Journal for Geometry and Graphics
Volume 19 (2015), No. 2, 219–226.

Rose Curves with Chebyshev Polynomials

László Németh

Institute of Mathematics, University of West Hungary
Ady E. u. 5, H-9400 Sopron, Hungary
email: nemeth.laszlo@emk.nyme.hu

Abstract. In this paper, we present a class of curves derived from a geometrical construction. We take points on two half-lines (or lines). The first point is on one of the half-lines and the second one is on the other half-line, while the next is again on the first half-line, and so on. The distance of two consecutive points is the unit. The orbits of these points when the angle of the lines goes from zero to 2π are similar to lemniscates and rose curves. For determining the parametric equation systems of the curves we use Chebyshev polynomials.

Key Words: lemniscate, rose curve, Chebyshev polynomials

MSC 2010: 51N20

1. Introduction

Let e be a half-line given by the equation $\cos\alpha \cdot y = \sin\alpha \cdot x$, where $0 \le \alpha \le \pi/2$ and $x \ge 0$. The initial point of e is the origin O. Let the point A_0 coincide with O and the point A_1 be given on e at the distance 1 to the point O. The point A_2 on the x-axis has the distance 1 to A_1 while $A_2 \ne O$. Recursively, we define the point A_i, $i \ge 2$, on the line e or on the x-axis depending on whether i is odd or even, respectively, at the distance $A_{i-1}A_i = 1$, where $A_i \ne A_{i-2}$. If α is less then $\pi/(2i-2)$ the point A_i exists. Figure 1 shows the first six points. We obtain a similar geometric construction when A_1 lies on the x-axis.

In [3] the author presented the parametric equation system of the orbits of the vertices $A_1, A_2, \ldots$, when α goes from 0 to $\pi/2$ — not only when A_1 is on the line e, but also when it is on the x-axis. The orbit of the vertex A_n, $n \ge 1$, satisfies

$$\begin{array}{rcl} x_n(\alpha) & = & \cos\alpha\, U_{n-1}(\cos\alpha) \\ y_n(\alpha) & = & \sin\alpha\, U_{n-1}(\cos\alpha) \end{array} \quad \text{for} \begin{cases} 0 \le \alpha \le \dfrac{\pi}{2} & \text{if } n = 1, \\ 0 \le \alpha < \dfrac{\pi}{2(n-1)} & \text{otherwise,} \end{cases} \tag{1}$$

where $U_{n-1}(x)$ is a Chebishev polynomial of the second kind (see Section 2.1 on page 222, or [4]). In Figure 1 the orbits of the vertices A_1, A_2, A_3 are displayed.

For $n \in \mathbb{Z}\backslash\{0\}$ let us extend the domain of the eq. (1) to $\alpha \in [0, 2\pi]$. Now the shapes of the curves satisfying by (1) with their different loops are similar to the rose curves as given [1].

ISSN 1433-8157/\$ 2.50

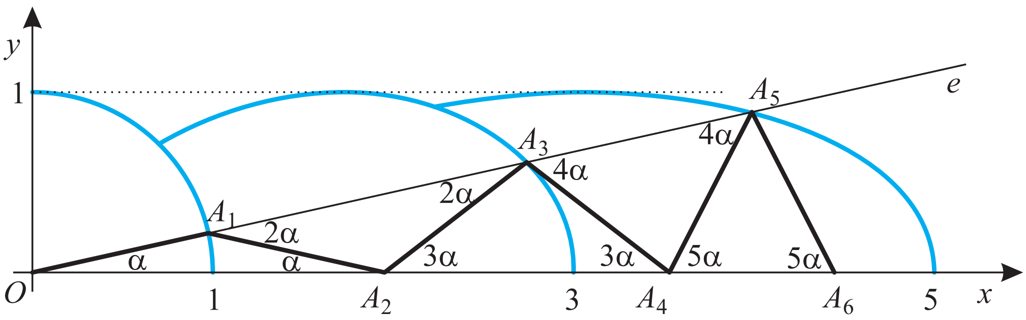

Figure 1: The geometric construction of the vertices $A_1, A_2, \ldots$, when the origin O is the inital point of the half-line e

Figures 2–5 show some of these curves. For odd n the two biggest loops are very similar to the loops of lemniscates. In Figure 4 we see the parts of the curve coming from the geometrical construction. The angles between any two lines m_i, $i = 1, \ldots, 2n$, are multiples of $\frac{\pi}{2n}$, where m_i is a tangent line of the curve at the origin or a line connecting the origin with maxima of the curve. For more details and more figures see [3].

