

# Vertex Positions of the Generalized Orthocenter and a Related Elliptic Curve

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**Abstract.** We study triangles  $ABC$  and points  $P$  for which the generalized orthocenter  $H$  corresponding to  $P$  coincides with a vertex. The set of all such points  $P$  is a union of three ellipses minus six points. If  $T_P$  is the affine map taking  $ABC$  to the cevian triangle of  $P$ ,  $P'$  is the isotomic conjugate of  $P$ , and  $K$  is the complement map for  $ABC$ , we also study the affine map  $M_P = T_P \circ K^{-1} \circ T_{P'}$  taking the circumconic of  $ABC$  for  $P$  to the inconic of  $ABC$  for  $P$ . We show that the locus of points  $P$  for which this map is a translation is an elliptic curve minus six points, and show how this locus can be synthetically constructed using the geometry of the triangle.

*Key Words:* generalized orthocenter, circumconic, inconic, affine maps, elliptic curve.

*MSC 2010:* 51A05, 51A20, 51M99, 51N10

## 1. Introduction

In the third of our series of papers on cevian geometry [9], we have studied the properties of the generalized orthocenter  $H$  of a point  $P$  with respect to an ordinary triangle  $ABC$  in the extended Euclidean plane, using synthetic techniques from projective geometry. This generalized orthocenter is defined as follows. Letting  $K$  denote the complement map with respect to  $ABC$  and  $\iota$  the isotomic map (see [1], [6]), the point  $Q = K \circ \iota(P)$  is called the isotomcomplement of  $P$ . Further, let  $D, E, F$  denote the traces of  $P$  on the sides of  $ABC$ . The generalized orthocenter  $H$  is defined to be the unique point  $H$  for which the lines  $HA, HB, HC$  are parallel to  $QD, QE, QF$ , respectively. We showed (synthetically) in [7] that  $H$  is given by the formula

$$H = K^{-1} \circ T_{P'}^{-1} \circ K(Q),$$

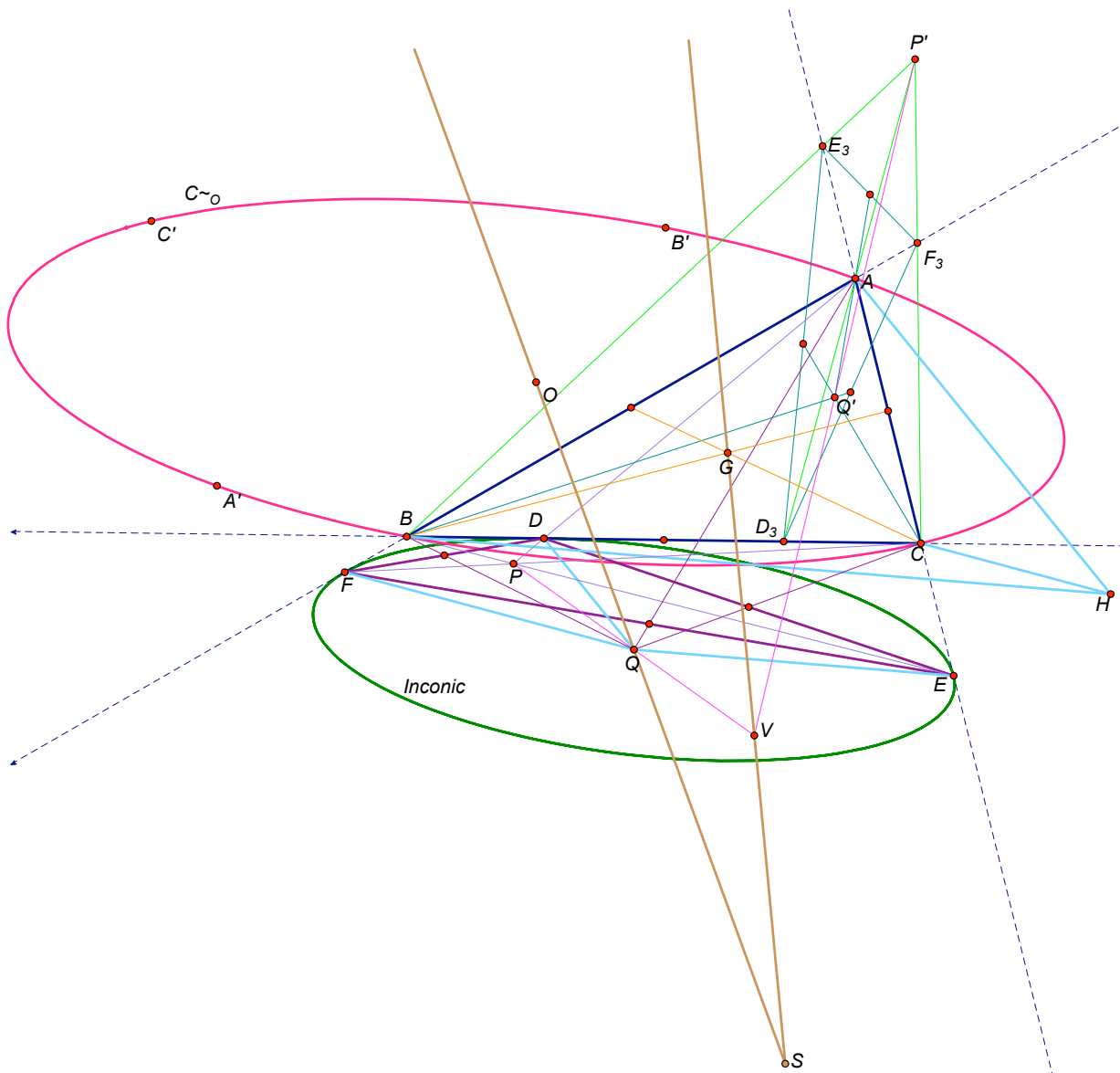


Figure 1: The conics  $\tilde{\mathcal{C}}_O$  (strawberry) and  $\mathcal{I}$  (green).

where  $T_{P'}$  is the unique affine map taking  $ABC$  to the cevian triangle  $D_3E_3F_3$  of the isotomic conjugate  $P' = \iota(P)$  of  $P$ . The related point

$$O = K(H) = T_{P'}^{-1} \circ K(Q)$$

is the generalized circumcenter (for  $P$ ) and is the center of the circumconic  $\tilde{\mathcal{C}}_O = T_{P'}^{-1}(\mathcal{N}_{P'})$ , where  $\mathcal{N}_{P'}$  is the nine-point conic for the quadrangle  $ABCP'$  (see [9, Theorems 2.2 and 2.4] and [2, p. 84]).

We also showed in [9, Theorem 3.4], that if  $T_P$  is the unique affine map taking  $ABC$  to the cevian triangle  $DEF$  of  $P$ , then the affine map  $M = T_P \circ K^{-1} \circ T_{P'}$  is a homothety or translation which maps the circumconic  $\tilde{\mathcal{C}}_O$  to the inconic  $\mathcal{I}$ , defined to be the conic with center  $Q$  which is tangent to the sides of  $ABC$  at the points  $D, E, F$ . In the classical case, when  $P = Ge$  is the Gergonne point of triangle  $ABC$ , the points  $O$  and  $H$  are the usual circumcenter and orthocenter, and the conics  $\tilde{\mathcal{C}}_O$  and  $\mathcal{I}$  are the circumcircle and incircle, respectively. In that case the map  $M$  taking  $\tilde{\mathcal{C}}_O$  to  $\mathcal{I}$  is a homothety, and its center is the

insimilicenter  $S$ . In general, if  $G$  is the centroid of  $ABC$ , and  $Q' = K(P)$ , then the center of the map  $\mathbf{M}$  is the point

$$S = OQ \cdot GV = OQ \cdot O'Q', \quad \text{where } V = PQ \cdot P'Q',$$

and  $O' = T_P^{-1} \circ K(Q')$  is the generalized circumcenter for the point  $P'$  (see Figure 1).

In this paper we first determine synthetically the locus of points  $P$  for which the generalized orthocenter is a vertex of  $ABC$ . This turns out to be the union of three conics minus six points. Excepting the points  $A, B, C$ , these three conics lie inside the Steiner circumellipse  $\iota(l_\infty)$  ( $l_\infty$  is the line at infinity), and each of these conics is tangent to  $\iota(l_\infty)$  at two of the vertices (see Figure 2). We also consider a special case in which  $H$  is a vertex of  $ABC$  and the map  $\mathbf{M}$  is a translation, so that the circumconic  $\tilde{\mathcal{C}}_O$  and the inconic are congruent (see Figures 3 and 4 in Section 2). In Section 3, we determine the locus of all points  $P$  for which  $\mathbf{M}$  is a translation, which is the set of  $P$  for which  $S \in l_\infty$ . We determine necessary and sufficient conditions for this to occur in Theorem 3.1; for example, we show  $\mathbf{M}$  is a translation if and only if the point  $P$  lies on the conic  $\tilde{\mathcal{C}}_O$ . (This situation does not occur in the classical situation, when  $P$  is the Gergonne point, since this point always lies *inside* the circumcircle.) Using barycentric coordinates we show that this locus is an elliptic curve minus six points (see Figure 5). We also show that there are infinitely many points  $P$  in this locus which can be defined over the quadratic field  $\mathbb{Q}(\sqrt{2})$ , i.e., whose barycentric coordinates can be taken to lie in this field. In particular, given two points in this locus, a third point can be constructed using the addition on the elliptic curve. In Section 4 we show how this elliptic curve, minus a set of 12 torsion points, may be constructed as the locus of points  $P = A(P_1)$ , where  $A$  runs over the affine mappings taking inscribed triangles (with a fixed centroid) on a subset  $\mathcal{A}$  of a hyperbola  $\mathcal{C}$  (consisting of six open arcs; see equation (5)) to a fixed triangle  $ABC$ , and where  $P_1$  is a fixed point on the hyperbola (an endpoint of one of the arcs making up  $\mathcal{A}$ ). In another paper [11] we will show that the locus of points  $P$ , for which the map  $\mathbf{M}$  is a half-turn, is also an elliptic curve, which can be synthetically constructed in a similar way using the geometry of the triangle.

We adhere to the notation of [6]–[10]:  $P$  is always a point not on the extended sides of the ordinary triangles  $ABC$  and  $K^{-1}(ABC)$ ;  $D_0E_0F_0 = K(ABC)$  is the medial triangle of  $ABC$ , with  $D_0$  on  $BC$ ,  $E_0$  on  $CA$ ,  $F_0$  on  $AB$  (and the same for further points  $D_i, E_i, F_i$ );  $DEF$  is the cevian triangle associated to  $P$ ;  $D_2E_2F_2$  the cevian triangle for  $Q = K \circ \iota(P) = K(P')$ ;  $D_3E_3F_3$  the cevian triangle for  $P' = \iota(P)$ . As above,  $T_P$  and  $T_{P'}$  are the unique affine maps taking triangle  $ABC$  to  $DEF$  and  $D_3E_3F_3$ , respectively, and  $\lambda = T_{P'} \circ T_P^{-1}$ . See [6] and [8] for the properties of these maps.

We also refer to the papers [6], [8], [9], and [10] as I, II, III, and IV respectively. See [1], [2], [3] for results and definitions in triangle geometry and projective geometry.

## 2. The special case $H = A$ , $O = D_0$

We now consider the set of all points  $P$  such that  $H = A$  and  $O = K(H) = K(A) = D_0$ . We start with a lemma.

**Lemma 2.1.** *Provided the generalized orthocenter  $H$  of  $P$  is defined, the following are equivalent:*

- (a)  $H = A$ .
- (b)  $QE = AF$  and  $QF = AE$ .

(c)  $F_3$  is collinear with  $Q$ ,  $E_0$ , and  $K(E_3)$ .

(d)  $E_3$  is collinear with  $Q$ ,  $F_0$ , and  $K(F_3)$ .

*Proof.* (See Figure 2.) We use the fact that  $K(E_3)$  is the midpoint of segment  $BE$  and  $K(F_3)$  is the midpoint of segment  $CF$  from I, Corollary 2.2. Statement (a) holds iff  $QE \parallel AB$  and  $QF \parallel AC$ , i.e., iff  $AFQE$  is a parallelogram, which is equivalent to (b). Suppose (b) holds. Let  $X = BE \cdot QF_3$ . Then triangles  $BXF_3$  and  $EXQ$  are congruent since  $QE \parallel BF_3 = AB$  and  $QE = AF = BF_3$ . Therefore,  $BX = EX$ , i.e.  $X$  is the midpoint  $K(E_3)$  of  $BE$ , so  $Q, F_3$ , and  $X = K(E_3)$  are collinear. The fact that  $E_0$  is also collinear with these points follows from  $K(BP'E_3) = E_0QK(E_3)$  and the collinearity of  $B, P', E_3$ . Similarly,  $Q, E_3, F_0$ , and  $K(F_3)$  are collinear. This shows (b)  $\Rightarrow$  (c), (d).

