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Concurrency in Hexagons — A Trigonometric Tale

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Abstract. It is shown that the concurrency of the main diagonals in a hexagon is equivalent to a nine-angle sine trigonometric identity. Applications and generalizations to hexagonal paths are also discussed.

Key Words: hexagon, main diagonals, concurrency, collinearity, cross ratio, Sine-theorems

MSC 2010: 51M04, 97G60, 51A20

1. Introduction

The geometry enthusiast may have sensed that the classical theorems (about concurrency of lines or collinearity of points in a plane) of Brianchon, Ceva, Desargues, Menelaus, Pappus, or Pascal [7] (the listing is in alphabetical order) are very much related, very much equivalent.

The hexagons are the best polygons: They have just enough sides/angles to accommodate diversity and versatility, yet not too many to induce confusion. Not surprisingly, Nature chose them for packing efficiency, from bees' honeycombs to the Giant's Causeway [8]. Let it be noted, for instance, that when probing the concurrency of three lines in a plane one deals practically with properties of hexagons, given that a line is uniquely determined by two points.

In plane geometry the trigonometric methods of proof are typically the most underrated: For noblesse oblige reasons one tries first synthetic methods, only to switch to analytic ones, if the former do not yield results. Trigonometry comes only as an afterthought, despite its undeniable efficiency in problems where shape and not size matters.

The purpose of this paper is to investigate the familiar theme of concurrency of three lines in plane geometry, via a trigonometric study of hexagonal paths. In the end a very nice equation emerges in hexagons (see (1) below), which surprisingly does not appear to be known.

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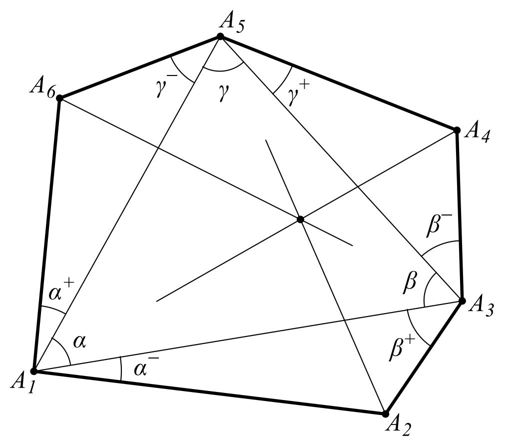

Figure 1: A convex hexagon with concurrent main diagonals, and the nine relevant angles.

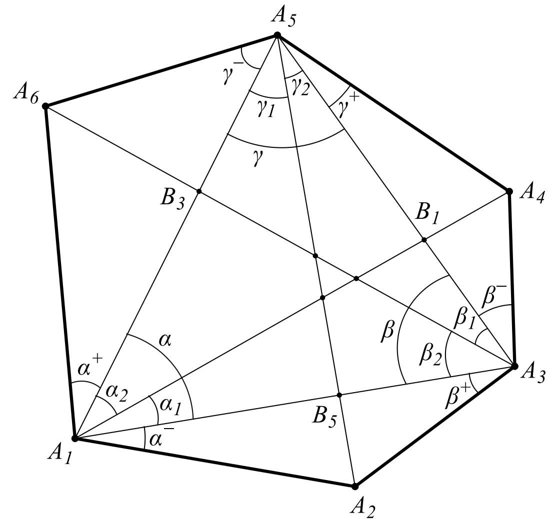

Figure 2: The angles associated to the trigonometry of hexagons.

2. Sine-Theorem

Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon. Its three main diagonals are the line segments $\overline{A_1A_4}$, $\overline{A_2A_5}$, and $\overline{A_3A_6}$, and a very natural problem is to find necessary and sufficient conditions guaranteeing their concurrency. More generally, given a hexagonal path in a plane, with distinct vertices A_1, $A_2, \ldots, A_6$ in reasonably generic position, when are its main diagonal lines $\overleftrightarrow{A_1A_4}$, $\overleftrightarrow{A_2A_5}$, and $\overleftrightarrow{A_3A_6}$ concurrent?

