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Two New Analytical and Two New Geometrical Solutions for the Weighted Fermat-Torricelli Problem in the Euclidean Plane

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Abstract. We obtain two analytic solutions for the weighted Fermat-Torricelli problem in the Euclidean Plane which states: given three points in the Euclidean plane and a positive real number (weight) which correspond to each point, find the point such that the sum of the weighted distances to these three points is minimized. Furthermore, we give two new geometrical solutions for the the weighted Fermat-Torricelli problem (weighted Fermat-Torricelli point), by using the floating equilibrium condition of the weighted Fermat-Torricelli problem (first geometric solution) and a generalization of Hofmann's rotation proof under the condition of equality of two given weights (second geometric solution).

Key Words: weighted Fermat-Torricelli point, floating case, absorbed case, median, ruler and compass construction.

MSC 2010: 51M04, 51M16, 51M15, 74P20

1. Introduction

We state the *weighted Fermat-Torricelli problem* in $\mathbb{R}^2$:

Problem 1. *Given a triangle $\triangle A_1A_2A_3$ with vertices $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $A_3 = (x_3, y_3)$, find a fourth point $A_F = (x_F, y_F)$ which minimizes the objective function*

$$f(x,y) = \sum_{i=1}^{3} B_i \sqrt{(x-x_j)^2 + (y-y_j)^2} \tag{1.1}$$

where B_i is a positive real number (weight) which corresponds to A_i.

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By replacing $B_1 = B_2 = B_3$ in (1.1), we obtain the (unweighted) Fermat-Torricelli problem which was first stated by Pierre DE FERMAT (1643).

The solution of the weighted Fermat-Torricelli problem (Problem 1) is called the *weighted Fermat-Torricelli point* A_F.

The existence and uniqueness of the weighted Fermat-Torricelli point and a complete characterization of the "floating case" and "absorbed case" has been established by Y.S. KUPITZ and H. MARTINI (see [11], theorem 1.1, reformulation 1.2 page 58, theorem 8.5 pages 76, 77). A particular case of this result for three non-collinear points in $\mathbb{R}^2$, is given by the following theorem:

Theorem 1. ([1], [11]) *Let there be given a triangle* $\triangle A_1A_2A_3$, $A_1, A_2, A_3 \in \mathbb{R}^2$ *with corresponding positive weights* B_1, B_2, B_3.

(a) *The weighted Fermat-Torricelli point* A_F *exists and is unique.*

(b) *If for each point* $A_i \in \{A_1, A_2, A_3\}$

$$\|\sum_{j=1, i\neq j}^{3} B_j \vec{u}(A_i, A_j)\| > B_i, \tag{1.2}$$

for $i, j = 1, 2, 3$ *holds, then*

(b$_1$) *the weighted Fermat-Torricelli point* A_F *(weighted floating equilibrium point) does not belong to* $\{A_1, A_2, A_3\}$ *and*

(b$_2$)
$$\sum_{i=1}^{3} B_i \vec{u}(A_F, A_i) = \vec{0}, \tag{1.3}$$

where $\vec{u}(A_k, A_l)$ *is the unit vector from* A_k *to* A_l*, for* $k, l \in \{0, 1, 2, 3\}$ *(Weighted Floating Case).*

(c) *If there is a point* $A_i \in \{A_1, A_2, A_3\}$ *satisfying*

$$\|\sum_{j=1, i\neq j}^{3} B_j \vec{u}(A_i, A_j)\| \leq B_i, \tag{1.4}$$

then the weighted Fermat-Torricelli point A_F *(weighted absorbed point) coincides with the point* A_i *(Weighted Absorbed Case).*

A direct consequence of the weighted floating case and the weighted absorbed case of Theorem 1 gives Corollary 1 (Torricelli's theorem) and Corollary 2 (Cavalieri's alternative) (see [1, p 236]).

Corollary 1. *If* $B_1 = B_2 = B_3$ *and all three angles of the triangle* $\triangle A_1A_2A_3$ *are less than* $120°$*, then* A_F *is the isogonal point (interior point) of* $\triangle A_1A_2A_3$ *whose sight angle to every side of* $A_1A_2A_3$ *is* $120°$ *(Torricelli's theorem).*

Corollary 2. *If* $B_1 = B_2 = B_3$ *and one of the angles of the triangle* $\triangle A_1A_2A_3$ *is equal or greater than* $120°$*, then* A_F *is the vertex of the obtuse angle of* $\triangle A_1A_2A_3$ *(Cavalieri's alternative).*

Concerning the solution of the weighted Fermat-Torricelli problem with the use of analytic geometry and trigonometry, we mention the works of [4], [5], [6], [10], [12], [16], and [19].

