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Concurrency and Collinearity in Hexagons

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Abstract. In a cyclic hexagon the main diagonals are concurrent if and only if the product of three mutually non-consecutive sides equals the product of the other three sides. We present here a vast generalization of this result to (closed) hexagonal paths (Sine-Concurrency Theorem), which also admits a collinearity version (Sine-Collinearity Theorem). The two theorems easily produce a proof of Desargues' Theorem. Henceforth we recover all the known facts about Fermat-Torricelli points, Napoleon points, or Kiepert points, obtained in connection with erecting three new triangles on the sides of a given triangle and then joining appropriate vertices. We also infer trigonometric proofs for two classical hexagon results of Pascal and Brianchon.

Key Words: Hexagon, Concurrency, Collinearity, Fermat-Torricelli Point, Napoleon Point, Kiepert Point, Desargues' Theorem, Pascal's Theorem, Brianchon's Theorem

MSC 2010: 51M04, 51A05, 51N15, 97G70

1. Two Sine-Theorems

Let $A_1A_2A_3A_4A_5A_6$ be a cyclic hexagon. A lesser known but nonetheless beautiful result states that the three main diagonals $\overline{A_1A_4}$, $\overline{A_2A_5}$, and $\overline{A_3A_6}$ are concurrent if and only if $A_1A_2 \cdot A_3A_4 \cdot A_5A_6 = A_2A_3 \cdot A_4A_5 \cdot A_6A_1$ [4]. Is there an equivalent of this result, holding for non-cyclic convex hexagons? The answer is yes, and it turns out to be true in much greater generality, for hexagons not necessarily convex, and not even simple, when viewed as closed (polygonal) curves. We will call such curves hexagonal paths. The only restriction in the hexagonal path is that the vertices be six mutually distinct points in general position: That is, no two lines through vertices of the hexagon may be identical or parallel (in particular, no three distinct vertices may be collinear). Even this hypothesis on vertices being in general position can be relaxed, see the Note following the Sine-Collinearity Theorem.

Our main results will then express the concurrency of the three main diagonal lines, $\overleftrightarrow{A_1A_4}$, $\overleftrightarrow{A_2A_5}$, and $\overleftrightarrow{A_3A_6}$, in terms of the measures of nine oriented angles, and it will also express the collinearity of the intersecting points of pairs of corresponding sides in two triangles, $\triangle A_1A_2A_3$ and $\triangle A_4A_5A_6$, in terms of those nine angles.

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In order not to be distracted by orientation issues, we state our results only when the hexagonal path is convex and the above vertex listing is consistent with traversing the sides of the hexagon in a counterclockwise manner. Fixing one of the two core internal triangles in the hexagon, say $\triangle A_1A_3A_5$ (the other being $\triangle A_2A_4A_6$), denote by α, β, and γ, the measures of its angles A_1, A_3, and A_5, respectively. Denote also by α^- and β^+ the measures of the angles A_1 and A_3, respectively, in $\triangle A_1A_2A_3$. Similarly, we have β^-, γ^+, and γ^-, α^+ (see Figure 1). Then the following holds true:

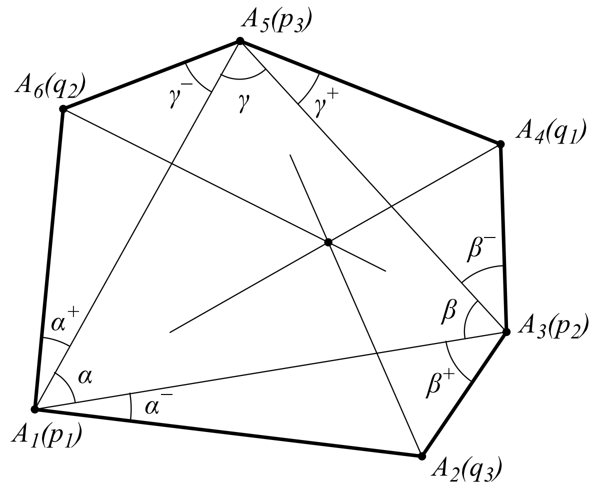

Figure 1: A convex hexagon with concurrent main diagonals, and the nine relevant angles

Sine-Concurrency Theorem. *Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon. With the above notations, the three main diagonals in the hexagon, $\overline{A_1A_4}$, $\overline{A_2A_5}$, and $\overline{A_3A_6}$, are concurrent if and only if*

$$\begin{aligned} &\sin(\alpha+\alpha^+)\sin(\beta+\beta^+)\sin(\gamma+\gamma^+)\sin\alpha^-\sin\beta^-\sin\gamma^- \\ =\ &\sin(\alpha+\alpha^-)\sin(\beta+\beta^-)\sin(\gamma+\gamma^-)\sin\alpha^+\sin\beta^+\sin\gamma^+ \end{aligned} \tag{1}$$

Note. For non-convex hexagonal paths $A_1A_2A_3A_4A_5A_6$ the theorem still holds true, however one needs to be more careful about the measures of the angles involved. The key here is the concept of oriented angle. For a proper angle, say $\widehat{BAC}$, with vertex A and rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ we define its oriented measure, $m(\widehat{BAC})=\theta$, as being the (real) angle θ (in radians), $0<|\theta|<\pi$, required to rotate (about vertex A) the ray $\overrightarrow{AB}$ over the ray $\overrightarrow{AC}$. The measure will be positive if this rotation is counterclockwise, and negative if it is clockwise. So for oriented angles, $m(\widehat{CAB})=-m(\widehat{BAC})$. Then, just as in the Sine-Concurrency Theorem, the main diagonal lines $\overleftrightarrow{A_1A_4}$, $\overleftrightarrow{A_2A_5}$, and $\overleftrightarrow{A_3A_6}$ will be concurrent if and only if Equation (1) holds, where $\alpha=m(\widehat{A_3A_1A_5})$, $\beta=m(\widehat{A_5A_3A_1})$, $\gamma=m(\widehat{A_1A_5A_3})$, $\alpha^-=m(\widehat{A_2A_1A_3})$, $\alpha^+=m(\widehat{A_5A_1A_6})$, $\beta^-=m(\widehat{A_4A_3A_5})$, $\beta^+=m(\widehat{A_1A_3A_2})$, $\gamma^-=m(\widehat{A_6A_5A_1})$, and $\gamma^+=m(\widehat{A_3A_5A_4})$. Notice that the same letter angle measures correspond to angles sharing the same vertex. For a more unorthodox implementation of these notations, see Figure 2.