Next, we show (c) and (d) are equivalent. Suppose (c) holds. The line  $F_3E_0 = E_0K(E_3) = K(BE_3)$  is the complement of the line  $BE_3$ , hence the two lines are parallel and

$$\frac{AF_3}{F_3B} = \frac{AE_0}{E_0E_3}. \tag{1}$$

Conversely, if this equality holds, then the lines are parallel and  $F_3$  lies on the line through  $K(E_3)$  parallel to  $P'E_3$ , i.e., the line  $K(P'E_3) = QK(E_3)$ , so (c) holds. Similarly, (d) holds if and only if

$$\frac{AE_3}{E_3C} = \frac{AF_0}{F_0F_3}. \tag{2}$$

A little algebra shows that (1) holds if and only if (2) holds. Using signed distances, and setting  $AE_0/E_0E_3 = x$ , we have  $AE_3/E_3C = (x + 1)/(x - 1)$ . Similarly, if  $AF_0/F_0F_3 = y$ ,

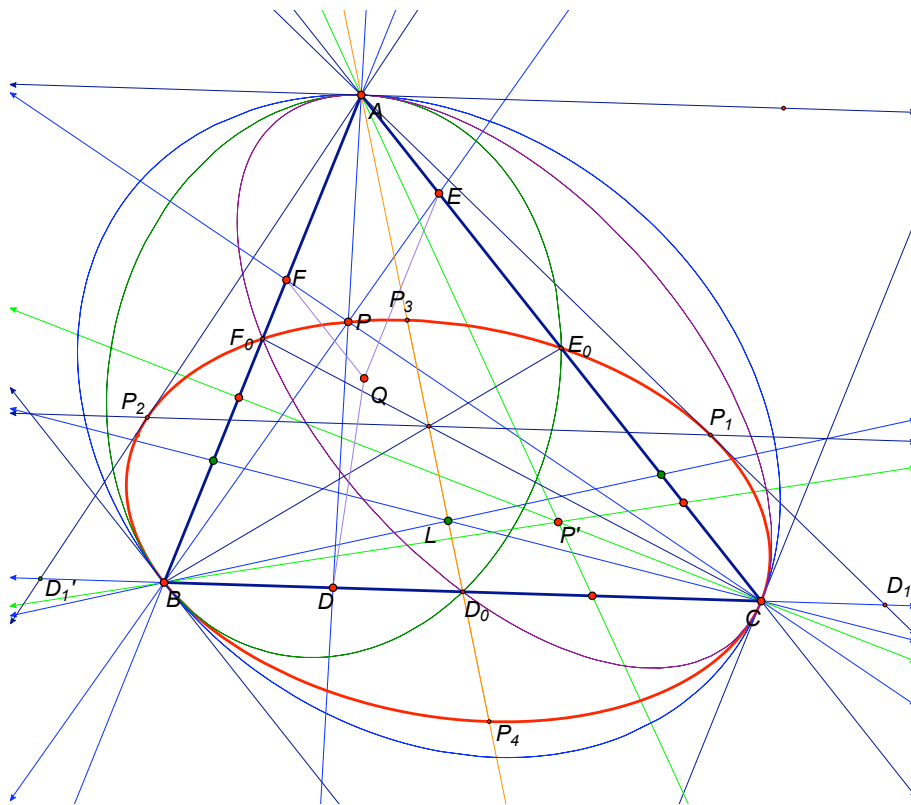


Figure 2: The conics  $\bar{C}_A$  (red),  $\bar{C}_B$  (purple),  $\bar{C}_C$  (green), and  $\iota(l_\infty)$  (blue).

then  $AF_3/F_3B = (y+1)/(y-1)$ . Now (1) is equivalent to  $x = (y+1)/(y-1)$ , which is equivalent to  $y = (x+1)/(x-1)$ , hence also to (2). Thus, (c) is equivalent to (d). Note that this part of the lemma does not use that  $H$  is defined.

Now assume (c) and (d) hold. We will show (b) holds in this case. By the reasoning in the previous paragraph, we have  $F_3Q \parallel E_3P'$  and  $E_3Q \parallel F_3P'$ , so  $F_3P'E_3Q$  is a parallelogram. Therefore,  $F_3Q = P'E_3 = 2 \cdot QK(E_3)$ , so  $F_3K(E_3) = K(E_3)Q$ . This implies the triangles  $F_3K(E_3)B$  and  $QK(E_3)E$  are congruent (SAS), so  $AF = BF_3 = QE$ . Similarly,  $AE = CE_3 = QF$ , so (b) holds.  $\square$

**Theorem 2.2.** *The locus  $\mathcal{L}_A$  of points  $P$  such that  $H = A$  is a subset of the conic  $\bar{\mathcal{C}}_A$  through  $B, C, E_0$ , and  $F_0$ , whose tangent at  $B$  is  $K^{-1}(AC)$  and whose tangent at  $C$  is  $K^{-1}(AB)$ . Namely,  $\mathcal{L}_A = \bar{\mathcal{C}}_A \setminus \{B, C, E_0, F_0\}$ .*

*Proof.* Given  $E$  on  $AC$  we define  $F_3$  as  $F_3 = E_0K(E_3) \cdot AB$ , and  $F$  to be the reflection of  $F_3$  in  $F_0$ . Then we have the following chain of projectivities ( $G$  is the centroid):

$$BE \frown E \frown E_3 \xrightarrow{G} K(E_3) \xrightarrow{E_0} F_3 \frown F \frown CF.$$

Then  $P = BE \cdot CF$  varies on a line or a conic. From the lemma it follows that:

- (a) for a point  $P$  thus defined,  $H = A$ ; and
- (b) if  $H = A$  for some  $P$ , then  $P$  arises in this way, i.e.,  $F_3$  is on  $E_0K(E_3)$ .

Now we list four cases in the above projectivity for which  $H$  is undefined, namely when  $P = B, C, E_0, F_0$ . Let  $A_\infty, B_\infty, C_\infty$  represent the points at infinity on the respective lines  $BC, AC$ , and  $AB$ .

1. For  $E = B_\infty = E_3 = K(E_3)$ , we have  $E_0K(E_3) = AC$  so  $F_3 = A, F = B$ , and  $P = BE \cdot CF = B$ .
2. For  $E = C$ , we have  $E_3 = A, K(E_3) = D_0, E_0K(E_3) = D_0E_0 \parallel AB, F = F_3 = C_\infty$ , so  $P = BE \cdot CF = C$ .
3. For  $E = E_0$ , we have  $E_3 = E_0$  and  $K(E_0)$  is the midpoint of  $BE_0$ , so  $F_3 = B, F = A$ , and  $P = BE \cdot CF = E_0$ .
4. For  $E = A$ , we have  $E_3 = C, K(E_3) = F_0, F_3 = F = F_0$ , and  $P = BE \cdot CF = F_0$ .

Since the four points  $B, C, E_0, F_0$  are not collinear, this shows that the locus of points  $P = BE \cdot CF$  is a conic  $\bar{\mathcal{C}}_A$  through  $B, C, E_0, F_0$ . Moreover, the locus  $\mathcal{L}_A$  of points  $P$  such that  $H = A$  is a subset of  $\bar{\mathcal{C}}_A \setminus \{B, C, E_0, F_0\}$ .

We claim that if  $E$  is any point on line  $AC$  other than  $A, C, E_0$ , or  $B_\infty$ , then  $P$  is a point for which  $H$  is well-defined. First,  $E_3$  is an ordinary point because  $E \neq B_\infty$ . Second, because  $E \neq B_\infty$ , the line  $E_0K(E_3)$  is not a sideline of  $ABC$ . The line  $E_0K(E_3)$  intersects  $AB$  in  $A$  if and only if  $K(E_3)$  lies on  $AC$ , which is true only if  $E_3 = B_\infty$ . The line  $E_0K(E_3)$  intersects  $AB$  in  $B$  iff  $K(E_3)$  is on  $BE_0$ , which holds iff  $E_3$  is on  $K^{-1}(B)B = BE_0$ , and this is the case exactly when  $E = E_3 = E_0$ . Since  $K(E_3)$  lies on  $K(AC) = D_0F_0$ , the line  $E_0K(E_3)$  is parallel to  $AB$  iff  $K(E_3) = D_0$ , giving  $E_3 = A$  and  $E = C$ . Thus, the line  $E_0K(E_3)$  intersects  $AB$  in an ordinary point which is not a vertex, so  $F_3$  and  $F$  are not vertices and  $P = BE \cdot CF$  is a point not on the sides of  $ABC$ .

It remains to show that  $P$  does not lie on the sides of the anticomplementary triangle of  $ABC$ . If  $P$  is on  $K^{-1}(AB)$  then  $F = F_3 = C_\infty$ , which only happens in the excluded case  $E = C$  (see Case 2 above). If  $P$  is on  $K^{-1}(AC)$  then  $E = B_\infty$ , which is also excluded. If  $P$  is on  $K^{-1}(BC)$  then  $P'$  is also on  $K^{-1}(BC)$  so  $Q = K(P')$  is on  $BC$ .

To handle the last case, suppose  $Q$  is on the same side of  $D_0$  as  $C$ . Then  $P'$  is on the opposite side of line  $AD_0$  from  $C$ , so it is clear that  $CP'$  intersects  $AB$  in the point  $F_3$  between  $A$  and  $B$ . If  $Q$  is between  $D_0$  and  $C$ , then  $F_3$  is between  $A$  and  $F_0$  (since  $F_0$ ,  $C$ , and  $G$  are collinear), and it is clear that  $F_3E_0$  can only intersect  $BC$  in a point outside of the segment  $D_0C$ , on the opposite side of  $C$  from  $Q$ . But this is a contradiction, since by construction  $F_3$ ,  $E_0$ , and  $K(E_3)$  are collinear, and  $Q = K(P')$  lies on  $K(BE_3) = E_0K(E_3)$ . On the other hand, if the betweenness relation  $D_0 * C * Q$  holds, then  $F_3$  is between  $B$  and  $F_0$ , and it is clear that  $F_3E_0$  can only intersect  $BC$  on the opposite side of  $B$  from  $C$ . This also applies when  $P' = Q$  is a point on the line at infinity, since then  $F_3 = B$ , and  $B$ ,  $E_0$  and  $Q = A_\infty$  (the point at infinity on  $BC$ ) are not collinear, contradicting part (c) of Lemma 2.1. A symmetric argument applies if  $Q$  is on the same side of  $D_0$  as  $B$ , using the fact that parts (c) and (d) of Lemma 2.1 are equivalent. Thus, no point  $P$  in  $\overline{\mathcal{C}}_A \setminus \{B, C, E_0, F_0\}$  lies on a side of  $ABC$  or its anticomplementary triangle, and the point  $H$  is well-defined; further,  $H = A$  for all of these points.

Finally, by the above argument, there is only one point  $P$  on  $\overline{\mathcal{C}}_A$  that is on the line  $K^{-1}(AB)$ , namely  $C$ , and there is only one point  $P$  on  $\overline{\mathcal{C}}_A$  that is on the line  $K^{-1}(AC)$ , namely  $B$ , so these two lines are tangents to  $\overline{\mathcal{C}}_A$ .  $\square$

This theorem shows that the locus of points  $P$ , for which the generalized orthocenter  $H$  is a vertex of  $ABC$ , is the union of the conics  $\overline{\mathcal{C}}_A \cup \overline{\mathcal{C}}_B \cup \overline{\mathcal{C}}_C$  minus the vertices and midpoints of the sides. The Steiner circumellipse is tangent to the sides of the anticomplementary triangle  $K^{-1}(ABC)$ , so the conic  $\overline{\mathcal{C}}_A$ , for instance, has the double points  $B$ ,  $C$  in common with  $\iota(l_\infty)$ . Since the conic  $\overline{\mathcal{C}}_A$  lies on the midpoints  $E_0$  and  $F_0$ , which lie inside  $\iota(l_\infty)$ , it follows from Bezout's theorem that the set  $\overline{\mathcal{C}}_A \setminus \{B, C\}$  lies entirely in the interior of  $\iota(l_\infty)$ , with similar statements for  $\overline{\mathcal{C}}_B$  and  $\overline{\mathcal{C}}_C$ .

In the next proposition and its corollary, we consider the special case in which  $H = A$  and  $D_3$  is the midpoint of  $AP'$ . We will show that, in this case, the map  $\mathbf{M}$  is a translation (see Figure 4). We first show that this situation occurs.

**Lemma 2.3.** *If the equilateral triangle  $ABC$  has sides of length 2, then there is a point  $P$  with  $AP \cdot BC = D$  and  $d(D_0, D) = \sqrt{2}$ , such that  $D_3$  is the midpoint of the segment  $AP'$  and  $H = A$ .*

*Proof.* (See Figure 3.) We will construct  $P'$  such that  $D_3$  is the midpoint of  $AP'$  and  $H = A$ , and then show that  $P$  satisfies the hypothesis of the lemma. The midpoint  $D_0$  of  $BC$  satisfies  $D_0B = D_0C = 1$  and  $AD_0 = \sqrt{3}$ . Let the triangle be positioned as in Figure 3. Let  $\tilde{A}$  be the reflection of  $A$  in  $D_0$ , and let  $D$  be a point on  $BC$  to the right of  $C$  such that  $D_0D = \sqrt{2}$ . In order to ensure that the reflection  $D_3$  of  $D$  in  $D_0$  is the midpoint of  $AP'$ , take  $P'$  on  $l = K^{-2}(BC)$  with  $P'\tilde{A} = 2\sqrt{2}$  and  $P'$  to the left of  $\tilde{A}$ . Then  $Q = K(P')$  is on  $K^{-1}(BC)$ , to the right of  $A$ , and  $AQ = \sqrt{2}$ . Let  $E_3$  and  $F_3$  be the traces of  $P'$  on  $AC$  and  $BC$ , respectively.