Recently, an analytic solution, which expresses explicitly the coordinates of the weighted Fermat-Torricelli point with respect to the coordinates of the three points A_i and the three weights B_1, B_2 B_3 for the weighted Fermat-Torricelli problem with respect to the weighted floating case of Theorem 1 has been derived in [15], and for the case $B_1 = B_2 = B_3 = 1$ has been derived in [13].

In this paper, we present two new analytical solutions for the weighted Fermat-Torricelli problem in $\mathbb{R}^2$ in the weighted floating case of Theorem 1. The first analytical solution gives the coordinates of the weighted Fermat-Torricelli point as a function of the coordinates of the three non collinear points and the three given weights (real positive numbers) in a different way from [15] and [13] (Theorem 2 in Section 2).

The second analytical solution gives the location of the weighted Fermat-Torricelli point as a function of two inscribed angles of the circumscribed circle which passes through the three non collinear points and the three given weights by applying a coordinate independent approach given in [19] (Theorem 3 in Section 3).

The first geometrical solution of the weighted Fermat-Torricelli point with ruler and compass focuses on constructing the intersection of two Simpson lines (weighted case) by applying the duality of the weighted Fermat-Torricelli problem which was introduced in [15] (Problem 2, Section 4).

Finally, the second geometric solution of the weighted Fermat-Torricelli point focuses on finding the angle of rotation of the three non collinear points about one of them and generalizes Hofmann's rotation proof ([1], [7], [14]) regarding the equality of the given weights (Problem 3 and Corollary 3 in Section 4).

2. Analytical solution of the weighted Fermat-Torricelli problem

We obtain an analytic solution for the floating case of Theorem 1, i.e., the weighted Fermat-Torricelli point A_F is an interior point of $\triangle A_1A_2A_3$, such that the coordinates x_F and y_F of A_F are expressed explicitly as a function of x_i, y_i and B_i, for $i = 1, 2, 3$, by using analytic geometry in $\mathbb{R}^2$.

We denote by a_{ij} the length of the linear segment A_iA_j and α_{ikj} the angle $\angle A_iA_kA_j$ for $i, j, k = 0, 1, 2, 2', 3, 3', i \neq j \neq k$ (see Figure 1). Without loss of generality, we set $A_1 = (0, 0)$, $A_2 = (a_{12}, 0)$, $A_3 = (x_3, y_3)$.

We need the following two lemmata:

Lemma 1. ([5], [16]) *Under the condition (1.2) and the weighted floating equilibrium condition (1.3) the following equation is satisfied:*

$$\frac{B_i}{\sin \alpha_{203}} = \frac{B_2}{\sin \alpha_{103}} = \frac{B_3}{\sin \alpha_{102}} = C, \tag{2.1}$$

where $C = \dfrac{2B_1B_2B_3}{\sqrt{(B_1 + B_2 + B_3)(B_2 + B_3 - B_1)(B_1 + B_3 - B_2)(B_1 + B_2 - B_3)}}$

Lemma 2. ([1], [6], [5], [16]) *Under the condition (1.2) and the weighted floating equilibrium condition (1.3) the angle α_{i0j} is expressed as a function of B_1, B_2 and B_3 as*

$$\alpha_{i0j} = \arccos\left(\frac{B_k^2 - B_i^2 - B_j^2}{2B_iB_j}\right) \tag{2.2}$$

for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$.

Theorem 2. *Under the condition (1.2) of the weighted floating case, the coordinates* (x_F, y_F) *of the weighted Fermat-Torricelli point* A_F *are given by the following relations:*

$$x_F = -\frac{(a_{12}-x_2')(x_3'y_3 - x_3y_3') + d_3(x_3 - x_3')}{(a_{12}-x_2')(y_3'-y_3) - (y_3 - x_3')y_2'}, \tag{2.3}$$

$$y_F = \frac{y_2'(a_{12}y_3 - x_3'y_3 - a_{12}y_3' + x_3y_3')}{x_3y_2' - x_3y_2' + a_{12}y_3 - x_2'y_3 - a_{12}y_3' + x_2'y_3'}, \tag{2.4}$$

where

$$x_2' = -\frac{B_3}{B_2}\left(x_3\frac{B_1^2 - B_2^2 - B_3^2}{2B_2B_3} + y_3\sqrt{1-\left(\frac{B_1^2-B_2^2-B_3^2}{2B_2B_3}\right)^2}\right), \tag{2.5}$$

$$y_2' = \frac{B_3}{B_2}\left(x_3\sqrt{1-\left(\frac{B_1^2-B_2^2-B_3^2}{2B_2B_3}\right)^2} - y_3\frac{B_1^2 - B_2^2 - B_3^2}{2B_2B_3}\right), \tag{2.6}$$

$$x_3' = -a_{12}\frac{B_2}{B_3}\left(\frac{B_1^2-B_2^2-B_3^2}{2B_2B_3}\right), \quad \textit{and} \tag{2.7}$$

$$y_3' = -a_{12}\frac{B_2}{B_3}\left(\sqrt{1-\left(\frac{B_1^2-B_2^2-B_3^2}{2B_2B_3}\right)^2}\right). \tag{2.8}$$

