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Rotor Coordinates, Vector Trigonometry and Quadrilaterals, with Applications to the Four Bar Linkage

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Abstract. The aim of this paper is to explain how the new theory of vector trigonometry which uses rotor coordinates instead of polar coordinates gives us new relations for quadrilaterals, and how these can be applied to the classical four bar linkages of kinematics.

Key Words: Rational trigonometry, vector trigonometry, half-slope, four bar linkage, quadrilateral

MSC 2010: 51N20, 70A05, 97G60, 53A17

1. Introduction

1.1. Quadrance and spreads for a general bilinear form

In this paper we apply a powerful recent variant of *rational trigonometry* called *vector trigonometry* [10] to establish some fundamental new trigonometric formulae for quadrilaterals. This theory is well suited to explicit computation and applications involving surveying, graphics, architecture and design. We will show that this theory is also potentially of importance in kinematics and robotics, by investigating some fundamental aspects of the four bar linkage from the point of view of the *rotor coordinates* that replace polar coordinates.

In this introduction we will first orient the reader towards rational trigonometry, and then explain how vector trigonometry is built from those ideas, but with an oriented aspect.

Rational trigonometry (RT) was introduced in author's 2005 book *Divine Proportions: Rational Trigonometry to Universal Geometry* [3], with further elaboration given by the YouTube series *WildTrig* at the author's YouTube channel *Insights into Mathematics*. RT provides a simple yet powerful alternative to classical trigonometry, eliminating the need for transcendental functions and calculators, simplifying many problems, and allowing a more careful and logical derivation of Euclidean geometry (see also [5]) which extends in many new directions. Rational trigonometry uses the entirely algebraic concepts of *quadrance* and *spread* instead

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of *distance* and *angle*. Hence the theory works over the rational numbers, as well as over general fields, including finite fields, and notably extends to arbitrary non-degenerate bilinear forms, embracing also relativistic geometry. Vector trigonometry is not entirely rational in this sense, as approximate square roots are involved, as we shall see.

The starting point, as described in [4], is an affine space $\mathbb{A}$ and the associated vector space $\mathbb{V}$ over a field (preferably not of characteristic two), together with a fixed symmetric bilinear form. By choosing an ordered basis, we may assume that $\mathbb{V}$ contains row vectors $\mathbf{v} = (x_1, x_2, \cdots, x_n)$ and that the bilinear form is given in the language of linear algebra by

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} A \,\mathbf{w}^T$$

for some non-degenerate (i.e. invertible) symmetric $n \times n$ matrix A.

The *quadrance* of a vector v is then defined to be the number

$$Q(\mathbf{v}) \equiv \mathbf{v} \cdot \mathbf{v}.$$

This quantity is an element of the field in which we are working.

The *spread* between two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined by

$$s(\mathbf{v}, \mathbf{w}) = 1 - \frac{(\mathbf{v} \cdot \mathbf{w})^2}{Q(\mathbf{v})\, Q(\mathbf{w})}.$$

Since the bilinear form is arbitrary, there is the possibility of having non-zero *null vectors* $\mathbf{v}$, that is vectors satisfying $Q(\mathbf{v}) = 0$, in which case a spread involving such a vector is undefined. Otherwise the spread is also an element of the field. Clearly spread is a symmetrical quantity, in that

$$s(\mathbf{v}, \mathbf{w}) = s(\mathbf{w}, \mathbf{v}).$$

The spread between vectors is unchanged when those vectors are multiplied by non-zero numbers, so that the spread extends to a well-defined property between lines (crucially not rays!). If lines l_1 and l_2 have direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then it makes sense to define the spread between them as $s(l_1, l_2) \equiv s(\mathbf{v}_1, \mathbf{v}_2)$. This purely algebraic approach works also for more general fields, even with finite characteristics, although the case of characteristic two has special properties and is best regarded somewhat separately.