Our result will express this concurrency in terms of the measures of nine oriented angles. In order not to be distracted by orientation issues, we will prove it only when the hexagonal path is convex and the above vertex listing is consistent with traversing the sides of the hexagon in a counterclockwise manner. Fixing one of the two core internal triangles in the hexagon, say $\triangle A_1A_3A_5$ (the other being $\triangle A_2A_4A_6$), denote by α, β, and γ, the measures of its angles A_1, A_3, and A_5, respectively. Denote also by α^- and β^+ the measures of the angles A_1 and A_3, respectively, in $\triangle A_1A_2A_3$. Similarly, we have β^-, γ^+, and γ^-, α^+ (see Figure 1). Then the following holds true:

Sine-Concurrency Theorem. *Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon. With the above notations, the three main diagonals in the hexagon, $\overline{A_1A_4}$, $\overline{A_2A_5}$, and $\overline{A_3A_6}$, are concurrent if and only if*

$$\begin{aligned}&\sin(\alpha+\alpha^+)\sin(\beta+\beta^+)\sin(\gamma+\gamma^+)\sin\alpha^-\sin\beta^-\sin\gamma^- =\\ &\sin(\alpha+\alpha^-)\sin(\beta+\beta^-)\sin(\gamma+\gamma^-)\sin\alpha^+\sin\beta^+\sin\gamma^+\end{aligned} \tag{1}$$

The proof of the Sine-Concurrency Theorem will be an immediate consequence of the following two Lemmas.

Lemma 1 – The Trigonometry of Hexagons. *The angle notations being those of Figure 2, in any convex hexagon $A_1A_2A_3A_4A_5A_6$ the following trigonometric identity holds true:*

$$\begin{aligned}&\sin(\alpha+\alpha^+)\sin(\beta+\beta^+)\sin(\gamma+\gamma^+)\sin\alpha^-\sin\beta^-\sin\gamma^-\sin\alpha_1\sin\beta_1\sin\gamma_1 =\\ &\sin(\alpha+\alpha^-)\sin(\beta+\beta^-)\sin(\gamma+\gamma^-)\sin\alpha^+\sin\beta^+\sin\gamma^+\sin\alpha_2\sin\beta_2\sin\gamma_2\end{aligned} \tag{2}$$

Proof of Lemma 1. Since the hexagon is convex, its main diagonals $\overline{A_1A_4}$, $\overline{A_2A_5}$, and $\overline{A_3A_6}$ intersect the sides $\overline{A_3A_5}$, $\overline{A_1A_3}$, and $\overline{A_5A_1}$ of the core triangle, $\triangle A_1A_3A_5$, at interior points B_1, B_5, and B_3, respectively. Consequently, for the angle A_1 in $\triangle A_1A_3A_4$ with measure α_1 and angle A_1 in $\triangle A_1A_4A_5$ with measure α_2, we have $\alpha_1 + \alpha_2 = \alpha$. Similarly, $\beta_1 + \beta_2 = \beta$ and $\gamma_1 + \gamma_2 = \gamma$ (cf. Figure 2).

The proof of Lemma 1 makes a judicious use of the Law of Sines in various triangles inside the hexagon. For instance, looking at triangles, $\triangle A_1A_3A_6$, $\triangle A_3A_5A_2$, and $\triangle A_5A_1A_4$, we get

$$\frac{\sin(\alpha+\alpha^+)}{A_3A_6} = \frac{\sin\beta_2}{A_6A_1}, \quad \frac{\sin(\beta+\beta^+)}{A_5A_2} = \frac{\sin\gamma_2}{A_2A_3}, \quad \text{and} \quad \frac{\sin(\gamma+\gamma^+)}{A_1A_4} = \frac{\sin\alpha_2}{A_4A_5}, \tag{3}$$

respectively. Similarly, from the triangles, $\triangle A_3A_4A_1$, $\triangle A_5A_6A_3$, and $\triangle A_1A_2A_5$, we get

$$\frac{\sin\alpha_1}{A_3A_4} = \frac{\sin(\beta+\beta^-)}{A_4A_1}, \quad \frac{\sin\beta_1}{A_5A_6} = \frac{\sin(\gamma+\gamma^-)}{A_6A_3}, \quad \text{and} \quad \frac{\sin\gamma_1}{A_1A_2} = \frac{\sin(\alpha+\alpha^-)}{A_2A_5}, \tag{4}$$

respectively. Finally, the triangles $\triangle A_1A_2A_3$, $\triangle A_3A_4A_5$, and $\triangle A_5A_6A_1$, give

$$\frac{\sin\alpha^-}{A_2A_3} = \frac{\sin\beta^+}{A_1A_2}, \quad \frac{\sin\beta^-}{A_4A_5} = \frac{\sin\gamma^+}{A_3A_4}, \quad \text{and} \quad \frac{\sin\gamma^-}{A_6A_1} = \frac{\sin\alpha^+}{A_5A_6}, \tag{5}$$

respectively. Multiplying now side by side the nine identities given by equations (3), (4), and (5) yields (2), after the requisite side-lengths cancellations. □