Proof: We apply the weighted Torricelli configuration which is similar to the configuration used in [3], and we construct two similar triangles $\triangle A_1A_2A_{3'}$ and $\triangle A_1A_3A_{2'}$, such that

$$\alpha_{13'2} = \alpha_{132'} = \pi - \alpha_{102}, \tag{2.9}$$

$$\alpha_{213'} = \alpha_{312'} = \pi - \alpha_{203}, \quad \text{and} \tag{2.10}$$

$$\alpha_{123'} = \alpha_{12'3} = \pi - \alpha_{103}. \tag{2.11}$$

From (2.9), (2.10) and (2.11), we derive that the point of intersection of the two circles which pass through $A_1, A_{3'}, A_2$ and $A_1, A_{2'}, A_3$, respectively, is the weighted Fermat-Torricelli point A_F (Figure 1). Therefore, we obtain that A_F is the intersection point of the lines (weighted Simpson lines) defined by $A_2A_{2'}$ and $A_3A_{3'}$.

Thus, we have:

$$x_2' = a_{12'}\cos(\alpha_{213} + \pi - \alpha_{203}), \tag{2.12}$$

$$y_2' = a_{12'}\sin(\alpha_{213} + \pi - \alpha_{203}), \tag{2.13}$$

$$x_3' = a_{13'}\cos(\pi - \alpha_{203}), \quad \text{and} \tag{2.14}$$

$$y_3' = -a_{13'}\sin(\pi - \alpha_{203}). \tag{2.15}$$

By applying the sine law to $\triangle A_1A_2A_{3'}$ and $\triangle A_1A_3A_{2'}$, we get, respectively,

$$a_{13'} = a_{12}\frac{B_2}{B_3} \quad \text{and} \tag{2.16}$$

$$a_{12'} = a_{13}\frac{B_3}{B_2}. \tag{2.17}$$

By substituting (2.16), (2.17) and (2.2) from Lemma 2 in (2.12), (2.13), (2.14), and (2.15), we obtain (2.5), (2.6), (2.7), and (2.8).

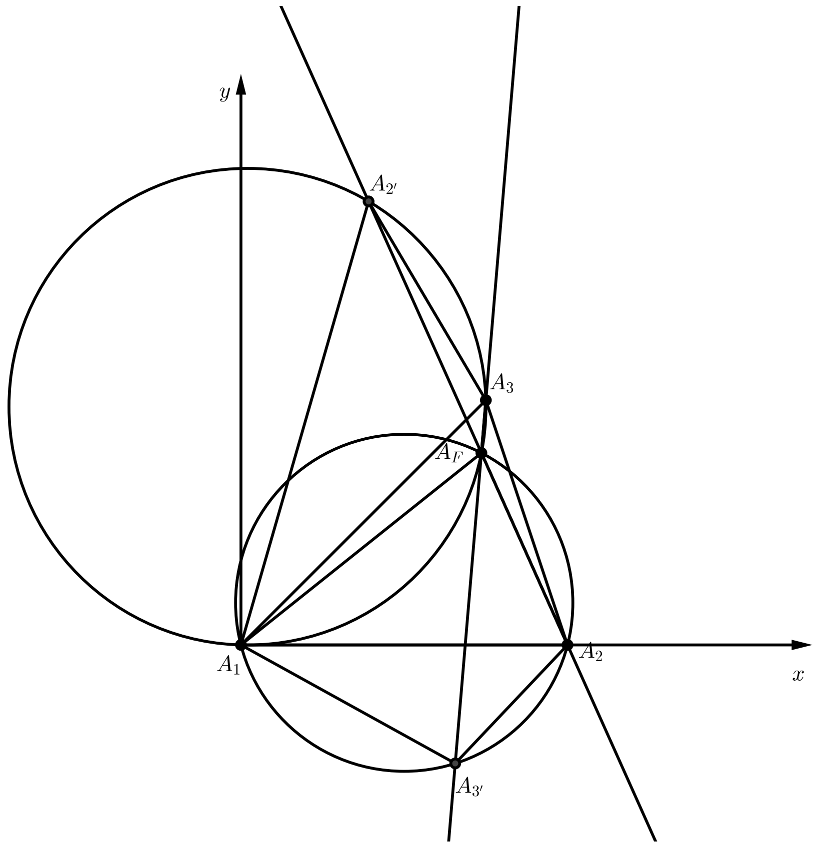

Figure 1: The weighted Fermat-Torricelli point A_F is the remaining point of intersection of the circumcircles of $\triangle A_1A_2A_{3'}$ and $A_1A_{2'}A_3$.

