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Generalization of the Pappus Theorem in the Plane and in Space

Victor Oxman [1,2], **Avi Sigler** [1], **Moshe Stupel** [1,3]

[1] *Shaanan College, P.O.B. 906, Haifa, Israel*
email: victor.oxman@gmail.com

[2] *Western Galilee College, Acre, Israel*

[3] *Gordon College, Haifa, Israel*

Abstract. One of the Pappus theorems states that if points F, E, D divide the sides of triangle ABC in the same ratio α, then the triangles ABC and FED have the same centroid. Therefore, the intersection $\mathcal{Q}$ of such triangles FED obtained for all non-negative α, is not empty. In this paper we will characterize the domain $\mathcal{Q}$ for general case of dividing the sides of a triangle (not necessary in the same ratio) and prove that $\mathcal{Q}$ is bound by conic sections. We will also present some surprising results concerning $\mathcal{Q}$ for the case of tetrahedron.

Key Words: Pappus Theorem, triangle, tetrahedron, conic sections

MSC 2010: 51M05

1. Introduction

One of the great ancient Greek mathematicians Pappus of Alexandria formulated and proved in his Book 8 the following statement [1, p. 10]:

> *"We make* ABG *a triangle and let its sides be divided in equal ratios at points* H, T *and* K*, such that the ratio of* BT *to* TG *is as the ratio of* AH *to* HB *and as that of* GK *to* KA*; let us join lines* HT, TK, KH*. Now I maintain that the centre of gravity of the two triangles* ABG *and* HTK *is one and the same."*

In other words, this Pappus theorem states: If the sides of some triangle ABC are divided by points F, E and D in the same ratio

$$\frac{\mathrm{BD}}{\mathrm{DC}} = \frac{\mathrm{CE}}{\mathrm{EA}} = \frac{\mathrm{AF}}{\mathrm{FB}} = \alpha,$$

then the triangles ABC and FED have the same centroid (see Figure 1). The triangle FED is sometimes called *Pappus' triangle*.

There are many ways to prove this theorem. It can be also generalized for n points in $\mathbb{R}^m$. We shall bring a simple proof for the generalized version:

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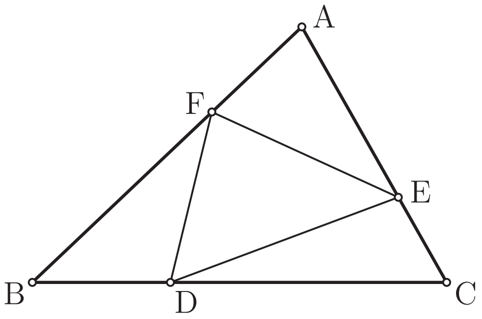

Figure 1: BD/DC = CE/EA = AF/FB.

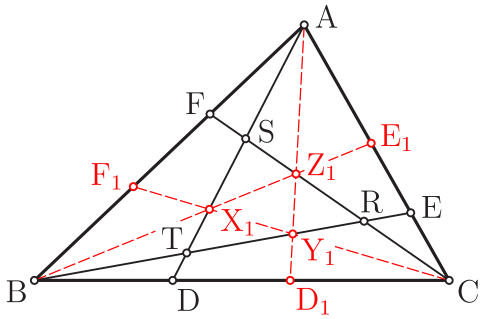

Figure 2: Illustration to the proof of Theorem 2.

Theorem 1. *Let* M *be the centroid of the points* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^m$. *And let the points* $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ *be on the line segments* $\mathbf{x}_1\mathbf{x}_2$, $\mathbf{x}_2\mathbf{x}_3$, $\ldots$, $\mathbf{x}_n\mathbf{x}_1$, *respectively, and divide this segments in the same ratio* α, *to say,* $\mathbf{x}_i\mathbf{w}_i/\mathbf{w}_i\mathbf{x}_{i+1} = \alpha$ *for all* i, $1 \leq i \leq n$, *and* $\mathbf{x}_{n+1} = \mathbf{x}_1$. *Then* M *is also the centroid of the points* $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$.