The polar equation of the curves is

$$r_n(\alpha) = U_{n-1}(\cos\alpha), \text{ where } \alpha \in [0, 2\pi], \tag{2}$$

and the corresponding Cartesian equation is

$$x^2 + y^2 = U_{n-1}^2\left(\sqrt{\frac{x^2}{x^2+y^2}}\right), \text{ where } x^2 + y^2 \neq 0\,. \tag{3}$$

Equation (3) covers the curves for n and $-n$ together (see Figure 6).

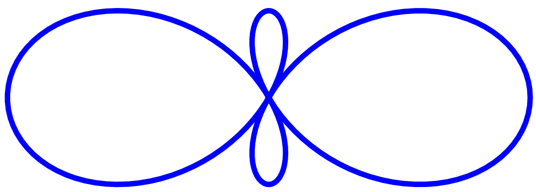

Figure 2: Curve in the case $n = 3$

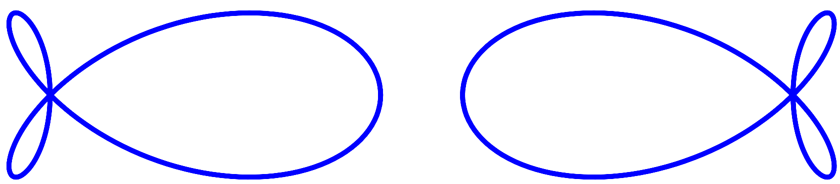

Figure 3: Curves in the case $n = 4$ and $n = -4$

2. Generalized curves

Now we generalize the geometric construction defined above in the introduction. Let the half-line e be given by the equation

$$y = \tan\alpha \cdot x + b, \text{ where } x \geq 0, \; -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \text{ and } 0 \leq b < 1. \tag{4}$$

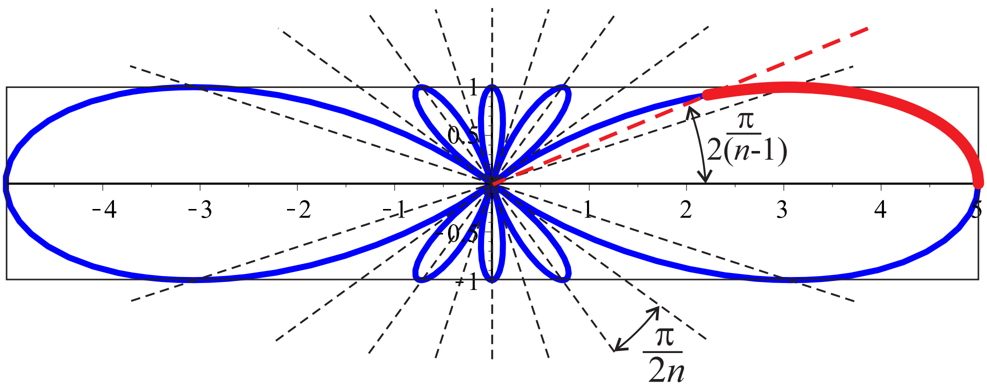

Figure 4: Curve with some properties for $n = 5$

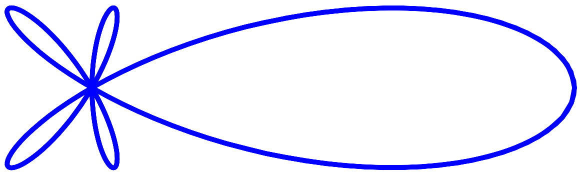

Figure 5: Curve in the case $n = 6$

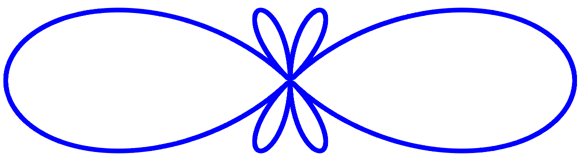

Figure 6: Union of curves for $n = 4$ and -4

Then, the intersection points of the line (4) with the x- and y-axis are $A = (-b \cot \alpha, 0)$ and $B = (0, b)$, respectively (see Figure 7).