The equations of the lines defined by $A_2A_{2'}$ and $A_3A_{3'}$, respectively, are as follows:

$$\frac{y_2'}{x_2' - a_{12}} = \frac{y}{x - a_{12}}, \quad \text{and} \tag{2.18}$$

$$\frac{y_3 - y_3'}{x_3 - x_3'} = \frac{y_3 - y}{x_3 - x}. \tag{2.19}$$

Solving (2.18) and (2.19) for (x, y) we derive the point of intersection $A_F = (x_F, y_F)$, and the coordinates x_F and y_F are given by (2.3) and (2.4), respectively. □

Remark 1. A similar system of equations with respect to (2.3), (2.4), which gives the coordinates of the weighted Fermat-Torricelli point, has also been obtained in [9, Formulas (1), (2), (3), (3a), (3b) (FERPOS case), pp. 78–79].

3. An explicit angular solution of the weighted Fermat-Torricelli problem

It is well known that the barycenter A_m of $\triangle A_1A_2A_3$ is constructed by the relation $a_{im} = \frac{2}{3}a_{i,jk} = \frac{1}{3}\sqrt{2a_{ij}^2 + 2a_{ik}^2 - a_{jk}^2}$, where $a_{i,jk}$ is the length of the midline that connects the vertex A_i with the midpoint of the line segment A_jA_k for $i, j, k = 1, 2, 3$, $i \neq j \neq k$, and the median minimizes the objective function $a_{m1}^2 + a_{m2}^2 + a_{m3}^2$.

A natural question is to ask if the location of the weighted Fermat-Torricelli problem could be given in terms of the lengths of the sides of $\triangle A_1A_2A_3$ and the constant positive weights B_1, B_2, B_3.

A positive answer to this question is given in the following lemma ([19, Corollary 2]).

Lemma 3. ([19, Corollary 2]) *The explicit solution of the weighted Fermat-Torricelli problem in $\mathbb{R}^2$, under the condition (1.2) (weighted floating case) is given below.*

$$\alpha_{013} = \operatorname{arccot}\left(\frac{\sin(\alpha_{213}) - \cos(\alpha_{213})\cot(\arccos\frac{B_3^2-B_1^2-B_2^2}{2B_1B_2}) - \frac{a_{13}}{a_{12}}\cot(\arccos\frac{B_2^2-B_1^2-B_3^2}{2B_1B_3})}{-\cos(\alpha_{213}) - \sin(\alpha_{213})\cot(\arccos\frac{B_3^2-B_1^2-B_2^2}{2B_1B_2}) + \frac{a_{13}}{a_{12}}}\right) \tag{3.1}$$

and

$$a_{10} = \frac{\sin\left(\alpha_{013} + \arccos\frac{B_2^2-B_1^2-B_3^2}{2B_1B_3}\right)a_{13}}{\sin\left(\arccos\frac{B_2^2-B_1^2-B_3^2}{2B_1B_3}\right)}, \tag{3.2}$$

where

$$\alpha_{213} = \arccos\left(\frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{2a_{12}a_{13}}\right), \tag{3.3}$$

α_{013} and a_{10} depend on B_1, B_2, B_3, a_{13}, a_{12} and a_{23}.

Proof: We assume that the weighted floating case occurs (see Theorem 1, Case b), in order to locate it in the interior of the $\triangle A_1A_2A_3$.

From the cosine law for $\triangle A_1A_0A_2$ and $\triangle A_1A_0A_3$ we get, respectively

$$a_{02}^2 = a_{01}^2 + a_{12}^2 - 2a_{01}a_{12}\cos(\alpha_{213} - \alpha_{013}) \quad \text{and} a_{03}^2 = a_{01}^2 + a_{13}^2 - 2a_{01}a_{13}\cos(\alpha_{013}). \tag{3.4}$$

By virtue of (3.4) and (3.4), a_{02} and a_{03} are expressed in terms of the two variables a_{01} and α_{013} as

$$a_{0i} = a_{0i}(a_1, \alpha_{013}) \quad \text{for} \quad i = 2, 3.$$

By differentiating (1.1) with respect to a_{01} and α_{013}, respectively, we get

$$B_1 + B_2\frac{\partial a_{02}}{\partial a_{01}} + B_3\frac{\partial a_{03}}{\partial a_{01}} = 0, \tag{3.5}$$

$$B_2\frac{\partial a_{02}}{\partial \alpha_{013}} + B_3\frac{\partial a_{03}}{\partial \alpha_{013}} = 0. \tag{3.6}$$

