf{a}, \quad \mathbf{k} \times \mathbf{l}' = \mathbf{b}, \quad \mathbf{k}' \times \mathbf{l}' = \mathbf{c}, \quad \mathbf{k}'' \times \mathbf{l}' = \mathbf{d}, \mathbf{k} \times \mathbf{l}'' = \mathbf{e},$$

the differential characteristics of the Minkowski product of two curves can be derived in the forms

$$\phi_1 = \|\mathbf{a}\|^2\, du^2 + 2\mathbf{a} \cdot \mathbf{b}\, dudv + \|\mathbf{b}\|^2\, dv^2, \quad D_1 = \|\mathbf{a} \times \mathbf{b}\|^2,$$

$$\phi_2 = D_1^{-1}(Ldu^2 + 2Mdudv + Ndv^2), \quad L = [\mathbf{d}, \mathbf{a}, \mathbf{b}], \quad M = [\mathbf{c}, \mathbf{a}, \mathbf{b}], \quad N = [\mathbf{e}, \mathbf{a}, \mathbf{b}]$$

$$D_2 = D_1^{-1}(D_{LN} - D_{M^2}), \quad D_{LN} = \begin{vmatrix} \mathbf{d} \cdot \mathbf{e} & \mathbf{d} \cdot \mathbf{a} & \mathbf{d} \cdot \mathbf{b} \\ \mathbf{a} \cdot \mathbf{e} & \|\mathbf{a}\|^2 & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{e} & \mathbf{a} \cdot \mathbf{b} & \|\mathbf{b}\|^2 \end{vmatrix}, \quad D_{M^2} = \begin{vmatrix} \|\mathbf{c}\|^2 & \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} \\ \mathbf{a} \cdot \mathbf{c} & \|\mathbf{a}\|^2 & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{b} & \|\mathbf{b}\|^2 \end{vmatrix}.$$

The Gauss curvature is given by the formula $K = (D_{LN} - D_{M^2})D_1^{-2}$.

Examples of Minkowski products of a line segment and a circle that are located in different super-positions are illustrated in Figure 10. For the patch on the left, the line segment is perpendicular to the osculating plane of the circle. The transition patch in the middle is the product of a line segment and a circle, both located in the same plane. The 2-sided strip on the right is the Minkowski product of a line segment parallel to the circle's osculating plane.

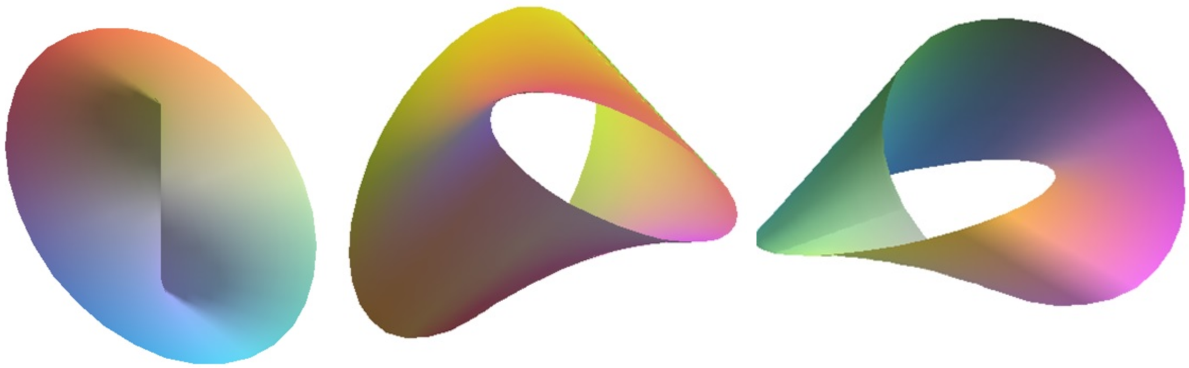

Figure 10: The Minkowski product of a circle and a line segment in various positions.

Other interesting examples arise as Minkowski sum and product of two circles. When the circles are placed in different planes, surface patches of interesting aesthetic forms can be generated. Figure 11 shows the Minkowski sum (disc) and product (torus) of circles in parallel planes. Figure 12, left, shows the Minkowski sum and next to it the product of two concentric circles which are placed in perpendicular planes. Different forms of the Minkowski product can be achieved with non-concentric circles, as illustrated in Figure 12, right. Their Minkowski sum displayed in the middle is identical for both positions of circles, up to its position in the space.

