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To the Volumes Theory of a Hyperbolic Space of Positive Curvature

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This paper is devoted to Nikolai Ivanovich Lobachevskiĭ

Abstract. In the Cayley-Klein model a hyperbolic space $\widehat{H}^3$ of positive curvature is realized on the ideal domain of the Lobachevskiĭ space, that is, on the exterior domain of the projective space P_3 with respect to an oval surface. In this paper the basic notions of the volumes theory of the space $\widehat{H}^3$ are introduced through projective invariants of the fundamental group of this space. The volume formulae for a monopolar tetrahedron and bodies bounded by a hypersphere of the space $\widehat{H}^3$ are obtained.

Key Words: Cayley-Klein model, hyperbolic space of positive curvature, volume, monopolar tetrahedron

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1. Introduction

1.1. The hyperbolic space $\widehat{H}^3$ of positive curvature

In the projective Cayley-Klein model the Lobachevskiĭ space Λ^3 can be realized on the interior domain of the projective space P_3 with respect to an oval surface [5, Chapter V, §15], [8, Chapter II, §4] γ. A hyperbolic space $\widehat{H}^3$ of positive curvature can be realized on the ideal domain of the Lobachevskiĭ space, that is, on the exterior domain with respect to the surface γ. The spaces Λ^3 and $\widehat{H}^3$ are connected components of the extended hyperbolic space H^3 [29, Chapter 4, §1]. The oval surface γ is called the *absolute* of the spaces H^3, $\widehat{H}^3$, and Λ^3. The group G of projective automorphisms of the oval surface γ is the *fundamental group* of transformations for H^3, $\widehat{H}^3$, and the Lobachevskiĭ space Λ^3. The space $\widehat{H}^3$ can be modelled in the Minkowski space $\mathbb{R}^4_1$ on the hypersphere of real radius. Therefore, the space $\widehat{H}^3$ is a projective model of the de Sitter 3-space [4, 32].

Nowadays the theory of volumes is developed mainly in classical spaces of constant curvature, that is, Euclidean, elliptic, and Lobachevskiĭ spaces. Main results and profound surveys

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on this topic are presented, for example, in [1, 2, 3, 7, 9, 10, 11, 12, 13, 30, 31, 33, 34]. For a number of reasons the volumes theory in the hyperbolic spaces of positive curvature is developed less successfully. We discuss some of these reasons in [16]. The main results on the areas theory of a hyperbolic plane $\widehat{H}$ of positive curvature are presented, for example, in [14, 20, 23, 25, 28]. The volumes of a finite light cone and a finite orthogonal h-cone of the space $\widehat{H}^3$ are calculated in [24] and [26] respectively, the volume formula for a Clifford surface layer can be found in [27].

In the underlying paper, by analogy with [24, 26, 27, 28], the foundations of the volumes theory for the space $\widehat{H}^3$ are provided. We introduce the basic notions of this theory through projective invariants of the fundamental group G and we calculate the volumes of the bodies bounded by hyperspheres. We pay special attention to the volume formula of a monopolar tetrahedron. Two opposite edges of such a tetrahedron lie on mutually polar lines with respect to the absolute. The volume formula for a monopolar tetrahedron is simpler than a general volume formula for an arbitrary tetrahedron and has the known analogue in the elliptic geometry. Unlike the Lobachevskiĭ space Λ^3, the space $\widehat{H}^3$ contains finite monopolar tetrahedrons. In [27] we obtained the volume formula only for a finite monopolar tetrahedron of the space $\widehat{H}^3$. In Theorem 1 of the presented work we prove this formula for any monopolar tetrahedron in $\widehat{H}^3$. To this end we classify the monopolar tetrahedrons of the space $\widehat{H}^3$ and introduce a new suitable orthogonal curvilinear coordinate system.

2. Preliminaries

2.1. Main notions

Assume that the absolute γ is added to the extended hyperbolic space H^3 and $\overline{H}^3 = H^3 \cup \gamma = \widehat{H}^3 \cup \Lambda^3 \cup \gamma$. The space $\overline{H}^3$ is homeomorphic to the projective space P_3. Let η be a surface which is homeomorphic to an oval surface and let $\eta \subset \overline{H}^3$. The surface η divides the space $\overline{H}^3$ into two connected components. One of these components is homeomorphic to the interior domain of an oval surface. We call it a *body* with boundary η. The second component is homeomorphic to the space $\widehat{H}^3$. We call it a *Möbius body* with boundary η. A body and a Möbius body with the same boundary we call *adjacent*. A body or a Möbius body F of the space $\overline{H}^3$ is called *finite* in the space $\widehat{H}^3$ or Λ^3 if $F \cap \widehat{H}^3 = F$ or $F \cap \Lambda^3 = F$, respectively.

Every line in the space $\widehat{H}^3$ belongs to one of three types depending on its position with respect to the absolute. Lines intersecting the absolute in two real or imaginary points are called *hyperbolic* or *elliptic*, respectively. Any tangent line to the absolute surface γ of the space $\widehat{H}^3$ is called *parabolic*. Main objects on lines of all types are introduced in [18, §4.2]. The type of a curve in its point M is determined by the type of the tangent line to this curve in the point M. We call the curve *elliptic* or *hyperbolic* if in each of its points the tangent line is elliptic or hyperbolic, respectively (see [28]).

Every real plane of the space $\widehat{H}^3$ belongs to one of three types. An *elliptic* plane crosses the absolute on a zero curve (see [5, Chapter V, §15], [8, Chapter II, §4]). A *hyperbolic plane of positive curvature* (see [18, 29]) crosses the absolute on an oval curve. A *co-Euclidean* plane (see [17, 29]) is tangent to the absolute and has a pair of imaginary conjugate lines from the absolute.

There are fifteen types of dihedrons of the space $\widehat{H}^3$. Dihedrons of six types are measurable, dihedrons of three types have real measures (see [16]). Every plane angle of the space $\widehat{H}^3$ belongs to one of twenty types. The type of a plane angle is determined by the types of its sides and the type of the plane containing this angle.

Assume that the lines a and b of the space $\widehat{H}^3$ belong to a plane α which is not co-Euclidean. The lines a and b are called *orthogonal* if each contains the pole of the other line with respect to the curve γ_α, where $\gamma_\alpha = \alpha \cap \gamma$. Two skew lines of the space $\widehat{H}^3$ are called *orthogonal* if they are mutually polar with respect to the absolute surface γ. Two segment of the space $\widehat{H}^3$ are called *orthogonal* if they lie on orthogonal lines. Two planes of the space $\widehat{H}^3$ are called *orthogonal* if each contains the absolute pole of the other plane. A line and a plane of the space $\widehat{H}^3$ are called *orthogonal* if the line contains the absolute pole of the given plane. A line and a plane are orthogonal if and only if the absolute polar line of the given line lies in the given plane.

2.2. Main metric formulae

A *canonical frame of the first type* of the spaces $\widehat{H}^3$, Λ^3, and H^3 is a projective frame $R^* = \{A_1, A_2, A_3, A_4, E\}$ whose vertices $A_1, \ldots, A_4$ form a nonplanar quadrilateral which is autopolar of the first order with respect to the absolute surface γ; the vertex A_4 lies in the space Λ^3, and the unit point E lies in each of three co-Euclidean planes α_{jk}, where $A_jA_k \subset \alpha_{jk}$, $j, k = 1, 2, 3$, $j \neq k$. In any canonical frame R^* of the first type the absolute surface γ is given by the equation

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0.$$

The quadratic form $\varphi(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 - x_4^2$ determines the metric of distances in the space $\widehat{H}^3$. It is the *metric form* of this space. Let $\overline{\varphi}$ be the symmetric bilinear form corresponding to the form φ. If points A and B of an elliptic or hyperbolic line have coordinates (a_p) and (b_p), $p = 1, 2, 3, 4$, then, in the frame R^*, the length σ of the segment between these points can be expressed by the formulae

$$\cos\frac{\sigma}{\rho} = \pm\frac{\overline{\varphi}(a_p, b_p)}{\sqrt{\varphi(a_p)\varphi(b_p)}} = \pm\frac{a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4}{\sqrt{a_1^2 + a_2^2 + a_3^2 - a_4^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2 - b_4^2}} \quad \text{or} \tag{2.1}$$

$$\cosh\frac{\sigma}{\rho} = \pm\frac{\overline{\varphi}(a_p, b_p)}{\sqrt{\varphi(a_p)\varphi(b_p)}} = \pm\frac{a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4}{\sqrt{a_1^2 + a_2^2 + a_3^2 - a_4^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2 - b_4^2}}, \tag{2.2}$$

respectively, where ρ, $\rho \in \mathbb{R}_+$, is a *curvature radius* of the space $\widehat{H}^3$.