1.2. Spreads, cross and related quantities in Euclidean geometry

In the special case of planar Euclidean geometry, where we use the familiar Euclidean dot product, the spread between the two non-zero vectors $\mathbf{v}_1 \equiv (x_1, y_1)$ and $\mathbf{v}_2 \equiv (x_2, y_2)$, can also be written as

$$s(\mathbf{v}_1, \mathbf{v}_2) \equiv \frac{(x_1 y_2 - x_2 y_1)^2}{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}. \tag{1}$$

This number can also be viewed as the square of the sine of the angle between the vectors, assuming that we have developed a number system in which these concepts make sense (which is actually much harder than is usually imagined). But crucially the algebraic nature of quadrance and spread does not require such a prior theory of "real numbers", and the theory makes sense even over the rational numbers (and in fact also over finite fields!)

There are some other secondary quantities that can be defined between lines. If l_1 and l_2 are lines with direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ then the *cross* between them is the number

$$c(l_1, l_2) \equiv \frac{(x_1 x_2 + y_1 y_2)^2}{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = 1 - s(l_1, l_2)$$

while the *twist* between them is

$$t(l_1,l_2) \equiv \frac{s(l_1,l_2)}{c(l_1,l_2)} = \frac{(x_1y_2 - x_2y_1)^2}{(x_1x_2 + y_1y_2)^2}.$$

Since the twist is always a square, we may define also the *turn*

$$u(l_1,l_2) \equiv \frac{x_1y_2 - x_2y_1}{x_1x_2 + y_1y_2}$$

which is an *oriented* quantity; in the sense that

$$u(l_2,l_1) = -u(l_1,l_2).$$

The main laws of rational trigonometry are universal, in that they apply in this general affine setting. Furthermore, a projective version of the entire theory allows one to introduce metrical structure also in a projective space over a general field, again with a general quadratic form (see [4]). This has led to a major new direction for hyperbolic geometry [6, 7, 8, 9].

1.3. Rotor coordinates and floating point numbers

In the paper [10] we have introduced a variant of rational trigonometry, called *vector trigonometry,* which is well suited for engineering, design, surveying and physics applications in the plane. This vector trigonometry is geared to problems in which direction figures prominently, and does not aspire, as rational trigonometry does, to provide a wide theoretical formulation. It is really an applied mathematical tool, and it rests on an underlying *floating point numerical system.* So in this theory when we discuss a quantity such as "$\sqrt{10}$" we mean a finite floating point decimal number whose square is approximately 10, for example $\sqrt{10} \approx 3.1623$. We will not take the position that there is an underlying "real number" arithmetic containing infinite precision objects. Instead we consider the floating point numbers with approximate arithmetical properties as the actual true objects, even if that introduces an element of ambiguity in the level of precision, or number of decimal digits, that we are considering.

The main idea is now to replace the usual polar coordinates r and θ of a planar vector $\mathbf{v} = (x,y)$ with *rotor coordinates* r and h, so that we write also

$$\mathbf{v} = |\, r, h\rangle.$$

The quantity $r \equiv |\mathbf{v}| = \sqrt{x^2+y^2}$ is the usual length, or approximate length, of $\mathbf{v}$, so this trigonometry does have a transcendental aspect in that square roots, or approximate square roots, will be needed. The key new point is to replace angle not with a spread, but rather *a directed quantity that allows us to deal with rays instead of just lines.* We will call this directed quantity *half-slope.*

This notion arises naturally from the rational parametrization of the unit circle c_U with equation $x^2+y^2=1$, given by

$$\mathbf{e}(h) \equiv \left(\frac{1-h^2}{1+h^2}, \frac{2h}{1+h^2}\right). \tag{2}$$

We say that the number h is the *half-slope* of the vector $\mathbf{v} = \mathbf{e}(h)$. The quantity h geometrically is the y-coordinate of the point which is the meet of the line joining the points $[-1,0]$ and $\mathbf{e}(h)$ on the unit circle with the y-axis, as shown in the Figure 1. In the special case when $\mathbf{e}(h)$ is the point $[-1, 0]$, this line is taken to be the tangent to the circle at that point and the quantity h is undefined, or declared to have the value $h = \infty$.

We then extend this notion to a general vector $\mathbf{v}$ by declaring its half-slope h to be equal to the half-slope of $\mathbf{v}/r$, which lies on the unit circle. It is worthwhile to remark that this discussion is taking place in the context of applied mathematics, where we are willing to consider approximate half-slopes on account of the necessarily approximate nature of length.