Proof. We have $\mathrm{M} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$. Since $\mathbf{w}_i = \dfrac{1}{1+\alpha}(\mathbf{x}_i + \alpha\mathbf{x}_{i+1})$, we obtain for the centroid of the points $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i = \frac{1}{n(1+\alpha)}\sum_{i=1}^{n}(\mathbf{x}_i + \alpha\mathbf{x}_{i+1}) = \frac{(1+\alpha)}{n(1+\alpha)}\sum_{i=1}^{n}\mathbf{x}_i = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i = \mathrm{M}. \qquad \square$$

Now we consider the general case for a triangle, where the points F, E, D divide the triangle's sides not necessarily in the same ratio

$$\frac{\mathrm{BD}}{\mathrm{DC}} = \alpha, \quad \frac{\mathrm{CE}}{\mathrm{EA}} = \beta, \quad \frac{\mathrm{AF}}{\mathrm{FB}} = \gamma.$$

Let us denote the triangle DEF by $\Delta(\alpha,\beta,\gamma)$ and $\bigcap_{k\geq 0}\Delta(k\alpha,k\beta,k\gamma)$ by $\mathcal{Q}(\alpha,\beta,\gamma)$.

Corollary 1. *Theorem 1 implies* $\mathcal{Q}(\alpha,\alpha,\alpha) \neq \emptyset$, *because the centroid* $\mathrm{M} \in \mathcal{Q}(\alpha,\alpha,\alpha)$.

2. The general case of dividing the sides of a triangle

Corollary 1 can be generalized in following way.

Theorem 2. *For given non-negative* α,β,γ *holds* $\mathcal{Q}(\alpha,\beta,\gamma) \neq \emptyset$.

Proof. Let $\mathrm{S} = \mathrm{AD} \cap \mathrm{FC}$, $\mathrm{T} = \mathrm{AD} \cap \mathrm{BE}$, $\mathrm{R} = \mathrm{BE} \cap \mathrm{FC}$. Then there are three points X_1, Y_1, Z_1 on ST, TR and RS, respectively, which satisfythe following conditions: A lies on line $\mathrm{Z}_1\mathrm{Y}_1$, B lies on line $\mathrm{X}_1\mathrm{Z}_1$, and C lies on line $\mathrm{X}_1\mathrm{Y}_1$ (see Figure 2).

To prove this statement let's choose points F_1, E_1, D_1 on AB, AC and BC, respectively, which divide the triangle's side in the ratios

$$\frac{\mathrm{BD}_1}{\mathrm{D}_1\mathrm{C}} = \sqrt{\frac{\alpha}{\beta\gamma}} = \alpha_1, \quad \frac{\mathrm{CE}_1}{\mathrm{E}_1\mathrm{A}} = \sqrt{\frac{\beta}{\alpha\gamma}} = \beta_1, \quad \frac{\mathrm{AF}_1}{\mathrm{F}_1\mathrm{B}} = \sqrt{\frac{\gamma}{\alpha\beta}} = \gamma_1\,.$$

Then $\dfrac{\mathrm{CE}_1}{\mathrm{E}_1\mathrm{A}}\cdot\dfrac{\mathrm{AF}_1}{\mathrm{F}_1\mathrm{B}}\cdot\dfrac{\mathrm{BD}}{\mathrm{DC}} = \sqrt{\dfrac{\beta}{\alpha\gamma}}\cdot\sqrt{\dfrac{\gamma}{\alpha\beta}}\cdot\alpha = 1$. Therefore, according to the Ceva's theorem [2, pp. 137–138], CF_1, BE_1 and AD meet at a common point X_1. Similarly,we obtain two other points Y_1 and Z_1.