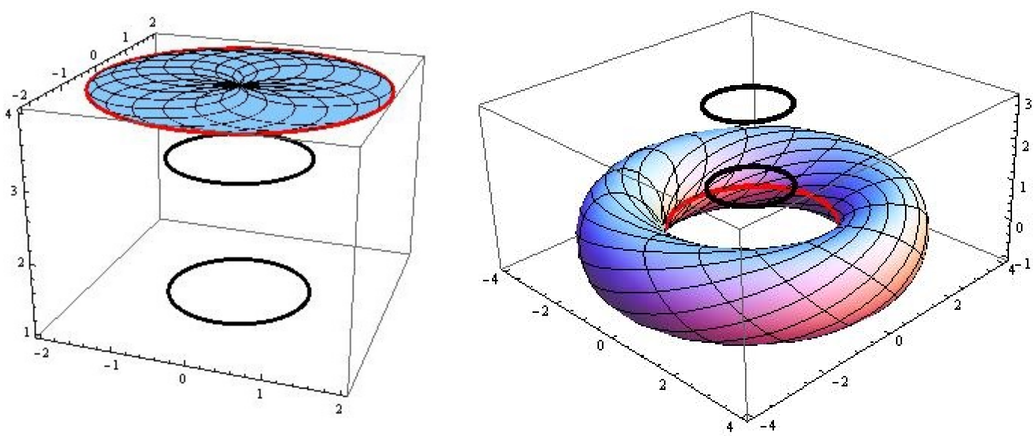

Figure 11: Minkowski sum (left) and product (right) of two circles in parallel planes.

Interesting questions arise in connection with the points on the resulting manifolds, whose curvilinear coordinates u and v are equal. Their point sets can be represented as so called *partial* Minkowski operation results. When both operand curves are parameterized by the same parameter and on the same interval $I \in \mathbb{R}$, the resulting manifold is a curve, represented by the vector map on I,

$$\mathbf{s}(u) = (x_k(u) + x - l(u),\ y_k(u) + y_l(u),\ z_k(u) + z_l(u)),$$

$$\mathbf{p}(u) = \begin{pmatrix} y_k(u)z_l(u) - z_k(u)y_l(u) \\ z_k(u)x_l(u) - x_k(u)z_l(u) \\ x_k(u)y_l(u) - y_k(u)x_l(u) \end{pmatrix}^T,$$

and located on the respective Minkowski sum or product of the two curves, when they are differently parameterized. The Figures 11 and 12 show examples in light colour. In some

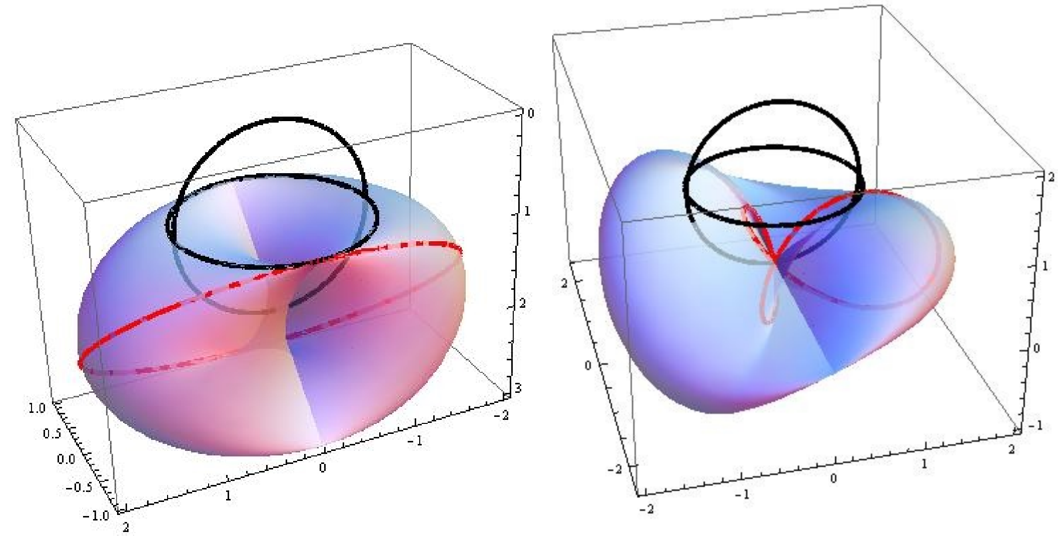

Figure 12: Minkowski sum and product of two circles in perpendicular planes I.

configurations the partial Minkowski operation results are sets of special points on surfaces, for example the outline of a planar region, the neck circle on a torus, or the set of parabolic points on a surface. Their properties are yet not clearly determined, and they are the topic of a current study.