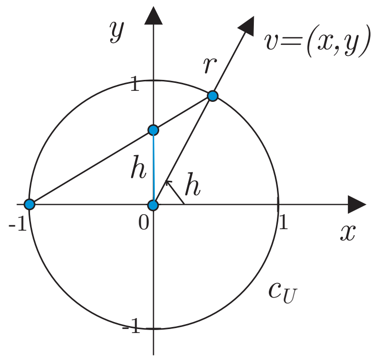

Figure 1: Rotor coordinates $|\,r, h\rangle$ for $\mathbf{v} = (x, y)$

Figure 1 shows a general vector $\mathbf{v}$, its half-slope h, and the notation we use to denote it in a diagram, namely a directed segment at the meet of the x axis and the vector $\mathbf{v}$ positioned at the origin. In terms of the usual polar coordinates, h is the tan of half of the polar angle θ. Rotor coordinates do not extend to general fields or arbitrary quadratic forms on account of the inclusion of length, so they should be viewed as tools of applied mathematics, where we approach geometry from the spirit of rational trigonometry, but allow an orientation to be defined.

It will be useful to introduce the rational functions

$$C(h) \equiv \frac{1-h^2}{1+h^2}, \quad S(h) \equiv \frac{2h}{1+h^2} \quad \text{and} \quad T(h) \equiv \frac{S(h)}{C(h)} = \frac{2h}{1-h^2}.$$

2. A review of vector trigonometry for triangles

2.1. The half-slope formula and transformations

We now summarize some of the main results from the paper [10] which we will need. The first formula shows that in fact the half-slope can be easily determined from the Cartesian coordinates and the length.

Theorem 1 (Half-slope formula). *If $\mathbf{v} \equiv (x, y)$ has length $r \equiv \sqrt{x^2 + y^2}$ and $y \neq 0$, then*

$$h(\mathbf{v}) = \frac{r - x}{y}. \tag{3}$$

Once we know the half-slope of a vector, we can quickly determine half-slopes of related vectors.

Theorem 2 (Half-slope transformations). *Suppose that the vector $\mathbf{v}$ has half-slope h. Then the reflection of $\mathbf{v}$ in the x-axis has half-slope $-h$, the reflection of $\mathbf{v}$ in the y-axis has half slope h^{-1}, the vector $-\mathbf{v}$ has half-slope $-h^{-1}$, while the reflection of $\mathbf{v}$ in the line $y = x$ and* *the rotation of* $\mathbf{v}$ *by a one-quarter of the full circle in the positive direction have respective half-slopes*

$$\frac{1-h}{1+h} \quad \text{and} \quad \frac{1+h}{1-h}.$$

2.2. Projective formulation and the circle sum

Since the half-slope h parametrizes points on the unit circle, it also parametrize rotations, as explained in more detail in [10]. For a number h we define the *rotation matrix*

$$\sigma_h \equiv \frac{1}{(1+h^2)}\begin{pmatrix} 1-h^2 & 2h \\ -2h & 1-h^2 \end{pmatrix} \quad \text{along with} \quad \sigma_\infty \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Such rotations act on a (row) vector $\mathbf{v} = (x, y)$ on the right by $\mathbf{v} \to \mathbf{v}\,\sigma_h$, and clearly the image of $(1,0)$ is $\mathbf{e}(h)$. The multiplicative structure of rotations is then given in terms of half-slopes by the following, which is clearly closely related to the addition formula for $\tan\theta$, but is logically independent of any transcendental interpretations.

Theorem 3 (Circle sum)**.** *For any numbers* h_1 *and* h_2

$$\sigma_{h_1}\sigma_{h_2} = \sigma_h \quad \text{where} \quad h \equiv h_1 \oplus h_2 = \frac{h_1+h_2}{1-h_1h_2}.$$