Lemma 2 – Ceva's Theorem-Trigonometric Form. *In a triangle $\triangle A_1A_3A_5$, let $B_1 \in \overline{A_3A_5}$, $B_1 \neq A_3$, $B_1 \neq A_5$, and similarly let $B_3 \in \overline{A_5A_1}$ and $B_5 \in \overline{A_1A_3}$. If the angle measures α, α_1, α_2, β, β_1, β_2, and γ, γ_1, γ_2, are cf. Figure 2 or Figure 3, then the line segments $\overline{A_1B_1}$, $\overline{A_3B_3}$, and $\overline{A_5B_5}$ are concurrent if and only if*

$$\sin\alpha_1 \sin\beta_1 \sin\gamma_1 = \sin\alpha_2 \sin\beta_2 \sin\gamma_2 \tag{6}$$

Proof of Lemma 2. Assume first that the line segments $\overline{A_1B_1}$, $\overline{A_3B_3}$, and $\overline{A_5B_5}$ intersect at point O (cf. Figure 3). Applying now the Law of Sines in triangles, $\triangle A_1A_3O$, $\triangle A_3A_5O$, and $\triangle A_5A_1O$, we get respectively

$$\frac{\sin\alpha_1}{A_3O} = \frac{\sin\beta_2}{OA_1}, \quad \frac{\sin\beta_1}{A_5O} = \frac{\sin\gamma_2}{OA_3}, \quad \text{and} \quad \frac{\sin\gamma_1}{A_1O} = \frac{\sin\alpha_2}{OA_5}. \tag{7}$$

Multiplying side by side the three identities given by (7) yields (6).

Conversely, assume that the identity (6) holds true. Let the line segments $\overline{A_3B_3}$ and $\overline{A_5B_5}$ intersect at point O', and let the line $\overleftrightarrow{A_1O'}$ intersect the line segment $\overline{A_3A_5}$ at the interior point B_1'. The line segment $\overline{A_1B_1'}$ splits the angle A_1 of $\triangle A_1A_3A_5$ into two angles with measures α_1' and α_2', and so $\alpha = \alpha_1' + \alpha_2'$. Applying now the *only if* part of Lemma 2 (which was proved above) to the concurrent line segments $\overline{A_1B_1'}$, $\overline{A_3B_3}$, and $\overline{A_5B_5}$ we conclude that

$$\sin\alpha_1' \sin\beta_1 \sin\gamma_1 = \sin\alpha_2' \sin\beta_2 \sin\gamma_2 \tag{8}$$

By division, the equations (6) and (8) yield

$$\frac{\sin\alpha_1}{\sin\alpha_1'} = \frac{\sin\alpha_2}{\sin\alpha_2'}, \tag{9}$$

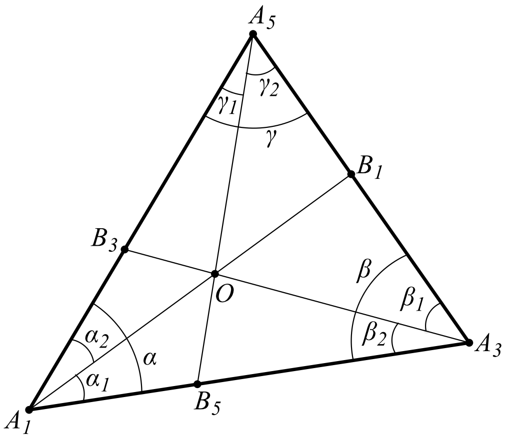

Figure 3: Ceva's Theorem – trigonometric form.