To the end of proving the Sine-Concurrency Theorem and its companion, the Sine-Collinearity Theorem, we take a complex number approach. Identifying the Euclidean plane $\mathcal{E}$ of the hexagonal path with the complex number system $\mathbb{C}$ any point $P\in\mathcal{E}$ will have an affix $p\in\mathbb{C}$. Although in the figures we sometimes indicate both points and affixes, as in $P(p)$, in

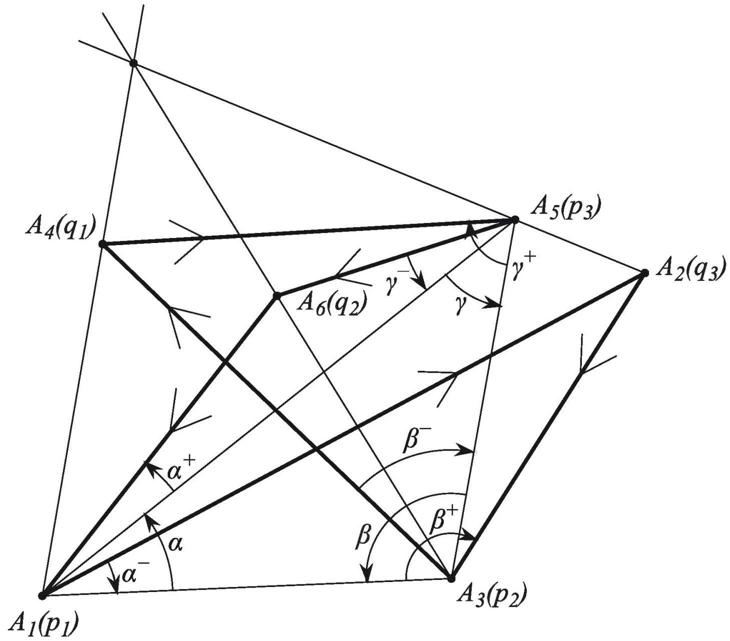

Figure 2: A non-convex, non-simple, hexagonal path in general position with concurrent main diagonals, and the nine relevant oriented angles.

all the other considerations the points and affixes will be identified and used interchangeably, as in 'the line determined by the points $p, q \in \mathbb{C}$'.

We recall now some key facts in complex Euclidean geometry. The reader can prove them easily, or approach them via the references [1, 2].

For two (distinct) points $p_1 \neq p_2$, the unique line determined by them, $\overleftrightarrow{p_1p_2}$, has the property that

$$z \in \mathbb{C} \text{ belongs to } \overleftrightarrow{p_1p_2} \iff \det \begin{bmatrix} z & \overline{z} & 1 \\ p_1 & \overline{p}_1 & 1 \\ p_2 & \overline{p}_2 & 1 \end{bmatrix} = 0. \tag{2}$$

Consequently, three points p_1, p_2 and p_3 will form the vertices of a (non-degenerate) triangle if and only if

$$\det \begin{bmatrix} p_1 & \overline{p}_1 & 1 \\ p_2 & \overline{p}_2 & 1 \\ p_3 & \overline{p}_3 & 1 \end{bmatrix} \neq 0.$$

Two lines as above, say $\overleftrightarrow{p_1q_1}$ and $\overleftrightarrow{p_2q_2}$ are non-parallel, and therefore intersect at an unique point, if and only if $\det \begin{bmatrix} p_1 - q_1 & \overline{p}_1 - \overline{q}_1 \\ p_2 - q_2 & \overline{p}_2 - \overline{q}_2 \end{bmatrix} \neq 0$. Moreover, via (2), the intersection point of the lines is

$$\overleftrightarrow{p_1q_1} \cap \overleftrightarrow{p_2q_2} = -\frac{\det \begin{bmatrix} p_1 - q_1 & p_1\overline{q}_1 - \overline{p}_1 q_1 \\ p_2 - q_2 & p_2\overline{q}_2 - \overline{p}_2 q_2 \end{bmatrix}}{\det \begin{bmatrix} p_1 - q_1 & \overline{p}_1 - \overline{q}_1 \\ p_2 - q_2 & \overline{p}_2 - \overline{q}_2 \end{bmatrix}}. \tag{3}$$

Finally, any affine transformation $\mathbb{C} \ni z \mapsto az + b \in \mathbb{C}$, $a, b \in \mathbb{C}$, $|a| = 1$, $a \neq 1$, can be viewed as a proper *rotation*, $R_{\theta, z_0}(z)$, of (oriented) angle $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ and center $z_0 \in \mathbb{C}$, via the identifications $a = e^{i\theta}$ and $z_0 = \dfrac{b}{1-a}$, i.e.,

$$R_{\theta, z_0}(z) = e^{i\theta} z + z_0 \left(1 - e^{i\theta}\right) = az + b. \tag{4}$$

Noticing that the center of rotation is the fixed point of the affine transformation, it follows that given a non-degenerate triangle, $\triangle p_1 q p_2$, with oriented angles $\theta_1 = m(\widehat{q p_1 p_2})$ at p_1 and $\theta_2 = m(\widehat{p_1 p_2 q})$ at p_2, the vertex q appears as the fixed point of a composition of two rotations, more exactly,

$$\begin{gathered} q = \operatorname{fix}\left(R_{2\theta_2, p_2} \circ R_{2\theta_1, p_1}\right) = \frac{\left(1-e^{2i\theta_2}\right)p_2 + e^{2i\theta_2}\left(1-e^{2i\theta_1}\right)p_1}{1-e^{2i(\theta_1+\theta_2)}} = s\,p_1 + (1-s)p_2, \\ \text{where} \quad s = \frac{e^{2i\theta_2}\left(1-e^{2i\theta_1}\right)}{1-e^{2i(\theta_1+\theta_2)}}. \end{gathered} \tag{5}$$

Lemma. *a) Let $p_1 \neq q_1$, $p_2 \neq q_2$, $p_3 \neq q_3$ be six points such that two of the three lines $\overleftrightarrow{p_1 q_1}$, $\overleftrightarrow{p_2 q_2}$, and $\overleftrightarrow{p_3 q_3}$, are non-identical and non-parallel. Then these three lines are concurrent if and only if*

$$\det \begin{bmatrix} p_1 - q_1 & \overline{p}_1 - \overline{q}_1 & p_1\overline{q}_1 - \overline{p}_1 q_1 \\ p_2 - q_2 & \overline{p}_2 - \overline{q}_2 & p_2\overline{q}_2 - \overline{p}_2 q_2 \\ p_3 - q_3 & \overline{p}_3 - \overline{q}_3 & p_3\overline{q}_3 - \overline{p}_3 q_3 \end{bmatrix} = 0. \tag{6}$$

b) Let p_1, p_2, p_3, s_1, s_2, s_3, and t_1, t_2, t_3, be nine complex numbers such that the first three, p_1, p_2, p_3, are non-zero. Set

$$\begin{aligned} &q_1 := s_1 p_2 + (1-s_1)p_3, \quad q_2 := s_2 p_3 + (1-s_2)p_1, \quad q_3 := s_3 p_1 + (1-s_3)p_2, \quad \textit{and} \\ &r_1 := t_1 \frac{1}{p_2} + (1-t_1)\frac{1}{p_3}, \quad r_2 := t_2 \frac{1}{p_3} + (1-t_2)\frac{1}{p_1}, \quad r_3 := t_3 \frac{1}{p_1} + (1-t_3)\frac{1}{p_2}. \end{aligned}$$

Then

$$\det \begin{bmatrix} p_1 - q_1 & \dfrac{1}{p_1} - r_1 & p_1 r_1 - \dfrac{1}{p_1} q_1 \\ p_2 - q_2 & \dfrac{1}{p_2} - r_2 & p_2 r_2 - \dfrac{1}{p_2} q_2 \\ p_3 - q_3 & \dfrac{1}{p_3} - r_3 & p_3 r_3 - \dfrac{1}{p_3} q_3 \end{bmatrix} = \frac{(p_1-p_2)(p_2-p_3)(p_3-p_1)}{p_1^2 p_2^2 p_3^2}(\xi - \eta), \tag{7}$$

where

$$\begin{aligned} \xi &= (t_1 p_1 - s_1 p_2)(t_2 p_2 - s_2 p_3)(t_3 p_3 - s_3 p_1), \\ \eta &= ((1-s_2)p_1 - (1-t_2)p_2)\,((1-s_3)p_2 - (1-t_3)p_3)\,((1-s_1)p_3 - (1-t_1)p_1). \end{aligned} \tag{8}$$

Proof. a) Denote by A the 3×3 complex matrix appearing in Equation (6). Assume that the three lines are concurrent at, say, $v \in \mathbb{C}$. Then, by Equation (2), the point $\mathbf{z}_0 = \begin{bmatrix} -\overline{v} \\ v \\ 1 \end{bmatrix} \in \mathbb{C}^3$ is a non-trivial solution of the homogeneous linear complex system $A\,\mathbf{z} = \mathbf{0}$. Consequently, Equation (6) holds.