Proof. We compute that

$$\begin{aligned} \sigma_{h_1}\sigma_{h_2} &= \frac{1}{(1+h_1^2)(1+h_2^2)}\begin{pmatrix} 1-h_1^2 & 2h_1 \\ -2h_1 & 1-h_1^2 \end{pmatrix}\begin{pmatrix} 1-h_2^2 & 2h_2 \\ -2h_2 & 1-h_2^2 \end{pmatrix} \\ &= \frac{1}{(1+h_1^2)(1+h_2^2)}\begin{pmatrix} (1-h_1h_2)^2-(h_1+h_2)^2 & 2(1-h_1h_2)(h_1+h_2) \\ -2(1-h_1h_2)(h_1+h_2) & (1-h_1h_2)^2-(h_1+h_2)^2 \end{pmatrix} \\ &= \frac{(1-h_1h_2)^2}{(1+h_1^2)(1+h_2^2)}\begin{pmatrix} 1-h^2 & 2h \\ -2h & 1-h^2 \end{pmatrix} = \sigma_h \end{aligned}$$

where

$$h \equiv h_1 \oplus h_2 = \frac{h_1+h_2}{1-h_1h_2}. \qquad \square$$

Define $h_1 \oplus h_2$ to be the *circle sum* of the numbers h_1 and h_2. We allow our number system to be extended to include ∞, in which case this circle sum operation extends to values of ∞ by limiting arguments, or by going back to a more fundamental projective formulation as described in [10]. The identity is $h = 0$, and the inverse of h is $-h$.

The extension of the circle sum to more than two inputs is also interesting, for example one can verify that

$$h_1 \oplus h_2 \oplus h_3 = \frac{h_1+h_2+h_3-h_1h_2h_3}{1-(h_1h_2+h_2h_3+h_1h_3)}.$$

The *Circle sum theorem* of [10] generalizes this to more than three values.

Example 1. The half-slope that corresponds to an angle of $45° + 30° = 75°$ is

$$h = \left(\sqrt{2}-1\right) \oplus \left(2-\sqrt{3}\right) = \frac{(\sqrt{2}-1)+(2-\sqrt{3})}{1-(\sqrt{2}-1)(2-\sqrt{3})} = \sqrt{3}+\sqrt{6}-\sqrt{2}-2.$$

2.3. Relative half-slopes

The notion of the half-slope of a single vector can be extended to the *relative half-slope* $h(\mathbf{v}_1, \mathbf{v}_2)$ between two vectors $\mathbf{v}_1 \equiv (x_1, y_1) = |\, r_1, h_1\rangle$ and $\mathbf{v}_2 \equiv (x_2, y_2) = |\, r_2, h_2\rangle$ by

$$h(\mathbf{v}_1, \mathbf{v}_2) \equiv h_2 \oplus (-h_1) = \frac{h_2 - h_1}{1 + h_1 h_2}.$$

For an oriented triangle $\overrightarrow{A_1A_2A_3}$ with side lengths r_1, r_2 and r_3, and corresponding half-slopes

$$h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right), \quad h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right) \quad \text{and} \quad h_3 \equiv h\left(\overrightarrow{A_3A_1}, \overrightarrow{A_3A_2}\right)$$

we use the pictorial conventions of (2).

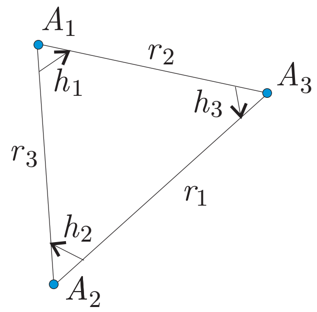

Figure 2: Lengths and relative half-turns of an oriented triangle $\overrightarrow{A_1A_2A_3}$

The *Relative half-slope formula* gives $h(\mathbf{v}_1, \mathbf{v}_2)$ in terms of the Cartesian coordinates x_1, y_1, x_2, y_2 and lengths $r_1 \equiv |\mathbf{v}_1|$ and $r_2 \equiv |\mathbf{v}_2|$:

$$h(\mathbf{v}_1, \mathbf{v}_2) = \frac{y_1(r_2 - x_2) - y_2(r_1 - x_1)}{y_1 y_2 + (r_1 - x_1)(r_2 - x_2)}.$$

What makes this notion important is that it is rotationally invariant.

Theorem 4 (Half-slope invariance)**.** *For vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and any number l*

$$h(\mathbf{v}_1, \mathbf{v}_2) = h(\mathbf{v}_1\sigma_l, \mathbf{v}_2\sigma_l).$$

Once we have relative notions, we can study the geometry of (oriented) triangles from a vector trigonometry point of view.