By the same way we can find the points X_2, Y_2 and Z_2 on the sides of the triangle $X_1Y_1Z_1$, whose appropriate points F_2, E_2, D_2 on the sides of triangle ABC divide them in the ratios

$$\alpha_2 = \frac{BD_2}{D_2C} = \sqrt{\frac{\alpha_1}{\beta_1\gamma_1}}, \quad \beta_2 = \frac{CE_2}{E_2A} = \sqrt{\frac{\beta_1}{\alpha_1\gamma_1}}, \quad \gamma_2 = \frac{AF_2}{F_2B} = \sqrt{\frac{\gamma_1}{\beta_1\alpha_1}}.$$

By continuing this process we get

$$\alpha_{n+1} = \frac{BD_{n+1}}{D_{n+1}C} = \sqrt{\frac{\alpha_n}{\beta_n\gamma_n}} = \alpha_n^{0.5}(\beta_n\gamma_n)^{-0.5}, \quad \beta_{n+1} = \frac{CE_{n+1}}{E_{n+1}A} = \sqrt{\frac{\beta_n}{\alpha_n\gamma_n}} = \beta_n^{0.5}(\alpha_n\gamma_n)^{-0.5},$$
$$\gamma_{n+1} = \frac{AF_{n+1}}{F_{n+1}B} = \sqrt{\frac{\gamma_n}{\beta_n\alpha_n}} = \gamma_n^{0.5}(\beta_n\alpha_n)^{-0.5}.$$

One can prove by mathematical induction that

$$\alpha_n = \left(\alpha^{p_n}(\beta\gamma)^{-p_{n-1}}\right)^{2^{-n}}, \quad \beta_n = \left(\beta^{p_n}(\alpha\gamma)^{-p_{n-1}}\right)^{2^{-n}}, \quad \gamma_n = \left(\gamma^{p_n}(\alpha\beta)^{-p_{n-1}}\right)^{2^{-n}},$$

where $p_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right)$. Then

$$\lim_{n\to\infty} \alpha_n = \sqrt[3]{\frac{\alpha^2}{\beta\gamma}}, \quad \lim_{n\to\infty} \beta_n = \sqrt[3]{\frac{\beta^2}{\alpha\gamma}}, \quad \lim_{n\to\infty} \gamma_n = \sqrt[3]{\frac{\gamma^2}{\alpha\beta}} \quad \text{and} \quad \lim_{n\to\infty} \alpha_n\beta_n\gamma_n = 1.$$

Therefore, while $n \to \infty$, the triangles $X_nY_nZ_n$ tend to a point which we denote by M_1. Obviously, $M_1 \in \Delta(\alpha,\beta,\gamma)$ and $M_1 \in \Delta(k\alpha,k\beta,k\gamma)$ for each $k \geq 0$. It means that $M_1 \in \bigcap_{k\geq 0} \Delta(k\alpha,k\beta,k\gamma) \neq \emptyset$. □

Note that for $\alpha = \beta = \gamma$ point M_1 is the centroid of the triangle ABC.

3. Characterization of the domain $\mathcal{Q}(\alpha,\beta,\gamma)$

We proved above that $\mathcal{Q}(\alpha,\beta,\gamma) \neq \emptyset$, because $M_1 \in \mathcal{Q}(\alpha,\beta,\gamma)$. The question arises whether there are any other points in this domain.

Theorem 3. *Let* AD, BE *and* CF *be three concurrent cevians of triangle* ABC *that meet at point* O *and let* $\frac{BD}{DC} = \alpha$, $\frac{CE}{EA} = \beta$ *and* $\frac{AF}{FB} = \gamma$. *If for arbitrary three points* $D_1 \in BC$, $E_1 \in AC$, $F_1 \in AB$ *with*

$$\frac{BD_1}{D_1C} = \alpha_1, \quad \frac{CE_1}{E_1A} = \beta_1, \quad \frac{AF_1}{F_1B} = \gamma_1$$

one of two following conditions holds, a) $\alpha_1 \leq \alpha$, $\beta_1 \leq \beta$, $\gamma_1 \leq \gamma$ *or b)* $\alpha \leq \alpha_1$, $\beta \leq \beta_1$, $\gamma \leq \gamma_1$, *then* $O \in \Delta(\alpha_1,\beta_1,\gamma_1)$.

Note: Geometrically the cases a) and b) mean that the three points D_1, E_1 and F_1 belong to the segments BD, CE, AF or DC, EA, FB, respectively (see Figures 3 and 4).