Let $A_0 = O$ and $A_1 \in e$ at unit distance to O. Furthermore, let point A_2 be on the x-axis such that the distance to A_1 is equal 1 while $A_2 \neq A_0$. Recursively, we define point A_i, $i \geq 2$, either on the half-line e or on the x-axis depending on whether i is odd or even, where $A_{i-1}A_i = 1$ and A_{i-1} is closer to O than A_i. Let α_i, $i \geq 1$, be the inner angle of the triangle $A_{i-1}A_iA_{i+1}$ at the vertices A_{i-1} and A_{i+1} and let $\varphi = \alpha_1$. If $\alpha_i < \pi/2$ then the point A_i exists. Figure 7 shows the first seven points.

In the following we write the equations of the orbits of the points A_n for $n = 2k+1$, $k \geq 0$, on the half-line e when φ goes from 0 to $\pi/2$ and $y \geq 0$. Let A_1' be the orthogonal projection of A_1 on the x-axis, and B' the orthogonal projection of B on the line A_1A_1' (Figure 7). Then from the right-angled triangle $BB'A_1$ and $A_1A_1' = \sin \varphi$ we obtain that

$$\tan \alpha = \frac{\sin \varphi - b}{\cos \varphi}, \tag{5}$$

where α goes from $\arctan(-b)$ to $\pi/2$. If $b = 0$, then we get the construction of the introduction and $\varphi = \alpha$.

Lemma 1. *If $n \geq 1$ and $A_0 = O$ then $\alpha_n = \varphi + (n-1)\alpha$.*

Proof: If $n = 1$ then $\alpha_1 = \varphi$. In case $n = 2$, $\alpha_2 = \varphi + \alpha$ (see Figure 7). We suppose that the lemma holds for any j from 3 to $n-1$. From the triangle $A_{n-2}A_{n-1}A_n$ we obtain at point A_{n-1} that $\alpha_{n-2} + \alpha_n = 2\alpha_{n-1}$. Thus

$$\alpha_n = 2\alpha_{n-1} - \alpha_{n-2} = 2(\varphi + (n-2)\alpha) - (\varphi + (n-3)\alpha) = \varphi + (n-1)\alpha. \qquad \square$$

Let $\overline{\alpha}_{2k}$ denote the angle α_{2k} if the point A_{2k+1} is on the x-axis, so $\alpha_{2k+1} = 0$ (see Figure 8). It is easy to see that if $\alpha_{2k} = \pi/2$ then A_{2k+1} coincides with A_{2k-1}. The coordinates of A_{2k+1}

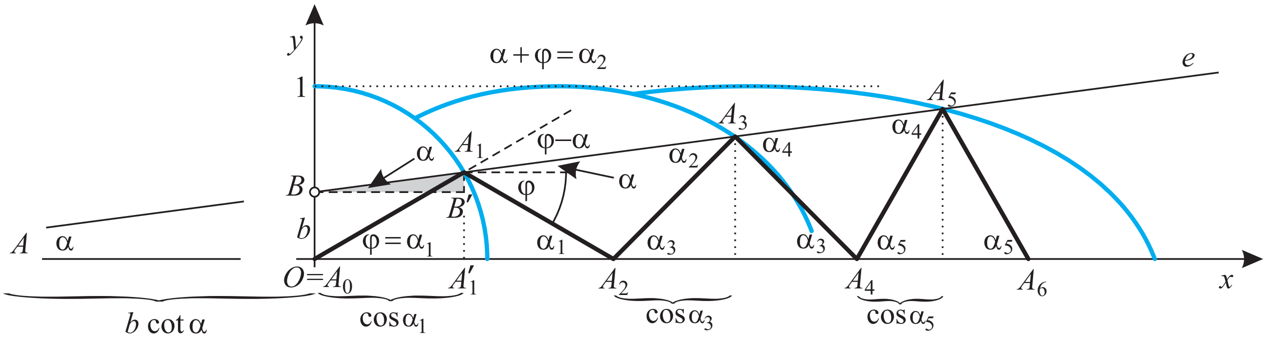

Figure 7: The construction when $A_0 = O$ and B is the initial point of the half-line e