2.4. Vector trigonometry for triangles

There are rotor analogs for most of the usual trigonometric laws. These are taken from [10]. For example the Cosine law is replaced by:

Theorem 5 (Cross law – rotor form)**.** *In an oriented triangle $\overrightarrow{A_1A_2A_3}$ with side lengths r_1, r_2 and r_3, and corresponding half-slopes*

$$h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right), \quad h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right) \quad \text{and} \quad h_3 \equiv h\left(\overrightarrow{A_3A_1}, \overrightarrow{A_3A_2}\right)$$

we have that

$$r_3^2 = r_1^2 + r_2^2 - 2r_1r_2C(h_3).$$

Corollary 6. *From this we can also deduce that*

$$h_3^2 = \frac{r_3^2 - (r_1 - r_2)^2}{(r_1 + r_2)^2 - r_3^2} = \frac{(r_1 - r_2 - r_3)(r_2 - r_1 - r_3)}{(r_1 + r_2 + r_3)(r_1 + r_2 - r_3)}.$$

The Sine law is replaced by:

Theorem 7 (Sine law – rotor form)**.** *With notation as above,*

$$\frac{S(h_1)}{r_1} = \frac{S(h_2)}{r_2} = \frac{S(h_3)}{r_3}.$$

We also get a relation between the half-slopes of three general vectors:

Theorem 8 (Triple half-slope formula)**.** *For any three vectors* $\mathbf{v}_1$*,* $\mathbf{v}_2$ *and* $\mathbf{v}_3$*, suppose that*

$$h_{12} \equiv h(\mathbf{v}_1, \mathbf{v}_2), \quad h_{23} \equiv h(\mathbf{v}_2, \mathbf{v}_3) \quad \text{and} \quad h_{31} \equiv h(\mathbf{v}_3, \mathbf{v}_1).$$

Then

$$h_{12} + h_{23} + h_{31} = h_{12}h_{23}h_{31}.$$

A variant of that gives us the relations between the half-slopes of an oriented triangle:

Theorem 9 (Triangle half-slope formula)**.** *Suppose that* $\overrightarrow{A_1A_2A_3}$ *is an oriented triangle with half turns*

$$h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right), \quad h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right) \quad \text{and} \quad h_3 \equiv h\left(\overrightarrow{A_3A_1}, \overrightarrow{A_3A_2}\right).$$

Then

$$h_1h_2 + h_1h_3 + h_2h_3 = 1.$$

3. Quadrilateral formulas

We now move to the novel formulas of the present paper, which concern four points or four vectors. While the previous two theorems have different formulas, the situation for four points is more symmetric.

Theorem 10 (Quadruple half-slope formula)**.** *For any four vectors* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *and* $\mathbf{v}_4$*, suppose that*

$$h_{12} \equiv h(\mathbf{v}_1, \mathbf{v}_2), \quad h_{23} \equiv h(\mathbf{v}_2, \mathbf{v}_3), \quad h_{34} \equiv h(\mathbf{v}_3, \mathbf{v}_4), \quad \text{and} \quad h_{41} \equiv h(\mathbf{v}_4, \mathbf{v}_1).$$

Then

$$h_{12} + h_{23} + h_{34} + h_{41} = h_{12}h_{23}h_{34} + h_{12}h_{23}h_{41} + h_{12}h_{34}h_{41} + h_{23}h_{34}h_{41}.$$

Proof. We suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ have half-slopes h_1, h_2, h_3 and h_4 respectively. Then the corresponding relative half-slopes are

$$h_{12} = \frac{h_2 - h_1}{1 + h_1h_2}, \quad h_{23} = \frac{h_3 - h_2}{1 + h_2h_3}, \quad h_{34} = \frac{h_4 - h_3}{1 + h_3h_4}, \quad h_{41} = \frac{h_1 - h_4}{1 + h_4h_1}.$$