Proof. Let $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that transforms the triangle ABC to the triangle $A_1B_1C_1$ with coordinates $A_1 = (0,1)$, $B_1 = (0,0)$, $C_1 = (1,0)$. Then

$$G(O) = O_1, \quad G(D_1) = D_2 \in B_1C_1, \quad \frac{B_1D_2}{D_2C_1} = \alpha_1, \quad G(E_1) = E_2 \in A_1C_1, \quad \frac{C_1E_2}{E_2A_1} = \beta_1,$$
$$G(F_1) = F_2 \in A_1B_1, \quad \frac{A_1F_2}{F_2B_1} = \gamma_1.$$

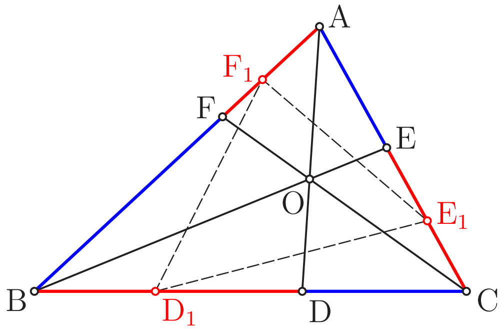

Figure 3: D_1, E_1 and F_1 belong to the segments BD, CE, AF.

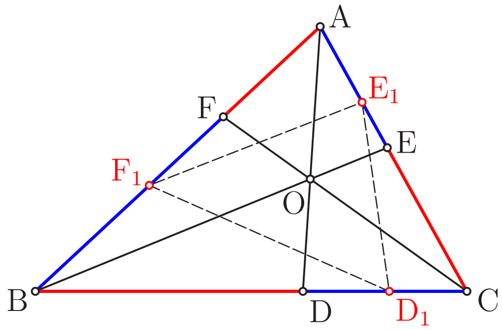

Figure 4: D_1, E_1 and F_1 belong to the segments DC, EA, FB.

Then the coordinates of the points are

$$D_2=\left(\frac{\alpha_1}{\alpha_1+1},\ 0\right),\quad E_2=\left(\frac{1}{\beta_1+1},\ \frac{\beta_1}{\beta_1+1}\right),\quad F_2=\left(0,\ \frac{1}{\gamma_1+1}\right).$$

O_1 is an interior point of $\Delta(\alpha_1,\beta_1,\gamma_1)$ (see Figure 5) if, and only if, the following system of equations has a non-negative solution $\lambda_1,\lambda_2,\lambda_3$, where $\lambda_1,\lambda_2,\lambda_3$ are barycentric coordinates of O_1 with respect to the triangle ABC [3, pp. 216–221]:

$$\begin{cases} \dfrac{\lambda_1\alpha_1}{\alpha_1+1}+\dfrac{\lambda_2}{\beta_1+1} = \dfrac{\alpha}{\alpha\beta+\alpha+1}, \\ \dfrac{\lambda_2\beta_1}{\beta_1+1}+\dfrac{\lambda_3}{\gamma_1+1} = \dfrac{\alpha\beta}{\alpha\beta+\alpha+1}, \\ \lambda_1+\lambda_2+\lambda_3 = 1. \end{cases}$$

From this system one can find that

$$\lambda_1=\frac{1+\frac{\gamma_1}{\gamma}\left(\frac{\beta_1-\beta}{\beta}\right)}{(\alpha\beta+\alpha+1)\left(\frac{\alpha_1\beta_1\gamma_1+1}{\alpha_1+1}\right)}.$$

Similar formulas can be obtained for λ_2 and λ_3. Thus we have two cases:

a) $\alpha_1\le\alpha$, $\beta_1\le\beta$, $\gamma_1\le\gamma$: In this case $-1\le\frac{\gamma_1}{\gamma}\left(\frac{\beta_1-\beta}{\beta}\right)\le 0$ and so $0\le\lambda_1$. Similarly $0\le\lambda_2$ and $0\le\lambda_3$.

b) $\alpha\le\alpha_1$, $\beta\le\beta_1$, $\gamma\le\gamma_1$: In this case $0\le\frac{\gamma_1}{\gamma}\left(\frac{\beta_1-\beta}{\beta}\right)$ and so $0<\lambda_1$. Similarly $0<\lambda_2$ and $0<\lambda_3$. □