Conversely, if Equation (6) holds then the homogeneous linear system $A\,\mathbf{z} = \mathbf{0}$ has non-trivial solutions. More precisely, since by the non-parallelism hypothesis the matrix A has rank 2, the solution set of the system is one-dimensional. Let $\begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathbb{C}^3$ be a non-zero vector spanning this solution set. Then $w \neq 0$ since otherwise, again by the non-parallelism hypothesis, the system cannot have non-trivial solutions. As a result, there is only one solution of the system $A\,\mathbf{z}=\mathbf{0}$ of type $\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$. However, then it is easy to see that $\begin{bmatrix} -\overline{v} \\ -\overline{u} \\ 1 \end{bmatrix}$ is also solution, and so $u=-\overline{v}$. In conclusion, by Equation (2) the three lines are concurrent at $v \in \mathbb{C}$.

b) The determinant appearing in Equation (6) is in fact a specialization of that appearing in Equation (7), when $|p_1|=|p_2|=|p_3|=1$, and q_1, q_2, q_3, are given by suitable linear combinations of type (5).

The identity (7) is not surprising, given the circular symmetries of the matrix involved. It probably can afford a more elegant proof than the one outlined below. While it can be easily checked by a brute force determinant expansion and lengthy algebraic manipulations, it is worthwhile explaining how one can arrive to the right hand side expression in (7).

Notice first that $p_1^2p_2^3p_3^2\det(B)$, where B is the matrix appearing in (7), is a homogeneous polynomial of degree 6 in p_1, p_2, p_3. Also, the elements of the first row, and subsequently the other two rows by circular permutations, can be expressed as

$$\begin{aligned} p_1-q_1 &= s_1(p_1-p_2)-(1-s_1)(p_3-p_1), \\ \frac{1}{p_1}-r_1 &= -\frac{t_1p_3}{p_1p_2p_3}(p_1-p_2)+\frac{(1-t_1)p_2}{p_1p_2p_3}(p_3-p_1), \\ p_1r_1-\frac{1}{p_1}q_1 &= \frac{s_1p_2p_3+t_1p_3p_1}{p_1p_2p_3}(p_1-p_2)-\frac{(1-s_1)p_2p_3+(1-t_1)p_1p_2}{p_1p_2p_3}(p_3-p_1). \end{aligned} \tag{9}$$

The expressions in (9) suggest that $p_1^2p_2^3p_3^2\det(B)$ should be divisible by $(p_1-p_2)(p_2-p_3)(p_3-p_1)$, and also that a homogeneity of degree 3 with respect to $s_i, t_i, (1-s_i), (1-t_i)$, $i=1,2,3$, be present. Indeed, when $p_1=p_2$, $\det(B)$ vanishes since then

$$B=(p_3-p_1)\begin{bmatrix} -(1-s_1) & \dfrac{1-t_1}{p_3p_1} & -\dfrac{(1-s_1)p_3+(1-t_1)p_1}{p_3p_1} \\ -s_2 & \dfrac{t_2}{p_3p_1} & -\dfrac{s_2p_3+t_2p_1}{p_3p_1} \\ 1 & -\dfrac{1}{p_3p_1} & \dfrac{p_3+p_1}{p_3p_1} \end{bmatrix},$$

and above the third column is obviously a linear combination of the first two columns.

The divisibility of $p_1^2p_2^3p_3^2\det(B)$ by $(p_1-p_2)(p_2-p_3)(p_3-p_1)$ shows that in the expansion of $\det(B)$, when the elements of B are expressed as in (9), s_i-containing terms multiplied by $(1-t_j)$-containing terms cancel out, and this and the degree 3 homogeneity mentioned above lead to the expressions of ξ and η. □

Proof of the Theorem. There is no loss of generality in assuming that the circumcenter of $\triangle A_1A_3A_5$ has affix 0, and the affixes p_1 of A_1, p_2 of A_3, and p_3 of A_5 are such that $|p_1|=|p_2|=|p_3|=1$. In $\triangle A_1A_2A_3$ the vertex A_2 has afix $q_3=\text{fix}\,(R_{2\beta^+,p_2}\circ R_{2\alpha^-,p_1})$. Similarly, A_4 has afix $q_1=\text{fix}\,(R_{2\gamma^+,p_3}\circ R_{2\beta^-,p_2})$ and A_6 has afix $q_2=\text{fix}\,(R_{2\alpha^+,p_1}\circ R_{2\gamma^-,p_3})$. By (5),

$$\begin{aligned} q_1 &= s_1p_2+(1-s_1)p_3, \quad \text{where} \quad s_1 = \frac{e^{2i\gamma^+}\left(1-e^{2i\beta^-}\right)}{1-e^{2i(\beta^-+\gamma^+)}}, \\ q_2 &= s_2p_3+(1-s_2)p_1, \quad \text{where} \quad s_2 = \frac{e^{2i\alpha^+}\left(1-e^{2i\gamma^-}\right)}{1-e^{2i(\gamma^-+\alpha^+)}}, \\ q_3 &= s_3p_1+(1-s_3)p_2, \quad \text{where} \quad s_3 = \frac{e^{2i\beta^+}\left(1-e^{2i\alpha^-}\right)}{1-e^{2i(\gamma^-+\alpha^+)}}. \end{aligned} \tag{10}$$

According to part a) of the Lemma the three segments $\overline{A_1A_4}$, $\overline{A_2A_5}$, and $\overline{A_3A_6}$ are concurrent if and only if the determinant in Equation (6) vanishes, for the choices of p_1, p_2, p_3, and q_1, q_2, q_3 given above. Since $\overline{p}_1 = \frac{1}{p_1}$, $\overline{p}_2 = \frac{1}{p_2}$, $\overline{p}_3 = \frac{1}{p_3}$, we can use part b) of the Lemma to evaluate the determinant in (6). It equals the determinant in (7) for the values of s_1, s_2, s_3 already indicated above in Equations (10), and for

$$\begin{aligned} t_1 &= \overline{s}_1 = \frac{1-e^{2i\beta^-}}{1-e^{2i(\beta^-+\gamma^+)}} = \frac{s_1}{e^{2i\gamma^+}}, \\ t_2 &= \overline{s}_2 = \frac{1-e^{2i\gamma^-}}{1-e^{2i(\gamma^-+\alpha^+)}} = \frac{s_2}{e^{2i\alpha^+}}, \\ t_3 &= \overline{s}_3 = \frac{1-e^{2i\alpha^-}}{1-e^{2i(\alpha^-+\beta^+)}} = \frac{s_3}{e^{2i\beta^+}}. \end{aligned} \tag{11}$$