Then a computation shows that

$$\begin{aligned}&h_{12}+h_{23}+h_{34}+h_{41}\\&=\frac{h_2-h_1}{1+h_1h_2}+\frac{h_3-h_2}{1+h_2h_3}+\frac{h_4-h_3}{1+h_3h_4}+\frac{h_1-h_4}{1+h_4h_1}\\&=\frac{(h_1-h_3)(h_2-h_4)(h_1-h_2+h_3-h_4+h_1h_2h_3-h_1h_2h_4+h_1h_3h_4-h_2h_3h_4)}{(h_1h_2+1)(h_2h_3+1)(h_3h_4+1)(h_1h_4+1)}.\end{aligned}$$

But a similar computation shows that

$$\begin{aligned}&h_{12}h_{23}h_{34}+h_{12}h_{23}h_{41}+h_{12}h_{34}h_{41}+h_{23}h_{34}h_{41}\\&=\frac{h_2-h_1}{1+h_1h_2}\frac{h_3-h_2}{1+h_2h_3}\frac{h_4-h_3}{1+h_3h_4}+\frac{h_2-h_1}{1+h_1h_2}\frac{h_3-h_2}{1+h_2h_3}\frac{h_1-h_4}{1+h_4h_1}\\&\quad+\frac{h_2-h_1}{1+h_1h_2}\frac{h_4-h_3}{1+h_3h_4}\frac{h_1-h_4}{1+h_4h_1}+\frac{h_3-h_2}{1+h_2h_3}\frac{h_4-h_3}{1+h_3h_4}\frac{h_1-h_4}{1+h_4h_1}\\&=\frac{(h_1-h_3)(h_2-h_4)(h_1-h_2+h_3-h_4+h_1h_2h_3-h_1h_2h_4+h_1h_3h_4-h_2h_3h_4)}{(h_1h_2+1)(h_2h_3+1)(h_3h_4+1)(h_1h_4+1)}.\end{aligned}$$ □

Theorem 11 (Quadrilateral turn formula)**.** *Suppose that $\overrightarrow{A_1A_2A_3A_4}$ is an oriented quadrilateral with half-slopes*

$$h_1\equiv h\left(\overrightarrow{A_1A_2},\overrightarrow{A_1A_4}\right),\qquad h_2\equiv h\left(\overrightarrow{A_2A_3},\overrightarrow{A_2A_1}\right),$$
$$h_3\equiv h\left(\overrightarrow{A_3A_4},\overrightarrow{A_3A_2}\right)\qquad\text{and}\quad h_4\equiv h\left(\overrightarrow{A_4A_1},\overrightarrow{A_4A_3}\right).$$

Then

$$h_1+h_2+h_3+h_4=h_1h_2h_3+h_1h_2h_4+h_1h_3h_4+h_2h_3h_4.$$

Proof. Apply the previous result to the vectors $v_1\equiv\overrightarrow{A_1A_2}$, $v_2\equiv\overrightarrow{A_2A_3}$, $v_3\equiv\overrightarrow{A_3A_4}$ and $v_4\equiv\overrightarrow{A_4A_1}$, so that $h_{12}=-1/h_2$, $h_{23}=-1/h_3$, $h_{34}=-1/h_4$ and $h_{41}=-1/h_1$. Then

$$-\frac{1}{h_2}-\frac{1}{h_3}-\frac{1}{h_4}-\frac{1}{h_1}=\frac{1}{h_1h_2h_3h_4}(-h_2-h_3-h_4-h_1)$$

and after clearing denominators we get the result. □

Example 2. Suppose that $v_1\equiv(3,4)$, $v_2\equiv(5,1)$, $v_3\equiv(1,3)$, and $v_4\equiv(-1,2)$. Then the respective half-slopes are, by the Half-slope formula,

$$h_1=\frac{5-3}{4}=\frac{1}{2},\quad h_2=\frac{\sqrt{26}-5}{1}=\sqrt{26}-5,\quad h_3=\frac{\sqrt{10}-1}{3},\quad h_4=\frac{\sqrt{5}+1}{2}.$$

Then we can verify that

$$h_{12}+h_{23}+h_{34}+h_{41}=\sqrt{2}-\tfrac{1}{2}\sqrt{5}-\tfrac{5}{17}\sqrt{26}+\tfrac{1}{7}\sqrt{65}+\frac{11}{238}$$

and also

$$h_{12}h_{23}h_{34}+h_{12}h_{23}h_{41}+h_{12}h_{34}h_{41}+h_{23}h_{34}h_{41}=\sqrt{2}-\tfrac{1}{2}\sqrt{5}-\tfrac{5}{17}\sqrt{26}+\tfrac{1}{7}\sqrt{65}+\frac{11}{238}.$$