We claim  $BF_3 = \sqrt{2}$ . Let  $M$  be the intersection of  $BC$  and the line through  $F_3$  parallel to  $AD_0$ . Then triangles  $BMF_3$  and  $BD_0A$  are similar, so  $F_3M = \sqrt{3} \cdot MB$ . Let  $N_1$  be the intersection of  $BC$  and the line through  $P'$  parallel to  $AD_0$ . Triangles  $P'N_1C$  and  $F_3MC$  are similar, so

$$\frac{F_3M}{MC} = \frac{P'N_1}{N_1C} = \frac{AD_0}{P'\tilde{A} + 1} = \frac{\sqrt{3}}{2\sqrt{2} + 1}.$$

Therefore,

$$\frac{\sqrt{3}}{2\sqrt{2} + 1} = \frac{F_3M}{MC} = \frac{\sqrt{3} \cdot MB}{MB + 2}$$

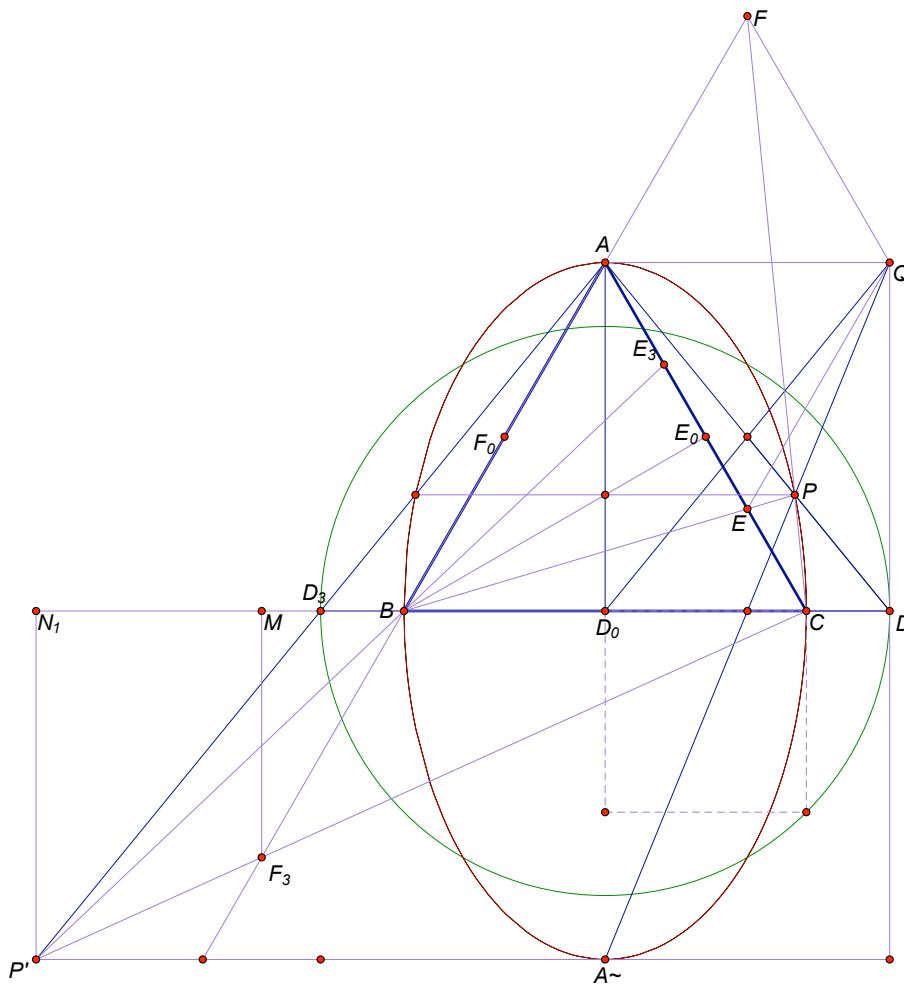


Figure 3: Proof of Lemma 2.3

which yields that  $MB = 1/\sqrt{2}$ . Then  $BF_3 = \sqrt{2}$  is clear from similar triangles.

Now, let  $F$  be the reflection of  $F_3$  in  $F_0$  (the midpoint of  $AB$ ). Then  $AQF$  is an equilateral triangle because  $m(\angle FAQ) = 60^\circ$  and  $AQ \cong BF_3 \cong AF$ , so  $\angle AQF \cong \angle AFQ$ . Therefore,  $QF \parallel AC$ . It follows that the line through  $F_0$  parallel to  $QF$  is parallel to  $AC$ , hence is a midline of triangle  $ABC$  and goes through  $D_0$ . Hence, the point  $O$ , which is the intersection of the lines through  $D_0, E_0, F_0$ , parallel to  $QD, QE, QF$ , respectively, must be  $D_0$ , giving that  $H = K^{-1}(O) = A$ . Clearly,  $P = AD \cdot CF$  is a point outside the triangle  $ABC$ , not lying on an extended side of  $ABC$  or its anticomplementary triangle, which satisfies the conditions of the lemma.  $\square$

The next proposition deals with the general case, and shows that the point  $P$  we constructed in the lemma lies on a line through the centroid  $G$  parallel to  $BC$ . In this proposition and in the rest of the paper, we will use various facts about the center  $Z$  of the cevian conic  $\mathcal{C}_P = ABCPQ$ , which we studied in detail in the papers [8] and [9]. Recall that  $Z$  lies on the nine-point conic  $\mathcal{N}_H$ . We also recall the definition of the affine reflection  $\eta$  from II, p. 27, which fixes the line  $GV$ , with  $V = PQ \cdot P'Q'$ , and moves points parallel to the line  $PP'$ .

**Proposition 2.4.** *Assume that  $H = A$ ,  $O = D_0$ , and  $D_3$  is the midpoint of  $AP'$ . Then the circumconic  $\tilde{\mathcal{C}}_O = \iota(l)$ , where  $l = K^{-1}(AQ) = K^{-2}(BC)$  is the line through the reflection  $\tilde{A}$*

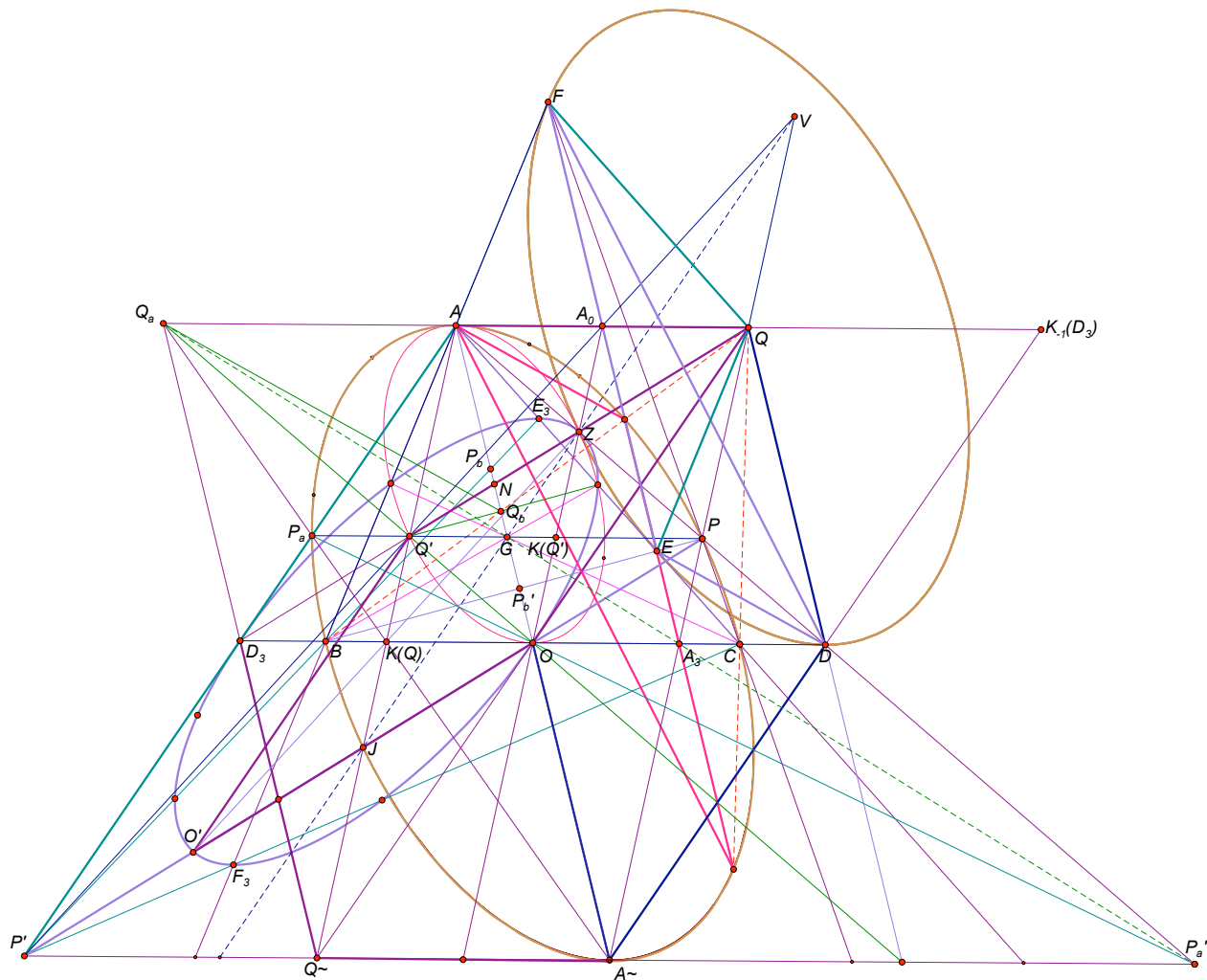


Figure 4: The case  $H = A$ ,  $O = D_0$ , and midpoint of  $AP' = D_3$ .

of  $A$  in  $O$  parallel to the side  $BC$ . The points  $O, O', P, P'$  are collinear, with  $d(O, P') = 3d(O, P)$ , and the map  $M$  taking  $\tilde{C}_O$  to the inconic  $\mathcal{I}$  is a translation. In this situation, the point  $P$  is one of the two points in the intersection  $l_G \cap \tilde{C}_O$ , where  $l_G$  is the line through the centroid  $G$  which is parallel to  $BC$ .

*Proof.* (See Figure 4.) Since the midpoint  $R'_1$  of segment  $AP'$  is  $D_3$ , lying on  $BC$ ,  $P'$  lies on the line  $l$  which is the reflection of  $K^{-1}(BC)$  (lying on  $A$ ) in the line  $BC$ . It is easy to see that this line is  $l = K^{-2}(BC)$ , and hence  $Q = K(P')$  lies on  $K^{-1}(BC)$ . From I, Corollary 2.6 we know that the points  $D_0, R'_1 = D_3$ , and  $K(Q)$  are collinear. Since  $K(Q)$  is the center of the conic  $\mathcal{N}_{P'}$  (the nine-point conic of quadrilateral  $ABCP'$ ; see III, Theorem 2.4), which lies on  $D_0$  and  $D_3$ ,  $K(Q)$  is the midpoint of segment  $D_0D_3$  on  $BC$ . Applying the map  $T_{P'}^{-1}$  gives that  $O = T_{P'}^{-1}(K(Q))$  is the midpoint of  $T_{P'}^{-1}(D_3D_0) = AT_{P'}^{-1}(D_0)$ . It follows that  $T_{P'}^{-1}(D_0) = \tilde{A}$  is the reflection of  $A$  in  $O$ , so that  $\tilde{A} \in \tilde{C}_O$ . Moreover,  $K(A) = O$ , so  $\tilde{A} = K^{-1}(A)$  lies on  $l = K^{-1}(AQ) \parallel BC$ .

Next we show that  $\tilde{C}_O = \iota(l)$ , where the image  $\iota(l)$  of  $l$  under the isotomic map is a circumconic of  $ABC$  (see Lemma 3.4 in [10]). It is easy to see that  $\iota(\tilde{A}) = \tilde{A}$ , since  $\tilde{A} \in AG$  and  $AB\tilde{A}C$  is a parallelogram. Therefore, both conics  $\tilde{C}_O$  and  $\iota(l)$  lie on the four points  $A, B, C, \tilde{A}$ . To show they are the same conic, we show they are both tangent to the line  $l$  at



the point  $\tilde{A}$ . From III, Corollary 3.5 the tangent to  $\tilde{\mathcal{C}}_O$  at  $\tilde{A} = T_{P'}^{-1}(D_0)$  is parallel to  $BC$ , and must therefore be the line  $l$ . To show that  $l$  is tangent to  $\iota(l)$ , let  $L$  be a point on  $l \cap \iota(l)$ . Then  $\iota(L) \in l \cap \iota(l)$ . If  $\iota(L) \neq L$ , this would give three distinct points,  $L, \iota(L)$ , and  $\tilde{A}$ , lying on the intersection  $l \cap \iota(l)$ , which is impossible. Hence,  $\iota(L) = L$ , giving that  $L$  lies on  $AG$  and therefore  $L = \tilde{A}$ . Hence,  $\tilde{A}$  is the only point on  $l \cap \iota(l)$ , and  $l$  is the tangent line. This shows that  $\tilde{\mathcal{C}}_O$  and  $\iota(l)$  share four points and the tangent line at  $\tilde{A}$ , proving that they are indeed the same conic.