The length of an elliptic or hyperbolic line equals $\pi\rho$ or $i\pi\rho$, respectively (see, for instance, [18, §§4.4.1, 4.4.3]). The orthogonality condition $A \perp B$ has the following form in R^*:

$$a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4 = 0. \tag{2.3}$$

The value of the form φ on the real coordinates (a_p) of the point A is called the *characteristic* of these coordinates. For the proper or ideal point A in $\widehat{H}^3$ we have, respectively,

$$\varphi(a_p) = a_1^2 + a_2^2 + a_3^2 - a_4^2 > 0 \quad \text{or} \quad \varphi(a_p) = a_1^2 + a_2^2 + a_3^2 - a_4^2 < 0. \tag{2.4}$$

The quadratic form $\Phi(X_1, X_2, X_3, X_4) = X_1^2 + X_2^2 + X_3^2 - X_4^2$, which is polar to the form φ, coincides with the form φ and determines the metric of dihedral angles in the space $\widehat{H}^3$. We call it the *tangential metric form* of this space. The value of the form Φ on the real coordinates (α_p) of a plane α is called the *characteristic* of these coordinates. For an elliptic or extended hyperbolic plane α we have, respectively,

$$\Phi(\alpha_p) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_4^2 < 0 \quad \text{or} \quad \Phi(\alpha_p) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_4^2 > 0. \tag{2.5}$$

The coordinate plane $A_1A_2A_3$ of any canonical frame R^* of the first type is elliptic. Any other coordinate plane of such frame is an extended hyperbolic plane.

2.3. Spheres of the space $\widehat{H}^3$

There are four types of nondegenerate spheres in the space $\widehat{H}^3$ of curvature radius ρ, $\rho \in \mathbb{R}_+$. These types are as follows:

(a) *hyperspheres* with centres at ideal points of the space $\widehat{H}^3$;

(b) *horospheres* with centres on the absolute;

(c) *elliptic spheres* with elliptic radiuses and centres at proper points of the space $\widehat{H}^3$;

(d) *hyperbolic spheres* with hyperbolic radiuses and centres at proper points of $\widehat{H}^3$.

In the projective sense, all hyperspheres, horospheres, and hyperbolic spheres are oval surfaces while elliptic spheres are anular surfaces (see the classification of surfaces of the second order in the projective space P_3 in [5, Chapter V, § 15], [8, Chapter II, § 4]). In this paper we consider hyperspheres of the space $\widehat{H}^3$ as the coordinate surfaces of the used orthogonal coordinate system C_1 of the first type. Let us formulate the metric definition of a hypersphere.

Let S be a proper point of the space Λ^3. The set of all points in $\widehat{H}^3$ so that the hyperbolic distance from it to the point S is a complex number $r = i\pi\rho/2 - h$, $h \in \mathbb{R}_+$, is called the *hypersphere* with *centre* at S and *radius* r and is denoted by $w(S; r)$. The elliptic plane α which is the absolute polar plane of the point S is called the *base* of the hypersphere ω. The distance from any point of the plane α to the point S is equal to $i\pi\rho/2$. Hence, a hypersphere with base α is the set of all points of the space $\widehat{H}^3$ so that the distance from the point to the plane α is a real number $h = i\pi\rho/2 - r$ which is called the *height* of the hypersphere. Thus, a hypersphere is an equidistant surface with an elliptic base plane.

Using Formula (2.2), we find the equation of the hypersphere $w(S; h)$ in the frame $R^* = \{A_1, A_2, A_3, S, E\}$

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 \coth^2 \frac{h}{\rho} = 0. \tag{2.6}$$

The equation (2.6) implies that a hypersphere of nonzero height is a nondegenerate surface of second order. The signature of the quadratic form on the left hand side of this equation with $h \neq 0$ is equal to 2. Therefore, in accordance with the classification of surfaces of the second order in the space P_3, a hypersphere is an oval surface. A hypersphere fully belongs to the space $\widehat{H}^3$ and intersects the absolute surface on a zero curve in the hypersphere base. The absolute surface of the space $\widehat{H}^3$ fully belongs to the interior domain with respect to a hypersphere. A hypersphere and the absolute have the common cone of imaginary tangent lines with the vertex at the centre of the hypersphere.

3. General provisions of the volume theory in the space $\widehat{H}^3$

3.1. Proper coordinates of points in $\widehat{H}^3$

Projective coordinates of points in the space $\widehat{H}^3$ are homogeneous and cannot uniquely provide the calculation of bodies volumes in this space. Let us choose in the space $\widehat{H}^3$ a coordinates normalization which is invariant under the transformations of the group G. Assume that in the frame R^* the real numbers (x_p), $p = 1, \ldots, 4$, are the coordinates of a proper point M of the space $\widehat{H}^3$ with curvature radius ρ, $\rho \in \mathbb{R}_+$. In the frame R^*, the quadruple of numbers

$$\bar{x}_p = \pm \frac{\rho x_p}{\sqrt{x_1^2 + x_2^2 + x_3^2 - x_4^2}}, \tag{3.1}$$

which are defined exactly up to the sign, are called the *proper coordinates* of point M.

The described normalization establishes a one-to-one dependence between proper points of the space $\widehat{H}^3$ and quadruples of real numbers, defined exactly up to their signs. According to the second condition from (2.4), the proper coordinates of ideal points of the space $\widehat{H}^3$ in this normalization are imaginary numbers. The proper coordinates of absolute points in the normalization (3.1) are infinitely large. For all point in space $\widehat{H}^3$, the proper coordinates satisfy

$$\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 - \bar{x}_4^2 = \rho^2. \tag{3.2}$$

3.2. The volume element of the space $\widehat{H}^3$

Assume that the domain Q of the space $\widehat{H}^3$ is homeomorphic to the interior domain of an oval surface, and a point M with the proper coordinates $(\bar{x}_p)$ lies in the domain Q. Let us set the curvilinear coordinate system C^* in the domain Q by the smooth functions

$$\bar{x}_p = \bar{x}_p(u, v, w), \quad p = 1, 2, 3, 4, \quad (u, v, w) \in \overline{Q} \subset \mathbb{R}^3.$$

When we calculate the volumes of infinite bodies of the space $\widehat{H}^3$, we generalize the system C^*. For the coordinates u, v, and w we consider besides real values also complex values in the form $i\pi\rho/2 + v$, $v \in \mathbb{R}$ (see Subsection 4.1).

We determine the system C^* so that all coordinate curves of the same family, u, v, or w, belong to the same type of curves. The system C^* is called *orthogonal* if every two coordinate curves are orthogonal at each point of the domain Q.

Differentiating sequentially the equality (3.2) with respect to the variables u, v, and w, we get the following conditions:

$$\begin{gathered} \bar{x}_1 \frac{\partial \bar{x}_1}{\partial u} + \bar{x}_2 \frac{\partial \bar{x}_2}{\partial u} + \bar{x}_3 \frac{\partial \bar{x}_3}{\partial u} - \bar{x}_4 \frac{\partial \bar{x}_4}{\partial u} = 0, \quad \bar{x}_1 \frac{\partial \bar{x}_1}{\partial v} + \bar{x}_2 \frac{\partial \bar{x}_2}{\partial v} + \bar{x}_3 \frac{\partial \bar{x}_3}{\partial v} - \bar{x}_4 \frac{\partial \bar{x}_4}{\partial v} = 0, \\ \bar{x}_1 \frac{\partial \bar{x}_1}{\partial w} + \bar{x}_2 \frac{\partial \bar{x}_2}{\partial w} + \bar{x}_3 \frac{\partial \bar{x}_3}{\partial w} - \bar{x}_4 \frac{\partial \bar{x}_4}{\partial w} = 0. \end{gathered} \tag{3.3}$$

According to condition (2.3), the equalities in (3.3) imply that each of the points

$$M_u \left(\frac{\partial \bar{x}_p}{\partial u} \right), \quad M_v \left(\frac{\partial \bar{x}_p}{\partial v} \right), \quad M_w \left(\frac{\partial \bar{x}_p}{\partial w} \right) \tag{3.4}$$

is orthogonal to the point M. Consequently, the plane $M_u M_v M_w$ is the polar plane of the point M with respect to the absolute surface γ. For the points M_u, M_v, and M_w the coordinates from (3.4) are not proper in the sense of condition (3.1). But these coordinates are uniquely determined by the proper coordinates $(\bar{x}_p)$ of the point M.