Clearly, from (10) and (11) we get

$$\begin{aligned} 1-s_1 &= \frac{1-e^{2i\gamma^+}}{1-e^{2i(\beta^-+\gamma^+)}}, \qquad & 1-t_1 &= e^{2i\beta^-}(1-s_1), \\ 1-s_2 &= \frac{1-e^{2i\alpha^+}}{1-e^{2i(\gamma^-+\alpha^+)}}, \qquad & 1-t_2 &= e^{2i\gamma^-}(1-s_2), \\ 1-s_3 &= \frac{1-e^{2i\beta^+}}{1-e^{2i(\alpha^-+\beta^+)}}, \qquad & 1-t_3 &= e^{2i\alpha^-}(1-s_3). \end{aligned} \tag{12}$$

The last piece of information required to finish the proof of the theorem concerns the angles α, β, and γ in $\triangle A_1A_3A_5$. It is not hard to see that they are related to p_1, p_2, and p_3 via the formulae

$$p_2 = e^{2i\gamma}p_1, \quad p_3 = e^{2i\alpha}p_2, \quad p_1 = e^{2i\beta}p_3. \tag{13}$$

From (11) and (13) it follows that

$$\begin{aligned} t_1p_1 - s_1p_2 &= t_1\left(1-e^{2i(\gamma+\gamma^+)}\right)p_1, \\ t_2p_2 - s_2p_3 &= t_2\left(1-e^{2i(\alpha+\alpha^+)}\right)p_2, \\ t_3p_3 - s_3p_1 &= t_3\left(1-e^{2i(\beta+\beta^+)}\right)p_3. \end{aligned} \tag{14}$$

From (12) and (13) it follows that

$$\begin{aligned} (1-s_2)p_1-(1-t_2)p_2 &= (1-s_2)\left(1-e^{2i(\gamma+\gamma^-)}\right)p_1, \\ (1-s_3)p_2-(1-t_3)p_3 &= (1-s_3)\left(1-e^{2i(\alpha+\alpha^-)}\right)p_2, \\ (1-s_1)p_3-(1-t_1)p_1 &= (1-s_1)\left(1-e^{2i(\beta+\beta^+)}\right)p_3. \end{aligned} \tag{15}$$

Consequently,

$$\begin{aligned} \xi &= (t_1p_1-s_1p_2)(t_2p_2-s_2p_3)(t_3p_3-s_3p_1) \\ &= t_1t_2t_3\left(1-e^{2i(\alpha+\alpha^+)}\right)\left(1-e^{2i(\beta+\beta^+)}\right)\left(1-e^{2i(\gamma+\gamma^+)}\right)p_1p_2p_3, \end{aligned}$$

and

$$\begin{aligned} \eta &= ((1-s_2)p_1-(1-t_2)p_2)((1-s_3)p_2-(1-t_3)p_3)((1-s_1)p_3-(1-t_1)p_1) \\ &= (1-s_1)(1-s_2)(1-s_3)\left(1-e^{2i(\alpha+\alpha^-)}\right)\left(1-e^{2i(\beta+\beta^-)}\right)\left(1-e^{2i(\gamma+\gamma^-)}\right)p_1p_2p_3, \end{aligned}$$

In conclusion, $\xi = \eta$ is equivalent, via (11) and (12), to

$$\left(1-e^{2i\alpha^-}\right)\left(1-e^{2i\beta^-}\right)\left(1-e^{2i\gamma^-}\right)\left(1-e^{2i(\alpha+\alpha^+)}\right)\left(1-e^{2i(\beta+\beta^+)}\right)\left(1-e^{2i(\gamma+\gamma^+)}\right)$$
$$=\left(1-e^{2i\alpha^+}\right)\left(1-e^{2i\beta^+}\right)\left(1-e^{2i\gamma^+}\right)\left(1-e^{2i(\alpha+\alpha^-)}\right)\left(1-e^{2i(\beta+\beta^-)}\right)\left(1-e^{2i(\gamma+\gamma^-)}\right),$$

which is easily seen to be equivalent to (1). The proof of the Sine-Concurrency Theorem is complete. □

Sine-Collinearity Theorem. *Given a convex hexagon $A_1A_2A_3A_4A_5A_6$ with vertices in general position, consider the three intersecting points of corresponding sides in $\triangle A_1A_2A_3$ and $\triangle A_4A_5A_6$. More precisely, let lines $\overleftrightarrow{A_1A_2}$ and $\overleftrightarrow{A_4A_5}$ intersect at M_1, lines $\overleftrightarrow{A_2A_3}$ and $\overleftrightarrow{A_5A_6}$ intersect at M_2, and lines $\overleftrightarrow{A_3A_1}$ and $\overleftrightarrow{A_6A_4}$ intersect at M_3 (cf. Figure 3). Then the points M_1, M_2, and M_3 are collinear if and only if for the angles α, α^+, α^-, β, β^+, β^-, and γ, γ^+, γ^- associated as before in connection with $\triangle A_1A_3A_5$ we have (Equation (1))*

$$\begin{aligned}&\sin(\alpha+\alpha^+)\sin(\beta+\beta^+)\sin(\gamma+\gamma^+)\sin\alpha^-\sin\beta^-\sin\gamma^-\\&=\sin(\alpha+\alpha^-)\sin(\beta+\beta^-)\sin(\gamma+\gamma^-)\sin\alpha^+\sin\beta^+\sin\gamma^+\end{aligned}\tag{16}$$

Proof. As the proof mimics that of the Sine-Concurrency Theorem we provide only its basic skeleton. Let m_1, m_2, and m_3 be the affixes of M_1, M_2, and M_3, respectively. Then, by (3)

$$\begin{aligned}
m_1 &= \overleftrightarrow{p_1q_3}\cap\overleftrightarrow{p_3q_1} = -\frac{\det\begin{bmatrix}p_1-q_3 & p_1\overline{q}_3-\overline{p}_1q_3\\ p_3-q_1 & p_3\overline{q}_1-\overline{p}_3q_1\end{bmatrix}}{\det\begin{bmatrix}p_1-q_3 & \overline{p}_1-\overline{q}_3\\ p_3-q_1 & \overline{p}_3-\overline{q}_1\end{bmatrix}},\\
m_2 &= \overleftrightarrow{p_2q_3}\cap\overleftrightarrow{p_3q_2} = -\frac{\det\begin{bmatrix}p_2-q_3 & p_2\overline{q}_3-\overline{p}_2q_3\\ p_3-q_2 & p_3\overline{q}_2-\overline{p}_3q_2\end{bmatrix}}{\det\begin{bmatrix}p_2-q_3 & \overline{p}_2-\overline{q}_3\\ p_3-q_2 & \overline{p}_3-\overline{q}_2\end{bmatrix}},\\
m_3 &= \overleftrightarrow{p_1p_2}\cap\overleftrightarrow{q_1q_2} = -\frac{\det\begin{bmatrix}p_1-p_2 & p_1\overline{p}_2-\overline{p}_1p_2\\ q_1-q_2 & q_1\overline{q}_2-\overline{q}_1q_2\end{bmatrix}}{\det\begin{bmatrix}p_1-p_2 & \overline{p}_1-\overline{p}_2\\ q_1-q_2 & \overline{q}_1-\overline{q}_2\end{bmatrix}},
\end{aligned}\tag{17}$$