From this we conclude that  $P = \iota(P')$  lies on  $\tilde{\mathcal{C}}_O$ . Hence,  $P$  is the fourth point of intersection of the conics  $\tilde{\mathcal{C}}_O$  and  $\mathcal{C}_P = ABCPQ$ . From III, Theorem 3.14 we deduce that  $P = \tilde{Z} = R_O K^{-1}(Z)$ , where  $R_O$  is the half-turn about  $O$ ; and we showed in the proof of that theorem that  $\tilde{Z}$  is a point on the line  $OP'$ . Hence,  $P, O, P'$  are collinear, and applying the affine reflection  $\eta$  gives that  $O' = \eta(O)$  lies on the line  $PP'$ , as well (see III, Theorem 2.4). Now,  $Z$  is the midpoint of  $HP = AP$ , since  $H = K \circ R_O$  is a homothety with center  $H = A$  and similarity factor  $1/2$ . Since  $Z$  lies on  $GV$ , where  $V = PQ \cdot P'Q'$  (II, Prop. 2.3), it is clear that  $P$  and  $Q$  are on the opposite side of the line  $GV$  from  $P', Q'$ , and  $A$ . The relation  $K(\tilde{A}) = A$  means that  $\tilde{A}$  and also  $O$  are on the opposite side of  $GV$  from  $A$  and  $O'$ . Also,  $J = K^{-1}(Z) = R_O(\tilde{Z}) = R_O(P)$  lies on the line  $GV$  and on the conic  $\tilde{\mathcal{C}}_O$ . This implies that  $O$  lies between  $J$  and  $P$ , and applying  $\eta$  shows that  $O'$  lies between  $J$  and  $P'$ . Hence,  $OO'$  is a subsegment of  $PP'$ , whose midpoint is exactly  $J = K^{-1}(Z)$ , since this is the point on  $GV$  collinear with  $O$  and  $O'$ . Now the map  $\eta$  preserves distances along lines parallel to  $PP'$  (see II, p. 27), so  $JO' \cong JO \cong OP \cong O'P'$ , implying that  $OO'$  is half the length of  $PP'$ . Furthermore, segment  $QQ' = K(PP')$  is parallel to  $PP'$  and half as long. Hence,  $OO' \cong QQ'$ , which implies that  $OQQ'O'$  is a parallelogram. Consequently,  $OQ \parallel O'Q'$ , and III, Theorems 3.4 and 3.9 show that  $\mathbf{M}$  is a translation. Thus, the circumconic  $\tilde{\mathcal{C}}_O$  and the inconic  $\mathcal{I}$  are congruent in this situation. This argument implies the distance relation  $d(O, P') = 3d(O, P)$ .

The relation  $O'Q' \parallel OQ$  implies, finally, that  $T_P(O'Q') \parallel T_P(OQ)$ , or  $K(Q')P \parallel A_0Q = AQ$ , since  $O' = T_P^{-1}K(Q')$  from [7, Theorem 6];  $T_P(Q') = P$  from I, Theorem 3.7;  $T_P(O) = T_P(D_0) = A_0$ ; and  $A_0$  is collinear with  $A$  and the fixed point  $Q$  of  $T_P$  by I, Theorem 2.4. Hence,  $PG = PQ' = PK(Q')$  is parallel to  $AQ$  and  $BC$ .  $\square$

There are many interesting relationships in the diagram of Figure 4. We point out several of these relationships in the following corollary.

**Corollary 2.5.** *Assume the hypotheses of Proposition 2.4.*

- a) *If  $Q_a$  is the vertex of the anticevian triangle of  $Q$  (with respect to  $ABC$ ) opposite the point  $A$ , then the corresponding point  $P_a$  is the second point of intersection of the line  $PG$  with  $\tilde{\mathcal{C}}_O$ .*
- b) *The point  $A_3 = T_P(D_3)$  is the midpoint of segment  $OD$  and  $P$  is the centroid of triangle  $ODQ$ .*
- c) *The ratio  $\frac{OD}{OC} = \sqrt{2}$ .*

*Proof.* The anticevian triangle of  $Q$  with respect to  $ABC$  is  $T_{P'}^{-1}(ABC) = Q_a Q_b Q_c$  (see I, Cor. 3.11 and III, Section 2). Since  $D_3$  is the midpoint of  $AP'$ , this gives that  $T_{P'}^{-1}(D_3) = A$  is the midpoint of  $T_{P'}^{-1}(AP') = Q_a Q$ . Therefore,  $Q_a$  lies on the line  $AQ = K^{-1}(BC)$ , so  $P'_a = K^{-1}(Q_a)$  lies on the line  $l$  and is the reflection of  $P'$  in the point  $\tilde{A}$ . Thus, the picture for the point  $P_a$  is obtained from the picture for  $P$  by performing an affine reflection about the line  $AG = A\tilde{A}$  in the direction of the line  $BC$ . This shows that  $P_a$  also lies on the line

$PG \parallel BC$ . The conic  $\tilde{\mathcal{C}}_O$  only depends on  $O$ , so this reflection takes  $\tilde{\mathcal{C}}_O$  to itself. This proves a).

To prove b) we first show that  $P$  lies on the line  $Q\tilde{A}$ . Note that the segment  $K(P'\tilde{A}) = AQ$  is half the length of  $P'\tilde{A}$ , so  $P'\tilde{A} \cong Q_aQ$ . Hence,  $Q_aQ\tilde{A}P'$  is a parallelogram, so  $Q\tilde{A} \cong Q_aP'$ . Suppose that  $Q\tilde{A}$  intersects line  $PP'$  in a point  $X$ . From the fact that  $K(Q)$  is the midpoint of  $D_3D_0$  we know that  $Q$  is the midpoint of  $K^{-1}(D_3)A$ . Also,  $D_3Q'$  lies on the point  $\lambda(A) = \lambda(H) = Q$ , by II, Theorem 3.4(b) and III, Theorem 2.7. It follows that  $K^{-1}(D_3)$ ,  $P = K^{-1}(Q')$ ,  $P' = K^{-1}(Q)$  are collinear and  $K^{-1}(D_3)QX \sim P'\tilde{A}X$ , with similarity ratio  $1/2$ , since  $K^{-1}(D_3)Q$  has half the length of  $P'\tilde{A}$ . Hence  $d(X, K^{-1}(D_3)) = \frac{1}{2}d(X, P')$ . On the other hand,  $d(O, P) = \frac{1}{3}d(O, P')$ , whence it follows, since  $O$  is halfway between  $P'$  and  $K^{-1}(D_3)$  on line  $BC$ , that  $d(P, K^{-1}(D_3)) = \frac{1}{2}d(P, P')$ . Therefore,  $X = P$  and  $P$  lies on  $Q\tilde{A}$ .

Now,  $P = AD_3OQ$  is a parallelogram, since  $K(AP') = OQ$ , so opposite sides in  $AD_3OQ$  are parallel. Hence,  $T_P(P) = DA_3A_0Q$  is a parallelogram, whose side  $A_3A_0 = T_P(D_3D_0)$  lies on the line  $EF = T_P(BC)$ . Applying the dilatation  $H = KR_O$  (with center  $H = A$ ) to the collinear points  $Q, P, \tilde{A}$  shows that  $H(Q), Z$ , and  $O$  are collinear. On the other hand,  $O = D_0, Z$ , and  $A_0$  are collinear by [7], Corollary 5 (since  $Z = R$  is the midpoint of  $AP$ ), and  $A_0$  lies on  $AQ$  by I, Theorem 2.4. This implies that  $A_0 = H(Q) = AQ \cdot OZ$  is the midpoint of segment  $AQ$ , and therefore  $A_3$  is the midpoint of segment  $OD$ . Since  $P$  lies on the line  $PG$ ,  $2/3$  of the way from the vertex  $Q$  of  $ODQ$  to the opposite side  $OD$ , and lies on the median  $QA_3$ , it must be the centroid of  $ODQ$ . This proves b).

To prove c), we apply an affine map taking  $ABC$  to an equilateral triangle. It is clear that such a map preserves all the relationships in Figure 4. Thus we may assume  $ABC$  is an equilateral triangle whose sidelengths are 2. By Lemma 2.3 there is a point  $P$  for which  $AP \cdot BC = D$  with  $D_0D = \sqrt{2}$ ,  $O = D_0$ , and  $D_3$  the midpoint of  $AP'$ . Now Proposition 2.4 implies the result, since the equilateral diagram has to map back to one of the two possible diagrams (Figure 4) for the original triangle.  $\square$

By Proposition 2.4 and III, Theorem 2.5 we know that the conic  $\bar{\mathcal{C}}_A$  lies on the points  $P_1, P_2, P_3, P_4$ , where  $P_1$  and  $P_2 = (P_1)_a$  are the points in the intersection  $\tilde{\mathcal{C}}_O \cap l_G$  described in Corollary 2.5, and  $P_3 = (P_1)_b, P_4 = (P_1)_c$  are the anti-isotomcomplements of the points  $(Q_1)_b, (Q_1)_c$ , since these points all have the same generalized orthocenter  $H = A$  (see Figure 2). It can be shown that the equation of the conic  $\bar{\mathcal{C}}_A$  in terms of the barycentric coordinates of the point  $P = (x, y, z)$  is  $xy + xz + yz = x^2$  (see [7]). Furthermore, the center of  $\bar{\mathcal{C}}_A$  lies on the median  $AG$ ,  $6/7$ -ths of the way from  $A$  to  $D_0$ .

**Remarks.** 1. The polar of  $A$  with respect to the conic  $\bar{\mathcal{C}}_A$  is the line  $l_G$  through  $G$  parallel to  $BC$ . This holds because the quadrangle  $BCE_0F_0$  is inscribed in  $\bar{\mathcal{C}}_A$ , so its diagonal triangle, whose vertices are  $A, G$ , and  $BC \cdot l_\infty$ , is self-polar. Thus, the polar of  $A$  is the line  $l_G$ .

2. The two points  $P$  in the intersection  $\bar{\mathcal{C}}_A \cap l_G$  have tangents which go through  $A$ . This follows from the first remark, since these points lie on the polar  $a = l_G$  of  $A$  with respect to  $\bar{\mathcal{C}}_A$ . As a result, the points  $D$  on  $BC$ , for which there is a point  $P$  on  $AD$  satisfying  $H = A$ , have the property that the ratio of unsigned lengths  $DD_0/D_0C \leq \sqrt{2}$ . This follows from the fact that  $\bar{\mathcal{C}}_A$  is an ellipse: since it is an ellipse for the equilateral triangle, it must be an ellipse for any triangle. Then the maximal ratio  $DD_0/D_0C$  occurs at the tangents to  $\bar{\mathcal{C}}_A$  from  $A$ ; and we showed above that for these two points  $P$ ,  $D = AP \cdot BC$  satisfies  $DD_0/D_0C = \sqrt{2}$ .

### 3. The locus of points $P$ for which $\mathbf{M}$ is a translation

We can characterize the points  $P$ , for which  $\mathbf{M}$  is a translation, as follows. We will have occasion to use the fact that  $\mathbf{M} = T_P \circ K^{-1} \circ T_{P'}$  is symmetric in the points  $P$  and  $P'$ , since  $T_P \circ K^{-1} \circ T_{P'} = T_{P'} \circ K^{-1} \circ T_P$ . This follows easily from the fact that the maps  $T_P \circ K^{-1}$  and  $T_{P'} \circ K^{-1}$  commute with each other (see III, Proposition 3.12 and IV, Lemma 5.2).

**Theorem 3.1.** *Let  $P$  and  $P'$  be ordinary points not on the sides or medians of  $ABC$  or  $K^{-1}(ABC)$ . Then the map  $\mathbf{M} = T_P \circ K^{-1} \circ T_{P'}$  is a translation if and only if any one of the following statements holds.*

1.  $OQQ'O'$  is a parallelogram;
2.  $P$  is on the circumconic  $\tilde{\mathcal{C}}_O$ ;
3.  $O$  and  $O'$  lie on  $PP'$ ;
4.  $Z$  lies on  $QQ'$ ;
5. The signed ratio  $\frac{GZ}{ZV} = \frac{1}{3}$ ;
6.  $U = K^{-1}(Z) = K(V)$ .

*Proof.* It is easy to see that  $\mathbf{M}$  is a translation if and only if  $OQQ'O'$  is a parallelogram, since  $\mathbf{M}(O) = Q$  and  $\mathbf{M}(O') = Q'$  and the center of  $\mathbf{M}$  is the point  $S = OQ \cdot GV = OQ \cdot O'Q'$  (see III, Theorem 3.4 and the proof of III, Theorem 3.9). Thus, we will prove that the statements (2)–(6) are equivalent to (1).