Corollary 2. *It is easy to see that Corollary 1 is a particular case of Theorem 3.*

Proof. Indeed, for given α,β,γ we can choose $k=\sqrt[3]{\alpha\beta\gamma}$ that satisfies $(k\alpha)(k\beta)(k\gamma)=1$, and so the point $S=(k\alpha,\ k\beta,\ k\gamma)$ is inside the triangle $\Delta(k\alpha,\ k\beta,\ k\gamma)$. Therefore S is also an interior point of all the triangles $\Delta(t\alpha,\ t\beta,\ t\gamma)$. □

Let M be the point of intersections of three concurrent cevians AD, BE and CF in the given triangle ABC, where $\frac{BD}{DC}=\alpha$, $\frac{CE}{EA}=\beta$ and $\frac{AF}{FB}=\gamma$. Let's denote M by $M=(\alpha,\beta,\gamma)$ and let's denote Q_ρ the locus of all the points (α,β,γ) which satisfy $\rho=\beta/\gamma$. In the following theorem we will characterize this curve Q_ρ.

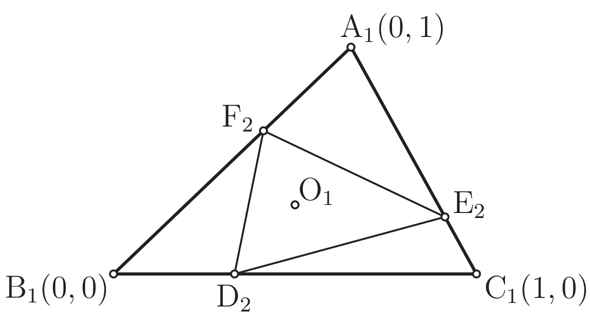

Figure 5: O_1 is an interior point of $\Delta F_2E_2D_2 = \Delta(\alpha_1, \beta_1, \gamma_1)$.

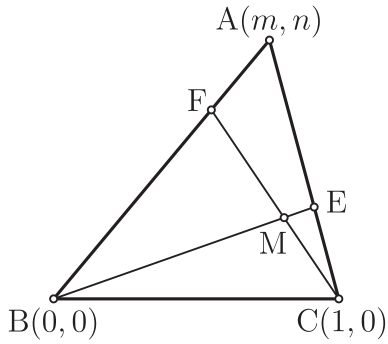

Figure 6: Illustration to the proof of Theorem 4.

Theorem 4. *Let $\rho = \beta/\gamma$ be a positive constant. Then the curve Q_ρ is* a) *an ellipse, when $\rho < 4$;* b) *a parabola, when $\rho = 4$; and* c) *a hyperbola, when $\rho > 4$.*

Proof. Without loss of generality we can assume that $A = (m, n)$, $B = (0, 0)$, $C = (1, 0)$ (see Figure 6). Since $\frac{CE}{EA} = \beta$ and $\frac{AF}{FB} = \gamma$, then

$$E = \left(\frac{1+\beta m}{\beta+1}, \frac{\beta n}{\beta+1}\right) \quad \text{and} \quad F = \left(\frac{m}{\gamma+1}, \frac{n}{\gamma+1}\right).$$

For $M = (x, y)$ we obtain the following system of equations:

$$\begin{cases} \dfrac{x}{y} = \dfrac{1+\beta m}{\beta n}, \\ \dfrac{x-1}{y} = \dfrac{m-1-\gamma}{n}. \end{cases}$$

This system and $\beta/\gamma = \rho$ imply

$$\rho n^2 x^2 + (\rho m^2 - \rho m + 1)y^2 + (\rho n - 2\rho mn)xy - \rho n^2 x + mn\rho y = 0.$$

This is a quadratic curve

$$ax^2 + 2hxy + by^2 + 2fx + 2gy + c = 0,$$

where $a = n^2\rho$, $b = \rho m^2 - \rho m + 1$, $c = 0$, $h = \frac{1}{2}(\rho n - 2\rho mn)$, $f = -\frac{1}{2}\rho n^2$, $g = \frac{1}{2}mn\rho$. The discriminant Δ of the quadratic equation is

$$\Delta = ab - h^2 = n^2\rho(\rho m^2 - \rho m + 1) - \left(\frac{\rho n - 2\rho mn}{2}\right)^2 = \frac{n^2\rho(4-\rho)}{4}.$$

Thus we obtain

- If $\Delta = 0$, i.e., $\rho = 4$, the locus is a parabola;
- If $\Delta > 0$, i.e., $\rho < 4$, the locus is an ellipse;
- If $\Delta < 0$, i.e., $\rho > 4$, the locus is a hyperbola.