From Appendix A, by replacing (A.1) and (A.2) in (3.5), we obtain

$$B_2\cos(\alpha_{102}) + B_3\cos(\alpha_{103}) = -B_1. \tag{3.7}$$

We substitute (A.5) and (A.6) in (3.6) and obtain:

$$-B_2\sin(\alpha_{102}) + B_3\sin(\alpha_{103}) = 0 \tag{3.8}$$

By squaring both parts of (3.7) and (3.8) and by adding the two derived equations, we get

$$\cos(\alpha_{203}) = \frac{B_1^2 - B_2^2 - B_3^2}{2B_2B_3}. \tag{3.9}$$

Similarly, by expressing the objective function in terms of the two variables a_2, α_{023}, and in terms of a_3, α_{031}, we derive, respectively,

$$\cos(\alpha_{103}) = \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_3} \quad \text{and} \tag{3.10}$$

$$\cos(\alpha_{102}) = \frac{B_3^2 - B_1^2 - B_2^2}{2B_1B_2}. \tag{3.11}$$

From the sine law applied to $\triangle A_1A_0A_2$ and $\triangle A_1A_0A_3$ we get, respectively,

$$\frac{a_{12}}{\sin(\alpha_{102})} = \frac{a_{01}}{\sin(\alpha_{213} - \alpha_{013} + \alpha_{102})} \quad \text{and} \tag{3.12}$$

$$\frac{a_{13}}{\sin(\alpha_{103})} = \frac{a_{01}}{\sin(\alpha_{013} + \alpha_{103})}. \tag{3.13}$$

The elimination of a_{01} from (3.12) and (3.13) yields

$$\alpha_{013} = \operatorname{arccot}\left(\frac{\sin(\alpha_{213}) - \cos(\alpha_{213})\cot(\alpha_{102}) - \frac{a_{31}}{a_{12}}\cot(\alpha_{103})}{-\cos(\alpha_{213}) - \sin(\alpha_{213})\cot(\alpha_{102}) + \frac{a_{31}}{a_{12}}}\right) \tag{3.14}$$

By substituting (3.11) and (3.10) in (3.14) we obtain (3.1). Finally, (3.2) follows by virtue of the sine law applied to $\triangle A_1A_0A_3$.
The values of a_{01} and α_{013} fix the location of the weighted Fermat-Torricelli point A_F. □

Remark 2. The explicit solution of the weighted Fermat-Torricelli problem is similar to the definition of a complex number in a polar form, $z = r\exp(i(\alpha_{213} - \alpha_{013}))$, where the absolute value of z is $r = a_1$ and the argument of z is $\arg z = \alpha_{213} - \alpha_{013}$.

Let $C(Q, R)$ be the the circumcircle of $\triangle A_1A_2A_3$ with center Q and radius R. Each of the three central angles is given by the relation:

$$c_{iQj} = 2\alpha_{imj} \tag{3.15}$$

such that

$$c_{1Q2} + c_{2Q3} + c_{1Q3} = 2\pi$$

or

$$c_{1Q2} = 2\pi - c_{1Q3} - c_{2Q3} \tag{3.16}$$

for $i \neq m \neq j$, $i, m, j = 1, 2, 3$. From the sine law for $\triangle A_1A_2A_3$ and taking (3.15) into account we get

$$\frac{a_{13}}{\sin(\frac{c_{1Q3}}{2})} = \frac{a_{12}}{\sin(\frac{c_{1Q2}}{2})} = 2R. \tag{3.17}$$

By substituting (3.15), (3.16), (3.17) in (3.1) and (3.2) of Lemma 3, we derive the following result.

Theorem 3. *Under the condition (1.2), an explicit angular solution of the weighted Fermat-Torricelli problem in $\mathbb{R}^2$ is as given below.*

$$\cot\alpha_{013} = \frac{\sin(\frac{c_{2Q3}}{2}) - \cos(\frac{c_{2Q3}}{2})\cot\left(\arccos\frac{B_3^2 - B_1^2 - B_2^2}{2B_1B_2}\right) - \frac{\sin(\frac{c_{1Q3}}{2})}{\sin(\frac{c_{1Q3}+c_{2Q3}}{2})}\cot\left(\arccos\frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_3}\right)}{-\cos(\frac{c_{2Q3}}{2}) - \sin(\frac{c_{2Q3}}{2})\cot\left(\arccos\frac{B_3^2 - B_1^2 - B_2^2}{2B_1B_2}\right) + \frac{\sin(\frac{c_{1Q3}}{2})}{\sin(\frac{c_{1Q3}+c_{2Q3}}{2})}} \tag{3.18}$$