First we prove that (1)  $\Rightarrow$  (4). If  $OQQ'O'$  is a parallelogram, then  $OQ \parallel O'Q'$ . By IV, Proposition 3.10,  $q = OQ$  is the tangent to the conic  $\mathcal{C}_P = ABCPQ$  at  $Q$  and  $q' = O'Q'$  is the tangent to  $\mathcal{C}_P$  at  $Q'$ . It follows that  $q \cdot q'$  is on  $l_\infty$ , which is the polar of the center  $Z$  of  $\mathcal{C}_P$ . Therefore,  $QQ'$  lies on  $Z$ .

Conversely, assume (4). Then  $Z \in QQ'$  implies that  $q \cdot q'$  lies on  $l_\infty$ , so  $OQ \parallel O'Q'$ , giving that  $S \in l_\infty$  and  $\mathbf{M}$  is a translation. Hence, (4)  $\Rightarrow$  (1). Furthermore, (1)  $\Rightarrow$  (3), as follows.  $\mathbf{M}$  is a translation so  $QM(Q) \cong OM(O) = OQ$ , i.e.,  $Q$  is the midpoint of  $OM(Q)$ , where  $\mathbf{M}(Q) = T_{P'} \circ K^{-1} \circ T_P(Q) = T_{P'}(P')$ . But  $Q$  is also the midpoint of  $PV$ , so triangles  $PQO$  and  $VQM(Q)$  are congruent, giving  $\mathbf{M}(Q)V \parallel OP$ . We know  $\mathbf{M}(Q)V = K^{-1}(PP')$  by II, Proposition 2.3(f) and IV, Theorem 3.11(7.). Since the line through  $P$  parallel to  $K^{-1}(PP')$  is  $PP'$ ,  $O$  lies on  $PP'$ . Hence,  $\eta(O) = O'$  also lies on  $PP'$ , giving (3).

Now (4) holds if and only if  $Z$  is the midpoint of  $QQ'$  ( $Z$  lies on  $GV$ , the fixed line of  $\eta$ , and  $\eta(Q) = Q'$ ). The point  $V$  is the midpoint of segment  $K^{-1}(PP')$  (II, Proposition 2.3), so  $K(V)$  is the midpoint of segment  $PP'$  and  $K^2(V)$  is the midpoint of  $K(PP') = QQ'$ . Hence, (4) holds if and only if  $K^2(V) = Z$ , which holds if and only if  $K^{-1}(Z) = K(V)$ . Thus, (4)  $\iff$  (6). This allows us to show (4)  $\Rightarrow$  (2), as follows. Since (4) also implies (1) and (3), we have that  $OO' = QQ' = \frac{1}{2}PP'$ . Also,  $K(V)$  on  $GV$  is the midpoint of  $PP'$  and  $OO'$ . Since  $OQ \parallel GV$  and  $QP'$  intersects  $GV$  at  $G$ , it is clear that  $O$  and  $P$  lie on the same side of line  $GV$ . Hence,  $O$  must be the midpoint of  $PK(V)$  (the dilation with center  $U = K(V)$  takes  $OO'$  to  $PP'$ ). Now  $K(V) = K^{-1}(Z) \in \tilde{\mathcal{C}}_O$ , since  $Z \in \mathcal{N}_H = K(\tilde{\mathcal{C}}_O)$ ; so  $P = \mathbf{R}_O(K^{-1}(Z))$  lies on  $\tilde{\mathcal{C}}_O$ , hence (2). Thus (4)  $\Rightarrow$  (2).

We next show that (2)  $\Rightarrow$  (3). Assume that  $P$  lies on  $\tilde{\mathcal{C}}_O$ . Then  $P$  is the fourth point of intersection of the conics  $\tilde{\mathcal{C}}_O$  and  $\mathcal{C}_P = ABCPQ$ . From III, Theorem 3.14 we deduce that  $P = \tilde{Z} = \mathbf{R}_O \circ K^{-1}(Z)$ . Furthermore,  $\tilde{Z}$  lies on  $OP'$ . Hence,  $P, O, P'$  are collinear, and applying the affine reflection  $\eta$  gives that  $O'$  lies on the line  $PP'$ , as well.

Now suppose that (3) holds, so that  $O$  lies on  $PP'$ . From II, Corollary 2.2 we know that  $T_P(P')$  lies on  $PP'$ , so that IV, Theorem 3.11 implies that  $O = OQ \cdot PP' = T_P(P')$ . Let  $\tilde{H} = T_P^{-1}(H) = T_{P'}^{-1}(Q)$ , as in III, Theorem 2.10. Note that

$$M(\tilde{H}) = T_P \circ K^{-1} \circ T_{P'}(\tilde{H}) = T_P \circ K^{-1}(Q) = T_P(P') = O.$$

Hence,  $M(\tilde{H}O) = OQ$ . Part III, Lemma 3.8 says that  $O$  is the midpoint of  $\tilde{H}Q$ , so  $M(\tilde{H}O) \cong \tilde{H}O$ . (Note that  $\tilde{H} = T_{P'}^{-1}(Q) \neq T_{P'}^{-1} \circ K(Q) = O$ .) The result of III, Theorem 3.4 says that  $M$  is a homothety or translation. A homothety expands or contracts all segments on lines by the same factor  $k$ , so if  $M$  were not a translation, the factor  $k = \pm 1$ . But  $k \neq 1$  since  $M$  is not the identity map and  $k \neq -1$  since it preserves the orientation of the segment  $\tilde{H}O$  on the line  $OQ$ . Hence,  $M$  must be a translation. This proves (3)  $\Rightarrow$  (1), and therefore (1)  $\Rightarrow$  (4)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (1).

Furthermore, (5)  $\iff$  (6). If  $U = K^{-1}(Z) = K(V)$ , then taking signed distances yields  $GZ = GK(U) = \frac{1}{2}UG = \frac{1}{4}GV$ , which gives that  $\frac{GZ}{ZV} = \frac{1}{3}$ . Conversely, if  $\frac{GZ}{ZV} = \frac{1}{3}$ , then  $\frac{GZ}{GV} = \frac{1}{4}$ , which implies  $Z = K^2(V)$ , since  $K^2(V) = K(K(V))$  is the unique point  $X$  on  $GV$  for which the signed ratio  $\frac{GX}{XV} = \frac{1}{4}$ . Thus, (5)  $\iff$  (6)  $\iff$  (4) (from above).

This completes the proof that (1)–(6) are equivalent. □

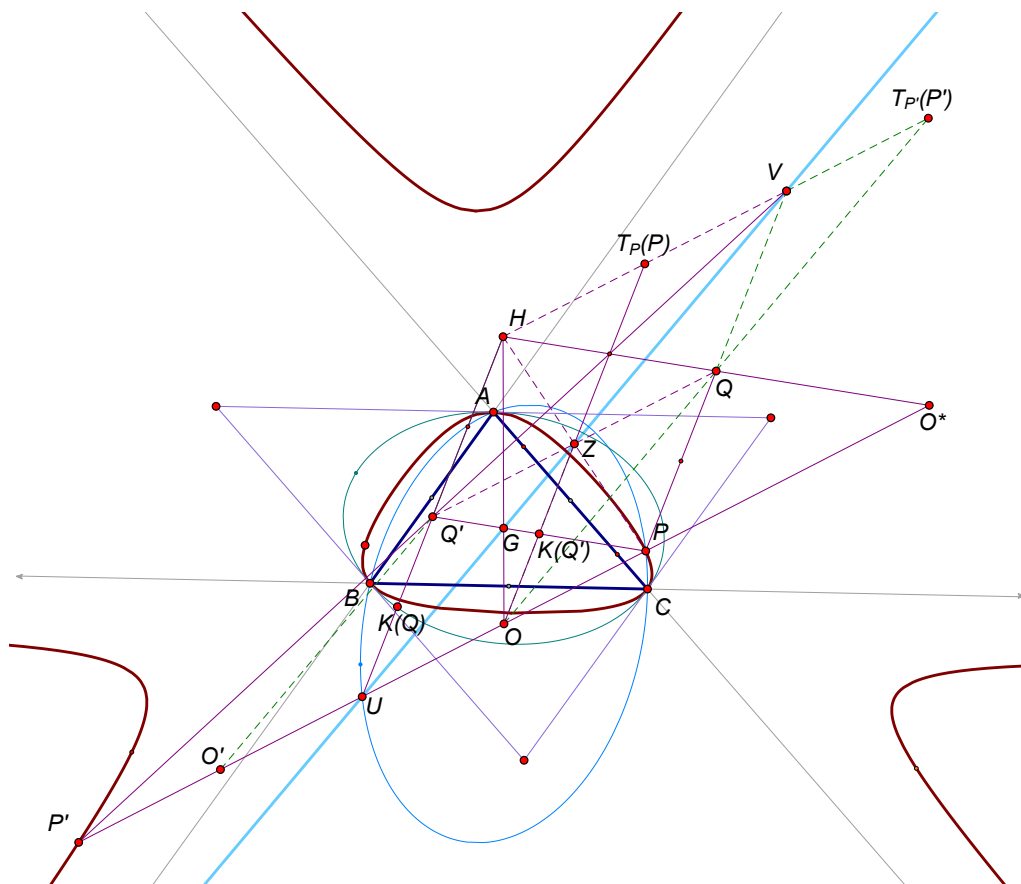


Figure 5: Locus of  $M$  a translation (curve  $\mathcal{E}_S$  in brown) with Steiner circumellipse (teal) and  $\tilde{C}_O$  (blue).

**Corollary 3.2.** *Under the hypotheses of Theorem 3.1, if  $M$  is a translation:*

1.  $HK^{-1}(Z)PV$  is a parallelogram;
2.  $T_P(P)$  is the midpoint of segment  $HV$ ;
3.  $T_P(P') = O$ ;
4. The points  $P', O', U = K^{-1}(Z), O, P$  are equally spaced on line  $PP'$ ;
5.  $OH$  is tangent to the conic  $\mathcal{C}_P = ABCPQ$  at  $H$ .

*Proof.* (See Figure 5.) Statements (3) and (4) were proved in the course of proving (3)  $\Rightarrow$  (1) and (4)  $\Rightarrow$  (2) above. With  $U = K^{-1}(Z) = K(V)$ , (4) gives that that  $UO = \frac{1}{2}UP$ , so  $UO \cong K(UP) = ZQ'$ . This shows that  $UO \parallel ZQ'$  and  $OZQ'U$  is a parallelogram. Thus,  $K^{-1}(OZQ'U) = HUPV$  is also a parallelogram. In particular,  $QQ' \parallel PP' \parallel HV$ . Since  $M(U) = M \circ K^{-1}(Z) = Z$  (by the Generalized Feuerbach Theorem in III) and  $Z$  is the midpoint of segment  $UV$ , the translation  $M$  maps parallelogram  $OZQ'U$  to  $QVM(Q')Z$ , where  $M(Q') = T_P \circ K^{-1} \circ T_{P'}(Q') = T_P(P)$ . As  $Z$  is the center of parallelogram  $HUPV$  and  $O$  is the midpoint of the side  $UP$ , while  $Q$  is the midpoint of side  $PV$ , it follows that  $Q'$  is the midpoint of side  $HU$  and  $M(Q') = T_P(P) = OZ \cdot HV$  is the midpoint of  $HV$ , proving statement (2). Finally, let  $O^* = PP' \cdot QH$ . Triangles  $PO^*Q$  and  $UO^*H$  are similar ( $UH \parallel PV = PQ$ ) and  $PQ = \frac{1}{2}UH$ , so  $PQ$  is the midline of triangle  $UO^*H$  and  $PO^* = UP = \frac{1}{2}PP'$ . This implies  $\frac{PO^*}{O^*P'} = -\frac{1}{3} = -\frac{PO}{OP'}$ , and therefore  $O^*$  is the harmonic conjugate of  $O$  with respect to  $P$  and  $P'$ . Thus,  $O^*$  is conjugate to  $O$  with respect to the polarity induced by  $\mathcal{C}_P$ . As in the proof of the theorem,  $q = OQ$  is tangent to  $\mathcal{C}_P$  at  $Q$ , so  $Q$  is also conjugate to  $O$ . Thus, the polar of  $O$  is  $o = QO^* = QH$  and since  $H$  and  $Q$  are on  $\mathcal{C}_P$  (III, Theorem 2.8), this implies that  $OH$  is the tangent to  $\mathcal{C}_P$  at  $H$ . This proves (5).  $\square$

**Remark.** The statements in Corollary 3.2 are actually all equivalent to the map  $M$  being a translation. For the sake of brevity, we leave this verification to the interested reader.