The Minkowski product of two circles sharing none, one and two common points is displayed in Figure 14. At the points, which correspond to common points of the operand curves, the resulting curve seems to have a double (triple) point.

For circles in parallel planes the partial Minkowski sum is an ellipse or a circle. Both curves are presented on the respective surface patches in Figure 13.

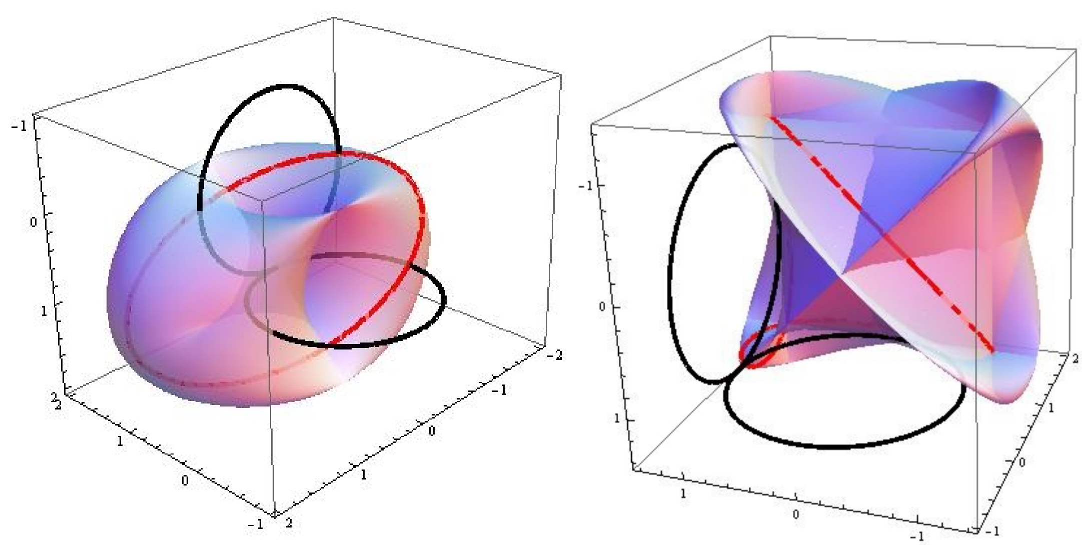

Figure 13: Minkowski sum and product of circles in perpendicular planes II.

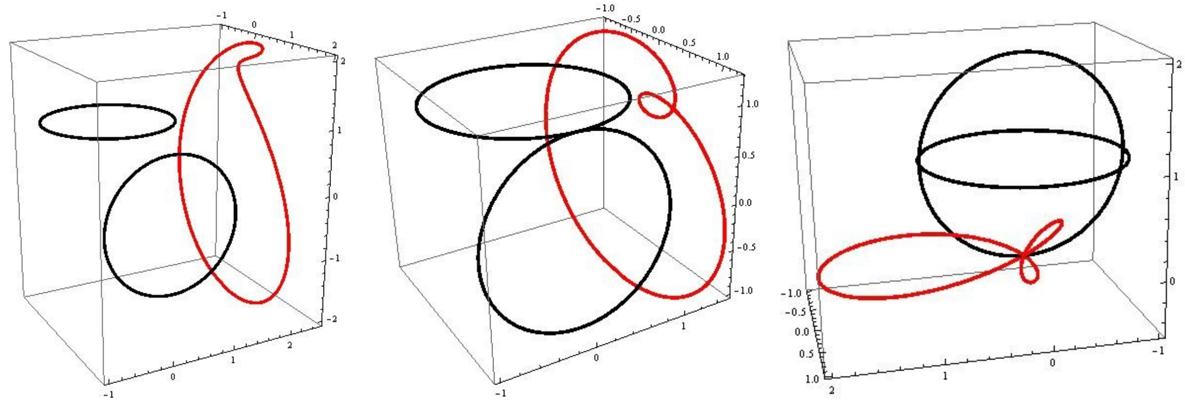

Figure 14: Minkowski product of two equally parameterized circles in perpendicular planes.