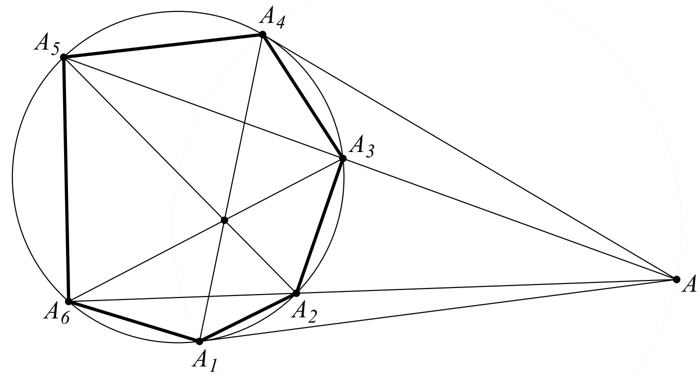

Figure 4: An example of a cyclic hexagon with concurrent main diagonals.

or equivalently,

$$\frac{\sin\alpha_1}{\sin(\alpha-\alpha_1)} = \frac{\sin\alpha_1'}{\sin(\alpha-\alpha_1')}. \tag{10}$$

The function

$$f(t) = \frac{\sin t}{\sin(\alpha - t)}, \qquad 0 < t < \alpha,$$

is strictly increasing, as its derivative

$$f'(t) = \frac{\sin\alpha}{\sin^2(\alpha - t)}$$

is strictly positive, and so equation (10) gives $\alpha_1 = \alpha_1'$. Thus, $B_1 = B_1'$, which proves the concurrency of $\overline{A_1B_1}$, $\overline{A_3B_3}$, and $\overline{A_5B_5}$ at the point O'. □

Proof of Sine-Concurrency Theorem. Cf. Figure 2, the line segments $\overline{A_1A_4}$, $\overline{A_2A_5}$, and $\overline{A_3A_6}$ are concurrent if and only if the line segments $\overline{A_1B_1}$, $\overline{A_3B_3}$, and $\overline{A_5B_5}$ are so, or, by Lemma 2, if and only if equation (6) holds true. Since the identity (2) of Lemma 1 is always true, dividing (2) by (6) gives that equation (6) holds true if and only if equation (1) does. □

We proceed with three applications to the Sine-Concurrency Theorem.

Corollary. *a) The notations being those of the Sine-Concurrency Theorem, the main diagonals of the convex hexagon $A_1A_2A_3A_4A_5A_6$ are concurrent if $\alpha^- = \alpha^+$, $\beta^- = \beta^+$, and $\gamma^- = \gamma^+$.*

b) Let $A_1A_2A_3A_4A_5A_6$ be a cyclic hexagon. Then its main diagonals are concurrent if and only if $A_1A_2 \cdot A_3A_4 \cdot A_5A_6 = A_2A_3 \cdot A_4A_5 \cdot A_6A_1$.

c) Let $B_1B_2B_3B_4B_5B_6$ be a cyclic hexagon. On its sides erect exterior triangles by extending these sides, and denote the additional vertices of these triangles by A_1, A_2, A_3, A_4, A_5, and A_6. Then the main diagonals in the convex hexagon $A_1A_2A_3A_4A_5A_6$ are concurrent.

Proof. *a)* is a result of de Villiers [4]. Its proof is an obvious consequence of the *if* part of the Sine-Concurrency Theorem, as the given hypotheses make the content of equation (1) plain. In the particular case $\alpha^+ = \alpha^- = \beta^+ = \beta^- = \gamma^+ = \gamma^-$, the concurrency point is known as the Kiepert point [5].

b) is a result of CARTENSEN [3]. Due to the cyclicity of the hexagon,

$$(\alpha + \alpha^+) + (\gamma + \gamma^-) = \pi, \quad (\beta + \beta^+) + (\alpha + \alpha^-) = \pi, \quad \text{and} \quad (\gamma + \gamma^+) + (\beta + \beta^-) = \pi.$$

Therefore,

$$\sin(\alpha + \alpha^+) = \sin(\gamma + \gamma^-), \quad \sin(\beta + \beta^+) = \sin(\alpha + \alpha^-), \quad \text{and} \quad \sin(\gamma + \gamma^+) = \sin(\beta + \beta^-).$$

In conclusion, equation (1) is equivalent to

$$\sin\alpha^+ \sin\beta^+ \sin\gamma^+ = \sin\alpha^- \sin\beta^- \sin\gamma^-,$$

and also to the metric property

$$A_1A_2 \cdot A_3A_4 \cdot A_5A_6 = A_2A_3 \cdot A_4A_5 \cdot A_6A_1,$$

via (5).