and so

$$\begin{aligned}
\overline{m}_1 &= -\frac{\det\begin{bmatrix}\overline{p}_1-\overline{q}_3 & p_1\overline{q}_3-\overline{p}_1q_3\\ \overline{p}_3-\overline{q}_1 & p_3\overline{q}_1-\overline{p}_3q_1\end{bmatrix}}{\det\begin{bmatrix}p_1-q_3 & \overline{p}_1-\overline{q}_3\\ p_3-q_1 & \overline{p}_3-\overline{q}_1\end{bmatrix}},\\
\overline{m}_2 &= -\frac{\det\begin{bmatrix}\overline{p}_2-\overline{q}_3 & p_2\overline{q}_3-\overline{p}_2q_3\\ \overline{p}_3-\overline{q}_2 & p_3\overline{q}_2-\overline{p}_3q_2\end{bmatrix}}{\det\begin{bmatrix}p_2-q_3 & \overline{p}_2-\overline{q}_3\\ p_3-q_2 & \overline{p}_3-\overline{q}_2\end{bmatrix}},\\
\overline{m}_3 &= -\frac{\det\begin{bmatrix}\overline{p}_1-\overline{p}_2 & p_1\overline{p}_2-\overline{p}_1p_2\\ \overline{q}_1-\overline{q}_2 & q_1\overline{q}_2-\overline{q}_1q_2\end{bmatrix}}{\det\begin{bmatrix}p_1-p_2 & \overline{p}_1-\overline{p}_2\\ q_1-q_2 & \overline{q}_1-\overline{q}_2\end{bmatrix}}.
\end{aligned}\tag{18}$$

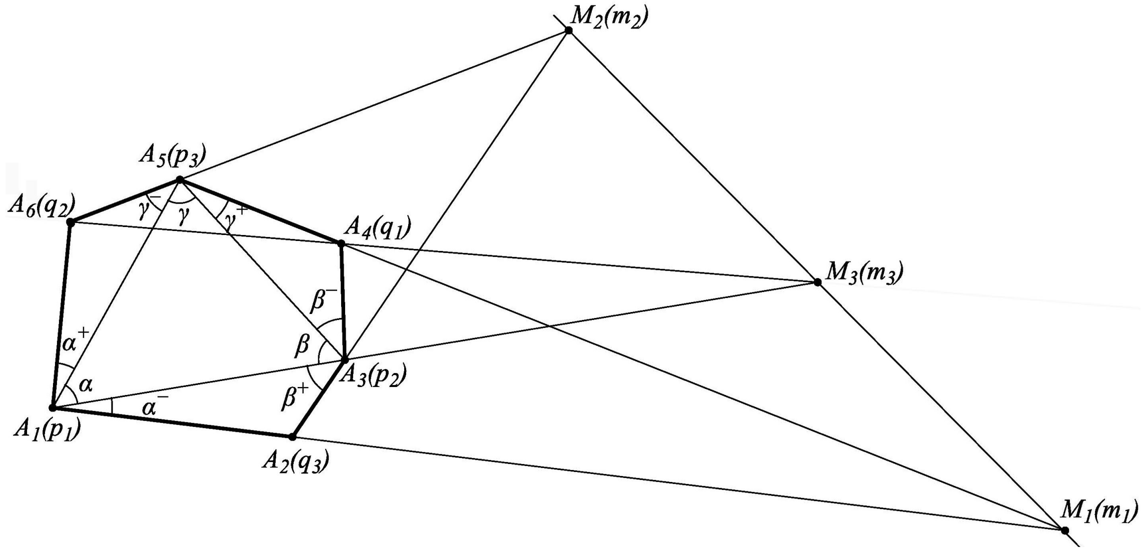

Figure 3: A convex hexagon exhibiting collinearity and the nine relevant angles, as in the Sine-Collinearity Theorem

M_1, M_2, and M_3 are then collinear if and only if

$$\det\begin{bmatrix} m_1 & \overline{m}_1 & 1 \\ m_2 & \overline{m}_2 & 1 \\ m_3 & \overline{m}_3 & 1 \end{bmatrix} = 0\,. \tag{19}$$

Make now the substitutions

$$\overline{p}_i \longrightarrow \frac{1}{p_i}, \quad \overline{q}_i \longrightarrow r_i,$$

in m_i and $\overline{m}_i$, $i = 1,2,3$. If as a result of the substitutions we let $m_i \longrightarrow u_i$ and $\overline{m}_i \longrightarrow v_i$ for $i = 1,2,3$, by further setting, as in the Lemma, $q_i := s_i p_{i+1} + (1-s_i)p_{i+2}$ and $r_i := t_i \dfrac{1}{p_{i+1}} + (1-t_i)\dfrac{1}{p_{i+2}}$, $i = 1,2,3$, the following identity holds true:

$$\frac{\det\begin{bmatrix} p_1 - q_3 & \dfrac{1}{p_1} - r_3 \\ p_3 - q_1 & \dfrac{1}{p_3} - r_1 \end{bmatrix} \det\begin{bmatrix} p_2 - q_3 & \dfrac{1}{p_2} - r_3 \\ p_3 - q_2 & \dfrac{1}{p_3} - r_2 \end{bmatrix} \det\begin{bmatrix} p_1 - p_2 & \dfrac{1}{p_1} - \dfrac{1}{p_2} \\ q_1 - q_2 & r_1 - r_2 \end{bmatrix}}{\det\begin{bmatrix} p_3 - q_2 & \dfrac{1}{p_3} - r_2 \\ p_3 - q_1 & \dfrac{1}{p_3} - r_1 \end{bmatrix}} \det\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{bmatrix} \tag{20}$$
$$= \frac{(p_1-p_2)^3(p_2-p_3)(p_3-p_1)(s_3-t_3)}{p_1^3 p_2^3 p_3^3}(\eta - \xi),$$

where ξ and η are those given by (8). Since in (20) the various determinants are non-vanishing, under the further hypothesis $|p_i| = 1$ and the specializations of s_i, and t_i for $i = 1,2,3$ given by (10) and (11), we see that

$$\det\begin{bmatrix} m_1 & \overline{m}_1 & 1 \\ m_2 & \overline{m}_2 & 1 \\ m_3 & \overline{m}_3 & 1 \end{bmatrix} = \det\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{bmatrix} = 0 \quad \Longleftrightarrow \quad \xi = \eta.$$

This proves the Sine-Collinearity Theorem. □

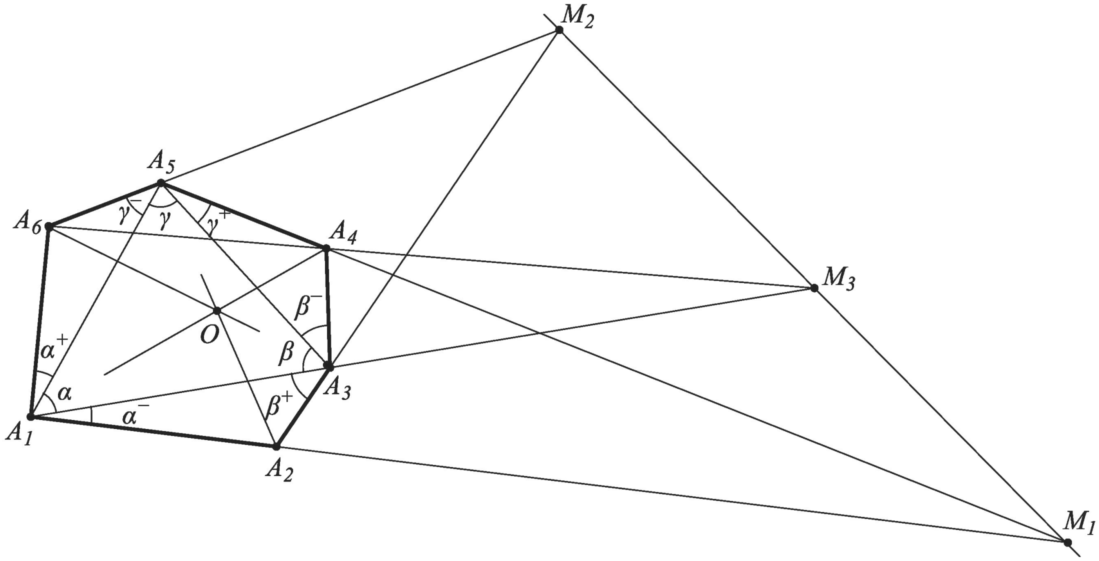

Figure 4: A nonstandard hexagonal implementation of the equivalence between concurrency and collinearity in Desargues' Theorem