Denote the values of the forms φ and $\overline{\varphi}$ on coordinates from (3.4) as follows:

$$\begin{gathered} \gamma_{uu} = \varphi \left(\frac{\partial \bar{x}_p}{\partial u} \right), \quad \gamma_{vv} = \varphi \left(\frac{\partial \bar{x}_p}{\partial v} \right), \quad \gamma_{ww} = \varphi \left(\frac{\partial \bar{x}_p}{\partial w} \right), \\ \gamma_{uv} = \overline{\varphi} \left(\frac{\partial \bar{x}_p}{\partial u}, \frac{\partial \bar{x}_p}{\partial v} \right), \quad \gamma_{vw} = \overline{\varphi} \left(\frac{\partial \bar{x}_p}{\partial v}, \frac{\partial \bar{x}_p}{\partial w} \right), \quad \gamma_{uw} = \overline{\varphi} \left(\frac{\partial \bar{x}_p}{\partial u}, \frac{\partial \bar{x}_p}{\partial w} \right), \end{gathered} \tag{3.5}$$

and let

$$J = \begin{vmatrix} \gamma_{uu} & \gamma_{uv} & \gamma_{uw} \\ \gamma_{uv} & \gamma_{vv} & \gamma_{vw} \\ \gamma_{uw} & \gamma_{vw} & \gamma_{ww} \end{vmatrix}. \tag{3.6}$$

The lines MM_u, MM_v, and MM_w are tangent lines to the coordinate curves in the point M. Therefore, in the case of an orthogonal system C^*, the points M_u, M_v, and M_w are pairwise orthogonal. This means that for the orthogonal coordinate system C^* the following conditions hold:

$$\gamma_{uv} = \gamma_{vw} = \gamma_{uw} = 0.$$

In the frame R^*, the plane $M_uM_vM_w$ is given by the equation

$$\alpha_1 x_2 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0,$$

where

$$\alpha_1 = \begin{vmatrix} \frac{\partial \bar{x}_2}{\partial u} & \frac{\partial \bar{x}_3}{\partial u} & \frac{\partial \bar{x}_4}{\partial u} \\ \frac{\partial \bar{x}_2}{\partial v} & \frac{\partial \bar{x}_3}{\partial v} & \frac{\partial \bar{x}_4}{\partial v} \\ \frac{\partial \bar{x}_2}{\partial w} & \frac{\partial \bar{x}_3}{\partial w} & \frac{\partial \bar{x}_4}{\partial w} \end{vmatrix}, \quad \alpha_2 = -\begin{vmatrix} \frac{\partial \bar{x}_1}{\partial u} & \frac{\partial \bar{x}_3}{\partial u} & \frac{\partial \bar{x}_4}{\partial u} \\ \frac{\partial \bar{x}_1}{\partial v} & \frac{\partial \bar{x}_3}{\partial v} & \frac{\partial \bar{x}_4}{\partial v} \\ \frac{\partial \bar{x}_1}{\partial w} & \frac{\partial \bar{x}_3}{\partial w} & \frac{\partial \bar{x}_4}{\partial w} \end{vmatrix},$$

$$\alpha_3 = \begin{vmatrix} \frac{\partial \bar{x}_1}{\partial u} & \frac{\partial \bar{x}_2}{\partial u} & \frac{\partial \bar{x}_4}{\partial u} \\ \frac{\partial \bar{x}_1}{\partial v} & \frac{\partial \bar{x}_2}{\partial v} & \frac{\partial \bar{x}_4}{\partial v} \\ \frac{\partial \bar{x}_1}{\partial w} & \frac{\partial \bar{x}_2}{\partial w} & \frac{\partial \bar{x}_4}{\partial w} \end{vmatrix}, \quad \alpha_4 = -\begin{vmatrix} \frac{\partial \bar{x}_1}{\partial u} & \frac{\partial \bar{x}_2}{\partial u} & \frac{\partial \bar{x}_3}{\partial u} \\ \frac{\partial \bar{x}_1}{\partial v} & \frac{\partial \bar{x}_2}{\partial v} & \frac{\partial \bar{x}_3}{\partial v} \\ \frac{\partial \bar{x}_1}{\partial w} & \frac{\partial \bar{x}_2}{\partial w} & \frac{\partial \bar{x}_3}{\partial w} \end{vmatrix}.$$

Direct calculations show that the equality $J = -\Phi(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ holds in the frame R^*. Therefore, the number J is invariant under the transformations of the group G.

For each proper point M of the space $\widehat{H}^3$ the absolute polar plane $M_uM_vM_w$ is extended hyperbolic. Hence, for the plane $M_uM_vM_w$ based on the second inequality from (2.5) we have the inequality $\Phi(\alpha_1, \alpha_2, \alpha_3, \alpha_4) > 0$. Thus, for each proper point of the space $\widehat{H}^3$ the inequality $J < 0$ holds in the frame R^*. Measuring volumes of finite bodies in the space $\widehat{H}^3$, we seek to use real positive numbers. Therefore, we accept the number

$$dV = \sqrt{-J}\, du\, dv\, dw \tag{3.7}$$

as the *volume element* of the space $\widehat{H}^3$.

In the case of an orthogonal coordinate system C^* the volume element has a simple expression through line elements. Indeed, the coordinate curves of such a system are mutually orthogonal in pairs. In the space $\widehat{H}^3$, it implies that two coordinate curves of the system C^* are elliptic, and the third coordinate curve is hyperbolic. Let, for example, u and v be the elliptic coordinate curves, and let w be the hyperbolic coordinate curve. From Formulae (3.5), (3.6), and (3.7) we obtain

$$dV = \sqrt{\varphi(M_u)}\, du \sqrt{\varphi(M_v)}\, dv \sqrt{-\varphi(M_w)}\, dw. \tag{3.8}$$

The numbers

$$dl_{\mathtt{e}} = \sqrt{\varphi(dM)}\, dt \quad \text{or} \quad dl_{\mathtt{h}} = \sqrt{-\varphi(dM)}\, dt$$

are *elliptic* or *hyperbolic* line elements, respectively, of the space $\widehat{H}^3$. Similar definitions of line elements of the plane $\widehat{H}$ have been given in [28]. The numbers

$$\sqrt{\varphi(M_u)}\,du, \quad \sqrt{\varphi(M_v)}\,dv, \quad \sqrt{-\varphi(M_w)}\,dw$$

are arc lengths of the coordinate curves corresponding to infinitesimal increments of the parameters u, v, and w, respectively. Based on the expression (3.8), we get the following assertion:

The volume element of the space $\widehat{H}^3$ is equal to the product of arc lengths of the coordinate curves which originate from a point of the space and correspond to infinitesimal increments of the coordinates of this point.

3.3. The volume formula in the space $\widehat{H}^3$ geometry

Let F be a body in the domain Q of the space $\widehat{H}^3$. Assume that the domain $\overline{F}$, where $\overline{F} \subset \overline{Q} \subset \mathbb{R}^3$, determines the body F in the coordinate system C^* given in the domain Q. According to Formula (3.7), the volume $V(F)$ of F can be expressed by the formula

$$V(F) = \iiint\limits_{\overline{F}} \sqrt{-J}\,du\,dv\,dw. \tag{3.9}$$

Note that the choice of the volume element in accordance with Formula (3.7) is convenient when we calculate volumes in the space $\widehat{H}^3$ because it provides real positive values of the volumes of finite bodies in this space. The volumes of finite bodies of the Lobachevskiĭ space calculated by Formula (3.9) are real negative numbers. Moreover, the equality holds

$$V_{\Lambda^3}(F) = -V_{\widehat{H}^3}(F), \tag{3.10}$$

where for each body F of the space H^3 the number $V_{\Lambda^3}(F)$ is the volume calculated in the geometry of the space Λ^3, and the number $V_{\widehat{H}^3}(F)$ is the volume calculated in the geometry of the space $\widehat{H}^3$ by Formula (3.9). The proof of Formula (3.10) is very unwieldy. Therefore, we confined ourselves to check this formula only at the calculation of the volume of the extended hyperbolic space H^3 in Subsection 5.2.