and

$$a_{10} = 2R\,\frac{\sin\left(\alpha_{013} + \arccos\frac{B_2^2-B_1^2-B_3^2}{2B_1B_3}\right)\sin\left(\frac{c_{1Q3}}{2}\right)}{\sin\left(\arccos\frac{B_2^2-B_1^2-B_3^2}{2B_1B_3}\right)}\,, \tag{3.19}$$

where α_{013} *and* a_{10} *depend on* B_1, B_2, B_3*,* c_{1Q3}*,* c_{2Q3}*, and* R*.*

Remark 3. We conclude that by setting $R = 1$ (unit radius of circumcircle) in (3.19), the explicit solution depends only on five given elements: B_1, B_2, B_3, c_{1Q3}, and c_{2Q3}. This unique result holds only if the inequalities (1.2) of the weighted floating case of Theorem 1, case (b), are satisfied.

Remark 4. We note that Lemma 3 and Theorem 3 provide an analytic solution for the weighted Fermat-Torricelli problem in $\mathbb{R}^2$ without using the coordinates of the points A_i, $i = 1, 2, 3$ (coordinate independent approach), taking into account only the given Euclidean elements, lengths and angles.

4. Two new geometrical constructions of the weighted Fermat-Torricelli point in the Euclidean plane

We present two new constructions for finding the weighted Fermat-Torricelli point in the weighted floating case. The first solution deals with the position of $A_{1'}$ and $A_{3'}$ which shall give the position of the two Simpson lines connecting $A_2, A_{2'}$ and A_3, $A_{3'}$, respectively (Figure 2). The second solution deals with the generalization of Hofmann's rotation proof ([14], [7]) for $B_1 = B_2$.

Problem 2. *Construct the solution of Problem 1 (Weighted Fermat-Torricelli problem) under the condition (1.2), using ruler and compass.*

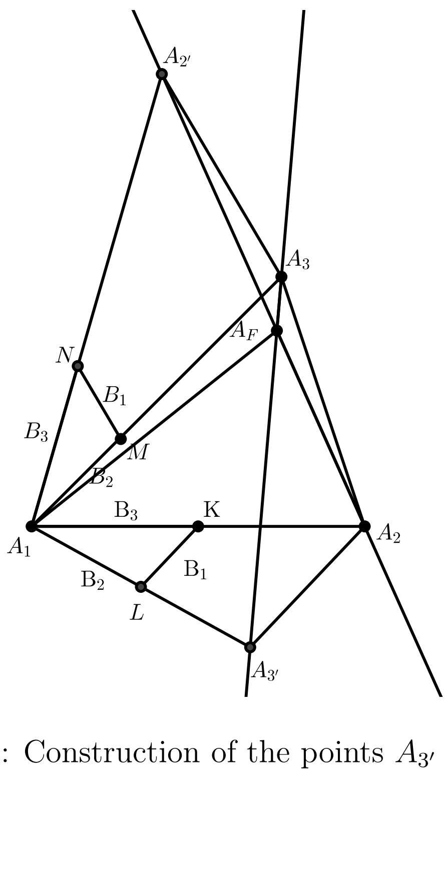

Figure 2: Construction of the points $A_{3'}$ and $A_{2'}$.