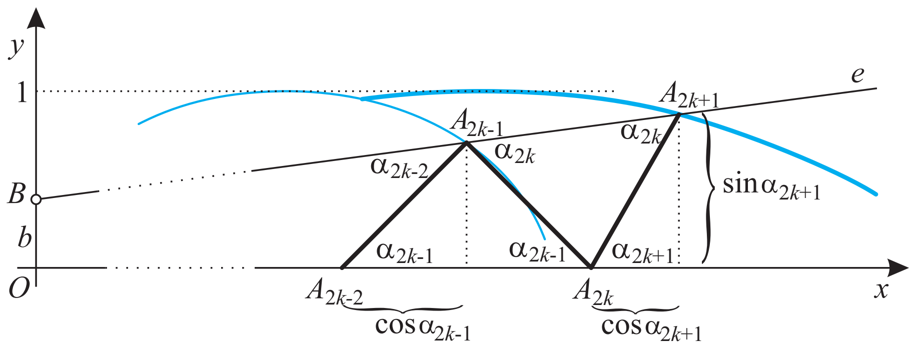

Figure 8: General orbit of the point A_{2k+1}

can be determined by the functions $\cos x$ and $\sin x$ in the following way.

$$\begin{aligned} x_{2k+1}(\varphi) &= 2(\cos\alpha_1 + \cos\alpha_3 + \cdots + \cos\alpha_{2k-1}) + \cos\alpha_{2k+1} \\ y_{2k+1}(\varphi) &= \sin\alpha_{2k+1}, \end{aligned} \tag{6}$$

where $\overline{\alpha}_{2k} \leq \alpha_{2k} < \pi/2$.

We can simplify (6) and write it in a closed form by using Chebishev polynomials.

2.1. Calculations with Chebyshev polynomials

The recursive definition of the Chebyshev polynomials of the first kind $T_n(x)$ is given by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x) \text{ for } n \geq 1,$$

and for the second kind $U_n(x)$

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2x\,U_n(x) - U_{n-1}(x), \text{ for } n \geq 1.$$

When $|x| \leq 1$, the substitution $x = \cos\varphi$ gives the expressions $\cos n\varphi = T_n(\cos\varphi)$ and $\sin n\varphi = \sin\varphi\, U_{n-1}(\cos\varphi)$ [4].

The identities $T_m(x) = U_m(x) - x\,U_{m-1}(x)$, $2T_m(x) = U_m(x) - U_{m-2}(x)$ can be used in the proof of Lemmas 2 and 3.

Lemma 2.

$$\begin{aligned} \sin(x + mz) &= \sin x\, U_m(\cos z) - \sin(x - z)\, U_{m-1}(\cos z), \\ \cos(x + mz) &= \cos x\, U_m(\cos z) - \cos(x - z)\, U_{m-1}(\cos z). \end{aligned}$$

Proof: Clearly,

$$\begin{aligned}
\sin(x+mz) &= \sin x\cos(mz)+\cos x\sin(mz)\\
&= \sin x\, T_m(\cos z)+\cos x\sin z\, U_{m-1}(\cos z)\\
&= \sin x\,\big(U_m(\cos z)-\cos z U_{m-1}(\cos z)\big)+\cos x\sin z\, U_{m-1}(\cos z)\\
&= \sin x\, U_m(\cos z)+\big(\cos x\sin z-\sin x\cos z\big)U_{m-1}(\cos z)\\
&= \sin x\, U_m(\cos z)-\sin(x-z)\, U_{m-1}(\cos z).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cos(x+mz) &= \cos x\cos(mz)-\sin x\sin(mz)\\
&= \cos x\, T_m(\cos(z))+\sin x\sin z\, U_{m-1}(\cos z)\\
&= \cos x\,\big(U_m(\cos z)-\cos(z)U_{m-1}(\cos z)\big)+\sin x\sin z\, U_{m-1}(\cos z)\\
&= \cos x\, U_m(\cos z)-\big(\cos x\cos z+\sin x\sin z\big)U_{m-1}(\cos z)\\
&= \cos x\, U_m(\cos z)-\cos(x-z)\, U_{m-1}(\cos z).
\end{aligned}$$ □

Lemma 3. $U_{2m-1}(x)=2\big(xU_{m-1}^2(x)-U_{m-1}(x)U_{m-2}(x)\big)$, *if* $|x|<1$.