The set of points  $P = (x, y, z)$ , for which  $M$  is a translation, can be determined using barycentric coordinates. This is just the set of points for which  $S \in l_\infty$ . It can be shown (see [5, eq. (8.1)]) that homogeneous barycentric coordinates of the point  $S$  are

$$S = (x(y+z)^2, y(x+z)^2, z(x+y)^2),$$

where  $P = (x, y, z)$ . Thus, the locus in question has the projective equation

$$\mathcal{E}_S: x(y+z)^2 + y(x+z)^2 + z(x+y)^2 = 0.$$

Setting  $z = 1 - x - y$ , so that  $(x, y, z)$  are absolute barycentric coordinates, the set of ordinary points for which  $M$  is a translation has the affine equation

$$(3x+1)y^2 + (3x+1)(x-1)y + x^2 - x = 0. \quad (3)$$

Since the discriminant of this equation, with respect to  $y$ , is  $D = (x-1)(3x+1)(3x^2-6x-1)$ , this curve is birationally equivalent to

$$Y^2 = (X-1)(3X+1)(3X^2-6X-1).$$

Using  $X = \frac{u-1}{u+3}$ ,  $Y = \frac{8v}{(u+3)^2}$ , this equation can be written in the form

$$\mathcal{E}'_S: v^2 = u(u^2 + 6u - 3),$$

which is an elliptic curve with  $j$ -invariant  $j = 54000 = 2^4 3^3 5^3$  and infinitely many points defined over real quadratic fields. Thus, we see that there are infinitely many points for which  $\mathbf{M}$  is a translation. Note that  $\mathcal{E}'_S$  is isomorphic to the curve (36A2) in Cremona's tables [4] (via the substitution  $u = x - 2$ ,  $v = y$ ). Consequently,  $\mathcal{E}'_S$  has the torsion points  $T = \{\tilde{O}, (0, 0), (1, \pm 2), (-3, \pm 6)\}$  ( $\tilde{O}$  is the base point) and rank  $r = 0$  over  $\mathbb{Q}$ . These six points correspond to the vertices of triangle  $ABC$  and the infinite points on its sides; the latter points are  $(0, 1, -1)$ ,  $(1, 0, -1)$ ,  $(1, -1, 0)$ .

It is not hard to calculate, using the equation (3) and the equation  $xy + xz + yz = x^2$  for  $\bar{\mathcal{C}}_A$  that the intersection  $\mathcal{E}_S \cap \bar{\mathcal{C}}_A$  consists of the points  $B = (0, 1, 0)$  and  $C = (0, 0, 1)$ , with intersection multiplicity 2 at both points, together with the points

$$P = \left( \frac{1}{3}, \frac{1 + \sqrt{2}}{3}, \frac{1 - \sqrt{2}}{3} \right) \quad \text{and} \quad P_a = \left( \frac{1}{3}, \frac{1 - \sqrt{2}}{3}, \frac{1 + \sqrt{2}}{3} \right), \quad (4)$$

where  $P$  and  $P_a$  are the points pictured in Figure 4. (These points are labeled  $P_1$  and  $P_2$  in Figure 2.) That these are the correct points follows from the fact that

$$P - G = P - \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) = \frac{\sqrt{2}}{3} (0, 1, -1),$$

and therefore  $PG \parallel BC$ . The affine coordinates of  $P$  on (3) are  $(x, y) = \left( \frac{1}{3}, \frac{1 + \sqrt{2}}{3} \right)$ , which corresponds to the point  $\tilde{P} = (u, v) = (3, 6\sqrt{2})$  on  $\mathcal{E}'_S$ . The double of the latter point is [2]  $\tilde{P} = \left( \frac{1}{2}, \frac{\sqrt{2}}{4} \right)$ , and [4]  $\tilde{P} = \left( \frac{169}{8}, -\frac{2483\sqrt{2}}{32} \right)$ . Using [12, Theorem VII.3.4, p. 193], with  $p = 2$  over the local field  $K = \mathbb{Q}_2(\sqrt{2})$ , the coordinates of the last point show that  $\tilde{P}$  is a point of infinite order on  $\mathcal{E}'_S$ , and therefore  $P$  is a point of infinite order on (3). Hence, there are infinitely many points on  $\mathcal{E}_S$  which have coordinates in the field  $\mathbb{Q}(\sqrt{2})$ .

It follows from this calculation that the only points, other than the vertices of  $ABC$ , in the intersection  $\mathcal{E}_S \cap \mathcal{L}$  of  $\mathcal{E}_S$  and the locus  $\mathcal{L} = \mathcal{L}_A \cup \mathcal{L}_B \cup \mathcal{L}_C$  of Section 2 are the 6 points obtained by permuting the coordinates of the point  $P$  in (4). There are, however, 6 more important points on the curve  $\mathcal{E}_S$ . These are the intersections of  $\mathcal{E}_S$  with the medians of triangle  $ABC$ , which are found by setting two variables equal to each other in the equation for  $\mathcal{E}_S$ . This yields the following six points on  $\mathcal{E}_S$ , paired with their isotomic conjugates:

$$\begin{aligned} P_1 &= (1, -2 + \sqrt{3}, -2 + \sqrt{3}), & P'_1 &= (1, -2 - \sqrt{3}, -2 - \sqrt{3}) \\ P_2 &= (-2 + \sqrt{3}, 1, -2 + \sqrt{3}), & P'_2 &= (-2 - \sqrt{3}, 1, -2 - \sqrt{3}) \\ P_3 &= (-2 + \sqrt{3}, -2 + \sqrt{3}, 1), & P'_3 &= (-2 - \sqrt{3}, -2 - \sqrt{3}, 1). \end{aligned}$$

These correspond to the six points

$$(x, y) = \left( 1 \pm \frac{2}{3}\sqrt{3}, \mp \frac{\sqrt{3}}{3} \right), \quad \left( \pm \frac{\sqrt{3}}{3}, 1 \mp \frac{2}{3}\sqrt{3} \right), \quad \left( \pm \frac{\sqrt{3}}{3}, \pm \frac{\sqrt{3}}{3} \right)$$

on (3); and to the six points

$$(u, v) = (-3 \pm 2\sqrt{3}, 0), \quad (3 \pm 2\sqrt{3}, 12 \pm 6\sqrt{3}), \quad (3 \pm 2\sqrt{3}, -12 \mp 6\sqrt{3})$$

on  $\mathcal{E}'_S$ . Together with the points in  $T$ , these points form a torsion group  $T_{12}$  of order 12 defined over  $\mathbb{Q}(\sqrt{3})$ , with  $T_{12} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ . The points in  $T_{12}$  are the points which are excluded in Theorem 3.1 and Corollary 3.2. In particular, there are only two excluded points on each median, for which  $\mathbf{M}$  is a translation.

## 4. Constructing the elliptic curve

In this section we will use the results of the previous section to give a geometric construction of the elliptic curve  $\mathcal{E}_S$ . We start with the following lemma.

**Lemma 4.1.** *Assume that  $P$  is a point for which the map  $\mathbf{M}$  is a translation. Then the line  $GZ = GV$  does not intersect the conic  $\mathcal{C}_P$ , which is a hyperbola.*

*Proof.* We will use the characterization of  $\mathcal{C}_P$  as the set of points  $Y$  for which  $P$ ,  $Y$ , and  $T_P(Y)$  are collinear (II, Corollary 2.2).

Let  $Y$  be a point on  $GV$  and  $Y' = PT_P(Y) \cdot GV$  the projection of  $Y_P = T_P(Y)$  onto  $GV$  from  $P$ . The mapping  $Y \rightarrow Y_P$  is projective, since  $T_P$  is an affine map, so the mapping  $\pi: Y \frown Y'$  is a projectivity from  $GV$  to itself. We will show that  $\pi$  has no invariant points. This will imply the lemma, since if  $Y \in \mathcal{C}_P$ , then  $Y$  lies on  $PT_P(Y)$ , implying that  $Y = Y'$ .

We will show that the projectivity  $\pi$  has order 3 by showing that  $\pi$  coincides with the projectivity  $UZV \frown ZVU$  on  $GV$ . First,  $\pi(U) = Z$ , because  $T_P(U) = T_P(K^{-1}(Z)) = Z$  is already on  $GV$ . Also, since  $Z$  is the midpoint of  $QQ'$ ,  $T_P(Z)$  is the midpoint of  $T_P(QQ') = QP$ . This implies that  $\pi(Z) = QP \cdot GV = V$ . Now,  $T_P(V)$  is the intersection of  $T_P(PQ) = QT_P(P)$  and  $T_P(P'Q') = OP = PP'$  by Corollary 3.2. Hence,  $\pi(V) = PT_P(V) \cdot GV = PP' \cdot GV = U$ .

Since  $\pi$  has order 3, it cannot have any invariant points (see [2, p. 43] or [3, p. 35, Exercises]). Finally, since  $GV$  lies on the center  $Z$  of  $\mathcal{C}_P$ , but does not intersect  $\mathcal{C}_P$ , the conic must be a hyperbola. This completes the proof.  $\square$

Thus, the line  $GV$  is an exterior line of  $\mathcal{C}_P$  ([2, p. 72]), so its pole  $V_\infty$  is an interior point, which implies that the line  $GV_\infty \parallel PP'$  is a secant for the conic  $\mathcal{C}_P$ , and therefore meets  $\mathcal{C}_P$  in two points  $E$  and  $F$ . (These are different points from the similarly named points in Figure 4.) Hence, as  $\eta$  fixes the line  $EF$  and maps the conic  $\mathcal{C}_P$  to itself (II, p. 27), we have  $\eta(E) = F$  and  $G$  on  $GV$  is the midpoint of segment  $EF$ . But  $EF = GV_\infty$  is the polar of  $V$  with respect to  $\mathcal{C}_P$ , so  $VE$  and  $VF$  are tangent to  $\mathcal{C}_P$  at  $E$  and  $F$ , respectively. We choose notation so that  $E$  is the intersection of  $GV_\infty$  with the branch of the hyperbola through  $P'$  and  $Q'$ , which exists since  $P'$  and  $Q'$  are on the same side of the line  $GV$ .

**Proposition 4.2.** *Assume  $P$  is a point for which  $\mathbf{M}$  is a translation. If  $E'$  and  $F'$  are the midpoints of segments  $EG$  and  $GF$ , then the lines  $ZE'$  and  $ZF'$  are the asymptotes of  $\mathcal{C}_P$ .*

*Proof.* (See Figure 6.) We know  $Z = K(U)$  is the midpoint of segment  $UV$  and the center of  $\mathcal{C}_P$ . If we rotate the tangents  $EV$  and  $FV$  by a half-turn about  $Z$ , we obtain two lines  $E''U$  and  $F''U$  through  $U$  which are also tangent at points  $E''$  and  $F''$  respectively. In particular,  $Z$  is the midpoint of segments  $EE''$ ,  $FF''$ , and  $UV$ . This implies  $F''V \parallel UF$  and  $Z$  is the center of the parallelogram  $VF''UF$ . Now define  $E'$  to be the midpoint of  $UF''$ . The line  $ZE'$  is halfway between  $F''V$  and  $UF$ , hence  $ZE' \parallel F''V$ .

Next we show that  $EF''G'G$  is a parallelogram, where  $G' = R_Z(G)$ . Now,  $Z$  is the midpoint of  $EE''$  and  $FF''$ , so  $EF''E''F$  is a parallelogram. Since  $G$  is the midpoint of  $EF$ ,  $G'$  is the midpoint of  $E''F''$ . This implies  $F''G' \cong EG$ , which proves  $EF''G'G$  is a parallelogram. Also, using part (6.) of Theorem 3.1 it is easy to see that  $G$  is the midpoint of  $UG'$  so  $UG \cong GG' \cong EF''$  and  $EF''GU$  is a parallelogram, with center  $E'$ . This verifies that  $E'$  is the midpoint of segment  $EG$ .

But  $E'$  on  $EG = v$  implies  $V$  lies on its polar  $e'$ . Also,  $E'$  is on  $UF'' = f''$ , so  $F''$  lies on  $e'$ . Together, this implies  $e' = F''V$ , so from the first paragraph of the proof,  $e' \parallel ZE'$ .

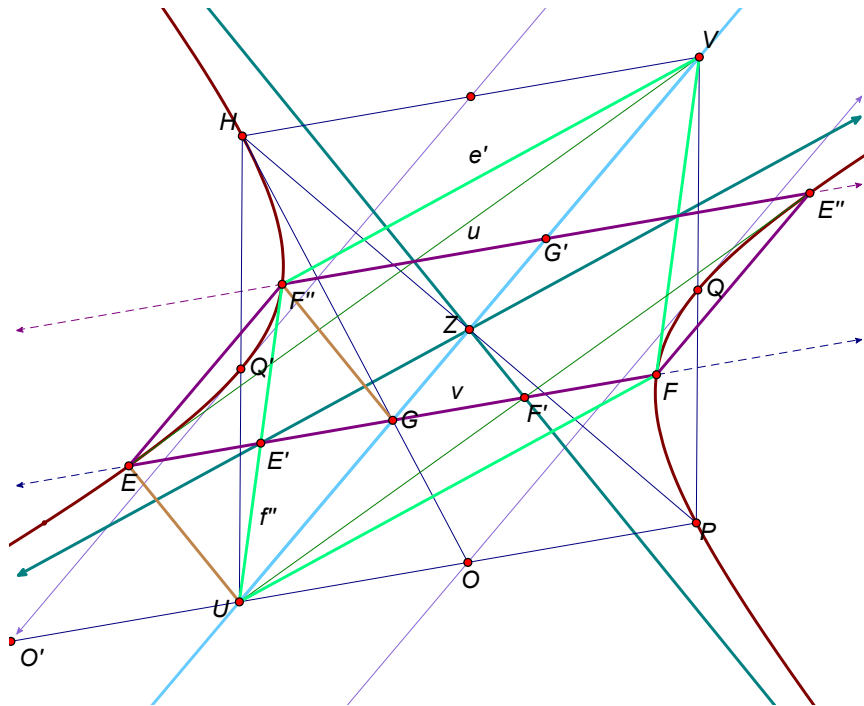


Figure 6: The parallelogram  $HVPV$  and conic  $\mathcal{C}_P$ .

