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Interpolations by Rational Motions Using Dual Quaternions

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Abstract. The main aim of this paper is to show an application of dual quaternions related to a rational spline motion. The interpolation by rational spline motions is an important part of technical practice, e.g., in robotics. Therefore, we will focus on most simple examples of piecewise rational motions with first and second order geometric continuity, in particular, a cubic G^2 Hermite interpolation. Consequently, it is shown that the new approach to rational spline motion design based on dual quaternions is an elegant mathematical method.

Key Words: Dual quaternion, rational spline motion, Hermite interpolation.

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1. Introduction

In computer graphics and animation, the rotational and translational motions have several important applications. In this paper we discuss the following interpolating problem: For given poses of a moving object in the three-dimensional space a continuous motion interpolating these positions shall be found. The solution of this problem is required, e.g., in robotics for the path planning. Techniques for solving this problem often deal separately with the translational and rotational components, i.e., with positions and orientations. As an innovation, we try to combine these two parts in one step.

Recently used algorithms solving this interpolation problem were based, e.g., on Euler angles, but then the trajectory of the moving object is a non-rational curve. Another approach is to interpolate rotations using normalized quaternion curves (see [11] or [9]). Hermite interpolation is often used to introduce an effective method for a smooth interpolation of given orientations of a rigid body motion. In this respect we can also refer, e.g., to [4] and [7].

This paper introduces dual quaternions as a tool for representing three-dimensional transformations of a rigid body. Such a transformation can uniquely be specified by a continuous path $q_d(t)$. Dual quaternions were invented to represent rigid transformations. Therefore,

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dual quaternions prove to be a very useful tool in computer graphics. Recently, the problem of a rational spline motion has been solved by Hermite interpolation based on quaternions (see [4]). This approach seems to be very interesting and efficient, and therefore we modify it by dual quaternions because of their unifying properties.

The remainder of the paper is organized as follows: Section 2 recalls some basic notions and facts about the rational spline motions based on quaternions. We introduce the definition of dual numbers and dual quaternions. Subsequently, the new approach using dual quaternion is introduced. The following part is devoted to a practical application of dual quaternions, i.e., to G^2 Hermite interpolation. Finally, we conclude the paper.

2. Preliminaries

In this section we present a brief tutorial on rational spline motions and dual quaternions. Dual quaternions can be considered as standard quaternions whose elements are dual numbers. This structure is mainly convenient for describing rigid motions, which are compositions of rotations and translations (see [10] or [6]). For a detailed description of dual quaternions we refer the reader, e.g., to [3]. This section provides a brief introduction to the theory of dual numbers and dual quaternions. First, we quickly review the basics of this algebra. More details can be found in [1], [2] or [12].

2.1. Rational spline motions

Rational spline motions are defined by the property that the trajectories of the points of the moving object are rational spline curves. A rigid body motion is described by the trajectory $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$ of the origin of the moving system and by the 3×3 rotation matrix $\mathbf{R}$. The trajectory of any other point P with position vector $\mathbf{p}$ in the moving system is then described by

$$\widehat{\mathbf{p}}(t) = \mathbf{c}(t) + \mathbf{R}(t)\mathbf{p}. \tag{1}$$

It is possible to use quaternions to describe rational spline motions. Then the rotation matrix $\mathbf{R}$ needs to be expressed with quaternion terms as

$$\mathbf{R} = \frac{1}{\|\mathcal{Q}\|}\begin{bmatrix} q_1^2 + q_0^2 - q_2^2 - q_3^2 & 2q_0q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_0q_3 + 2q_1q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & -2q_0q_1 + 2q_2q_3 \\ -2q_0q_2 + 2q_1q_3 & 2q_0q_1 + 2q_2q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}, \tag{2}$$

where $\mathcal{Q} = (q_0, q_1, q_2, q_3)$ is a point of $\mathbb{R}^4$. The trajectory of the point can be expressed as a quaternion curve (see, e.g., [8] for more details).