Obviously, the same result can be obtained for the loci of points for which $\gamma/\alpha = \rho$ or $\alpha/\beta = \rho$. □

Theorem 5. *Point* $\mathrm{O} = (\alpha_1, \beta_1, \gamma_1) \in \mathcal{Q}(\alpha, \beta, \gamma)$ *if and only if the three following conditions are satisfied:*

$$\frac{\beta_1}{\gamma_1} \le \frac{4\beta}{\gamma}, \quad \frac{\gamma_1}{\alpha_1} \le \frac{4\gamma}{\alpha}, \quad \frac{\alpha_1}{\beta_1} \le \frac{4\alpha}{\beta}.$$

Proof. As it was mentioned in the proof of Theorem 3, $\mathrm{O} = (\alpha_1, \beta_1, \gamma_1)$ is an interior point of the triangle $\Delta(k\alpha, k\beta, k\gamma)$ if and only if the three following inequalities are satisfied:

$$1 + \frac{k\gamma}{\gamma_1}\left(\frac{k\beta - \beta_1}{\beta_1}\right) \ge 0, \quad 1 + \frac{k\alpha}{\alpha_1}\left(\frac{k\gamma - \gamma_1}{\gamma_1}\right) \ge 0, \quad 1 + \frac{k\beta}{\beta_1}\left(\frac{k\alpha - \alpha_1}{\alpha_1}\right) \ge 0.$$

Therefore $\mathrm{O} = (\alpha_1, \beta_1, \gamma_1) \in \mathcal{Q}(\alpha, \beta, \gamma)$ if and only if these inequalities are satisfied for each $k \ge 0$. Since the first inequality is equivalent to $\gamma\beta k^2 - \beta_1\gamma k + \beta_1\gamma_1 \ge 0$, we obtain that $(\beta_1\gamma)^2 - 4\gamma\beta\beta_1\gamma_1 \le 0$, i.e., $\beta_1/\gamma_1 \le 4\beta/\gamma$. Similarly, the two other inequalities imply $\gamma_1/\alpha_1 \le 4\gamma/\alpha$ and $\alpha_1/\beta_1 \le 4\alpha/\beta$. □

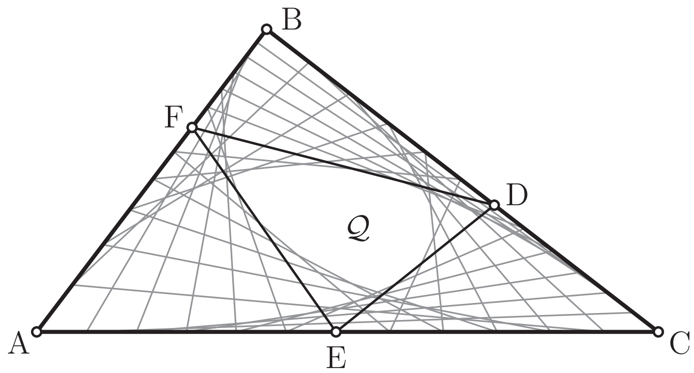

Figure 7: Illustration to Corollary 3.

Corollary 3. *Let's denote* $\rho_1 = 4\beta/\gamma$, $\rho_2 = 4\gamma/\alpha$, $\rho_3 = 4\alpha/\beta$. *Let* D_{ρ_1}, D_{ρ_2}, D_{ρ_3} *be the domains inside the triangle* ABC, *bound by conic sections* Q_{ρ_1}, Q_{ρ_2} *and* Q_{ρ_3}, *respectively. Then* $\mathcal{Q}(\alpha, \beta, \gamma) = D_{\rho_1} \cap D_{\rho_2} \cap D_{\rho_3}$ *(see Figure 7).*

This corollary follows directly from the two previous theorems.