Hence,  $e'$ ,  $ZE'$ , and  $l_\infty$  are concurrent. The dual of this statement says that  $E'$ ,  $l_\infty \cdot e'$ , and  $Z$  are collinear. Thus, the infinite point  $l_\infty \cdot e'$  lies on  $ZE'$ , which is its own polar! Hence,  $l_\infty \cdot e'$  must lie on the conic and  $ZE'$  must be an asymptote. Applying the map  $\eta$  shows that  $ZF' = \eta(ZE')$  is also an asymptote.  $\square$

We now consider a fixed configuration of points, as in Figure 6, consisting of the parallelogram  $HVPV$ , its center  $Z$ , the point  $O$  which is the midpoint of side  $UP$ , the point  $G = UV \cdot HO$ , the midpoints  $Q, Q'$  of opposite sides  $HU$  and  $PV$ , and the points  $O', P'$  which are the affine reflections of the points  $O, P$  through the line  $UV = GZ$  in the direction of the line  $UP$ , together with the conic  $\mathcal{C} = PQHQ'P'$ . By Theorem 3.1 and Corollary 3.2 this configuration arises from a triangle  $ABC$  and the point  $P$  (not on the sides of  $ABC$  or  $K^{-1}(ABC)$ ), for which the map  $M$  is a translation, and such a configuration certainly exists because it can be taken to be the image under an affine map of the configuration constructed in Lemma 2.3 and Proposition 2.4. For this configuration the conclusions of Lemma 4.1 and Proposition 4.2 hold, so that  $\mathcal{C}$  is a hyperbola. Our focus now is on finding all triangles  $A_1B_1C_1$  inscribed in the conic  $\mathcal{C} = \mathcal{C}_P$  for which the map  $M_P$  corresponding to  $A_1B_1C_1$  is a translation. This will lead us to a synthetic construction of the elliptic curve  $\mathcal{E}_S$  discussed in Section 3.

Let  $A_1$  be any point on the conic  $\mathcal{C} = PQHQ'P'$ , and define  $D_0 = K(A_1)$ , where  $K$  is the dilation about  $G$  with signed ratio  $-1/2$ . Further, let  $\mathcal{C}(A_1)$  be the reflection of the conic  $\mathcal{C}$  in the point  $D_0$ . If the conics  $\mathcal{C}(A_1)$  and  $\mathcal{C}$  intersect in two points  $B_1, C_1$ , then  $A_1B_1C_1$  is the unique triangle with vertex  $A_1$  and centroid  $G$  which is inscribed in  $\mathcal{C}$ . This is because  $D_0$  must be the midpoint of side  $B_1C_1$  in any such triangle, and lying on  $\mathcal{C}$ ,  $B_1$  and  $C_1$  must both lie on  $\mathcal{C}(A_1)$ . Since  $\mathcal{C}(A_1)$  is the reflection of  $\mathcal{C}$  in  $D_0$ , its asymptotes  $c = R_{D_0}(a)$  and  $d = R_{D_0}(b)$  are parallel to the respective asymptotes  $a$  and  $b$  of  $\mathcal{C}$ . It follows that  $\mathcal{C} \cap \mathcal{C}(A_1)$  can consist of at most two points other than the infinite points on the asymptotes.



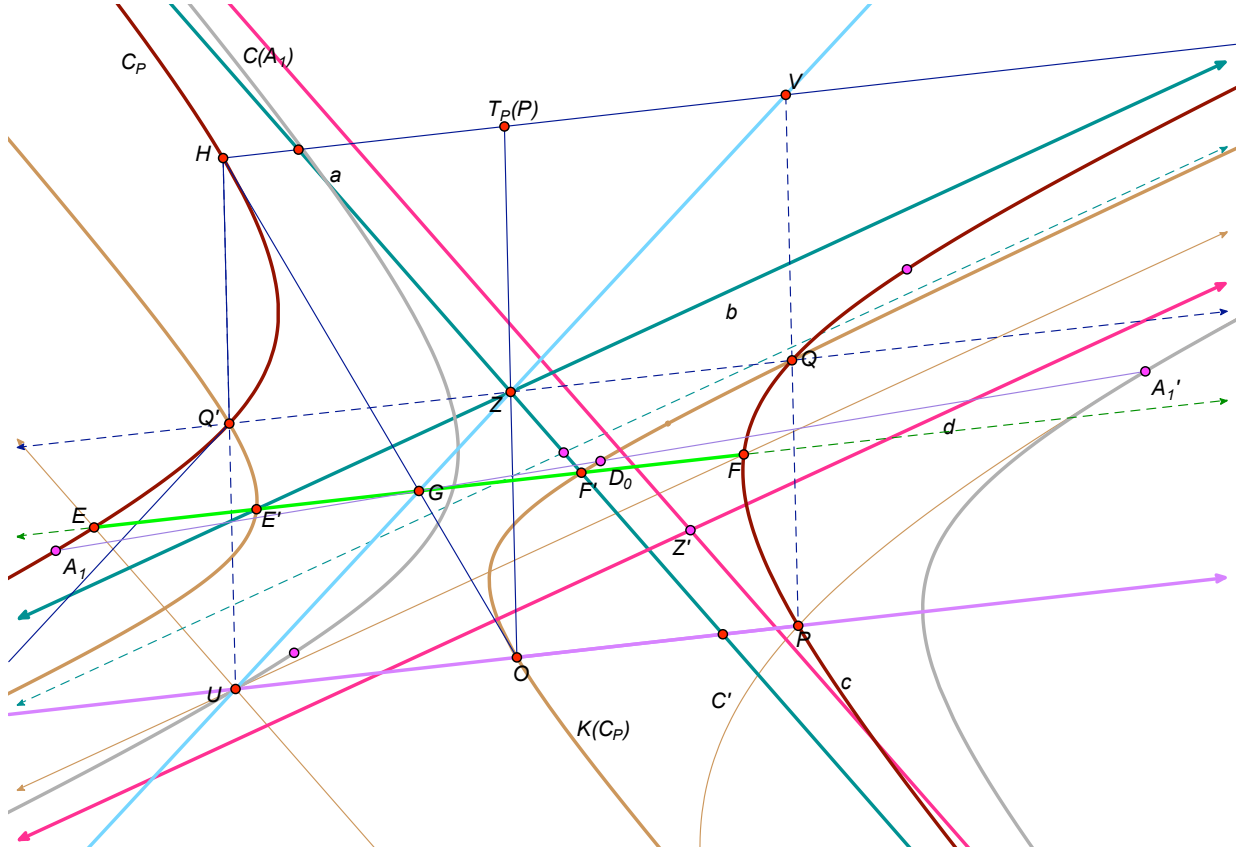


Figure 7: Conics  $\mathcal{C}$  (brown),  $K(\mathcal{C}_P)$  (tan),  $\mathcal{C}(A_1)$  (grey),  $\mathcal{C}'$  (light tan), with  $A_1$  on the arc  $\mathcal{E}$ .

**Lemma 4.3.** *The conics  $\mathcal{C}(A_1)$  and  $\mathcal{C} = PQHQ'P'$  intersect in two ordinary points if and only if  $A_1$  does not lie on either of the closed arcs of  $\mathcal{C}$  between the lines  $EF = GV_\infty$  and  $P'P$ .*

*Proof.* It suffices to prove the lemma for the configuration pictured in Figures 7 and 8, since any two configurations for which  $M$  is a translation are related by an affine map. In particular, Corollary 3.2 shows that one configuration can be mapped to any other by an affine map taking the parallelogram  $HUPV$  for the one configuration to the corresponding parallelogram for the other.

Let  $c = R_{D_0}(a)$  and  $d = R_{D_0}(b)$  be the asymptotes of  $\mathcal{C}(A_1)$ , where  $a = ZF'$  and  $b = ZE'$  are the asymptotes of  $\mathcal{C}$ . When  $A_1 = E$ , then  $D_0 = K(E) = F'$  lying on  $a = ZF'$ , so the lines  $a, c$  coincide. Then  $\mathcal{C}$  and  $\mathcal{C}(A_1)$  have the common tangent  $a = c$ , so they intersect with multiplicity at least 2 at  $a \cdot l_\infty$ . They also intersect with multiplicity 1 at  $b \cdot l_\infty$ , since they have different tangents at that point ( $D_0 = F'$  is on  $a$  but not  $b$ , so  $b \neq d$ ). Hence, they can have at most one ordinary point in common. However, reflecting in  $D_0$  (lying on  $a = c$  and therefore in the exterior of  $\mathcal{C}$ ), any ordinary intersection of  $\mathcal{C}$  and  $\mathcal{C}(A_1)$  would yield a second intersection, so the two conics can't have any ordinary points in common. On the other hand, if  $A_1 = P'$  on arc  $EP'$ , then  $D_0 = K(P') = Q$  lies on  $\mathcal{C} \cap \mathcal{C}(A_1)$  and also on the tangent  $OQ$  to  $\mathcal{C}$ , so that  $\mathcal{C}$  and  $\mathcal{C}(A_1)$  touch at  $D_0 = Q$ . Thus, they don't intersect in any other ordinary point.

We also claim that there are no ordinary points on  $\mathcal{C} \cap \mathcal{C}(A_1)$  when  $A_1$  lies between  $E$  and  $P'$  on the open arc  $\mathcal{E} = EP'$ . In this case  $D_0 = K(A_1)$  lies on the open arc of the conic  $K(\mathcal{C})$  between  $F'$  and  $Q$ . For any  $A_1$  on the left branch of  $\mathcal{C}$ , let  $A'_1$  be the reflection of the point  $A_1$  in  $D_0$ . Using  $D_0 = K(A_1)$  it is easy to see that  $A'_1 = K^{-1}(A_1)$ , so the tangent  $\ell$  to  $\mathcal{C}$  at

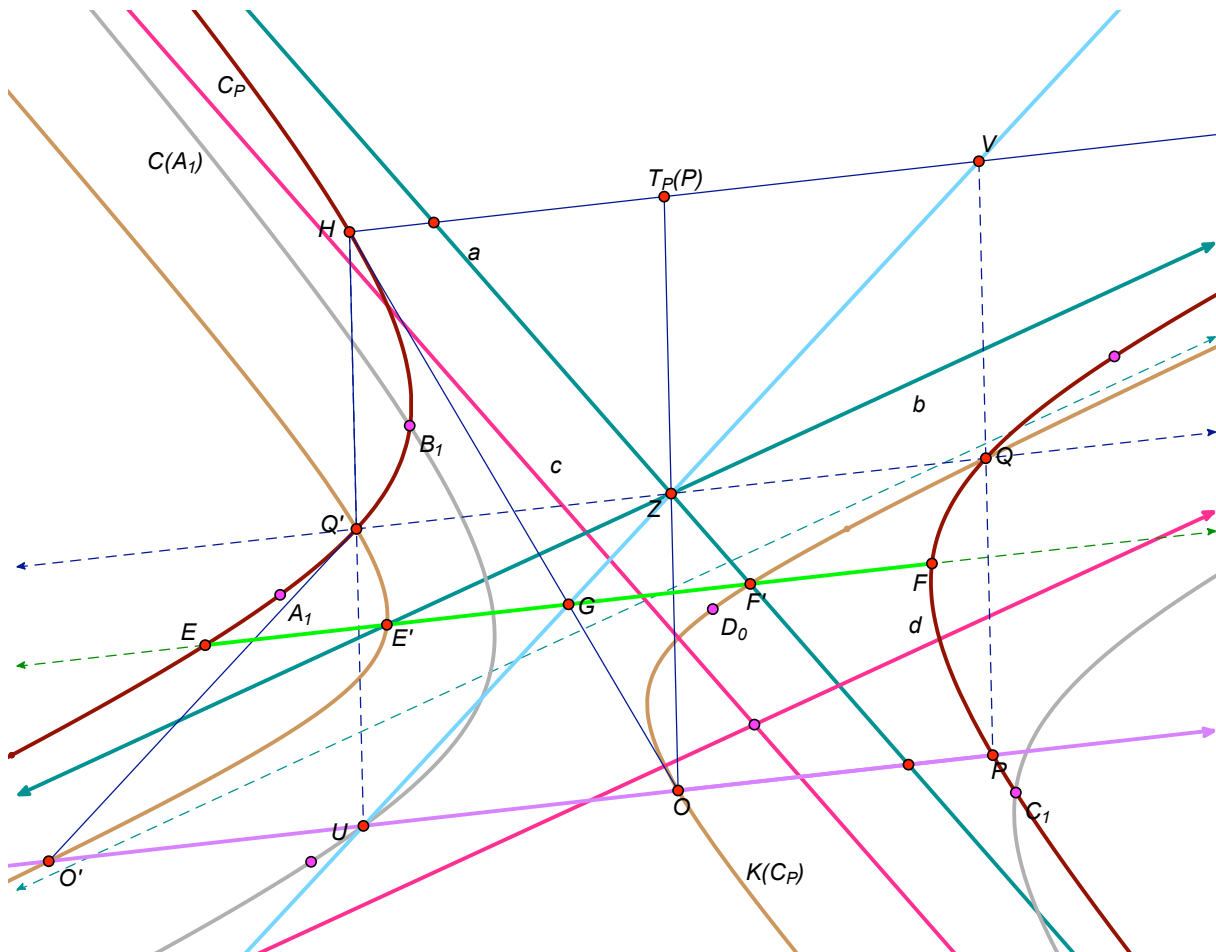


Figure 8: Conics  $\mathcal{C}$  (brown),  $K(\mathcal{C}_P)$  (tan),  $\mathcal{C}(A_1)$  (grey), with  $A_1$  above  $E$ .