4. The orthogonal curvilinear coordinate system

4.1. A construction

Let $R^* = \{A_1, A_2, A_3, A_4, E\}$ be a canonical frame of the first type in the space $\widehat{H}^3$ of curvature radius ρ, $\rho \in \mathbb{R}_+$, and let E_4 be the orthogonal projection of the point E in the elliptic plane $A_1A_2A_3$, i.e., $E_4 = A_4E \cap A_1A_2A_3$ (Figure 1a). Denote the quasiangle between the planes $A_1A_2A_3$ and $A_1A_2A_4$ containing the point E by ψ and denote the angle between the lines A_1A_2 and A_1A_3 (A_1A_3 and A_2A_3) containing the point E_4 by ψ_{23} (ψ_{12}). Let M be an arbitrary point of the domain Q in the space $\overline{H}^3$, and let Q be a topological equivalent to the interior domain of an oval surface. Denote the orthogonal projection of M in the plane $A_1A_2A_3$ by M_4, i.e., $M_4 = A_4M \cap A_1A_2A_3$. Let $\widetilde{w}$ be the part of the hyperbolic line between the points M, M_4 which completely or by its bigger part belongs to the quasiangle ψ. Denote the orthogonal projection of M_4 on the line A_1A_2 in the plane $A_1A_2A_3$ by M_{12}, hence $M_{12} = A_3M_4 \cap A_1A_2$. Let

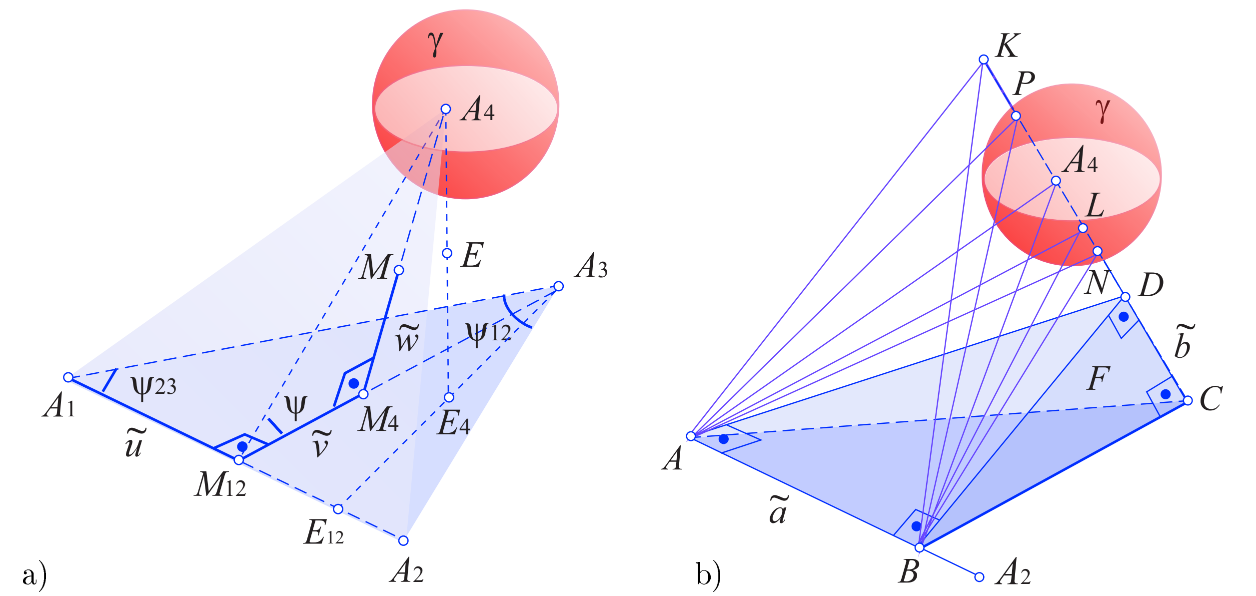

Figure 1: a) The orthogonal coordinate system $C_1 = \{A_1, A_1A_2, A_1A_2A_3, \psi_{12}, \psi_{23}, \psi\}$ and the absolute surface γ. b) The finite monopolar tetrahedron F and the absolute oval surface γ.

$\widetilde{u}$ ($\widetilde{v}$) be the elliptic segment between the points A_1 and M_{12} (M_4 and M_{12}) which completely or by its bigger part belongs to the angle ψ_{12} (ψ_{23}). Let us agree that

$$u = \frac{|\widetilde{u}|}{\rho}, \quad v = \frac{|\widetilde{v}|}{\rho}, \quad w = \frac{|\widetilde{w}|}{\rho}. \tag{4.1}$$

The set of figures $C_1 = \{A_1, A_1A_2, A_1A_2A_3, \psi_{12}, \psi_{23}, \psi\}$ is called a *coordinate system of the first type* in the space $\overline{H}^3$. We say that the canonical frame R^* is *attached* to the system C_1. The point A_1, the line A_1A_2, and the plane $A_1A_2A_3$ are called the *origin*, the *axis*, and the *base* of the system C_1, respectively. The point A_4 is called the *pole* of this system. The three numbers (u, v, w) from (4.1) are the *coordinates* of the point M in the system C_1.

For the coordinates u and v from (4.1) we have $u \in [0, \pi)$ and $v \in [0, \pi)$. The value of the coordinate w depends on the object $\widetilde{w}$ type. Later the following alternatives will be used.

1. If $\widetilde{w}$ is a segment of the space $\widehat{H}^3$, then $w \in \mathbb{R}_+$. In this case the point M lies in $\widehat{H}^3$.
2. If $\widetilde{w}$ is a beam of the space $\widehat{H}^3$, then $w = \infty$. In this case M lies on the absolute γ.
3. If $\widetilde{w}$ is a quasisegment of the space $\widehat{H}^3$ (see [18, Chapter 4]), then $w = i\pi/2 + w^*$, where $w^* \in \mathbb{R}$. In this case the point M lies in the space Λ^3.

Assume that the point M passes all possible locations on the line M_4A_4 from the point M_4 to an arbitrary point of the space Λ^3. Then all possibilities for the coordinate w of the point M are described in cases 1 – 3. We formally denote such a change of the coordinate w in the following manner: $w \in [0,\, i\pi/2 + w^*]$.

4.2. The relation between proper and curvilinear coordinates

Assume that the coordinate system C_1 of the first type is given on the domain Q in the space $\overline{H}^3$, and the canonical frame R^* is attached to this system. Let (u, v, w) be the coordinates of an arbitrary point M from the domain Q in the system C_1 and let $(x_1 : x_2 : x_3 : x_4)$ be the projective coordinates of M in the frame R^*. Then we have

$$A_4M \cap A_1A_2A_3 = M_4(x_1 : x_2 : x_3 : 0), \quad A_3M_4 \cap A_1A_2 = M_{12}(x_1 : x_2 : 0 : 0).$$

By Formulae (2.1), (2.2), (4.1) we obtain:

$$\cos u = \pm\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad \cos v = \pm\frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \quad \cosh w = \pm\frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2 - x_4^2}}. \tag{4.2}$$

From the expressions (4.2) under the condition (3.2) we find the proper coordinates of the point M in the frame R^*:

$$\begin{array}{ll} \bar{x}_1 = \rho \cos u \cos v \cosh w, & \bar{x}_2 = \rho \sin u \cos v \cosh w, \\ \bar{x}_3 = \rho \sin v \cosh w, & \bar{x}_4 = \rho \sinh w. \end{array} \tag{4.3}$$

Formulae (4.3) set the *parametrization of the first type* on the domain Q.

4.3. The volume element in the coordinate system C_1

Assume that the proper coordinates of the point M are given by Formulae (4.3). Let us calculate the coordinates of the points M_u, M_v, and M_w in the frame R^*:

$$\begin{array}{l} M_u \, (-\rho \sin u \cos v \cosh w : \rho \cos u \cos v \cosh w : 0 : 0), \\ M_v \, (-\rho \cos u \sin v \cosh w : -\rho \sin u \sin v \cosh w : \rho \cos v \cosh w : 0), \\ M_w \, (\rho \cos u \cos v \sinh w : \rho \sin u \cos v \sinh w : \rho \sin v \sinh w : \rho \cosh w). \end{array}$$

Using these coordinates and designations from (3.5), we obtain

$$\gamma_{uu} = \rho^2 \cos^2 v \cosh^2 w, \quad \gamma_{vv} = \rho^2 \cosh^2 w, \quad \gamma_{ww} = -\rho^2, \quad \gamma_{uv} = \gamma_{uw} = \gamma_{vw} = 0. \tag{4.4}$$

The last three equalities from (4.4) imply the orthogonality of the coordinate system C_1. From expressions (4.4), via Formulae (3.7) and (3.9), we find the volume element of the space $\overline{H}^3$ and the volume formula for bodies of this space in the coordinate system C_1:

$$dV = \rho^3 \cos v \cosh^2 w \, du \, dv \, dw, \quad V = \rho^3 \iiint\limits_{\overline{\Phi}} \cos v \cosh^2 w \, du \, dv \, dw. \tag{4.5}$$

4.4. Coordinate surfaces in the system C_1

Via the first two formulae from (4.3) we find the equation of the coordinate surface $u = u_0 =$ const. in the system C_1 as

$$x_1 \tan u_0 - x_2 = 0. \tag{4.6}$$

This equation determines a plane containing the hyperbolic line A_3A_4. The coordinates $(\tan u_0 : -1 : 0 : 0)$ of this plane satisfy the second condition from (2.5). Therefore, the vw coordinate surfaces in C_1 are hyperbolic planes. They form a pencil with an axis which is polar with respect to the absolute to the axis of the system C_1.

Via the first three formulae from (4.3) we find the equation of the coordinate surface $v = v_0 =$ const. in the system C_1 as

$$x_1^2 + x_2^2 - x_3^2 \cot^2 v_0 = 0. \tag{4.7}$$

Equation (4.7) determines a cone with vertex A_4 and axis A_3A_4. This cone crosses the base $A_1A_2A_3$ of the system C_1 along a circular curve with centre A_3. Hence, the uw coordinate surfaces in C_1 are circular cones with the vertex at the system pole and an axis which is polar with respect to the absolute to the system axis.