Note. Although the Sine-Theorems were stated for convex hexagons the above proofs are valid, as already mentioned, for arbitrary hexagonal paths with vertices in general position and oriented angles. In fact, even the requirement that the hexagon vertices be in general position can be removed if the usual convention that parallel lines meet at infinity is allowed. Only the algebraic limitations of our proof prevented us from stating the result at this level of generality. However, it is clear how to get this more general result from ours by a limiting argument. The natural habitat for matters involving concurrency and collinearity being projective and not affine geometry, all this is normal.

2. Consequences of the Sine-Theorems

We conclude this paper with few applications to the two Sine-Theorems.

Corollary.

a) Desargues' Theorem – Indirect Trigonometric Proof. Given a convex hexagon $A_1A_2A_3$ $A_4A_5A_6$ *with vertices in general position, let* M_1, M_2, *and* M_3 *be the three intersecting points of the corresponding sides in* $\triangle A_1A_2A_3$ *and* $\triangle A_4A_5A_6$. *Then the main diagonals in the hexagon,* $\overline{A_1A_4}$, $\overline{A_2A_5}$, *and* $\overline{A_3A_6}$ *are concurrent if and only if* M_1, M_2, *and* M_3 *are collinear (cf. Figure 4).*

b) Assume that on the sides of a given triangle, $\triangle A_1A_3A_5$, *with angles* α, β, *and* γ, *three new triangles,* $\triangle A_1A_2A_3$, $\triangle A_3A_4A_5$, *and* $\triangle A_5A_6A_1$ *are erected, with oriented angles,* α^- *and* β^+, β^- *and* γ^+, γ^- *and* α^+ *respectively, as described after the statement of the Sine-Concurrency Theorem. If* $\alpha^- = \alpha^+$, $\beta^- = \beta^+$, *and* $\gamma^- = \gamma^+$ *then the main diagonal lines of the hexagonal path* $A_1A_2A_3A_4A_5A_6$ *are concurrent.*

c) Let $A_1A_2A_3A_4A_5A_6$ *be a cyclic hexagon. Then its main diagonals are concurrent if and only if* $A_1A_2 \cdot A_3A_4 \cdot A_5A_6 = A_2A_3 \cdot A_4A_5 \cdot A_6A_1$.

d) Let $B_1B_2B_3B_4B_5B_6$ *be a cyclic hexagon. On its sides erect exterior triangles by extending these sides, and denote the additional vertices of these triangles by* A_1, A_2, A_3, A_4, A_5, *and* A_6. *Then the main diagonals in the convex hexagon* $A_1A_2A_3A_4A_5A_6$ *are concurrent.*

Proof. *a*) This is a standard instance of transitivity in mathematics. The concurrency, at O, of the main diagonals $\overline{A_1A_4}$, $\overline{A_2A_5}$ and $\overline{A_3A_6}$, respectively the collinearity of M_1, M_2, and M_3, is equivalent via the Sine-Concurrency Theorem, respectively the Sine-Collinearity Theorem, to the same trigonometric identity (1), involving the nine angles α, α^+, α^-, β, β^+, β^-, and γ, γ^+, γ^- associated as before in connection with $\triangle A_1A_3A_5$.

Notice that there are three more hexagons with the same vertex set and the same main diagonals as $A_1A_2A_3A_4A_5A_6$, for which Desargues' Theorem holds true, namely $A_1A_2A_6A_4A_5A_3$, $A_1A_5A_3A_4A_2A_6$, and $A_1A_5A_6A_4A_2A_3$. Evidently, they generate different sets of collinear points.

b) is a result of de Villiers [8]. Its proof is an obvious consequence of the Sine-Concurrency Theorem, as the given hypotheses make the content of Equation (1) plain. Sub-particular cases reveal important concurrency points:

- A point on the Kiepert hyperbola [9], if $\alpha^+ = \alpha^- = \beta^+ = \beta^- = \gamma^+ = \gamma^- = \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
- The first/second Fermat-Torricelli point [3], if $\alpha^+ = \alpha^- = \beta^+ = \beta^- = \gamma^+ = \gamma^- = +\frac{\pi}{3}/-\frac{\pi}{3}$.
- The first/second Napoleon point [6], if $\alpha^+ = \alpha^- = \beta^+ = \beta^- = \gamma^+ = \gamma^- = +\frac{\pi}{6}/-\frac{\pi}{6}$.
- The centroid of $\triangle A_1A_3A_5$, in the limiting case $\alpha^+ = \alpha^- = \beta^+ = \beta^- = \gamma^+ = \gamma^- = 0$.
- The orthocenter of $\triangle A_1A_3A_5$, in the limiting case $\alpha^+ = \alpha^- = \beta^+ = \beta^- = \gamma^+ = \gamma^- = \frac{\pi}{2}$.

c) is a result of Cartensen [4]. To the end of proving it we rely on the notations of Figure 1. To show that Equation (1) is equivalent to the metric property given by *c*) we employ the Law of Sines in various triangles with vertices among the vertices of the hexagon. By hypothesis, all these triangles have the same circumcircle, of radius, say, R. For instance, in $\triangle A_1A_3A_6$, $\frac{\sin(\alpha+\alpha^+)}{A_3A_6} = \frac{1}{2R}$ and in $\triangle A_3A_5A_6$, $\frac{\sin(\gamma+\gamma^-)}{A_3A_6} = \frac{1}{2R}$, give $\sin(\alpha+\alpha^+) = \sin(\gamma+\gamma^-)$. Similarly, $\sin(\beta+\beta^+) = \sin(\alpha+\alpha^-)$ and $\sin(\gamma+\gamma^+) = \sin(\beta+\beta^-)$. Therefore, Equation (1) is equivalent to $\sin\alpha^+ \sin\beta^+ \sin\gamma^+ = \sin\alpha^- \sin\beta^- \sin\gamma^-$.

Now, in $\triangle A_1A_2A_3$, $\frac{\sin\beta^+}{A_1A_2} = \frac{\sin\alpha^-}{A_2A_3} = \frac{1}{2R}$. Similarly, $\frac{\sin\gamma^+}{A_3A_4} = \frac{\sin\beta^-}{A_4A_5} = \frac{1}{2R}$ and $\frac{\sin\alpha^+}{A_5A_6} = \frac{\sin\gamma^-}{A_6A_1} = \frac{1}{2R}$. They all lead to the equivalence of $\sin\alpha^+ \sin\beta^+ \sin\gamma^+ = \sin\alpha^- \sin\beta^- \sin\gamma^-$ to $A_1A_2 \cdot A_3A_4 \cdot A_5A_6 = A_2A_3 \cdot A_4A_5 \cdot A_6A_1$.