Corollary 4. *If* $\alpha = \beta = \gamma$, *then* $\mathcal{Q}(\alpha, \alpha, \alpha)$ *is the intersection of all Pappus triangles (for all* $\alpha \ge 0$*). In this case* $\rho_1 = \rho_2 = \rho_3 = 4$, *and so* $\mathcal{Q}(\alpha, \alpha, \alpha)$ *is bound by three identical parabolas. This is a well-known fact because for a given angle* AOB *the envelope of the family of segments* XY *satisfying* AX/XO = OY/YB *is a parabola touching the angle's rays at points* A *and* B *(see Figure 7).*

Theorem 6. *For any two domains* $\mathcal{Q}_1 = \mathcal{Q}(\alpha_1, \beta_1, \gamma_1)$ *and* $\mathcal{Q}_2 = \mathcal{Q}(\alpha_2, \beta_2, \gamma_2)$ *with* $\alpha_1\beta_1\gamma_1 = \alpha_2\beta_2\gamma_2 = 1$, *there is a homography (projective transformation)* $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ *so that* $G(\mathcal{Q}_1) = \mathcal{Q}_2$.

Proof. It is known that there is a unique homography of a plane which transforms given four points not including any three collinear ones into four points. Let $\mathrm{D}_1(\alpha_1, \beta_1, \gamma_1)$ and $\mathrm{D}_2(\alpha_2, \beta_2, \gamma_2)$ be two points in the triangle ABC and let $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homography such that $G(\mathrm{A}) = \mathrm{A}$, $G(\mathrm{B}) = \mathrm{B}$, $G(\mathrm{C}) = \mathrm{C}$, and $G(\mathrm{D}_1) = \mathrm{D}_2$. Then $G(\mathrm{E}_1) = \mathrm{E}_2$ where

$$\frac{\mathrm{BE}_1}{\mathrm{E}_1\mathrm{C}} = \alpha_1, \quad \frac{\mathrm{BE}_2}{\mathrm{E}_2\mathrm{C}} = \alpha_2, \quad (\mathrm{E}_1 = \mathrm{AD}_1 \cap \mathrm{BC} \quad \mathrm{E}_2 = \mathrm{AD}_2 \cap \mathrm{BC}).$$

Keeping in the mind that for any point $M = M(\alpha, \beta, \gamma)$ in the triangle ABC holds $\alpha\beta\gamma = 1$, it is easy to check that

$$G(M) = M_1 = \left(\alpha \cdot \frac{\alpha_2}{\alpha_1},\ \beta \cdot \frac{\beta_2}{\beta_1},\ \gamma \cdot \frac{\gamma_2}{\gamma_1}\right).$$

This means that if point E_k on the side BC satisfies $k\alpha_1 = BE_k/E_kC$, then $G(E_k) = F_k$, and F_k satisfies $k\alpha_2 = BF_k/F_kC$. Therefore each triangle with vertices dividing the sides BC, CA, AB in ratios $k\alpha_1$, $k\beta_1$, $k\gamma_1$, respectively, is transformed to a triangle whose vertices divide BC, CA, AB in ratios $k\alpha_2$, $k\beta_2$, $k\gamma_2$, i.e., $G(\mathcal{Q}_1) = \mathcal{Q}_2$. □

Corollary 5. *It is impossible that* $\mathcal{Q}_1 \subset \mathcal{Q}_2$ *or* $\mathcal{Q}_2 \subset \mathcal{Q}_1$.