4. Minkowski sum and product of curves in higher dimensions

The Minkowski sum of two curve segments in the space $\mathbf{E}^4$, determined parametrically as

$\mathbf{k}(u) = (x_{k1}(u),\, x_{k2}(u),\, x_{k3}(u),\, x_{k4}(u)),\ u \in I,\quad \mathbf{l}(v) = (x_{l1}(v),\, x_{l2}(v),\, x_{l3}(v),\, x_{l4}(v)),\ v \in K,$

can be considered as a surface patch in $\mathbf{E}^4$ with the parametric vector map

$$\mathbf{p}(u,v) = \mathbf{k}(u) + \mathbf{l}(v) = \begin{pmatrix} x_{k1}(u) + x_{l1}(v) \\ x_{k2}(u) + x_{l2}(v) \\ x_{k3}(u) + x_{l3}(v) \\ x_{k4}(u) + x_{l4}(v) \end{pmatrix}$$

defined on $\Omega = I \times L$. Orthographic views of the resulting manifolds in four possible coordinate subspaces in $\mathbf{E}^4$ provide four different forms of surface patches in 3D. Figure 15 shows as examples the Minkowski sums of an ellipse and a conical helix.

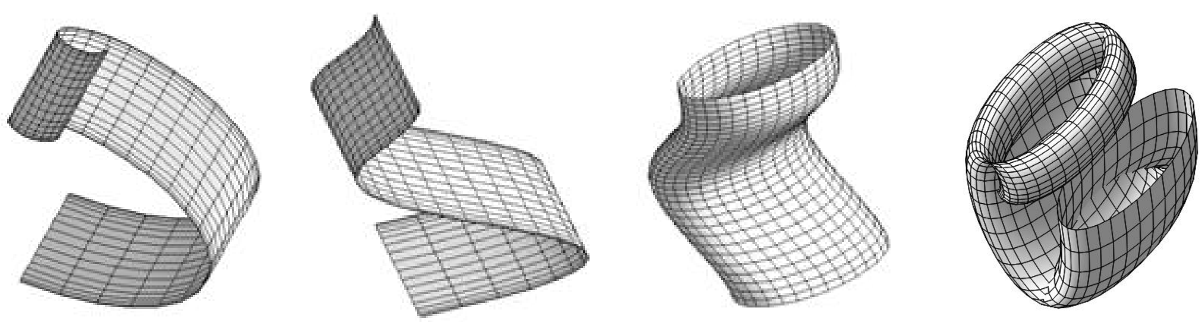

Figure 15: The Minkowski sum of two curves in $\mathbf{E}^4$.

The Minkowski product of the curves $\mathbf{k}(u)$ and $\mathbf{l}(v)$ in $\mathbf{E}^4$ generates a two-parametric manifold in the space $\mathbf{E}^6$ with a vector map, defined on $\Omega = I \times L$, as

$$\mathbf{p}(u,v) = \mathbf{k}(u) \wedge \mathbf{l}(v) = \begin{pmatrix} x_{k1}(u)x_{l2}(v) - x_{k2}(u)x_{l1}(v) \\ x_{k1}(u)x_{l3}(v) - x_{k3}(u)x_{l1}(v) \\ x_{k1}(u)x_{l4}(v) - x_{k4}(u)x_{l1}(v) \\ x_{k2}(u)x_{l3}(v) - x_{k3}(u)x_{l2}(v) \\ x_{k2}(u)x_{l4}(v) - x_{k4}(u)x_{l2}(v) \\ x_{k3}(u)x_{l4}(v) - x_{k4}(u)x_{l3}(v) \end{pmatrix}.$$