Proof: Suppose $\sin\alpha\neq 0$ and use the substitution $x=\cos\alpha$, so $|x|\neq 1$. Since

$$\sin(2m\alpha)=2\sin(m\alpha)\cos(m\alpha),$$

we have

$$\sin\alpha\, U_{2m-1}(\cos\alpha)=2\sin\alpha\, U_{m-1}(\cos\alpha)\, T_m(\cos\alpha),$$

and then

$$\begin{aligned}
U_{2m-1}(\cos\alpha) &= U_{m-1}(\cos\alpha)\big(2T_m(\cos\alpha)\big)\\
&= U_{m-1}(\cos\alpha)\big(U_m(\cos\alpha)-U_{m-2}(\cos\alpha)\big)\\
&= U_{m-1}(\cos\alpha)\big(2\cos\alpha\, U_{m-1}(\cos\alpha)-2U_{m-2}(\cos\alpha)\big)\\
&= 2\big(\cos\alpha\, U_{m-1}^2(\cos\alpha)-U_{m-1}(\cos\alpha)\, U_{m-2}(\cos\alpha)\big).
\end{aligned}$$ □

2.2. Parametric equation system of curves when n is odd

In order to determine the orbits of points A_n for $n=2k+1$ and $k\geq 0$, when φ goes from 0 to $\pi/2$, we use the Lemmas 2 and 3. Moreover, from (5) we obtain $\alpha=\alpha(\varphi)$.

Theorem 1. *The equation system of the orbit of* A_n, $n=2k+1$, $k\geq 0$, *is given by*

$$\begin{aligned}
x_{2k+1}(\varphi) &= \cos\varphi\, U_{2k}(\cos\alpha)+2\big(\cos\varphi-\cos(\varphi-\alpha)\cos\alpha\big)U_{k-1}^2(\cos\alpha)\\
y_{2k+1}(\varphi) &= \sin\varphi\, U_{2k}(\cos\alpha)+\sin(\varphi-\alpha)\, U_{2k-1}(\cos\alpha),
\end{aligned} \tag{7}$$

where $\overline{\alpha}_{2k}\leq\varphi+(2k-1)\alpha<\pi/2$ *and* $\alpha=\alpha(\varphi)$ *comes from* (5).

Proof: In this proof we use the formula for the sum of $\cos(a+mb)$ from [2] and Lemmas 2 and 3.

$$\begin{aligned}
x_{2k+1}(\varphi) &= 2(\cos\alpha_1+\cos\alpha_3+\dots+\cos\alpha_{2k-1})+\cos\alpha_{2k+1}\\
&= 2\textstyle\sum_{j=0}^{k-1}\cos(\varphi+2j\alpha)+\cos(\varphi+2k\alpha)\\
&= 2\,\dfrac{\sin(k\alpha)}{\sin\alpha}\cos(\varphi+(k-1)\alpha)+\cos(\varphi+2k\alpha)\\
&= 2U_{k-1}(\cos\alpha)\big(\cos\varphi\,U_{k-1}(\cos\alpha)-\cos(\varphi-\alpha)\,U_{k-2}(\cos\alpha)\big)\\
&\quad+\big(\cos(\varphi)\,U_{2k}(\cos(\alpha))-\cos(\varphi-\alpha)\,U_{2k-1}(\cos(\alpha))\big)\\
&= \cos\varphi\big(2U_{k-1}^2(\cos\alpha)+U_{2k}(\cos\alpha)\big)\\
&\quad-\cos(\varphi-\alpha)\big(2U_{k-1}(\cos\alpha)\cdot U_{k-2}(\cos\alpha)+U_{2k-1}(\cos\alpha)\big)\\
&= \cos\varphi\big(2U_{k-1}^2(\cos\alpha)+U_{2k}(\cos\alpha)\big)-\cos(\varphi-\alpha)\big(2\cos\alpha\,U_{k-1}^2(\cos\alpha)\big)\\
&= \cos\varphi\,U_{2k}(\cos\alpha)+2\big(\cos\varphi-\cos(\varphi-\alpha)\cos\alpha\big)U_{k-1}^2(\cos\alpha).\\
y_{2k+1}(\varphi) &= \sin\alpha_{2k+1}=\sin(\varphi+2k\alpha)\\
&= \sin\varphi\,U_{2k}(\cos\alpha)+\sin(\varphi-\alpha)\,U_{2k-1}(\cos\alpha).
\end{aligned}$$