Formulae (4.3) with $w = w_0 = \text{const.}$ yield

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 \coth^2 w_0 = 0. \tag{4.8}$$

Equation (4.8) has the form of Eq. (2.6) and determines the hypersphere with centre A_4 and height $\rho|w_0|$. Consequently, the uv coordinate surfaces in C_1 are hyperspheres centered at the system pole.

5. The calculation of the bodies' volumes in the space $\widehat{H}^3$

5.1. The volume of a monopolar tetrahedron

5.1.1. The notion of a monopolar tetrahedron

A tetrahedron having two opposite edges on mutually polar lines relative to the absolute γ of the space $\widehat{H}^3$ is called *monopolar* [27]. The opposite edges of a monopolar tetrahedron on the mutually polar lines relative to the absolute are called the *base edges* of the tetrahedron. Let us prove that base edges of a monopolar tetrahedron are non-parabolic.

Let $\widetilde{a}$ and $\widetilde{b}$ be the base edges of a monopolar tetrahedron F. Assume that $\widetilde{a} \subset a_0$ and $\widetilde{b} \subset b_0$, where the lines a_0 and b_0 are mutually polar with respect to γ. Denote the absolute polarity of the space $\overline{H}^3$ by ξ. Let us assume that, for example, the line a_0 is parabolic with the absolute point Z. The image $\xi(Z)$ of the point Z under the transformation ξ is a co-Euclidean plane. Since $Z \in a_0 \subset \xi(Z)$ and $\xi(a_0) = b_0$, the line b_0 belongs to the plane $\xi(Z)$, too. Then the edges $\widetilde{a}$ and $\widetilde{b}$ lie in the same co-Euclidean plane. But these edges are opposite in the tetrahedron F. These two facts contradict each other. Consequently, our assumption is incorrect, and the edges $\widetilde{a}$ and $\widetilde{b}$ are non-parabolic.

Two non-parabolic lines which are mutually polar with respect to the absolute of the space $\overline{H}^3$ belong to different types. The base edge of a monopolar tetrahedron is called *elliptic* (*hyperbolic*) if it belongs to an elliptic (hyperbolic) line.

The elliptic base edge of a monopolar tetrahedron lies completely in the space $\widehat{H}^3$. It is likely that therefore the monopolar tetrahedrons have not been studied in detail at the research of finite bodies and some infinite polyhedra in the Lobachevskiĭ space Λ^3. In the Lobachevskiĭ geometry the simplest polyhedra are presented by orthoschemes.

A tetrahedron $P_1P_2P_3P_4$ is called an *orthoscheme* or, in other terms, a *biorthogonal tetrahedron* if under the corresponding designation of its vertices the conditions $P_1P_2 \perp P_2P_3P_4$ and $P_1P_2P_3 \perp P_3P_4$ hold (see [6, 7, 34]). Under these conditions the vertices P_1 and P_4 are called *principal vertices* of $P_1P_2P_3P_4$.

Let us compare the notions of a monopolar tetrahedron and an orthoscheme: Consider a monopolar tetrahedron $ABCD$ with edges AB and CD on the mutually polar lines relative to the absolute γ. According to the definitions from Section 2.1, the tetrahedron $ABCD$ satisfies the following conditions: $AB \perp BCD$, $CD \perp ABC$. Under these conditions the tetrahedron $ABCD$ is an orthoscheme. Thus, *each monopolar tetrahedron is an orthoscheme.*

Now consider the nonplanar quadrilateral $A_1A_2A_3A_4$ of a canonical frame R^* of the first type. Let B be a point on the line A_2A_3, where $B \neq A_2$, $B \neq A_3$, and let C be a point on the line A_3A_4, where $C \neq A_3$, $C \neq A_4$. Since the point A_1 is the absolute pole of the plane $A_2A_3A_4$, the line A_1B is orthogonal to the plane BA_3C. Since the point A_4 is the absolute pole of the plane $A_1A_2A_3$, the line CA_3 is orthogonal to the plane BA_1A_3. Consequently, the tetrahedron A_1BA_3C is an orthoscheme. The lines A_2A_4, A_1A_4, and A_1A_2 are the respective absolute polar lines of the lines A_1A_3, BA_3, and CA_3. Since $BC \neq A_2A_4$, $A_1C \neq A_1A_4$, and $A_1B \neq A_1A_2$, the tetrahedron A_1BA_3C is not monopolar. Thus, *there are orthoschemes which are not monopolar tetrahedrons.*

The following properties of a monopolar tetrahedron in the space $\widehat{H}^3$ of curvature radius ρ, $\rho \in \mathbb{R}_+$, are obtained directly from its definition and the definitions from Section 2.1. In general, an orthoscheme has none of these properties (see [7]).

1. Any two vertices of a monopolar tetrahedron are its principal vertices.
2. Each face of a monopolar tetrahedron contains at least two right angles.
3. At least two planar angles at any vertex of a monopolar tetrahedron are right.
4. Each nonbasic edge of a monopolar tetrahedron equals half of the line containing this edge. Therefore, the lenght of the elliptic (hyperbolic) nonbasic edge is equal to $\pi\rho/2$ ($i\pi\rho/2$).
5. The elliptic (hyperbolic) base edge of a monopolar tetrahedron is the common perpendicular of the planes of its faces through its hyperbolic (elliptic) edge. Owing to this, we find the equalities $a = \rho\widehat{A}$ and $b = \rho\widehat{B}$, where the number $\widehat{A}$ ($\widehat{B}$) is the measure of an interior dihedron at the hyperbolic (elliptic) base edge, and the number a (b) is the lenght of the elliptic (hyperbolic) base edge of a monopolar tetrahedron (see [16, Subsection 3.6], where the linear measure of a dihedron of the space $\widehat{H}^3$ is considered).
6. The interior dihedron at each nonbasic edge of a monopolar tetrahedron is right.

Note that the introduction of the notion of an interior dihedron at the edge of a tetrahedron is a painstaking task in the space $\widehat{H}^3$ geometry (see, for instance, [22], where the definition of an interior angle for a polygon of the plane $\widehat{H}^3$ is given). Therefore we present here only assertions 5 and 6, without the introduction of rigorous definitions and specifying the dihedrons' types.

5.1.2. The classification of monopolar tetrahedrons

We classify monopolar tetrahedrons of the space $\widehat{H}^3$, considering all possibilities for the types of their faces and the type of the hyperbolic base edge.

Let F be a monopolar tetrahedron of the space $\widehat{H}^3$. Two faces of the tetrahedron F at the hyperbolic base edge are congruent and lie in hyperbolic planes forming a semispace (see [16]). The face planes through the elliptic base edge can be of any type. Hence, for monopolar tetrahedrons there are six sets of the face types. These sets are as follows: *EEHH*, *EHHH*, *HHHH*, *ECHH*, *HCHH*, *CCHH*. Denoting the face type, we use the symbols E, H or C if the type is elliptic, hyperbolic or co-Euclidean, respectively (see [15, 16]).

Let us consider all possibilities for each set, depending on the type of the hyperbolic base edge of the tetrahedron F. We accompany the description by Figure 1b, assuming that the points A, B, C, D, and K lie in the space $\widehat{H}^3$, the points N and P belong to the absolute γ while the points L and A_4 lie in the space Λ^3.

1. If the tetrahedron F has the set *EEHH*, then its hyperbolic base edge can be a segment of the space $\widehat{H}^3$ or the sum of two beams of this space and a full line of the space Λ^3. In these cases we denote the respective types of the tetrahedron F by *EEHH*(I)

or $EEHH(II)$. In Figure 1b the tetrahedrons $ABCD$ and $ABCK$ of respective types $EEHH(I)$ and $EEHH(II)$ are shown.