4. A four bar linkage

4.1. Classical and rotor forms

The theoretical framework for viewing quadrilaterals above can be applied to a famous configuration from mechanical engineering. Four bar linkages have a rich history in kinematics and play an important role also in the foundations of robotics. This is a four-revolute mechanical linkage which consists of a fixed base $\overline{OC}$, two arms $\overline{OA}$ and $\overline{CB}$ which are free to rotate about the ends of this fixed base $\overline{OC}$, and a coupler arm $\overline{AB}$ connecting the two moveable ends A and B. These kinds of configurations have been used to transfer power for centuries, and then were adopted by WATT to generate (approximately) linear motion from circular motion [2].

We assume the arms have specified fixed lengths, and are interested in the possible range of motions as the arms move around the base. It is known that there are in fact a variety of possible movements, depending on the relative lengths of the four segments comprising the mechanism. It is common to consider the locus of the midpoint of the coupler arm $\overline{AB}$ to form a coupler curve, but in more general circumstances a specified point on a plate afixed to the arm $\overline{AB}$ is also considered.

The usual analysis of such four bar linkages is traditionally expressed in terms of the *polar coordinates* of the three moveable arms as vectors, following current practice to base robotic analysis on angles. This analysis can be found in many places, for example the above reference or [1].

If the arms are given by the vectors $\overrightarrow{OC}$, $\overrightarrow{OA}$, $\overrightarrow{AB}$ and $\overrightarrow{CB}$ with polar coordinates

$$\overrightarrow{OC} = (u, 0)\,, \quad \overrightarrow{OA} = (r, \phi)\,, \quad \overrightarrow{AB} = (t, \theta)\,, \quad \overrightarrow{CB} = (s, \varphi)$$

then there is, for example, an important relation between the angles ϕ and φ involving inverse circular functions.

With the advent of rational trigonometry, and now vector trigonometry, some new opportunities for analysing robotics arise. For example we can replace the polar coordinate description of the four bar linkage above with an equivalent rotor coordinate description.

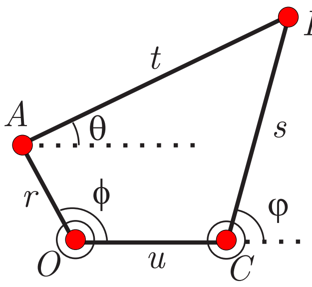

Figure 3: A 4 bar linkage in polar coordinates

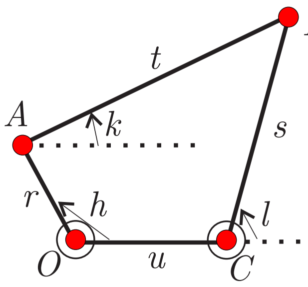

Figure 4: A 4 bar linkage in rotor coordinates

Assume then that we have a four bar linkage with horizontal base $\overrightarrow{OC}$ and arms $\overrightarrow{OA}$, $\overrightarrow{AB}$ and $\overrightarrow{CB}$, described now with rotor coordinates

$$\overrightarrow{OC} = |\,u, 0\rangle\,, \quad \overrightarrow{OA} = |\,r, h\rangle\,, \quad \overrightarrow{AB} = |\,t, k\rangle\,, \quad \overrightarrow{CB} = |\,s, l\rangle$$

as in Figure 4.

The relation between the two possible revolute motions at O and C can now be expressed in the language of rotor coordinates.