□

If $b=0$ then $\alpha=\varphi$ and we get back the equation system (1) with the appropriate domain.

3. Parametric equation system of curves when n is even

In this section we specify the first vertex as the starting point of the half-line e, so $A_0=B$ (Figure 9). The definitions of A_i and α_i are the same as in the previous sections. Let $\varphi=\alpha_2$. Now we can prove Lemma 4 in a very similar way to Lemma 1.

Lemma 4. *If $n\ge 2$ and $A_0=B$ then $\alpha_n=\varphi+(n-2)\alpha$.*

From the right-angled triangle $BB'A_2$ we get, similarly to (5),

$$\tan\alpha=\frac{\sin\varphi-b}{\cos\varphi+\sqrt{1-b^2}}. \tag{8}$$

Let $\overline{\alpha}_{2k+1}$ denote the angle α_{2k+1} if the point A_{2k+2} is on the x-axis, so $\alpha_{2k+2}=0$.

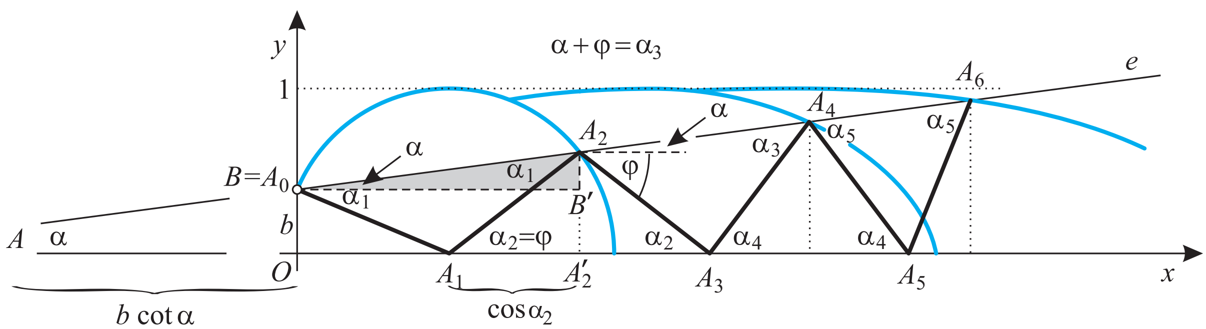

Figure 9: Geometric construction when $A_0=B$ and B is the initial point of the half-line e

Theorem 2. *The equation system of the orbit of A_n $(n = 2k+2, k \geq 0)$ is*

$$\begin{aligned} x_{2k+2}(\varphi) &= \cos\varphi\, U_{2k}(\cos\alpha) + 2\big(\cos\varphi - \cos(\varphi+\alpha)\cos\alpha\big)U_{k-1}^2(\cos\alpha) + \sqrt{1-b^2} \\ y_{2k+2}(\varphi) &= \sin\varphi\, U_{2k}(\cos\alpha) + \sin(\varphi-\alpha)\, U_{2k-1}(\cos\alpha), \end{aligned} \tag{9}$$

where $\overline{\alpha}_{2k+1} \leq \varphi + 2k\alpha < \pi/2$ and $\alpha = \alpha(\varphi)$ comes from (8).