Here are now two natural implementations of *b*).

b_1). Let A_1, A_2, A_3, A_4, and A_5 be five distinct points, distributed in a counterclockwise manner on a given circle. If $\widehat{A_5A_1}$ is the counterclockwise oriented arc of the circle (with initial point A_5 and terminal point A_1), the continuous function

$$f\colon \widehat{A_5A_1} \to \mathbb{R}, \quad f(A) = A_1A_2 \cdot A_3A_4 \cdot A_5A - A_2A_3 \cdot A_4A_5 \cdot AA_1,$$

is strictly increasing as A advances along the arc, $f(A_5) < 0$, and $f(A_1) > 0$. By the Intermediate Value Property there is an unique point $A = A_6 \in \widehat{A_5A_1}$ such that the main diagonals in the cyclic hexagon $A_1A_2A_3A_4A_5A_6$ are concurrent. Clearly, A_6 is the intersection point of the arc $\widehat{A_5A_1}$ with the line $\overleftrightarrow{A_3I}$, where I is the intersection point of the line segments $\overline{A_1A_4}$ and $\overline{A_2A_5}$.

b_2) Let A be a point exterior to a given circle, and let A_1 and A_4 be the points where the two tangents to the circle through the point A intersect the circle. Let also two secants through A intersect the circle at A_2 and A_6, respectively A_3 and A_5 (cf. Figure 4). Then the main diagonals in the cyclic hexagon $A_1A_2A_3A_4A_5A_6$ are concurrent.

This can be seen by using similarity in three pairs of triangles. For instance $\triangle AA_1A_2 \sim \triangle AA_6A_1$ gives

$$\frac{A_1A_2}{A_6A_1} = \frac{AA_1}{AA_6} = \frac{AA_2}{AA_1}, \quad \text{which implies} \quad \frac{(A_1A_2)^2}{(A_6A_1)^2} = \frac{AA_2}{AA_6}.$$

Similarly,

$$\frac{A_3A_4}{A_4A_5} = \frac{AA_3}{AA_4} = \frac{AA_4}{AA_5}, \quad \text{gives} \quad \frac{(A_3A_4)^2}{(A_4A_5)^2} = \frac{AA_3}{AA_5},$$

and

$$\frac{A_5A_6}{A_2A_3} = \frac{AA_5}{AA_2} = \frac{AA_6}{AA_3}, \quad \text{gives} \quad \frac{(A_5A_6)^2}{(A_2A_3)^2} = \frac{AA_5 \cdot AA_6}{AA_2 \cdot AA_3}.$$

Therefore,

$$\frac{(A_1A_2)^2}{(A_6A_1)^2}\frac{(A_3A_4)^2}{(A_4A_1)^5}\frac{(A_5A_6)^2}{(A_2A_3)^2} = 1,$$

which proves the validity of b_2).

c) Referring to Figure 5, by the Sine-Concurrency Theorem we have to establish the validity of equation (1) for the choices of angles indicated. The content of equation (5) is equivalent to

$$\frac{\sin\alpha^-}{\sin\beta^+}=\frac{A_2A_3}{A_1A_2},\quad \frac{\sin\beta^-}{\sin\gamma^+}=\frac{A_4A_5}{A_3A_4},\quad \text{and}\quad \frac{\sin\gamma^-}{\sin\alpha^+}=\frac{A_6A_1}{A_5A_6}. \tag{11}$$

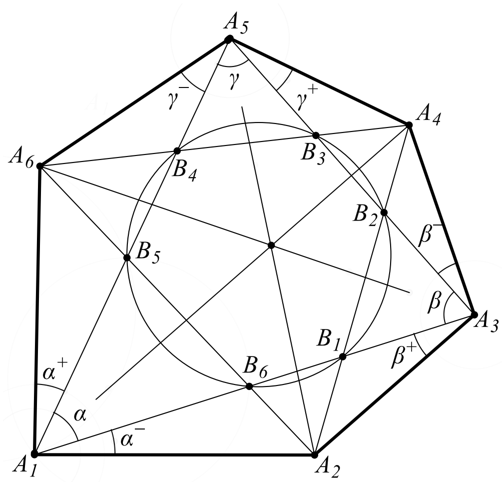

Figure 5: The main diagonals in the convex hexagon $A_1A_2A_3A_4A_5A_6$ are always concurrent, while those in the cyclic hexagon $B_1B_2B_3B_4B_5B_6$ may not be.