Rational spline motions are obtained by rational spline functions $q_i(t)$ and $c_i(t)$ for $i = 0, \dots 4$, where the $q_i(t)$ represent the coordinates of the quaternion and the $c_i(t)$ represent the trajectory of the moving frame's origin. Rational splines can be classified by the degree of their trajectories. If quadratic polynomials $q_i(t)$ are used then the rational spherical motion is of degree four or higher (see [5]). In order to obtain a rational spline motion (1) of degree four or higher, the functions $c_i(t)$ should be chosen as

$$c_i(t) = \frac{d_i(t)}{\|\mathcal{Q}\|}, \quad i = 1, 2, 3, \tag{3}$$

where d_i are polynomials of degree four or higher.

2.2. Dual numbers and dual quaternions

Dual numbers were invented by Clifford in 1873 (see [1] for more details). They are similar to complex numbers, because any *dual number* z_d can be written as

$$z_d = a + \varepsilon a_\varepsilon, \tag{4}$$

where a is the non-dual part, a_ε the dual part and ε is a basis element called *dual unit*. The defining condition for the dual unit is $\varepsilon^2 = 0$. The *dual conjugate* is analogous to the complex conjugate, i.e.,

$$\overline{z_d} = a - \varepsilon a_\varepsilon. \tag{5}$$

The *multiplication* of two dual numbers is given as

$$z_d \hat{z}_d = a\hat{a} + \varepsilon(a\hat{a}_\varepsilon + a_\varepsilon \hat{a}). \tag{6}$$

Finally, note that *pure* dual numbers, i.e., dual numbers with $a = 0$, do not have an inverse. This is a fundamental difference to complex numbers because every non-zero complex number has an inverse.

A *dual quaternion* $\mathcal{Q}_d$ can be written as the sum of two standard quaternions

$$\mathcal{Q}_d = \mathcal{Q} + \varepsilon \mathcal{Q}_\varepsilon, \tag{7}$$

where

$$\mathcal{Q} = q_0 + q_1\,\mathbf{i} + q_2\,\mathbf{j} + q_3\,\mathbf{k} \quad \text{and} \quad \mathcal{Q}_\varepsilon = q_{0\varepsilon} + q_{1\varepsilon}\,\mathbf{i} + q_{2\varepsilon}\,\mathbf{j} + q_{3\varepsilon}\,\mathbf{k}, \tag{8}$$

are real quaternions and $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are the usual quaternion units. The dual unit ε commutes with the quaternion units, for example $\mathbf{i}\,\varepsilon = \varepsilon\mathbf{i}$. A dual quaternion can also be considered as an 8-tuple of real numbers, or as

$$\begin{aligned} \mathcal{Q}_d &= q_{0d} + q_{1d}\,\mathbf{i} + q_{2d}\,\mathbf{j} + q_{3d}\,\mathbf{k} \\ &= (q_0 + \varepsilon q_{0\varepsilon}) + (q_1 + \varepsilon q_{1\varepsilon})\mathbf{i} + (q_2 + \varepsilon q_{2\varepsilon})\mathbf{j} + (q_3 + \varepsilon q_{3\varepsilon})\mathbf{k}, \end{aligned} \tag{9}$$

where q_{0d} is the *scalar part* (a dual number) and (q_{1d}, q_{2d}, q_{3d}) is the *vector part* (a dual vector) (see [12]). The *product* of two dual quaternions $\mathcal{Q}_d$ and $\widehat{\mathcal{Q}}_d$ is defined as

$$\mathcal{Q}_d \widehat{\mathcal{Q}}_d = \widehat{\mathcal{Q}}\mathcal{Q} + \varepsilon(\mathcal{Q}\widehat{\mathcal{Q}}_\varepsilon + \mathcal{Q}_\varepsilon \widehat{\mathcal{Q}}). \tag{10}$$