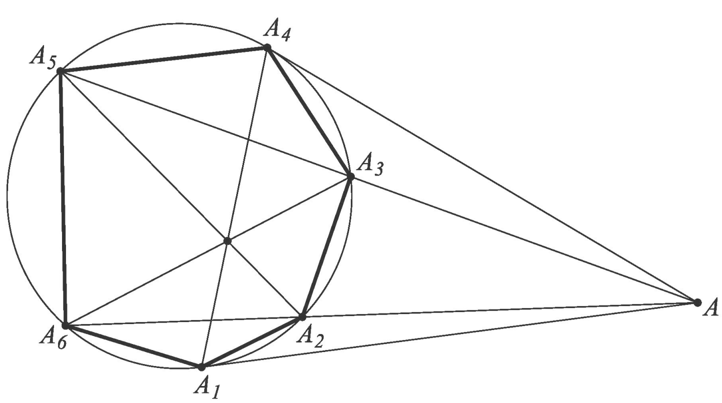

Figure 5: An example of a cyclic hexagon with concurrent main diagonals

Here are now two natural implementations of c).

c_1) Let A_1, A_2, A_3, A_4, and A_5 be five distinct points, distributed in a counterclockwise manner on a given circle. If $\overset{\frown}{A_5A_1}$ is the counterclockwise oriented arc of the circle (with initial point A_5 and terminal point A_1), the continuous function

$$f\colon\ \overset{\frown}{A_5A_1} \to \mathbb{R}, \quad f(A) = A_1A_2 \cdot A_3A_4 \cdot A_5A - A_2A_3 \cdot A_4A_5 \cdot AA_1\,,$$

is strictly increasing as A advances along the arc, $f(A_5) < 0$, and $f(A_1) > 0$. By the Intermediate Value Property there is an unique point $A = A_6 \in \overset{\frown}{A_5A_1}$ such that the main diagonals in the cyclic hexagon $A_1A_2A_3A_4A_5A_6$ are concurrent. Clearly, A_6 is the intersection point of the arc $\overset{\frown}{A_5A_1}$ with the line $\overleftrightarrow{A_3I}$, where I is the intersection point of the line segments $\overline{A_1A_4}$ and $\overline{A_2A_5}$.

c_2) Let A be a point exterior to a given circle, and let A_1 and A_4 be the points where the two tangents to the circle through the point A intersect the circle. Let also two secants through A intersect the circle at A_2 and A_6, respectively A_3 and A_5 (cf. Figure 5). Then the main diagonals in the cyclic hexagon $A_1A_2A_3A_4A_5A_6$ are concurrent.

This can be seen by using similarity in three pairs of triangles. For instance $\triangle AA_1A_2 \sim \triangle AA_6A_1$ gives $\frac{A_1A_2}{A_6A_1} = \frac{AA_1}{AA_6} = \frac{AA_2}{AA_1}$, which implies $\frac{(A_1A_2)^2}{(A_6A_1)^2} = \frac{AA_2}{AA_6}$. Similarly, $\frac{A_3A_4}{A_4A_5} = \frac{AA_3}{AA_4} = \frac{AA_4}{AA_5}$ gives $\frac{(A_3A_4)^2}{(A_4A_5)^2} = \frac{AA_3}{AA_5}$ and $\frac{A_5A_6}{A_2A_3} = \frac{AA_5}{AA_2} = \frac{AA_6}{AA_3}$ gives $\frac{(A_5A_6)^2}{(A_2A_3)^2} = \frac{AA_5 \cdot AA_6}{AA_2 \cdot AA_3}$. Therefore, $\frac{(A_1A_2)^2}{(A_6A_1)^2}\frac{(A_3A_4)^2}{(A_4A_1)^5}\frac{(A_5A_6)^2}{(A_2A_3)^2} = 1$, which proves the validity of c_2).

c_2) also holds true in the more general case when the circle is replaced by an ellipse. This follows easily from the circle case since the plane transformation which projects an ellipse onto its associated great circle preserves lines. In fact, the elliptic c_2) case can be viewed as a variant of Brianchon's Theorem [5]. We let the reader sort out the details with the help of Figure 6.

d) Referring to Figure 7, by the Sine-Concurrency Theorem we have to establish the validity of Equation (1) for the choices of angles indicated. The Law of Sines applied to $\triangle A_1A_2A_3$

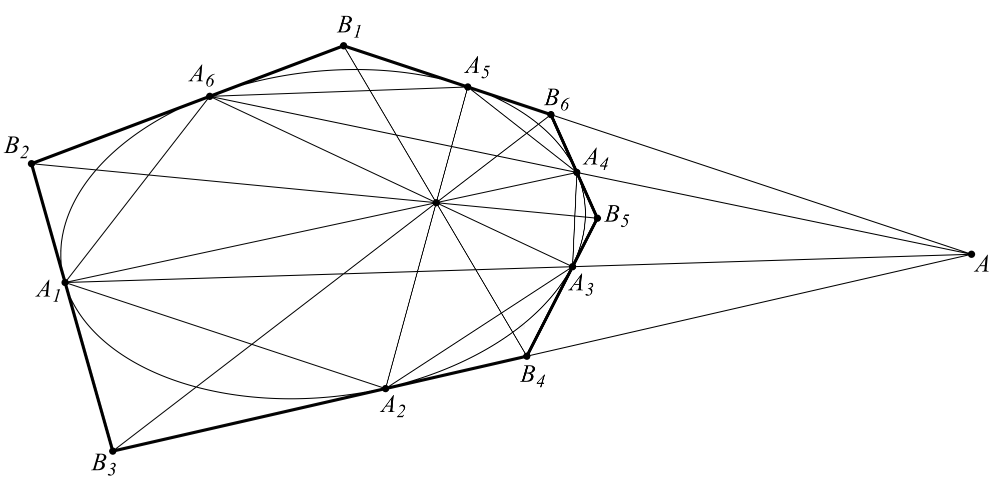

Figure 6: The concurrency point of the main diagonals in the Brianchon hexagon $B_1B_2B_3B_4B_5B_6$ is the same as that in the elliptic c_2) case

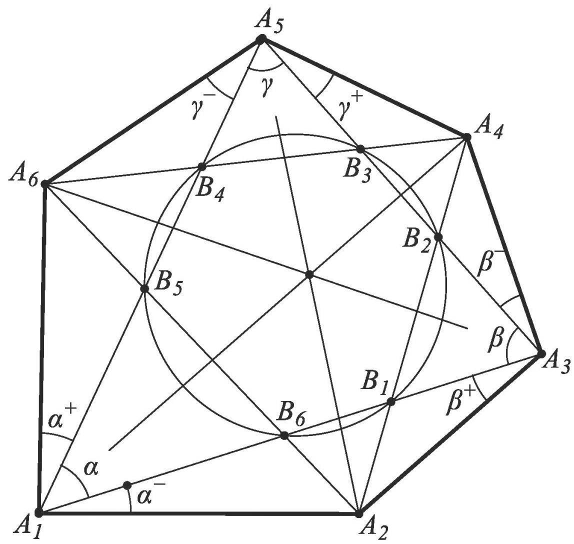

Figure 7: The main diagonals in the convex hexagon $A_1A_2A_3A_4A_5A_6$ are always concurrent, while those in the cyclic hexagon $B_1B_2B_3B_4B_5B_6$ may not be

gives $\dfrac{\sin\alpha^-}{\sin\beta^+} = \dfrac{A_2A_3}{A_1A_2}$. Similarly, we have

$$\frac{\sin\beta^-}{\sin\gamma^+} = \frac{A_4A_5}{A_3A_4} \qquad \text{and} \qquad \frac{\sin\gamma^-}{\sin\alpha^+} = \frac{A_6A_1}{A_5A_6}. \tag{21}$$