Proof. Let's suppose that $\mathcal{Q}_1 \subset \mathcal{Q}_2$. Since G is a continuous function that transforms the vertices A,B,C to themselves and point $M_1(\alpha_1, \beta_1, \gamma_1)$ to $M_2(\alpha_2, \beta_2, \gamma_2)$, then according to Brouwer's fixed-point theorem [4, p. 170] G has a fixed-point in $\mathcal{Q}_1$. This means that G is the identity function (four points are transformed to themselves) which contradicts $G(M_1) = M_2$. □

4. Generalization of Pappus theorem for tetrahedra

The Pappus theorem can be formulated with the help of barycentric coordinates. Let the vertexes of the triangle ABC have barycentric coordinates $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. Then the points F, E, D are dividing the sides AB, AC and BC in the same ratio, and they have barycentric coordinates $F = (\lambda_1, \lambda_2, 0)$, $E = (\lambda_2, 0, \lambda_1)$ and $D = (0, \lambda_1, \lambda_2)$, where λ_1, λ_2 are non-negative numbers with $\lambda_1 + \lambda_2 = 1$. Let's denote the triangle DEF by

$$\begin{bmatrix} 0 & \lambda_1 & \lambda_2 \\ \lambda_2 & 0 & \lambda_1 \\ \lambda_1 & \lambda_2 & 0 \end{bmatrix}.$$

Then, according to the Pappus theorem, point M (the triangle's centroid) with barycentric coordinates

$$M = \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right) \in \bigcap_{\substack{\lambda_1, \lambda_2 \geq 0 \\ \lambda_1 + \lambda_2 = 1}} \begin{bmatrix} 0 & \lambda_1 & \lambda_2 \\ \lambda_2 & 0 & \lambda_1 \\ \lambda_1 & \lambda_2 & 0 \end{bmatrix}.$$

This result can be generalized for tetrahedra, i.e., for the case of 3-space. Let

$$L = (\lambda_1, \lambda_2, \lambda_3, 0), \quad N = (0, \lambda_1, \lambda_2, \lambda_3), \quad P = (\lambda_3, 0, \lambda_1, \lambda_2), \quad Q = (\lambda_2, \lambda_3, 0, \lambda_1)$$

be points on the faces of a tetrahedron ABCD, where $\lambda_1, \lambda_2, \lambda_3 \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Let's denote the tetrahedron LNPQ by

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 & 0 & \lambda_1 \end{bmatrix} \quad \text{and} \quad \bigcap_{\substack{\lambda_1, \lambda_2, \lambda_3 \geq 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 1}} \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 & 0 & \lambda_1 \end{bmatrix} \quad \text{by } \mathcal{T}.$$

Theorem 7. $M = \left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right) \in \mathcal{T}.$

Proof. $\begin{pmatrix} x_L \\ x_N \\ x_P \\ x_Q \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 & 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \\ x_D \end{pmatrix}$. So

$$\frac{x_L + x_N + x_P + x_Q}{4} = \frac{(x_A + x_B + x_C + x_D)(\lambda_1 + \lambda_2 + \lambda_3)}{4} = \frac{x_A + x_B + x_C + x_D}{4}.$$

By the same way a similar equality can be obtained for the other coordinates of L, N, P, and Q. Thus the tetrahedra ABCD and LNPQ have a common centroid $\mathrm{M} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. □

Similarly to the triangle's case we can ask if there are other points in $\mathcal{T}$. The following theorem gives a surprising answer to this question.

Theorem 8. $\mathcal{T} = \{\mathrm{M}\}$ *for* $\mathrm{M} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

Proof. For $\lambda_1 = \lambda_2 = \frac{1}{2}$, $\lambda_3 = 0$ we obtain a degenerated flat tetrahedron with the vertices at the midpoints of AB, BC, CD, and DA. These points form parallelogram $\mathcal{P}$. Thus $\mathcal{T} \subseteq \mathcal{P}$. For $\lambda_1 = \lambda_3 = \frac{1}{2}$, $\lambda_2 = 0$ the tetrahedron degenerates to the line segment $\mathcal{S}$ with the ends at the midpoints of DB and AC. Therefore $\mathcal{T} \subseteq \mathcal{P} \cap \mathcal{S}$. Since the segment $\mathcal{S}$ does not lie in the plane of the parallelogram $\mathcal{P}$, $\{\mathrm{M}\} = \mathcal{P} \cap \mathcal{S}$ and thus $\{\mathrm{M}\} = \mathcal{T}$. □

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