The multiplication of dual quaternions is associative, distributive, but not commutative. The *conjugation* of a dual quaternion is defined using the classical quaternion conjugation

$$\mathcal{Q}_d{}^* = \mathcal{Q}^* + \varepsilon \mathcal{Q}_\varepsilon^*, \tag{11}$$

where $\mathcal{Q}^*$ denotes the conjugate quaternion. However, the dual number conjugation (5) can be applied to dual quaternion conjugation and we get the dual conjugate dual quaternion

$$\overline{\mathcal{Q}_d^*} = \mathcal{Q}^* - \varepsilon \mathcal{Q}_\varepsilon^*. \tag{12}$$

The *norm* of a dual quaternion is a dual scalar and defined as

$$\|\mathcal{Q}_d\| = \sqrt{\mathcal{Q}_d^* \mathcal{Q}_d}. \tag{13}$$

A dual quaternion is called a *unit dual quaternion* if $\|\mathcal{Q}_d\| = 1$. Note that a dual quaternion $\mathcal{Q}_d$ is unit if and only if

$$\|\mathcal{Q}\| = 1 \quad \wedge \quad \mathcal{Q} \cdot \mathcal{Q}_\varepsilon = 0, \tag{14}$$

where $\cdot$ denotes the standard dot product. If we have a vector $\mathbf{p}^T = (p_1, p_2, p_3) \in \mathbb{R}^3$, we define the associated unit dual quaternion as

$$\mathcal{P}_d = 1 + \varepsilon(p_1\,\mathbf{i} + p_2\,\mathbf{j} + p_3\,\mathbf{k}), \tag{15}$$

which satisfies the previous statement.

A new method to represent rigid transformations is based on using dual quaternions. Dual quaternions capture in their inner structure the basic information about this transformations — namely, the axis of rotation, the rotation angle about the axis and the translation along it. A composition of these transformations corresponds to the multiplication of dual quaternions.

Suppose that $\mathbf{p}^T = (p_1, p_2, p_3)$ is the position vector of a point P, $\mathbf{t}^T = (t_1, t_2, t_3)$ is a translation vector and $\mathcal{Q}$ the unit quaternion representing a rotation. Then we can express the image of the point P after this translation and rotation as

$$\widehat{\mathcal{P}}_d = \mathcal{Q}_d\,\mathcal{P}_d\,\overline{\mathcal{Q}_d^*}, \tag{16}$$

where $\mathcal{Q}_d$ is the unit dual quaternion

$$\mathcal{Q}_d = \mathcal{Q} + \varepsilon\frac{\mathcal{T}\mathcal{Q}}{2} \quad \text{with} \quad \mathcal{T} = t_1\,\mathbf{i} + t_2\,\mathbf{j} + t_3\,\mathbf{k}. \tag{17}$$

To sum up, unit dual quaternions naturally represent rotations when the dual part $\mathcal{Q}_\varepsilon = 0$.

3. Hermite interpolation by rational G^2 motions

This part of the paper is devoted to the cubic G^2 Hermite interpolation. The cubic geometric interpolation is chosen to show the advantages of various applications of dual quaternions considering their properties. Notice, that the main emphasis is laid on the Hermite interpolation by rational Bézier curves in space. Further, the method based on quaternions studied in [4] will be extended using dual quaternions. It is easily shown that they are excellent tools to describe the G^2 rational spline motions.

3.1. Cubic G^2 Hermite interpolation using quaternions

Assume that a curve is determined by two points, i.e., P_0 and P_1, and two associated velocities, i.e., U_0 and U_1. The objective is to find a cubic Bézier curve $r(t)$ which interpolates the given data as

$$r(0) = P_0, \qquad r'(0) = U_0, \tag{18}$$
$$r(1) = P_1, \qquad r'(1) = U_1. \tag{19}$$