Combining now three applications of the Law of Sines respectively to $\triangle A_6A_1B_6$, $\triangle A_2B_1B_6$ and $\triangle A_3A_4B_1$ we have $\dfrac{\sin(\alpha+\alpha^+)}{\sin(\beta+\beta^-)} = \dfrac{A_6B_6}{A_6A_1}\,\dfrac{A_2B_1}{A_2B_6}\,\dfrac{A_3A_4}{A_4B_1}$ and similarly,

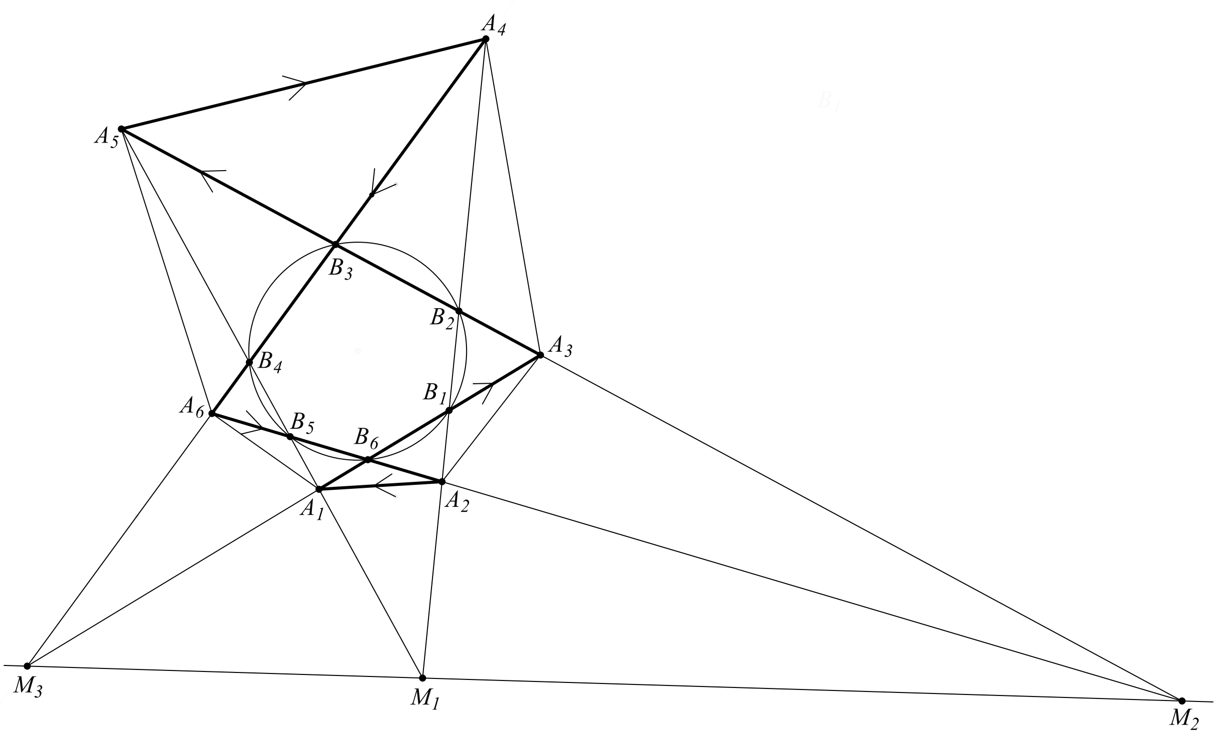

Figure 8: Pascal's Theorem for the cyclic hexagon $B_1B_2B_3B_4B_5B_6$ is an example of a Sine-Collinearity Theorem applied to the non-convex hexagonal path $A_1A_3A_5A_4A_6A_2$

$$\frac{\sin(\beta+\beta^+)}{\sin(\gamma+\gamma^-)} = \frac{A_2B_2}{A_2A_3}\frac{A_4B_3}{A_4B_2}\frac{A_5A_6}{A_6B_3} \qquad \text{and} \qquad \frac{\sin(\gamma+\gamma^+)}{\sin(\alpha+\alpha^-)} = \frac{A_4B_4}{A_4A_5}\frac{A_6B_5}{A_6B_4}\frac{A_1A_2}{A_2B_5}. \tag{22}$$

Multiplying together Equations (21) and (22) and simplifying yields now

$$\begin{aligned} &\frac{\sin(\alpha+\alpha^+)\sin(\beta+\beta^+)\sin(\gamma+\gamma^+)\sin\alpha^-\sin\beta^-\sin\gamma^-}{\sin(\alpha+\alpha^-)\sin(\beta+\beta^-)\sin(\gamma+\gamma^-)\sin\alpha^+\sin\beta^+\sin\gamma^+} \\ &= \frac{A_6B_5\cdot A_6B_6}{A_6B_4\cdot A_6B_3}\frac{A_2B_1\cdot A_2B_2}{A_2B_6\cdot A_2B_5}\frac{A_4B_3\cdot A_4B_4}{A_4B_2\cdot A_4B_1}. \end{aligned} \tag{23}$$

However, each one of the three ratios contained on the right hand side of Equation (23) equals 1, due to the well-known invariance of the power of a point exterior to a circle.

A similar approach proves also the concurrency of the main diagonals in the convex hexagon $O_1O_2O_3O_4O_5O_6$, with vertices the circumcenters of the triangles erected, e.g., O_1 the circumcenter of $\triangle A_6B_5B_4$, etc. This is a result of Dao [7].

Referring now to Figure 8 we know by the above that the main diagonals in the convex hexagon $A_1A_2A_3A_4A_5A_6$ are concurrent. Thus, so are the main diagonals of the hexagonal path $A_1A_3A_5A_4A_6A_2$. As a result, Equation (1) holds for this hexagonal path and the nine oriented angles associated to $\triangle A_1A_5A_6$, and so the Sine-Collinearity Theorem applies. However, this yields exactly Pascal's Hexagon Theorem [10] for the cyclic hexagon $B_1B_2B_3B_4B_5B_6$, since (Figure 8), $\overleftrightarrow{A_1A_3} = \overleftrightarrow{B_6B_1}$, $\overleftrightarrow{A_3A_5} = \overleftrightarrow{B_2B_3}$, $\overleftrightarrow{A_5A_1} = \overleftrightarrow{B_4B_5}$, $\overleftrightarrow{A_4A_6} = \overleftrightarrow{B_3B_4}$, $\overleftrightarrow{A_6A_2} = \overleftrightarrow{B_5B_6}$, and $\overleftrightarrow{A_2A_4} = \overleftrightarrow{B_1B_2}$. Of course we could have shortened the argument by applying Desargues' Theorem *a*). □

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