2. If the tetrahedron F has the set $EHHH$, then its hyperbolic base edge can be only a quasisegment. Depending on the types of quasisegments (see [18, Chapter 4]), there are three types of monopolar tetrahedrons with the set $EHHH$. We denote the types by $EHHH(h)$, $EHHH(r)$ or $EHHH(e)$ in the cases when the hyperbolic base edge is a hyperbolic, right or elliptic quasisegment, respectively. In Figure 1b the tetrahedrons $ABCL$, $ABDL$, and $ABDA_4$ of respective types $EHHH(h)$, $EHHH(r)$, and $EHHH(e)$ are shown under the conditions $L \perp D$ and $A_4 \perp C$.
   In the particular case, when the vertices on the elliptic base edge of a monopolar tetrahedron of type $EHHH(r)$ are orthogonal, the tetrahedron's vertices form a nonplanar quadrilateral which is autopolar of the first order with respect to the absolute.
3. If the tetrahedron F has the set $HHHH$, then its hyperbolic base edge can be a segment of the space Λ^3 or the sum of two beams of this space and a full line of the space $\widehat{H}^3$. In these cases we denote the respective types of the tetrahedron F by $HHHH(I)$ or $HHHH(II)$. In Figure 1b the tetrahedron $ABLA_4$ of type $HHHH(I)$ is shown.
4. If the tetrahedron F has the set $ECHH$, then its hyperbolic base edge can be a beam of the space $\widehat{H}^3$ or the sum of a beam of this space and a full line of the space Λ^3. In these cases we denote the respective types of the tetrahedron F by $ECHH(I)$ or $ECHH(II)$. In Figure 1b the tetrahedrons $ABCN$ and $ABCP$ of types $ECHH(I)$ and $ECHH(II)$, respectively, are shown.
5. If the tetrahedron F has the set $HCHH$, then its hyperbolic base edge can be a beam of the space Λ^3 or the sum of a beam of this space and a full line of the space $\widehat{H}^3$. In these cases we denote the respective types of the tetrahedron F by $HCHH(I)$ or $HCHH(II)$. In Figure 1b the tetrahedron $ABLN$ of type $HCHH(I)$ is shown.
6. If the tetrahedron F has the set $CCHH$, then its hyperbolic base edge can be a full line of the space $\widehat{H}^3$ or a full line of the space Λ^3. In these cases we denote the respective types of the tetrahedron F by $CCHH(I)$ or $CCHH(II)$. In Figure 1b the tetrahedron $ABNP$ of type $CCHH(II)$ is shown.

Thus, there are 13 types of monopolar tetrahedrons in the space $\widehat{H}^3$. The finite monopolar tetrahedrons of this space belong to the type $EEHH(I)$. The Lobachevskiĭ space has no finite monopolar tetrahedrons.

5.1.3. The main theorem on the volume of a monopolar tetrahedron

Note that the length of the elliptic base edge of a monopolar tetrahedron in the space $\widehat{H}^3$ is limited by the length $\pi\rho$ of an elliptic line, while the length of the hyperbolic base edge can infinitely increase by keeping real values or accept complex values. Based on properties of a monopolar tetrahedron (see Subsection 5.1.1), it is determined exactly up to a motion by its base edges. Hence, the volume of a monopolar tetrahedron is uniquely determined by the lengths of these edges. The volume formula for a finite monopolar tetrahedron has been proved in [27] with the use of the coordinate Clifford system C_{eCl} introduced for studying Clifford surfaces of elliptic type. The system C_{eCl} cannot be generalized for the research of objects outside the space $\widehat{H}^3$. However, the introduced coordinate system C_1 (see Section 4) allows us to prove the volume formula for any monopolar tetrahedron of the space $\widehat{H}^3$.

Theorem 1. *In the space $\widehat{H}^3$ of curvature radius ρ, $\rho \in \mathbb{R}_+$, the volume V of a monopolar tetrahedron with the lengths a and b of the base edges is given by*

$$V = \frac{1}{2}\,\rho ab. \tag{5.1}$$

Proof. We carry out the proof in two stages. At first we prove Formula (5.1) for monopolar tetrahedrons of the types $EEHH(I)$, $ECHH(I)$, and $EHHH(h)$. Then, we prove this formula for other types of monopolar tetrahedrons by the method of addition. Note that we can finish the proof of Formula (5.1) on the first stage, changing the values interval of the variable w from (5.4) in accordance to the type of the monopolar tetrahedron.

1. Assume that a monopolar tetrahedron F with the elliptic base edge AB of length a, where $a \in (0, \pi\rho)$, and the hyperbolic base edge CD of length b belongs to the types $EEHH(I)$, $ECHH(I)$ or $EHHH(h)$. For the number b there are three possibilities: $b \in \mathbb{R}_+$ if F belongs to the type $EEHH(I)$, $b = \infty$ if F belongs to the type $ECHH(I)$, and $b = i\pi/2 + b^*$, where $b^* \in \mathbb{R}_+$, if F belongs to the type $EHHH(h)$. Let us agree that in any considered case the plane ABC is elliptic.

Choose the canonical frame $R^* = \{A, A_2, C, A_4, E\}$ of the first type, where the point A_2 (A_4) lies on the line AB (CD) and $A_2 \perp A$ ($A_4 \perp C$) (see Figure 1b). Assume that the unit point E of the frame R^* satisfies the following requirements:

(i) the point E lies in three co-Euclidean planes each of which contains one of the lines AB, A_2C or AC;

(ii) for the point E_{12}, where $E_{12} = CE_4 \cap AB$ and $E_4 = A_4E \cap ABC$, the following conditions hold: $(BE_{12}AA_2) \geq 0$ if $a \in (0, \pi\rho/2]$, $(BE_{12}AA_2) < 0$ if $a \in (\pi\rho/2, \pi\rho)$;

(iii) for the point E_{34}, where $E_{34} = A_2E_1 \cap A_3A_4$ and $E_1 = AE \cap A_2CD$, the inequality $(DE_{34}CA_4) > 0$ holds.

Attach the coordinate system C_1 to the frame R^*. Assume that every point M of the domain bounded by the tetrahedron F has the coordinates (u, v, w) from the numerical domain $\overline{F}$. Under the conditions (i) – (iii) the proper coordinates $(\bar{x}_p)$ of the point M are given in (4.3), where $u \in [0, a/\rho]$ and $v \in [0, \pi/2]$, in the frame R^*. The hyperbolic coordinate w runs through values from zero in the plane ABC to w_0 in the plane ABD. To express the value w_0 in terms of the lengths a and b of the base edges of the tetrahedron F, we find the equation of the plane ABD.

Since the point D lies on the line A_3A_4, it has the coordinates $(0 : 0 : 1 : d)$, $d \in \mathbb{R}$, in the frame R^*. Moreover, the points A and B satisfy the following conditions: $A = A_1(1 : 0 : 0 : 0)$ and $B \in A_1A_2$. Therefore, the plane ABD is given by the equation

$$d\bar{x}_3 - \bar{x}_4 = 0. \tag{5.2}$$

Writing down Eq. (5.2) in the coordinates of the system C_1 via Formulae (4.3), we obtain $\tanh w = d \sin v$. This equation of the plane ABD yields the equality $w_0 = \tanh^{-1}(d \sin v)$, where $\tanh^{-1}(x)$ is the inverse hyperbolic tangent function.

Writing down the inequality from (iii) in the coordinates of the points $D(0 : 0 : 1 : d)$, $E_{34}(0 : 0 : 1 : 1)$, $C(0 : 0 : 1 : 0)$, and $A_4(0 : 0 : 0 : 1)$, we find $(DE_{34}CA_4) = d > 0$. Thus, under the condition (iii) the inequality $d > 0$ holds. Let us refine the valuation of d, using conditions from (2.5) for the coordinates $(0 : 0 : d : -1)$ of the plane ABD.

When F is a tetrahedron of types $EEHH(I)$, $ECHH(I)$ or $EHHH$, the plane ABD is elliptic, co-Euclidean or hyperbolic, respectively. Then we have $d < 1$, $d = 1$ or $d > 1$, respectively. By Formula (2.2) we obtain

$$\cosh\frac{|CD|}{\rho} = \cosh\frac{b}{\rho} = \frac{1}{\sqrt{1-d^2}}\,. \tag{5.3}$$

Since $d > 0$, the expression (5.3) implies $d = \tanh\frac{b}{\rho}$. Note that $d = \coth\frac{b^*}{\rho} > 1$ in the case when $b = i\pi/2 + b^*$, $b^* \in \mathbb{R}_+$. Thus, the domain $\overline{F}$ can be given as follows:

$$u \in \left[0, \frac{a}{\rho}\right], \; v \in \left[0, \frac{\pi}{2}\right], \; w \in [0, w_0], \text{ where } w_0 = \tanh^{-1}(d\sin v), \; d = \tanh\frac{b}{\rho}. \tag{5.4}$$

Using conditions (5.4) for the domain $\overline{F}$, we calculate the volume V of the tetrahedron F via the second formula in (4.5) by means of WolframAlpha:

$$\begin{aligned}
V(F) &= \rho^3 \iiint\limits_{\overline{F}} \cos v\, \cosh^2 w\, du\, dv\, dw = \rho^3 \int_0^{\frac{a}{\rho}} du \int_0^{\frac{\pi}{2}} \cos v\, dv \int_0^{w_0} \cosh^2 w\, dw \\
&= \frac{\rho^3}{2} \int_0^{\frac{a}{\rho}} du \int_0^{\frac{\pi}{2}} (w_0 + \sinh w_0\, \cosh w_0)\, \cos v\, dv \\
&= \frac{\rho^3}{2} \int_0^{\frac{a}{\rho}} du \int_0^{\frac{\pi}{2}} \cos v\, [\tanh^{-1}(d\, \sin v) + \sinh(\tanh^{-1}(d\, \sin v)) \\
&\quad \times \cosh(\tanh^{-1}(d\, \sin v))]\, dv = \frac{\rho^3}{2} \tanh^{-1}\left(\tanh\frac{b}{\rho}\right) \int_0^{\frac{a}{\rho}} du = \frac{\rho a b}{2}.
\end{aligned}$$

So, in the considered cases Formula (5.1) is proved.