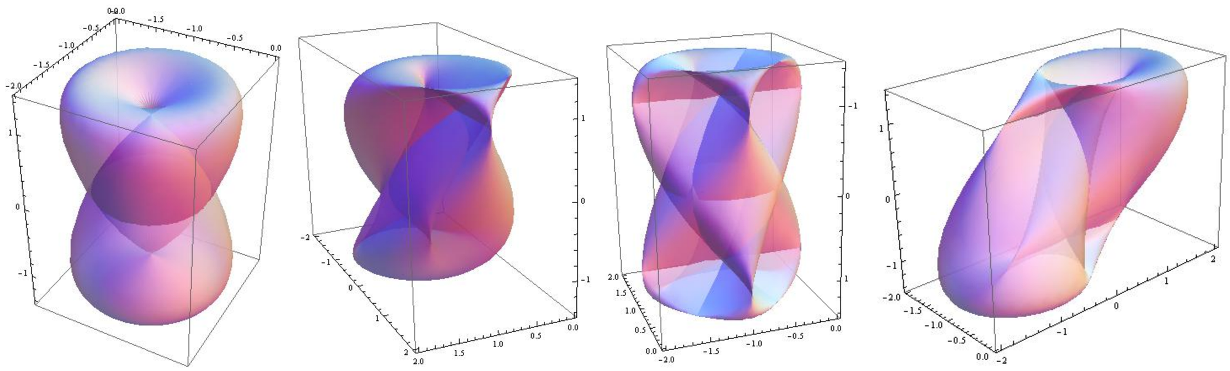

Figure 16: 3D orthographic views of the Minkowski product of two circles in parallel planes in $\mathbf{E}^4$.

The differential characteristics of both manifolds in higher dimensions and their dependence on the characteristics of the operand curve segments are topics for further research.

Surface patches in $\mathbf{E}^6$, which are generated as Minkowski product of two curves from $\mathbf{E}^4$, can be successively orthographically mapped onto various 5-, 4- and 3-dimensional subspaces, leading thus to plenty of new interesting forms of surfaces in $\mathbf{E}^3$ with specific geometric properties.

Two circles located in different coordinate planes in $\mathbf{E}^4$ as operands of the Minkowski product determine a 2-dimensional manifold in $\mathbf{E}^6$. Some of its orthographic views in different 3-dimensional coordinate subspaces are illustrated in the Figures 16, 17 and 18.

The Shamrock curve

$$\mathbf{r}(u) = \left(a_1 \cos u \sin^2 u,\ b_1 \sin u \cos^2 u,\ 0,\ 0\right), \quad u \in \langle 0, 2\pi \rangle$$

and the versiera ('Witch of Agnesi')

$$\mathbf{s}(v) = \left(0,\ a_2(2v-1),\ \frac{b_2}{c_2 + d_2(2v-1)^2},\ 0\right), \quad v \in \langle 0, 1 \rangle$$

determine as Minkowski sum and product surfaces in different 3-dimensional subspaces of $\mathbf{E}^4$ and $\mathbf{E}^6$ (see Figure 19), represented by the vector maps

$$\mathbf{s}(u,v) = \begin{pmatrix} a_1 \cos u \sin^2 u \\ b_1 \sin u \cos^2 u + a_2(2v-1) \\ \dfrac{b_2}{c_2 + d_2(2v-1)^2} \\ 0 \end{pmatrix},$$

$$\mathbf{p}(u,v) = \begin{pmatrix} a_1 a_2 (2v-1) \cos u \sin^2 u \\ \dfrac{a_1 b_2}{c_2 + d_2(2v-1)^2} \cos u \sin^2 u \\ 0 \\ \dfrac{b_1 b_2}{c_2 + d_2(2v-1)^2} \sin u \cos^2 u \\ 0 \\ 0 \end{pmatrix}.$$

5. Conclusions

In this paper results of the Minkowski sum and product of basic geometric sets like points, lines and planes were presented together with illustrations of the resulting manifolds depending on the position of the two operands. We derived the basic differential characteristics of the manifolds which result as Minkowski sum and Minkowski product of two curves in the space $\mathbf{E}^3$. Furthermore, illustrations of the Minkowski sum and product of two curves positioned in different planes in $\mathbf{E}^3$ or $\mathbf{E}^4$ are included — as inspiration ideas. The presented set operations might be applied in design or architecture for modelling unusual shapes, in computer art for smooth modelling and morphing of aesthetic objects, and in various other areas of computer graphics and applications in art and science.