The sought curve can be represented as a cubic Bézier curve (see Figure 1)

$$r(t) = \sum_{i=0}^{3} R_i B_i^3(t), \tag{20}$$

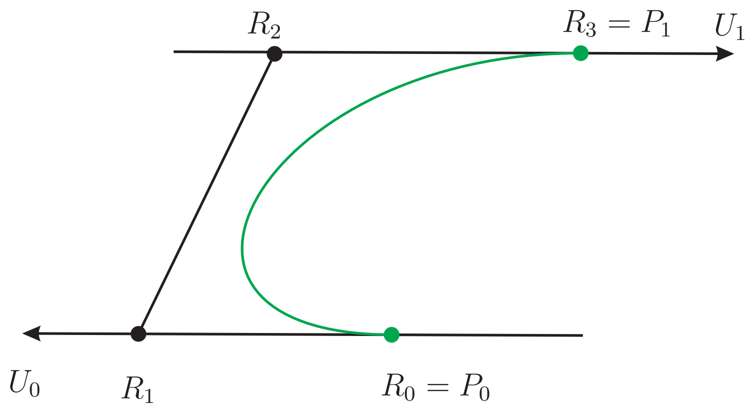

Figure 1: Illustrative figure — Hermite interpolation using a cubic.

where R_0, R_1, R_2, R_3 are the control points which satisfy $R_0 = P_0$, $R_1 = P_0 + \frac{U_0}{3}$, $R_2 = P_1 - \frac{U_1}{3}$, and $R_3 = P_1$, while $B_i^3(t)$ are the Bernstein polynomials of degree 3, i.e.,

$$B_i^3(t) = \binom{3}{i} t^i (1-t)^{3-i}. \tag{21}$$

A cubic Bézier quaternion curve can be used to define a Hermite quaternion curve which interpolates two end unit quaternions. Let $\mathcal{Q}_i$ be the unit quaternion, $\mathcal{U}_i = \mathcal{Q}_i^{(1)}$ the velocity quaternion and $\mathcal{V}_i = \mathcal{Q}_i^{(2)}$ the acceleration quaternion at orientation $\mathcal{Q}_i$ for $i = 0, 1$. Then the cubic quaternion interpolation curve $q\colon [0,1] \to \mathbb{H}$ can be found as

$$q(t) = \sum_{j=0}^{3} \mathcal{B}_j B_j^3(t), \tag{22}$$

where $\mathcal{B}_j$ are unknown control quaternions and $B_j^3(t)$ are the Bernstein polynomials of degree 3.

The quaternion curve is G^2 continuous if the following conditions are satisfied:

$$q(j) = \lambda_j \mathcal{Q}_j, \tag{23}$$
$$q'(j) = \lambda_j^{(1)} \mathcal{Q}_j + \lambda_j \phi_j^{(1)} \mathcal{U}_j, \tag{24}$$
$$q''(j) = \lambda_j^{(2)} \mathcal{Q}_j + 2\lambda_j^{(1)} \phi_j^{(1)} \mathcal{U}_j + \lambda_j \phi_j^{(2)} \mathcal{U}_j + \lambda_j (\phi_j^{(1)})^2 \mathcal{U}_j^{(2)}, \tag{25}$$
$$\lambda_0 = \lambda_1 = 1, \tag{26}$$

where $j = 0, 1$. The parameters λ_j and ϕ_j for $j = 0, 1$ correspond to the function λ and the reparametrization ϕ, respectively. Their n-th derivatives, where $n = 1, 2$, are denoted by $\lambda_j^{(n)}$ and $\phi_j^{(n)}$. Moreover, the following conditions have to be satisfied:

$$\phi_0^{(1)} > 0 \quad \text{and} \quad \phi_1^{(1)} > 0, \tag{27}$$

for guaranteeing that the reparametrization ϕ is regular. To obtain the following equations, we have to use some basic properties of the Bézier curves mentioned at the beginning of this section:

$$\mathcal{B}_0 = \mathcal{Q}_0, \qquad \mathcal{B}_3 = \mathcal{Q}_1, \tag{28}$$
$$3\Delta \mathcal{B}_{2i} = \lambda_i^{(1)} \mathcal{Q}_i + \phi_i^{(1)} \mathcal{U}_i, \quad i = 0, 1, \tag{29}$$
$$6\Delta^2 \mathcal{B}_i = \lambda_i^{(2)} \mathcal{Q}_i + (2\lambda_i^{(1)} \phi_i^{(1)} + \phi_i^{(2)}) \mathcal{U}_i + (\phi_i^{(2)})^2 \mathcal{V}_i, \quad i = 0, 1. \tag{30}$$