$A_1$  is mapped to a parallel tangent  $\ell' = K^{-1}(\ell)$  to  $\mathcal{C}' = K^{-1}(\mathcal{C})$  at  $A'_1$ . On the other hand,  $\ell$  is mapped by reflection in  $D_0$  to a line through  $A'_1$  parallel to  $\ell$ , so it must also be mapped to  $\ell'$ . Therefore,  $\ell'$  is tangent to both conics  $\mathcal{C}'$  and  $\mathcal{C}(A_1) = R_{D_0}(\mathcal{C})$  at  $A'_1$ . Hence, these conics intersect with multiplicity 2 at  $A'_1$ , and since their asymptotes are parallel, this is the only ordinary point where they can intersect. This holds for any point  $A_1$  on the left branch of  $\mathcal{C}$ , and therefore the right branch of the conic  $\mathcal{C}'$  is an *envelope* for the right branches of the conics  $\mathcal{C}(A_1)$ . For all  $A_1$  on the left branch of  $\mathcal{C}$ ,  $D_0$  lies below the asymptote  $K(b)$  of  $K(\mathcal{C})$ , which is parallel to and lies halfway between  $b$  and the asymptote  $b' = K^{-1}(b) = UF$  of  $\mathcal{C}'$ ; hence, the asymptote  $d$  of  $\mathcal{C}(A_1)$  lies below  $b'$ , implying that the right branch of  $\mathcal{C}(A_1)$  lies in the interior of the right branch of  $\mathcal{C}'$ . Since  $\mathcal{C}'$  intersects  $\mathcal{C}$  at  $K^{-1}(Q') = P$  and  $K^{-1}(Q) = P'$ , it crosses  $\mathcal{C}$  at  $P$ , and points on  $\mathcal{C}'$  to the right of  $P$  lie in the interior of  $\mathcal{C}$ . For  $A_1 = E$ , the right branches of  $\mathcal{C}$  and  $\mathcal{C}(A_1)$  are also asymptotic in the direction of line  $c = a$ . Therefore, as  $A_1 \in \mathcal{E}$  moves from  $E$  to  $P'$ , the right branch of  $\mathcal{C}(A_1)$  moves to the right, separated from the right branch of  $\mathcal{C}$  by the asymptote  $c$  and the conic  $\mathcal{C}'$  and remaining in the interior of  $\mathcal{C}$ . It follows that the right branch of  $\mathcal{C}(A_1)$  contains no ordinary points of  $\mathcal{C}$  for  $A_1 \in \mathcal{E}$ . Since  $D_0$  is in the exterior of  $\mathcal{C}$  for these points  $A_1$ , any intersection of the left branch of  $\mathcal{C}(A_1)$  and the left branch of  $\mathcal{C}$  would reflect through  $D_0$  to an intersection of the right branches. The left branch of  $\mathcal{C}(A_1)$  also does not intersect the right branch of  $\mathcal{C}$  since it is the reflection of that branch in  $D_0$ . This proves the claim.

Now assume  $A_1$  lies outside the closed arc  $\overline{\mathcal{E}}$  on the left branch of  $\mathcal{C}$ . First, if  $A_1$  lies above

the point  $E$ , then  $D_0$  lies below the point  $F'$  on  $K(\mathcal{C})$ , and since  $c$  lies on the same side of the line  $a$  as  $D_0$ ,  $c$  lies to the left of the line  $a$  in Figure 8. Since  $c$  lies on  $a \cdot l_\infty$  and is not the tangent  $a$  at that point, it must intersect the conic  $\mathcal{C}$  in a second, ordinary, point. It cannot intersect the right branch of  $\mathcal{C}$  because that branch is on the other side of the line  $a$ . Hence,  $c$  intersects the left branch of  $\mathcal{C}$ , whence it follows that the left branch of  $\mathcal{C}(A_1)$  intersects  $\mathcal{C}$  as well (because this branch of  $\mathcal{C}(A_1)$  is asymptotic to an exterior ray of line  $d$  in one direction and to  $c$  in the other direction). At the same time, this shows that the asymptote  $a$  of  $\mathcal{C}$  intersects the right branch of  $\mathcal{C}(A_1)$ , so the right branch of  $\mathcal{C}$  intersects the latter. It is easy to see that these two intersection points are reflections of each other in the point  $D_0$ .

On the other hand, if  $A_1$  lies below the point  $P'$  on the left branch of  $\mathcal{C}$ , then  $D_0$  lies above  $Q$  on  $K(\mathcal{C})$ . Since  $Q \in \mathcal{C} \cap K(\mathcal{C})$ , points to the right of  $Q$  on  $K(\mathcal{C})$  are in the interior of  $\mathcal{C}$ , so the reflection  $Q_0$  of the point  $Q$  in  $D_0$  lies on the left branch of  $\mathcal{C}(A_1)$  in the interior of  $\mathcal{C}$ . It follows that the left branch of  $\mathcal{C}(A_1)$  must intersect the right branch of  $\mathcal{C}$  in two points. The same arguments apply to points  $A_1$  on the right branch of  $\mathcal{C}$ , and this completes the proof.  $\square$

**Lemma 4.4.** *The points  $A_1 = Q, Q'$  are the only points on  $\mathcal{C} = PQHQ'P'$  for which  $A_1$  lies on  $\mathcal{C}(A_1)$ .*

*Proof.* Certainly  $Q' \in \mathcal{C}(Q')$  because  $D_0 = K(Q')$  is the midpoint of segment  $Q'P$ , so  $Q' = \mathbf{R}_{D_0}(P)$  lies on  $\mathcal{C}(Q')$ , the reflection of  $\mathcal{C}$  in  $D_0$ . The same argument holds for  $Q$ . If  $A_1$  is any point lying on  $\mathcal{C}(A_1)$ , then  $A_1$  and its reflection  $A'_1$  in  $D_0 = K(A_1)$  both lie on the conic  $\mathcal{C}$  and are collinear with the point  $G$ . Since  $A'_1 = K^{-1}(A_1)$ , the locus of points  $A'_1$  coincides with the conic  $\mathcal{C}' = K^{-1}(\mathcal{C})$ , whose asymptotes are parallel to the asymptotes of  $\mathcal{C}$ . Hence,  $\mathcal{C}'$  intersects  $\mathcal{C}$  in only the two points  $P = K^{-1}(Q')$  and  $P' = K^{-1}(Q)$ . This proves the lemma.  $\square$

We now fix a parallelogram  $H_1U_1P_1V_1$  with center  $Z_1$ , distinguished point  $G_1 = U_1V_1 \cdot H_1O_1$ , and its corresponding conic  $\mathcal{C} = P_1Q_1H_1Q'_1P'_1$ , as in Figure 6. We will call this configuration the  $P_1$  configuration, and consider it fixed for the following discussion.

Let  $ABC$  be a given triangle. For any point  $P$ , not on a median of  $ABC$ , for which the map  $\mathbf{M}$ , defined relative to  $ABC$  and  $P$ , is a translation, there is an affine map  $\mathbf{A}^{-1}$  taking the parallelogram  $HUPV$  for  $ABC$  to the parallelogram  $H_1U_1P_1V_1$ . (We avoid points on the medians of  $ABC$ , because for these points, the conic  $\mathcal{C}_P = ABCPQ = AP \cup BC$  and parallelogram  $HUPV$  are degenerate.) Since  $ABC$  is inscribed in the cevian conic  $\mathcal{C}_P = ABCPQ = PQHQ'P'$  for  $P$ , and the points  $P', Q, Q'$  are defined by simple affine relationships in terms of the parallelogram  $HUPV$ , the image triangle  $\mathbf{A}^{-1}(ABC) = A_1B_1C_1$  under the map  $\mathbf{A}^{-1}$  is a triangle inscribed in the conic  $\mathcal{C} = P_1Q_1H_1Q'_1P'_1$ . By Theorem 3.1 and Corollary 3.2 all the same relationships hold for the two configurations. Hence, the centroid  $G$  maps to the centroid  $G_1$  in the  $P_1$  configuration. It follows from Lemma 4.3 and Lemma 4.4 that the image  $A_1$  of the point  $A$  must lie in the complement of the union of closed arcs  $\overline{\mathcal{E}}$  (from  $E$  to  $P'_1$ ) and  $\overline{\mathcal{F}}$  (from  $F$  to  $P_1$ ) on  $\mathcal{C}$ , and that  $A_1$  is also distinct from the points  $Q_1, Q'_1$  (as there is no triangle  $A_1B_1C_1$  for these two points). Thus,

$$A_1 \in \mathcal{A} = \mathcal{C} - (\overline{\mathcal{E}} \cup \overline{\mathcal{F}} \cup \{Q_1, Q'_1, A_\infty, B_\infty\}), \quad (5)$$

where  $A_\infty = a \cdot l_\infty$  and  $B_\infty = b \cdot l_\infty$  are the infinite points on the asymptotes. The set  $\mathcal{A}$  is a union of six open arcs on  $\mathcal{C}$ .

Conversely, let  $A_1 \in \mathcal{A}$  and let  $A_1B_1C_1$  be the corresponding triangle inscribed in  $\mathcal{C}_{P_1}$ . Then the centroid of  $A_1B_1C_1$  is  $G_1$ , and the cevian conic for  $A_1B_1C_1$  and  $P_1$  is  $A_1B_1C_1P_1Q'_1 = \mathcal{C}_{P_1}$ , coinciding with the conic  $\mathcal{C} = P_1Q_1H_1Q'_1P'_1$ . Moreover, the point  $P_1$  does not lie on a median of  $A_1B_1C_1$ ; otherwise one of the vertices of the triangle would be collinear with  $P_1$  and  $G_1$ , implying that this vertex would have to coincide with  $Q_1$  or  $Q'_1$ . This conic has center  $Z_1$ , and the pole of  $G_1Z_1$  is the point  $V_\infty = P_1P'_1 \cdot l_\infty$ . Now we use the characterization of the isotomic conjugate  $\iota(P_1)$  (with respect to  $A_1B_1C_1$ ) as the unique point different from  $P_1$  lying in the intersection  $\mathcal{C}_{P_1} \cap P_1V_\infty = \mathcal{C} \cap P_1V_\infty$  to deduce that  $\iota(P_1) = P'_1$  (see II, p. 26). Theorem 3.1 shows that  $M_1 = M_{P_1}$  for the triangle  $A_1B_1C_1$  must be a translation. If  $A$  is an affine map taking  $A_1B_1C_1$  to  $ABC$ , then Theorem 3.1 shows once again that the map  $M = AM_1A^{-1}$  is a translation for the point  $P = A(P_1)$ . Hence,  $P$  lies on the elliptic curve  $\mathcal{E}_S$  of Section 3. The argument of the previous paragraph shows that every point  $P$  on  $\mathcal{E}_S$ , other than the 12 points in its torsion group  $T_{12}$ , is the image  $P = A(P_1)$  for some triangle  $A_1B_1C_1$  inscribed in  $\mathcal{C}$  and an affine map  $A$  for which  $A(A_1B_1C_1) = ABC$ . This proves the following theorem.

**Theorem 4.5.** *Fix a parallelogram  $H_1U_1P_1V_1$  and the corresponding hyperbola  $\mathcal{C} = P_1Q_1H_1Q'_1P'_1$ , as in Figure 6. The elliptic curve  $\mathcal{E}_S$ , minus the torsion subgroup  $T_{12}$ , corresponding to the vertices of  $ABC$ , the infinite points on its sides, and the points lying on the medians of  $ABC$ , is the locus of images  $A(P_1)$ , where  $A_1$  is a point in the set  $\mathcal{A} \subset \mathcal{C}$  (a union of six open arcs on the hyperbola  $\mathcal{C}$ ),  $B_1$  and  $C_1$  are the unique points on  $\mathcal{C}$  for which triangle  $A_1B_1C_1$  has centroid  $G_1$ , and  $A$  is one of the two affine maps for which  $A_1(A_1B_1C_1) = ABC$  or  $A_2(A_1C_1B_1) = ABC$ .*

By virtue of the above discussion, we have taken the situation of Figure 8, where  $P_1$  is fixed and the triangle  $A_1B_1C_1$  varies, and transformed it, via the locus of maps  $A$  corresponding to  $A_1 \in \mathcal{A}$ , to the fixed triangle  $ABC$  and varying point  $A(P_1) = P$  lying on the elliptic curve  $\mathcal{E}_S$ .

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