The concepts of the Minkowski sum and Minkowski product of two point sets A and B can be also generalized to the concept of Minkowski summative and multiplicative operators defined on pairs of subsets in the Euclidean space. Here any pair of point sets is attached a linear summative combination or a multiplicative combination of two point sets

$$LS_{k,1}(A,B) = kA \oplus lB$$
$$PS_{k,1}(A,B) = kA \otimes lB.$$

Thus a more powerful modelling tool is provided, in which k- and l-fold scalar multiples of point sets A and B are summed or multiplied in the Minkowski way thus generating a variety of shapes resembling natural forms, applicable, e.g., for bionic purposes.

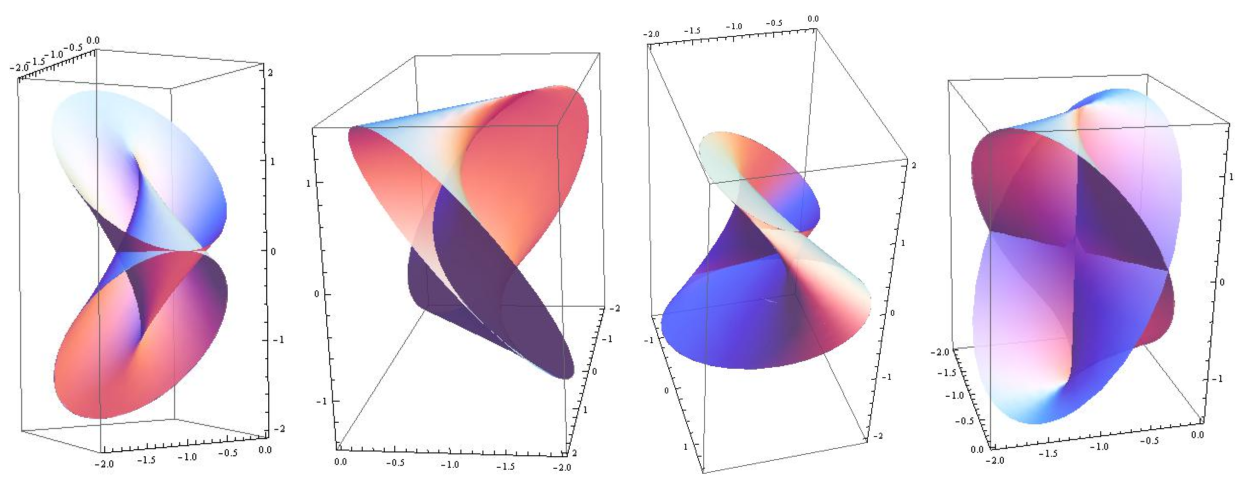

Figure 17: 3D orthographic views of the Minkowski product (in $\mathbf{E}^6$) of two circles located in perpendicular planes in $\mathbf{E}^4$.

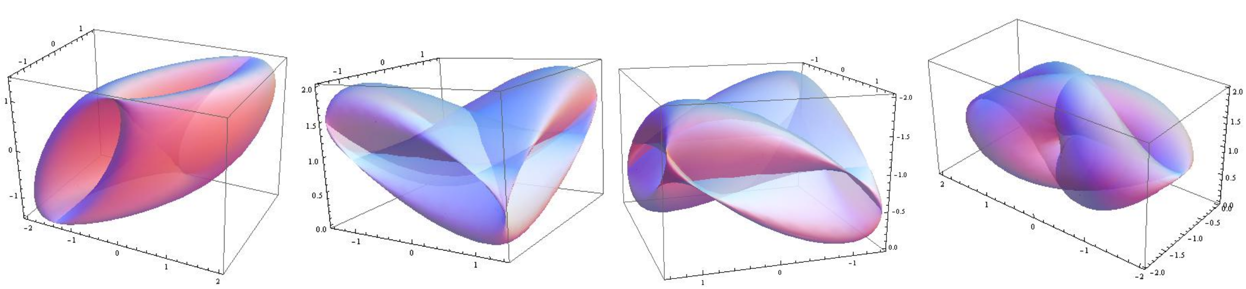

Figure 18: 3D orthographic views of the Minkowski product of two circles in $\mathbf{E}^4$.