Note that the volume of a monopolar tetrahedron of type $EEHH(I)$ is a real positive number. If a monopolar tetrahedron belongs to the types $ECHH(I)$ or $EHHH(h)$, then its respective volume is infinite or is a complex number with a positive real part.

2. Now we consider all other possibilities for the types of a monopolar tetrahedron. In our reasonings we use the additivity property for the bodies' volumes and the lengths of objects on a hyperbolic line (see [18, Subsections 4.4.2, 4.4.3]). We still denote the length of the elliptic base edge AB of a monopolar tetrahedron by a.

2.1. Assume that the monopolar tetrahedron $F^* = ABCC^*$ belongs to the type $EHHH(r)$. In this case the conditions $C \perp C^*$ and thereby $b = |CC^*| = i\pi\rho/2$ hold. Assume that an arbitrary point X from the space Λ^3 lies on the hyperbolic base edge CC^* of the tetrahedron F^*. Then the quasisegment CX, which fully lies in the edge CC^*, is a hyperbolic quasisegment. Hence, $|CX| = i\pi\rho/2 + b^*$, where $b^* \in \mathbb{R}_+$. Consider a monopolar tetrahedron $F = ABCX$ of type $EHHH(h)$ with the hyperbolic base edge CX. For tetrahedrons of this type Formula (5.1) has been proved at stage **1**. Via this formula we find

$$V(F) = \frac{\rho a |CX|}{2} = \frac{\rho a}{2}\left(\frac{i\pi\rho}{2} + b^*\right).$$

When the point X tends to the point C^*, the tetrahedron F tends to the tetrahedron F^*. Under this condition the number b^* vanishes. Consequently,

$$V(F^*) = \lim_{b^* \to 0} V(F) = \lim_{b^* \to 0} \left(\frac{\rho a}{2}\left(\frac{i\pi\rho}{2} + b^*\right)\right) = \frac{\rho a b}{2}.$$

Thus, Formula (5.1) holds for monopolar tetrahedrons of type $EHHH(r)$.

2.2. Assume that a monopolar tetrahedron $F = ABA_4L$ with the hyperbolic base edge A_4L belongs to the type $HHHH(I)$ (see Figure 1b). In this case the edge A_4L is a segment of the space Λ^3 and $|A_4L| = -l^*$, where $l^* \in \mathbb{R}_+$. Let C be the point on the line A_4L which is orthogonal to the point A_4. Since for the right quasisegment A_4C and the hyperbolic quasisegment LC we have $|A_4C| - |LC| = |A_4L| = -l^*$, the equality $|LC| = i\pi\rho/2 + l^*$ holds. Consider the monopolar tetrahedrons $F_1 = ABA_4C$ of type $EHHH(r)$ and $F_2 = ABLC$ of type $EHHH(h)$. For the monopolar tetrahedrons of such types Formula (5.1) is proved. Therefore, via the equality $V(F) = V(F_1) - V(F_2)$ we obtain

$$V(F) = \frac{\rho a|A_4C|}{2} - \frac{\rho a|LC|}{2} = \frac{\rho a}{2}\left(\frac{i\pi\rho}{2} - \left(\frac{i\pi\rho}{2} + l^*\right)\right) = \frac{-\rho a l^*}{2} = \frac{\rho a|A_4L|}{2}.$$

Thus, Formula (5.1) is proved for monopolar tetrahedrons of type $HHHH(I)$.

Remark. Before continuing the proof of Theorem 1, let us discuss an interesting result. A monopolar tetrahedron of type $HHHH(I)$ has two vertices in the space $\widehat{H}^3$ and two vertices in the space Λ^3. Such a tetrahedron is infinite, but has a finite negative volume. This fact gives us new ideas about bodies of the space H^3. Compare, for example, with the following description of a finite orthoscheme in [6, p. 180]: *"Now we extend the class of orthoschemes by further allowing one or both of the principal vertices to lie beyond infinity, i.e., outside the absolute quadric. In this way we obtain polyhedra of infinite volume, ...".*

2.3. When the length of the hyperbolic base edge of a monopolar tetrahedron of type $HHHH(I)$ tends to infinity, having negative values, the volume of this tetrahedron tends to infinity, too. This proves Formula (5.1) for a monopolar tetrahedron of type $HCHH(I)$. For the monopolar tetrahedrons of types $HCHH(II)$, $CCHH(I)$, $CCHH(II)$, and $ECHH(II)$ we can prove Formula (5.1) in the same way.

2.4. Assume that the monopolar tetrahedron $F = ABCD$ with the hyperbolic base edge CD, where $C \in \widehat{H}^3$, belongs to the type $EHHH(e)$. Let C^* be the point on the line CD which is orthogonal to the point C. Then the tetrahedron F is the sum of the monopolar tetrahedrons $ABDC^*$ of type $HHHH(I)$ and $ABCC^*$ of type $EHHH(r)$. Using Formula (5.1) for these tetrahedrons, we prove this formula for F and thereby for each monopolar tetrahedron of type $EHHH(e)$.

When the tetrahedron F belongs to the types $EEHH(II)$ or $HHHH(II)$, we similarly divide it into two monopolar tetrahedrons $ABCC^*$ of type $EHHH(r)$ and $ABDC^*$ of types $EHHH(e)$ or $EHHH(h)$, respectively. Using Formula (5.1) for these tetrahedrons, we prove this formula for F and thereby for each monopolar tetrahedron of types $EEHH(II)$ or $HHHH(II)$. This completes the proof of Theorem 1. □

5.1.4. Corollary of the main theorem

Theorem 2. *In the space $\widehat{H}^3$ of curvature radius ρ, $\rho \in \mathbb{R}_+$, the volume V of a monopolar tetrahedron can be expressed by the formulae*

$$V = \frac{1}{2}\rho^3\widehat{A}\widehat{B}, \qquad V = \frac{1}{2}aS_b, \qquad V = \frac{1}{2}bS_a, \tag{5.5}$$

where $\widehat{A}$ and $\widehat{B}$ are the measures of the dihedrons at the base edges, a (b) is the length of the elliptic (hyperbolic) base edge, and S_a (S_b) is the area of the elliptic (hyperbolic) face containing the elliptic (hyperbolic) base edge of the tetrahedron.

Proof. Assume that the numbers $\widehat{A}$ and $\widehat{B}$ are the measures of the dihedrons at the base edges with the respective lengths b and a in a monopolar tetrahedron F of the space $\widehat{H}^3$. By definition of the dihedron measure (see [16]), we have $a = \rho\widehat{A}$ and $b = \rho\widehat{B}$. From these equalities via Formula (5.1) we get the first formula of the theorem.

In the hyperbolic plane $\widehat{H}$ of curvature radius ρ, $\rho \in \mathbb{R}_+$, the following equalities hold for the area S of a twice rectangular triangle with the length b of the hyperbolic base and the measure B of the opposite angle (see [23, Theorem 3.1], [25, Theorem 4])

$$S = b\rho, \qquad S = B\rho^2. \tag{5.6}$$

Using the parametrization (4.3) with $w = 0$, we can prove similar classical formulae for the area S of a twice rectangular triangle in the elliptic plane of the curvature radius ρ (see, for instance, [29, Chapter 2]). Applying these formulae and Formulae (5.1) and (5.6), we find the two last formulae from (5.5). Thus, the theorem is proved. □

5.2. The volume of the extended hyperbolic space H^3

Let points A_1, A_2, A_3, and A_4 be the vertices of the canonical frame R^* of the first type in the space H^3. Consider the tetrahedron $F = A_1A_2A_3A_4$ of type $EHHH(r)$ with the base edges A_1A_2 and A_3A_4. Since $|A_1A_2| = \pi\rho/2$ and $|A_3A_4| = i\pi\rho/2$, we obtain by Formula (5.1)

$$V(F) = \frac{i\pi^2\rho^3}{8}.$$

The faces planes of the tetrahedron F divide the space H^3 into eight congruent tetrahedrons. Hence, the volume of the space H^3 is equal to $i\pi^2\rho^3$. When we calculate the space H^3 volume by means of Lobachevskiĭ geometry, we find the number $-i\pi^2\rho^3$ (see, for instance, [29, Section 3.3.2]). Thus, the equality $V_{\Lambda^3}(H^3) = -V_{\widehat{H}^3}(H^3)$ holds. This result is fully in line with the volume definitions accepted for the spaces Λ^3 and $\widehat{H}^3$ (see Formula (3.10)).