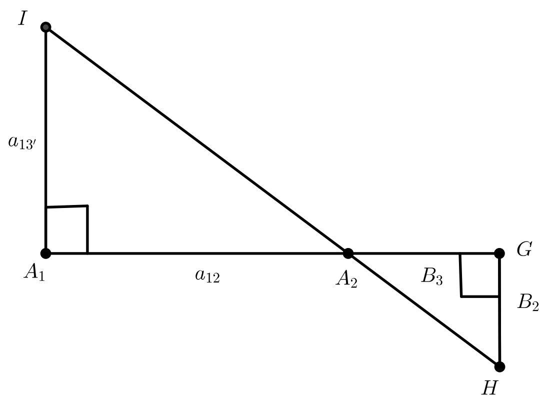

Figure 3: Construction of $a_{13'}$

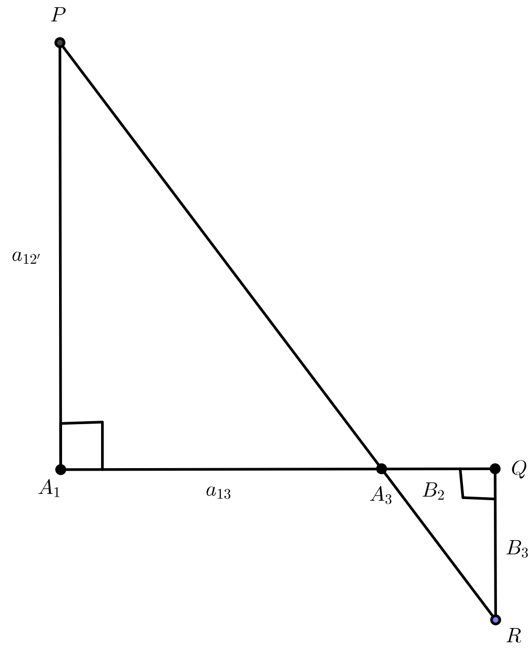

Figure 4: Construction of $a_{12'}$

Solution of Problem 2: We need to construct the vertices $A_{3'}$ and $A_{2'}$ (see Figure 2).

First, we construct the vertex $A_{3'}$. We select a point K which belongs to the linear segment A_1A_2, such that $\|A_1K\| = B_3$, and we construct a triangle $\triangle A_1KL$ with the other two sides $\|A_1L\| = B_2$ and $\|KL\| = B_3$. Thus, Lemma 1 yields $\angle A_2A_1A_{3'} = \alpha_{213'} = \pi - \alpha_{203}$.
We need to calculate $a_{13'}$, in order to find the location of A'_3. To this end, take a point G on the line A_1A_2 such that $\|A_1A_2\| = a_{12}$ and $\|A_2G\| = B_3$, where $\|A_1G\| = a_{12} + B_3$. Furthermore, construct with ruler and compass the line segment through G perpendicular to the line A_1A_2 and take on it a point H such that $\|GH\| = B_2$. We denote by I the point of intersection of the line A_2H and the perpendicular line to A_1A_2 through the point A_1 (Figure 3).
Taking into account the similar triangles $\triangle A_1A_2I$ and $\triangle A_2GH$, we get

$$a_{13'} = a_{12}\,\frac{B_2}{B_3}\,. \tag{4.1}$$

Similarly, we construct the vertex $A_{2'}$. We select a point M on the line segment A_1A_3, such that $\|A_1M\| = B_2$, and we construct a triangle $\triangle A_1MN$ with the other two sides $\|A_1N\| = B_3$ and $\|MN\| = B_1$. Thus, Lemma 1 yields $\angle A_3A_1A_{2'} = \alpha_{312'} = \pi - \alpha_{203}$.
We need to calculate $a_{12'}$, in order to find the location of A'_2: Take a point Q on the line A_1A_3 such that $\|A_1A_3\| = a_{13}$, $\|A_2G\| = B_3$, where $\|A_1Q\| = a_{13} + B_3$, and construct with a ruler and compass through Q the linear segment which is perpendicular to A_1A_3; specify on it a point R such that $\|RQ\| = B_3$.
We denote by P the point of intersection of the line A_3R and the perpendicular to A_1A_3 through A_1 (Figure 4). Taking into account the similar triangles $\triangle A_1A_3P$ and $\triangle A_3QR$, we get

$$a_{12'} = a_{13}\,\frac{B_3}{B_2}\,. \tag{4.2}$$

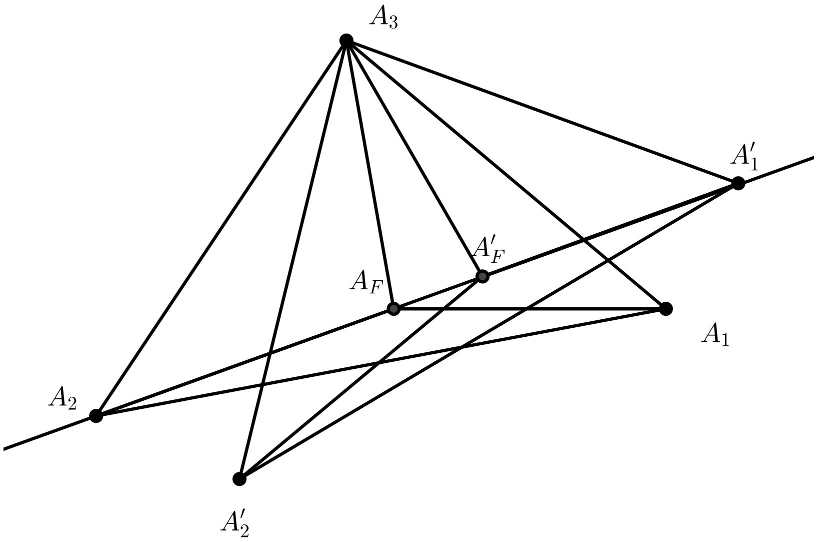

Figure 5: The weighted Fermat-Toricelli points A_F and A'_F are collinear.