Proof: Based on the proof of Theorem 1, we obtain

$$\begin{aligned} x_{2k+2}(\varphi) &= \sqrt{1-b^2} + 2\left(\cos\alpha_2 + \cos\alpha_4 + \cdots + \cos\alpha_{2k}\right) + \cos\alpha_{2k+2} \\ &= \sqrt{1-b^2} + 2\big(\cos\varphi + \cos(\varphi+2\alpha) + \cos(\varphi+4\alpha) + \cdots + \cos(\varphi+(2k-2)\alpha)\big) \\ &\quad + \cos(\varphi+2k\alpha) \\ &= x_{2k+1}(\varphi) + \sqrt{1-b^2}, \\ y_{2k+2}(\varphi) &= \sin\alpha_{2k+2} = \sin(\varphi+2k\alpha) = y_{2k+1}(\varphi). \end{aligned}$$

□

4. Generalization of the curves with domain extension

Let us extend the domain of parametric equations (7) and (9) in Theorems 1 and 2, respectively, to $\varphi \in [0, 2\pi]$. In that way we get the generated rose curves with Chebishev polynomials. The Figures 10–13 show some special cases of these curves.

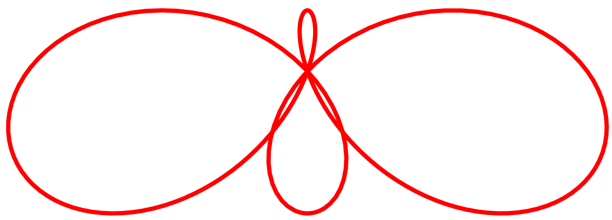

Figure 10: Curve in the case $n = 3$, $b = 0.4$

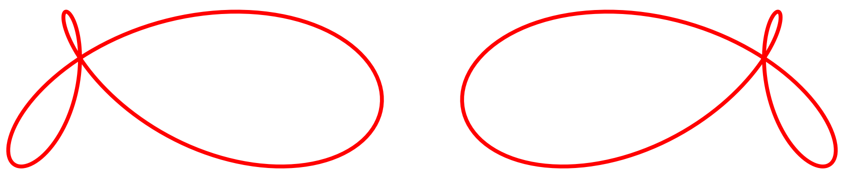

Figure 11: Curve in the cases $n = 4$ and $n = -4$, $b = 0.4$

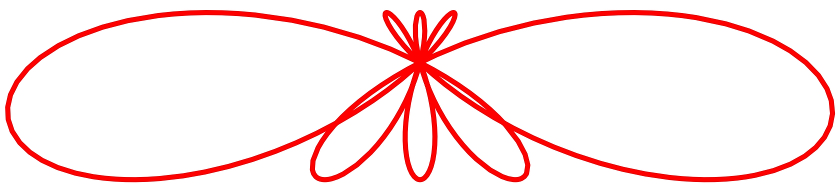

Figure 12: Curve in the case $n = 5$, $b = 0.4$

If we consider e as a line instead of a half-line and we require only that any two consecutive points are on different lines, on e or on the x-axis, the orbits of the points give the generalized

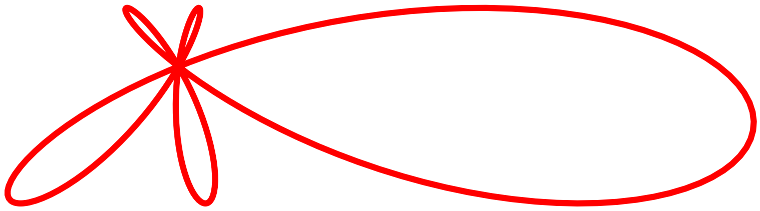

Figure 13: Curve in the case $n = 6$, $b = 0.4$

curves. In this case the angles α_i can be larger then $\pi/2$, moreover larger than 2π, but the properties of the functions $\cos x$ and $\sin x$ ensure that the equation systems (7) and (9) give the curves defined by the geometric construction in case of the whole line e.

Unfortunately, we cannot combine the two equation systems (7) and (9) in the same way as in [3], because in (9) there is an additional term, and the connection between α and φ is not the same in these two cases.

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Received September 11, 2014; final form May 27, 2015