The previous set of equations forms a system of 24 nonlinear equations for the unknown control quaternions $\mathcal{B}_j$ for $j = 0, 1, 2, 3$ and unknown scalar parameters $\phi_i^{(1)}$, $\phi_i^{(2)}$, $\lambda_i^{(1)}$, $\lambda_i^{(2)}$ for $i = 0, 1$. The unknowns $\phi_0^{(1)}$, $\phi_0^{(2)}$ have to be positive (see equation (27)). The set of 22 equations can be reduced to a system of 8 nonlinear equations (see [4])

$$\left(\frac{2(-1)^i}{3}\lambda_i^{(1)} + \frac{1}{6}\lambda_i^{(2)} + 1\right)\mathcal{Q}_i + \left(\frac{(-1)^i}{3}\lambda_{1-i}^{(1)} - 1\right)\mathcal{Q}_{1-i} + \frac{1}{6}(\phi_i^{(1)})^2\mathcal{V}_i + \left(\frac{2(-1)^i}{3}\phi_i^{(1)} + \frac{1}{3}\lambda_i^{(1)}\phi_i^{(1)} + \frac{1}{6}\phi_i^{(2)}\right)\mathcal{U}_i + \frac{(-1)^i}{3}\phi_{1-i}^{(1)}\mathcal{U}_{1-i} = 0 \quad , \tag{31}$$

where $i = 0, 1$. Let us denote

$$\mathbf{D}_{i,j} = \frac{|\mathbf{A}_i^{(j)}(\mathcal{U}_{1-i})|}{|\mathbf{A}_i|}, \quad \text{for} \quad j = 1, \ldots, 4, \quad \text{and} \quad i = 0, 1, \tag{32}$$

where $|\mathbf{A}_i|$ denotes the determinant of the matrix $\mathbf{A}_i = (\mathcal{Q}_i, \mathcal{Q}_{1-i}, \mathcal{U}_i, \mathcal{V}_i)$ which is composed from quaternions, and $\mathbf{A}^{(j)}$ denotes the matrix $\mathbf{A}$ with the i-th column replaced by the quaternion $\mathcal{U}_{1-i}$. Then we can mention that $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{U}_0, \mathcal{U}_1, \mathcal{V}_0$ and $\mathcal{V}_1$ are given quaternions such that $\mathbf{A}_0$ and $\mathbf{A}_1$ are nonsingular and $\mathbf{D}_{0,4} < 0$, $\mathbf{D}_{1,4} > 0$. Then there exists a unique cubic interpolating quaternion curve $q(t)$ defined by equations (22), (28), (29) and (30) where

$$\phi_i^{(1)} = 2(-1)^i\sqrt[3]{\mathbf{D}_{i,4}^2\mathbf{D}_{1-i,4}}, \quad \lambda_i^{(1)} = -1^{(1-i)}(3 + 2\mathbf{D}_{1-i,2}\sqrt[3]{\mathbf{D}_{i,4}^2\mathbf{D}_{1-i,4}}) \tag{33}$$

for $i = 0, 1$ (see [4] for more details).

This approach solves only the rotational part, but the construction of the rational spline motion is the combination of rotations and translations (see equation (1)). Therefore, the curve needs to be translated to get the final view of the motion (see Section 2.1 for more details).

3.2. Improved method using dual quaternions

Dual quaternions will be used in this section in a similar way as for G^2 Hermite interpolation due to their amazing properties. All their advantages and suitability were introduced in previous part. Therefore we will focus now on their application.