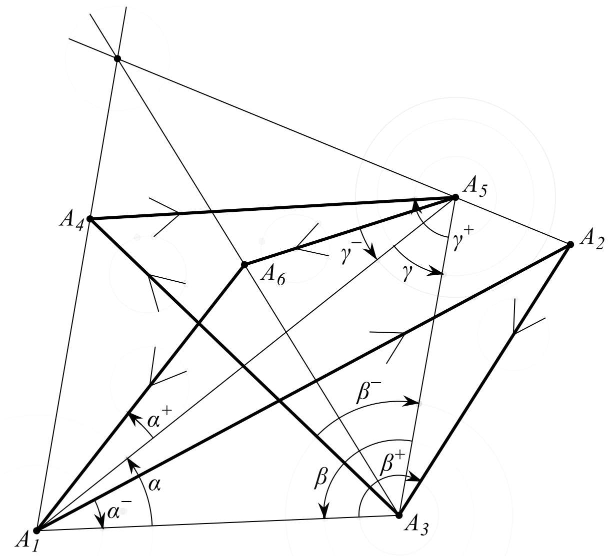

Figure 6: A non-convex, non-simple, hexagonal path in general position with concurrent main diagonals, and the nine relevant oriented angles.

Combining now three implementations of the Law of Sines respectively to triangles, $\triangle A_6A_1B_6$, $\triangle A_2B_1B_6$ and $\triangle A_3A_4B_1$, we have

$$\frac{\sin(\alpha+\alpha^+)}{\sin(\beta+\beta^-)}=\frac{A_6B_6}{A_6A_1}\frac{A_2B_1}{A_2B_6}\frac{A_3A_4}{A_4B_1}, \tag{12}$$

and similarly,

$$\frac{\sin(\beta+\beta^+)}{\sin(\gamma+\gamma^-)}=\frac{A_2B_2}{A_2A_3}\frac{A_4B_3}{A_4B_2}\frac{A_5A_6}{A_6B_3}\quad\text{and}\quad\frac{\sin(\gamma+\gamma^+)}{\sin(\alpha+\alpha^-)}=\frac{A_4B_4}{A_4A_5}\frac{A_6B_5}{A_6B_4}\frac{A_1A_2}{A_2B_5}. \tag{13}$$

Multiplying together equations (11), (12), and (13), and simplifying yields now

$$\begin{aligned}&\frac{\sin(\alpha+\alpha^+)\sin(\beta+\beta^+)\sin(\gamma+\gamma^+)\sin\alpha^-\sin\beta^-\sin\gamma^-}{\sin(\alpha+\alpha^-)\sin(\beta+\beta^-)\sin(\gamma+\gamma^-)\sin\alpha^+\sin\beta^+\sin\gamma^+}=\\&\frac{A_6B_5\cdot A_6B_6}{A_6B_4\cdot A_6B_3}\frac{A_2B_1\cdot A_2B_2}{A_2B_6\cdot A_2B_5}\frac{A_4B_3\cdot A_4B_4}{A_4B_2\cdot A_4B_1}.\end{aligned} \tag{14}$$

However, each one of the three ratios contained on the right hand side of equation (14) equals 1, due to the well-known invariance of the power of a point exterior to a circle. □

3. Final Remarks

We conclude the paper with some substantive remarks.

1. Sine-Concurrency Theorem for Hexagonal Paths. The Sine-Concurrency Theorem still holds true for non-convex, in fact even non-simple (when viewed as closed polygonal curves), hexagonal paths $A_1A_2A_3A_4A_5A_6$, however we need to be more careful how we measure the angles involved. First, the hexagonal path has to be in general position, that is no lines through any two of its vertices may be identical or parallel (in particular, no three vertices may be collinear). Second, all the angles considered have to be oriented angles.

For a proper angle, say $\widehat{BAC}$, with vertex A and rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ we define its oriented measure, $m(\widehat{BAC}) = \theta$, as being the (real) angle θ (in radians), $0 < |\theta| < \pi$, required to rotate (about vertex A) the ray $\overrightarrow{AB}$ over the ray $\overrightarrow{AC}$. The measure will be positive if this rotation is counterclockwise, and negative if it is clockwise. So for oriented angles, $m(\widehat{CAB}) = -m(\widehat{BAC})$. Then, just as in the Sine-Concurrency Theorem, the main diagonal lines $\overleftrightarrow{A_1A_4}$, $\overleftrightarrow{A_2A_5}$, and $\overleftrightarrow{A_3A_6}$ will be concurrent if and only if equation (1) holds, where $\alpha = m(\widehat{A_3A_1A_5})$, $\beta = m(\widehat{A_5A_3A_1})$, $\gamma = m(\widehat{A_1A_5A_3})$, $\alpha^- = m(\widehat{A_2A_1A_3})$, $\alpha^+ = m(\widehat{A_5A_1A_6})$, $\beta^- = m(\widehat{A_4A_3A_5})$, $\beta^+ = m(\widehat{A_1A_3A_2})$, $\gamma^- = m(\widehat{A_6A_5A_1})$, and $\gamma^+ = m(\widehat{A_3A_5A_4})$. Notice that the same letter angle measures correspond to angles sharing the same vertex. For a more unorthodox implementation of these notations, see Figure 6.