Theorem 12 (Four bar half-slope relation). *The relation between the half-slopes h and l in the above four bar linkage is given by:*

$$\begin{aligned}0 = &(r-s+t+u)(r-s-t+u)h^2l^2 + (r+s+t+u)(r+s-t+u)h^2 \\ &+ (-8rs)hl + (r+s+t-u)(r+s-t-u)l^2 + (r-s+t-u)(r-s-t-u).\end{aligned}$$

Proof. We may suppose that OC is horizontal with $O = [0,0]$ and $C = [u,0]$ in a usual Cartesian coordinate system. This means that the moveable ends of the arms are

$$A = [rC(h), rS(h)] \quad \text{and} \quad B = [u + sC(l), sS(l)]$$

so that

$$\overrightarrow{AB} = (u + sC(l) - rC(h), sS(l) - rS(h)).$$

The fact that the quadrance of $\overrightarrow{AB}$ is by definition t^2 means that

$$(u + sC(l) - rC(h))^2 + (sS(l) - rS(h))^2 = t^2.$$

When these terms are expanded out, and suitably factored, we get a quartic relation between the half-spreads h and l. □

Clearly the relative values of the lengths u, r, s, and t will determine properties of this motion, and in particular situations where some of these linear factors appearing here are zero can be expected to play an important role.

Example 3. While it is not possible for $r - s + t + u = 0$ without the quadrilateral becoming linear, it is possible for $r + u = s + t$. In this case the relation reduces to a quadratic one between h and l:

$$2(rs + su)h^2 - 4rshl + r(r-u)l^2 = u(r-u).$$

The discriminant of the quadratic form on the left is

$$(4rsl)^2 - 4(2(rs+su))(r(r-u)) = 8rs\left(u^2 + 2sr - r^2\right).$$

4.2. The coupler half-slope

Theorem 13 (Four bar coupler half-slope). *The half-slope of the coupler arm AB, as in Figure 4, is given in terms of h and l by*

$$k = \frac{(s-r-t+u)h^2l^2 + (u-s-t-r)h^2 + (r+s-t+u)l^2 + (r-s-t+u)}{2(-sh^2l + rhl^2 + rh - sl)}.$$

Proof. Using the above notation, we have that $\overrightarrow{AB} = (u + sC(l) - rC(h), sS(l) - rS(h))$ has assumed length t. So by the Half-slope formula the half-slope k of $\overrightarrow{AB}$ is

$$k = \frac{t - u + sC(l) - rC(h)}{sS(l) - rS(h)}.$$

When we substitute and simplify we get the quantity in the theorem. □

4.3. Coupler curves

When we choose a point X on AB, or on a plate rigidly affixed to AB, then the locus of that point traces a coupler curve. For simplicity we will stick to the simplest case, when the point X is the midpoint of AB, namely

$$X = \frac{1}{2}\left([rC(h),\, rS(h)] + [u + sC(l),\, sS(l)]\right) \tag{4}$$

$$= \frac{\begin{bmatrix} (u-s-r)\,h^2l^2 + (s-r+u)\,h^2 + (r-s+u)\,l^2 + (r+s+u)\,, \\ 2\,(sh^2l + rhl^2 + rh + sl) \end{bmatrix}}{2\,(h^2+1)\,(l^2+1)}. \tag{5}$$

In rotor coordinates, the quadrance of $\overrightarrow{OX}$ turns out to be, rather remarkably, the quantity

$$\frac{(u-s-r)^2\,h^2l^2 + (s-r+u)^2\,h^2 + 8rshl + (r-s+u)^2\,l^2 + (r+s+u)^2}{4\,(h^2+1)\,(l^2+1)}.$$

Another interesting case is when the point X is chosen to be the reflection of A in B, namely

$$X = -\,[rC(h),\, rS(h)] + 2\,[u + sC(l),\, sS(l)]$$

$$= \frac{\begin{bmatrix} (r-2s+2u)\,h^2l^2 + (r+2s+2u)\,h^2 + (2u-2s-r)\,l^2 + (2s-r+2u)\,, \\ 2\,(2sh^2l - rhl^2 - rh + 2sl) \end{bmatrix}}{(h^2+1)\,(l^2+1)}.$$

In this case the quadrance of $\overrightarrow{OX}$ is

$$\frac{(r-2s+2u)^2\,h^2l^2 + (r+2s+2u)^2\,h^2 + (-16rs)\,hl + (2u-2s-r)^2\,l^2 + (2s-r+2u)^2}{(h^2+1)\,(l^2+1)}.$$

Such formulas are intriguing due to the pleasant symmetries they contain, and the suggestion of further remarkable algebraic relations. So perhaps rotor coordinates and vector trigonometry can shed some new light on this old topic, and contribute to the modern development of robotics as well.

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