We can modify equations (22) according to the dual quaternion description (16). Then we get the following dual quaternion curve

$$q_d(t) = \mathcal{Q}_{d_0}B_0^3(t) + \mathcal{Q}_{d_1}B_1^3(t) + \mathcal{Q}_{d_2}B_2^3(t) + \mathcal{Q}_{d_3}B_3^3(t), \tag{34}$$

where $\mathcal{Q}_{d_0}$, $\mathcal{Q}_{d_1}$, $\mathcal{Q}_{d_2}$, $\mathcal{Q}_{d_3}$ are unit dual quaternions and B_0^3, B_1^3, B_2^3, B_3^3 are again Bernstein polynomials. Of course, the equations (22), (28), (29) and (30) can be also used to find the rotational part of the dual quaternion.

We solve the following interpolation problem: There are given several poses P_i, $i = 0, \ldots, m$ of a rigid body. The pose P_i is composed of the position of the moving frame's origin (c_{1i}, c_{2i}, c_{3i}) and by the associated rotation matrix. The rotations can be represented by the unit quaternions $\mathcal{Q}_i$. Since we usually have a non-unit quaternion describing the rotations, this quaternion has to be normalized, i.e.,

$$\mathcal{Q}_i = \mp\frac{\widehat{\mathcal{Q}_i}}{\|\widehat{\mathcal{Q}_i}\|}, \tag{35}$$

where $\widehat{\mathcal{Q}_i}$ is an arbitrary quaternion which is not unit, in general. The appropriate sign in equation (35) is chosen to satisfy

$$\mathcal{Q}_i \cdot \mathcal{Q}_{i+1} > 0, \quad i = 0, \dots, m, \tag{36}$$

which provides that both quaternions lie on the same hemisphere. The translation can be described by a pure quaternion. We use equation (17)

$$\mathcal{T}_i = c_{1i}\,\mathbf{i} + c_{2i}\,\mathbf{j} + c_{3i}\,\mathbf{k}. \tag{37}$$

Algorithm 1 Rational spline motion with continuity G^2 using dual quaternion

Input: The rotation motion defined by the quaternion curve $\hat{q}(t)$ and the trajectory $c(t)$ of the center, where $t_i = i$ for $i = 0, \dots, m$.

1: Normalize the rotational quaternions by equation (35) to get $\mathcal{Q}_i$.
2: Compute the velocity $\mathcal{U}_i$ as $\mathcal{U}_i = \mathcal{Q}_i^{(1)}$.
3: Compute the quaternion $\mathcal{V}_i$ as $\mathcal{V}_i = \mathcal{Q}_i^{(2)}$.
4: Compute the translation quaternion $\mathcal{T}_i$ using (37).
5: Determine the dual quaternion curve using (34).

Output: Dual quaternion curves q_{d_i} describing a rational spline motion.

Due to the dual quaternion description (16) we can combine both quaternions and we get the unit dual quaternion

$$\mathcal{Q}_{d_i} = \mathcal{Q}_i + \varepsilon \frac{\mathcal{T}_i \mathcal{Q}_i}{2}, \tag{38}$$

that will be used in equations (34), and so we get two dual quaternion curves containing the translation and the rotational part. The interpolating algorithm based on the dual quaternion curve can be computed in the following steps (see Algorithm 1). For an example and figures please see [4], Section 8, especially Figures 2–4 and Table 1.

4. Conclusion

This paper is focused on one particular example of practical applications of dual quaternions. Since the algebra of this structure is very popular and frequently used in various mathematical fields nowadays, we try to show some of their applications. As we know, in computer graphics, animation and robotics the rotational and translational motions have several important applications. Therefore a simple algorithm for the interpolation by the rational spline motion based on dual quaternions is presented in this paper. The construction of the algorithm was motivated by [4] where a G^2 Hermite interpolation based on quaternions was investigated. We have modified an algorithm for rational spline motion using the dual quaternions approach. The rational spline motions are composed of a rotational and a translational part, i.e., this motion can be easily described by dual quaternions. The main advantage of this approach is that the dual quaternions allow us to use these two transformations in only one operation, which simplifies the original method.

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