A proof similar to that given in the convex case may be attempted. It requires a 'Signed Law of Sines' and many particular cases need to be considered. For a compact analytic proof, see [1].

2. Sine-Collinearity Theorem. It is well-known that Desargues' Theorem [7] renders the concurrency of three line equivalent to the collinearity of three points. When teamed up with the Sine-Concurrency Theorem it produces the following

Sine-Collinearity Theorem. *Given a convex hexagon $A_1A_2A_3A_4A_5A_6$ with vertices in general position, consider the three intersecting points of corresponding sides in $\triangle A_1A_2A_3$ and $\triangle A_4A_5A_6$. More precisely, let lines $\overleftrightarrow{A_1A_2}$ and $\overleftrightarrow{A_4A_5}$ intersect at M_1, lines $\overleftrightarrow{A_2A_3}$ and $\overleftrightarrow{A_5A_6}$ intersect at M_2, and lines $\overleftrightarrow{A_3A_1}$ and $\overleftrightarrow{A_6A_4}$ intersect at M_3 (cf. Figure 7). Then the points M_1, M_2, and M_3 are collinear if and only if for the angle measures α, α^+, α^-, β, β^+, β^-, and γ, γ^+, γ^- associated as before in connection with $\triangle A_1A_3A_5$ equation (1) holds true.*

A direct proof (without Desargues' Theorem) of the Sine-Collinearity Theorem can be found in [1].

3. Sine-Cross Ratio Theorem. Recall that the cross ratio [6, 7] of four (distinct) *collinear* points C_1, C_2, C_3, and C_4 in some plane, denoted $[C_1, C_2, C_3, C_4]$, is the real number

$$[C_1, C_2, C_3, C_4] := \frac{\vec{C_1C_3}}{\vec{C_3C_2}} \Bigg/ \frac{\vec{C_1C_4}}{\vec{C_4C_2}}, \tag{15}$$

where for two points A and B we denote by $\vec{AB}$ the vector with origin A and end B (different from $\overrightarrow{AB}$, by which we denote the ray originating at A through the point B). In general two vectors cannot be divided, except when they are proportional, as in (15), in which case by their ratio we mean the proportionality constant. Then the following theorem holds true:

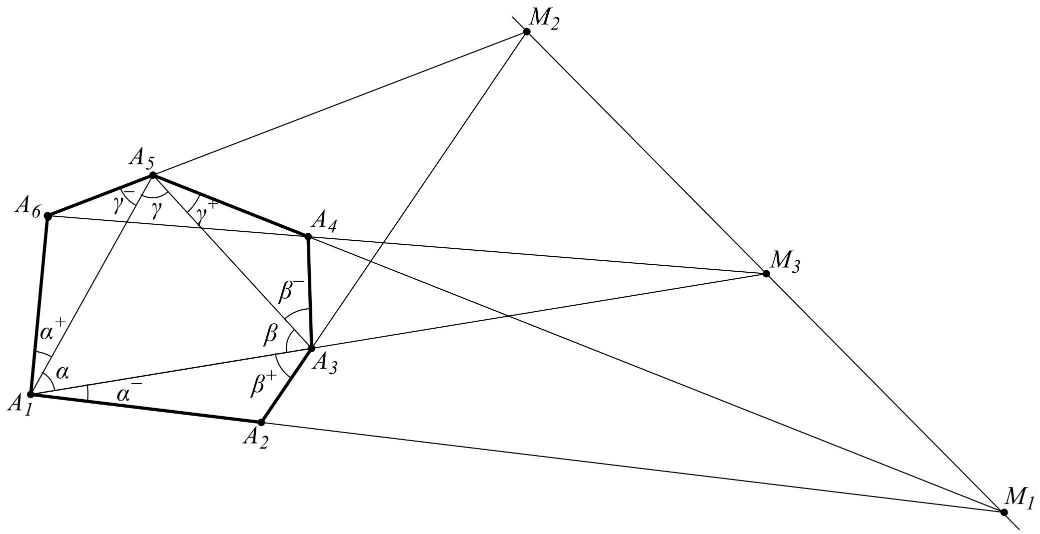

Figure 7: A convex hexagon exhibiting collinearity and the nine relevant angles, as in the Sine-Collinearity Theorem.