5.3. Volumes of the bodies bounded by hyperspheres

Consider an arbitrary elliptic plane α and its pole A relative to the absolute surface γ of the space H^3. Let ς be a closed two-sided curve in the plane α and let ς_0 be the domain bounded by ς in the plane α. Assume that the domain ς_0 is homeomorphic to the interior domain with respect to an oval curve. The plane α divides the interior domain of the cone with the vertex A and the base curve ς into two components. The closings of these components are called *adjacent orthogonal e-cones* with the *vertex* A and the *polar base* ς_0. The symbol e indicates the elliptic type of the base plane α.

Let eC be an orthogonal e-cone with vertex A and a polar base ς_0 in the plane α. Each line passing through the point A is orthogonal to the plane α. In the space $\widehat{H}^3$ the hypersphere ω with centre A divides the e-cone eC into two components. The component, which contains the polar base ς_0, is called the *hyperspherical keg* of the e-cone eC. Denote it by hK. The second component is called the *hyperspherical sector* of the e-cone eC. Denote it by hS. This component consists of two parts. One of them belongs to the space $\widehat{H}^3$ while the second part belongs to the space Λ^3. The height (radius) of the hypersphere ω is called the *height* (*radius*) of the hyperspherical keg hK (hyperspherical sector hS). If the polar base ς_0 is a circle, then the e-cone, the hyperspherical keg, and the hyperspherical sector are called *circular*. In this case denote them by eC^o, hK^o, and hS^o, respectively.

To calculate the volumes of the defined bodies, we choose a canonical frame $R^* = \{A_1, A_2, A_3, A, E\}$ of the first type. In this frame the plane $A_1A_2A_3$ coincides with the base plane α of the e-cone eC. We attach the orthogonal coordinate system C_1 to the frame R^*. In the system C_1 the coordinate pair (u, v) belongs to the domain $\overline{\varsigma}$ which determines the polar base ς_0 of eC. The area S of the domain ς_0 can be expressed by the formula

$$S = \rho^2 \iint\limits_{\overline{\varsigma}} \cos v \, du \, dv \tag{5.7}$$

which can be obtained by analogy with Formula (32) from [25] via Formulae (4.3) with $w = 0$.

Let h, $h \in \mathbb{R}_+$, be the height of the hypersphere ω. Then for the body Φ the hyperbolic coordinate w of the system C_1 accepts the following values:

1) $w \in [0, h/\rho]$ if Φ is the hyperspherical keg hK;
2) $w \in [h/\rho, i\pi/2]$ if Φ is the hyperspherical sector hS of radius $r = i\pi\rho/2 - h$;
3) $w \in [0, i\pi/2]$ if Φ is the e-cone eC.

Therefore, we obtain for the volumes of the bodies hK, hS, and eC:

$$\begin{aligned}
V(hK) &= \rho^3 \iiint\limits_{\overline{\Phi}} \cos v \, \cosh^2 w \, du \, dv \, dw = \rho S \int_0^{\frac{h}{\rho}} \cosh^2 w \, dw \\
&= \frac{\rho S}{2}\left(\frac{h}{\rho} + \sinh\frac{h}{\rho}\cosh\frac{h}{\rho}\right) = \frac{\rho S}{4}\left(\frac{2h}{\rho} + \sinh\frac{2h}{\rho}\right), \\
V(hS) &= \rho^3 \iiint\limits_{\overline{\Phi}} \cos v \, \cosh^2 w \, du \, dv \, dw = \rho S \int_{\frac{h}{\rho}}^{i\frac{\pi}{2}} \cosh^2 w \, dw \\
&= \frac{\rho S}{2}\left(i\frac{\pi}{2} + \sinh\frac{i\pi}{2}\cosh\frac{i\pi}{2} - \frac{h}{\rho} - \sinh\frac{h}{\rho}\cosh\frac{h}{\rho}\right) = \frac{\rho S}{4}\left(\frac{2r}{\rho} - \sinh\frac{2r}{\rho}\right), \\
V(eC) &= \rho^3 \iiint\limits_{\overline{\Phi}} \cos v \, \cosh^2 w \, du \, dv \, dw = \rho S \int_0^{i\frac{\pi}{2}} \cosh^2 w \, dw = \frac{i\pi\rho S}{4}.
\end{aligned}$$

Based on the obtained results, let us formulate the following theorem.

Theorem 3. *In the space $\widehat{H}^3$ of curvature radius ρ, $\rho \in \mathbb{R}_+$, the following formulae hold for the respective volumes $V(hK)$, $V(hS)$, and $V(eC)$ of a hyperspherical keg hK, a hyperspherical sector hS and an orthogonal e-cone eC:*

$$V(hK) = \frac{\rho S}{4}\left(\frac{2h}{\rho} + \sinh\frac{2h}{\rho}\right), \quad V(hS) = \frac{\rho S}{4}\left(\frac{2r}{\rho} - \sinh\frac{2r}{\rho}\right), \quad V(eC) = \frac{i\pi\rho S}{4}, \tag{5.8}$$

where h (r) is the height $($radius$)$ of hK (hS) and S is the area of the polar base of the body.

The third formula of (5.8) determines, in particular, the volume of any infinite pyramid of the space $\widehat{H}^3$ with the ideal vertex which is orthogonal to the base.

In the elliptic plane the area of the circle of radius r_o is equal to $4\pi\rho^2\sin^2(r_o/2\rho)$ (see, for instance, the first formula from (2.64) in [29]). Hence, the volumes of circular bodies can be expressed by the formulae

$$\begin{gathered}
V(hK^o) = \pi\rho^3 \sin^2\frac{r_o}{2\rho}\left(\frac{2h}{\rho} + \sinh\frac{2h}{\rho}\right), \\
V(hS^o) = \pi\rho^3 \sin^2\frac{r_o}{2\rho}\left(\frac{2r}{\rho} - \sinh\frac{2r}{\rho}\right), \qquad V(eC^o) = i\pi^2\rho^3 \sin^2\frac{r_o}{2\rho},
\end{gathered} \tag{5.9}$$

where h (r) is the height (radius) of hK^o (hS^o) and r_o is the base radius of the circular body.

Via Formulae (5.9), we find the following volumes formulae for sectors hK^o_s, hS^o_s, and eC^o_s of the respective circular bodies hK^o, hS^o, and eC^o with the angle α of the base sector:

$$V(hK^o_s) = \frac{\alpha\rho^3}{2}\sin^2\frac{r_o}{2\rho}\left(\frac{2h}{\rho} + \sinh\frac{2h}{\rho}\right),$$

$$V(hS^o_s) = \frac{\alpha\rho^3}{2}\sin^2\frac{r_o}{2\rho}\left(\frac{2r}{\rho} - \sinh\frac{2r}{\rho}\right), \qquad V(eC^o_s) = \frac{i\alpha\pi\rho^3}{2}\sin^2\frac{r_o}{2\rho}.$$

Assume that the polar base ς_0 of a hyperspherical keg hK^1 is the triangle $A_1A_2A_3$, where A_1, A_2, and A_3 are the vertices from the frame R^*. Then we have $S(\varsigma_0) = \pi\rho^2/2$. In this case, according to the first formula from (5.8), we obtain for the volume $V\left(hK^1\right)$

$$V\left(hK^1\right) = \frac{\pi\rho^3}{8}\left(\frac{2h}{\rho} + \sinh\frac{2h}{\rho}\right). \tag{5.10}$$

Eight hyperspherical kegs, each of which is congruent to the keg hK^1, form the exterior domain ω_{ext} of the space $\widehat{H}^3$ relative to the hypersphere ω. Via Formula (5.10) the volume of the Möbius body ω_{ext} with boundary ω can be expressed by the formula

$$V(\omega_{ext}) = \pi\rho^3\left(\frac{2h}{\rho} + \sinh\frac{2h}{\rho}\right).$$

The volume $V(\omega_{ext})$ is a real positive number while the volume $V(\omega_{int})$ of the interior domain ω_{int} of the space H^3 with respect to the hypersphere ω, that is, the volume of the full hypersphere ω, is a complex number. Indeed, the third formula from (5.8) yields

$$V(\omega_{int}) = \pi\rho^3\left(\frac{2r}{\rho} - \sinh\frac{2r}{\rho}\right), \tag{5.11}$$

where the complex number $r = i\pi\rho/2 - h$, $h \in \mathbb{R}_+$, is the radius of the hypersphere ω.

After the coordination via Formula (3.10), Formula (5.11) coincides with the known formula of the sphere volume in Lobachevskiĭ geometry (see, for instance, the second formula from (3.83) in [29]).

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