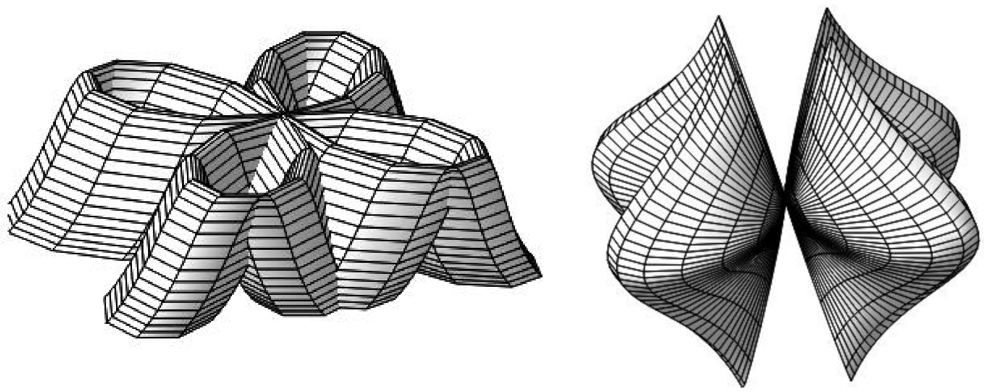

Figure 19: The Minkowski sum and product of the Shamrock curve and the versiera in $\mathbf{E}^4$.

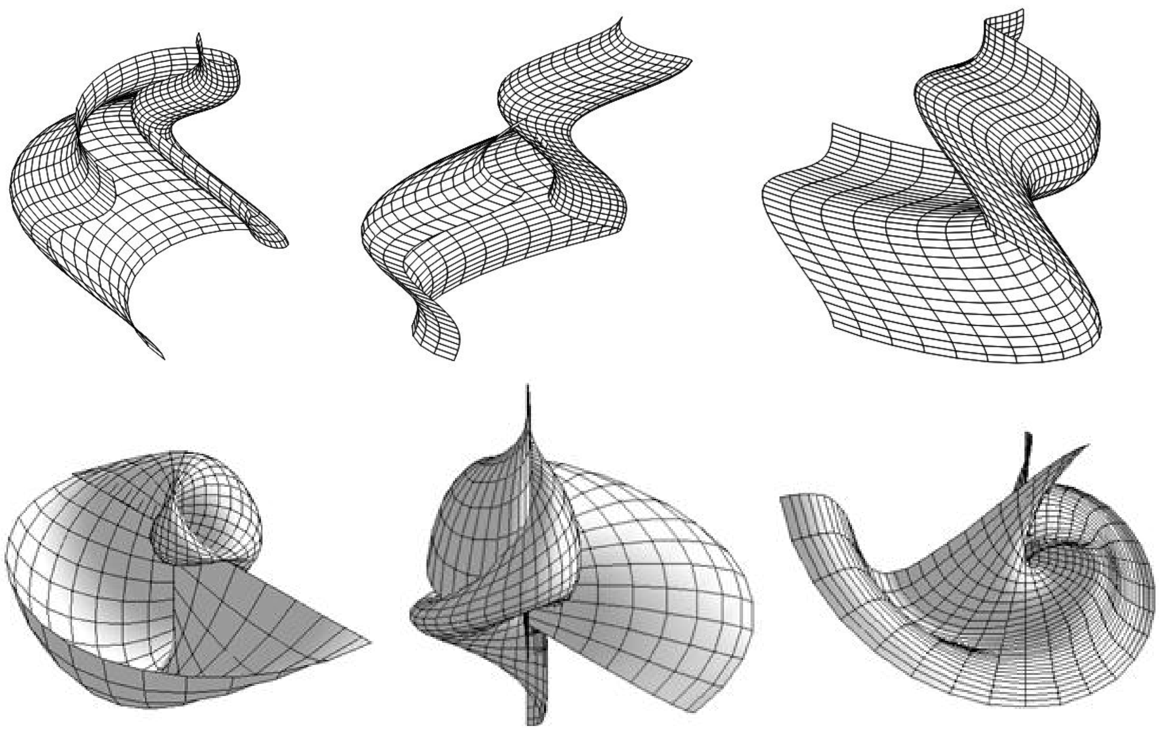

Figure 20: Minkowski sum and product combinations of two helices.

Images of the Minkowski linear summative operator applied to two cylindrical helices for various coefficients are presented in Figure 20 on top, while illustration of images of their Minkowski multiplicative operator are given in Figure 20, bottom.

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