Sine-Cross Ratio Theorem. *In the convex hexagon in standard position* $A_1A_2A_3A_4A_5A_6$, *let* E_1, E_2, E_3, *and* E_4, *be the intersection points of* $\overleftrightarrow{A_1A_2}$, $\overleftrightarrow{A_2A_3}$, $\overleftrightarrow{A_3A_4}$, *and* $\overleftrightarrow{A_4A_5}$ *with* $\overleftrightarrow{A_5A_6}$, *respectively, and similarly let* F_1, F_2, F_3, *and* F_4, *be the intersection points of those same four lines with* $\overleftrightarrow{A_6A_1}$ *(cf. Figure 8). Then the trigonometric equation* (1), *associated as before to* $\triangle A_1A_3A_5$ *and to the angle measures* α, α^+, α^-, β, β^+, β^-, *and* γ, γ^+, γ^-, *holds true if and only if*

$$[E_1, E_2, E_3, E_4] = [F_1, F_2, F_3, F_4], \tag{16}$$

where $[E_1, E_2, E_3, E_4]$ *and* $[F_1, F_2, F_3, F_4]$ *stand for the cross ratios of those respective points.*

For a proof of the Sine-Cross Ratio Theorem, see [2].

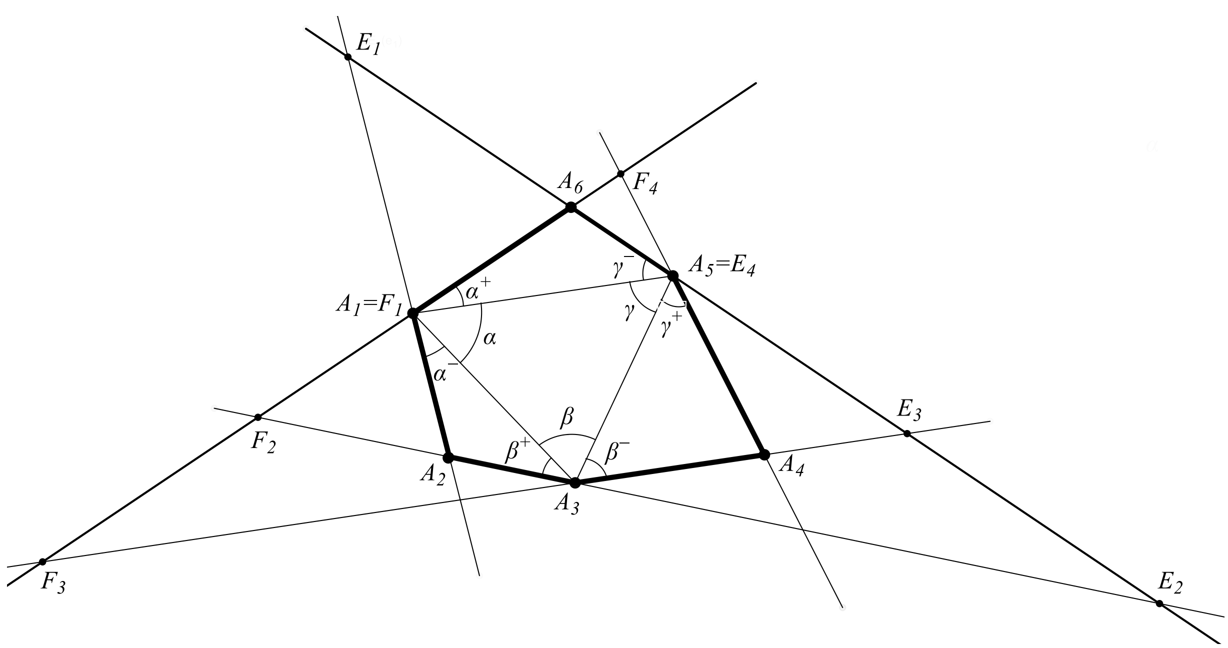

Figure 8: A hexagon with cross-ratio points E_1, E_2, E_3, E_4, and F_1, F_2, F_3, F_4

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