Problem 3. *Solve Problem 1 (Weighted Fermat-Torricelli problem) by generalizing* HOFMANN*'s rotation, under the condition (1.2) and* $B_1 = B_2$*, such that* $B_1 + B_2 + B_3 = 1$*,* $B_1 > \frac{1}{4}$ *and* $\alpha_{132} > \pi - \arccos\left(-1 + \frac{(1-2B_1)^2}{2B_1^2}\right)$.

Solution of Problem 3: We consider a weight B_i which corresponds to the vertex A_i in $\mathbb{R}^2$for $i = 1, 2, 3$. By substituting $B_1 = B_2$ in (2.2) of Lemma 2, we derive that

$$\alpha_{203} = \alpha_{103} = \arccos\left(1 - \frac{1}{2B_1}\right) \quad \text{and} \tag{4.3}$$

$$\alpha_{102} = \arccos\left(-1 + \frac{(1-2B_1)^2}{2B_1^2}\right), \tag{4.4}$$

which yields $B_1 > \frac{1}{4}$.
Taking into account (4.3) and (4.4), we rotate the triangle $\triangle A_1A_2A_3$ about A_3 through $\pi - \alpha_{102} = 2\alpha_{103} - \pi$ and obtain the triangle $\triangle A_3A'_1A'_2$. Let A'_F be the corresponding weighted Fermat-Torricelli point of $\triangle A'_1A'_2A_3$, for $B'_1 = B_1$ and $B_2 = B'_2$ (Figure 5). Thus, the points A_2, A_F, A'_F, and A'_3 are collinear, because $\triangle A_FA'_FA_3$ is an isosceles triangle and

$$\angle A_FA'_FA_3 = \angle A'_FA_FA_3 = \pi - \alpha_{103}.$$

Corollary 3. ([7], [14], [1]) *For* $B_1 = B_2 = B_3$ *the solution of Problem 3 is given by rotating the* $\triangle A_1A_2A_3$ *about* A_3 *through* $60°$.

Proof. By plugging $B_1 = B_2 = B_3 = 1$ into the solution of Problem 3, we deduce that the rotation about A_3 needs to be through $\pi - 120° = 60° < \alpha_{132}$ (HOFMANN's rotation). □

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A. Appendix

We mention two methods of the length of a linear segment with respect to (I) a variable length and (II) a variable angle, which have been used, in order to find the weighted Fermat-Torricelli point.

I. A method of differentiating the length of a linear segment with respect to the length of a variable linear segment has been given first in [16, Proposition 2.6(b), Remark 2.4, Corollary 3.3], [17, formula (4), p. 413] and has been explained in detail in [2], and [18, Corollary 2]. Specifically, by differentiating (3.4) with respect to a_{01} and by replacing in the derived equation $\cos(\alpha_{213} - \alpha_{013})$ taken from (3.4), we obtain

$$\frac{\partial a_{02}}{\partial a_{01}} = \cos(\alpha_{102}). \tag{A.1}$$

Similarly, by differentiating (3.4) with respect to a_{01} and by replacing in the derived equation $\cos(\alpha_{013})$ taken from (3.4), we obtain

$$\frac{\partial a_{03}}{\partial a_{01}} = \cos(\alpha_{103}). \tag{A.2}$$

We mention a method of differentiating the length of a linear segment with respect to a variable angle, which has been used in order to find the weighted Fermat-Torricelli point ([16, Proposition 2.6 (b)]) in $\mathbb{R}^2$. By mentioning this technique of differentiation, we correct some typographical errors which appear in [16]. Specifically, by differentiating (3.4) with respect to α_{013}, we get

$$\frac{\partial a_{02}}{\partial \alpha_{013}} = -a_{01}\,\frac{a_{12}}{a_{02}}\sin(\alpha_{213} - \alpha_{013})\,. \tag{A.3}$$

From the sine law for $\triangle A_1A_0A_2$ we get

$$\frac{a_{12}}{\sin(\alpha_{102})} = \frac{a_{02}}{\sin(\alpha_{213} - \alpha_{013})}\,. \tag{A.4}$$

By substituting (A.4) in (A.3) we obtain

$$\frac{\partial a_{02}}{\partial \alpha_{013}} = -a_{01}\sin(\alpha_{102}). \tag{A.5}$$

Similarly, by differentiating (3.4) with respect to α_{013}, we obtain

$$\frac{\partial a_{03}}{\partial \alpha_{013}} = a_{01}\sin(\alpha_{103}). \tag{A.6}$$

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