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A Set of Rectangles Inscribed in an Orthodiagonal Quadrilateral and Defined by Pascal-Points Circles

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Abstract. In this paper, we define the concept of "a rectangle defined by a circle that forms Pascal points and a Pascal-points circle" in an orthodiagonal quadrilateral. This rectangle is inscribed in the given orthodiagonal quadrilateral. We show that every orthodiagonal quadrilateral has an infinite set of such rectangles. We also investigate the properties of this set of rectangles: we show that the angle between the diagonals is equal in all the rectangles and we find the rectangle with the smallest area and perimeter, and more. We also examine the intersection of this set of rectangles with another set of rectangles inscribed in the orthodiagonal quadrilateral. This second set satisfies the condition that the sides of the rectangles are parallel to the diagonals of the given quadrilateral. Finally, we prove the uniqueness of the set of rectangles defined by circles that form Pascal points and Pascal-points circles.

Key Words: Rectangles inscribed in orthodiagonal quadrilateral, circle that forms Pascal points, Pascal-points circle

MSC 2010: 51M04, 51N20

1. Introduction: General properties and definitions

First, we shall recall some definitions and properties that we shall use in the present paper: *"Pascal points on the sides of a quadrilateral"* and *"a circle that forms Pascal points"*. For a convex quadrilateral $ABCD$ in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD, a circle that forms Pascal points is any circle that passes through points E and F and also through interior points of sides BC and AD (see Figure 1).

Let ω be a circle that forms Pascal points, and let $M = \omega \cap BC$, $N = \omega \cap AD$, and K and L be the points of intersection of ω with the extensions of diagonals BD and AC, respectively (see Figure 2).

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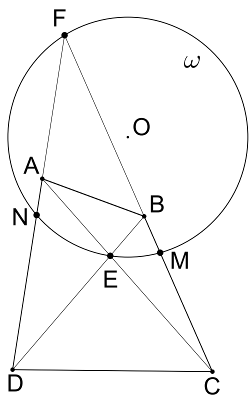

Figure 1

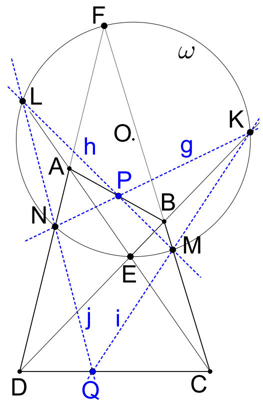

Figure 2

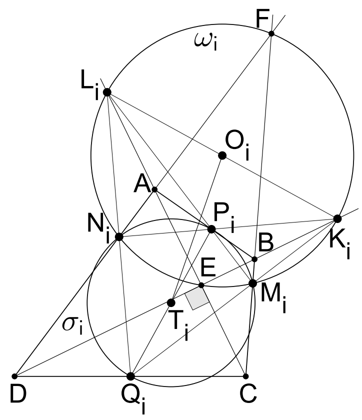

Figure 3

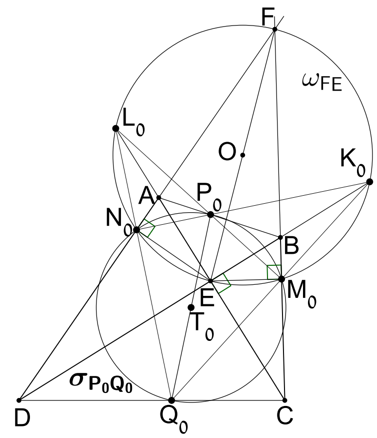

Figure 4

We further denote $P = KN \cap LM$ and $Q = KM \cap LN$. According to Pascal's Theorem, in the crossed hexagon $EKNFML$ inscribed in the circle ω, the points A, P, and B are collinear and, in addition, it can be proven that P lies between A and B (see [2]). In a similar manner, we can prove that in the crossed hexagon $EKMFNL$ that is also inscribed in the circle ω, the point Q belongs to the segment CD. Therefore, the points P and Q are called *"Pascal points" formed by the circle ω on the sides AB and CD*. Different circles that pass through the points E and F and through interior points of the sides BC and AD form different pairs of Pascal points P and Q. We denote this set of circles by $\{\omega_i\}$.

For a pair of Pascal points P and Q, the *"Pascal-points circle σ_{PQ}"* is the circle whose diameter is the segment PQ (see [5]). We denote this set of circles by $\{\sigma_{P_iQ_i}\}$.

In the present paper, we consider the case in which the quadrilateral $ABCD$ is orthodiagonal, in other words, its diagonals intersect at a right angle.

General data

Throughout this paper, *$ABCD$ is an orthodiagonal quadrilateral in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD.*

1. *ω_i is an arbitrary circle (whose center is at O_i) that passes through points E and F and through interior points of sides BC and AD (we denote $M_i = \omega_i \cap BC$, $N_i = \omega_i \cap AD$, $K_i = \omega_i \cap BD$, and $L_i = \omega_i \cap AC$);*
   P_i and Q_i are a pair of Pascal points formed by the circle ω_i on the sides AB and CD;
   $\sigma_{P_iQ_i}$ is a Pascal-points circle P_i and Q_i (see Figure 3).
2. *ω_{EF} is a circle whose diameter is the segment EF that passes through the interior points M_0 and N_0 of the sides BC and AD (we denote $M_0 = \omega_{EF} \cap BC$, $N_0 = \omega_{EF} \cap AD$, $K_0 = \omega_{EF} \cap BD$, and $L_0 = \omega_{EF} \cap AC$);*
   P_0 and Q_0 are a pair of Pascal points formed by the circle ω_{EF} on the sides BC and AD;
   $\sigma_{P_0Q_0}$ is a Pascal-points circle P_0 and Q_0 (see Figure 4).

Property 1. *For an orthodiagonal quadrilateral $ABCD$, there are an infinite number of circles that form Pascal points* (see [5, Theorem 1]).

Property 2. *The Pascal-points circle $\sigma_{P_iQ_i}$ intersects the sides BC and AD at points M_i and V_i, N_i and W_i, respectively* (see Figure 5)*, and there holds* (see [5, Theorem 2])

(i) *the segment V_iW_i is a diameter of the circle $\sigma_{P_iQ_i}$;*

(ii) *the quadrilateral $P_iV_iQ_iW_i$ is a rectangle.*

Note: in a particular case in which one of the sides BC or AD is tangent to the circle $\sigma_{P_iQ_i}$, the pairs of points M_i and V_i or N_i and W_i coincide. In this case, one of the segments, V_iN_i or M_iW_i, will be a diameter.

From Property 2 follows that for every orthodiagonal quadrilateral $ABCD$, each Pascal-points circle defines a rectangle inscribed in this quadrilateral.

Property 3. *In addition to item 2 of the general data, let G be the point of intersection of the sides AB and CD, and let ψ_{EG} be a circle whose diameter is the segment EG. Then there holds* (see [5, Theorems 3-4])

(i) *the circle $\sigma_{P_0Q_0}$ intersects the sides of the quadrilateral $ABCD$ at 8 points;*

(ii) *the angle FEG between the diameters EF and EG of the circles ω_{EF} and ψ_{EG} equals the angle $V_0T_0Q_0$ between the diameters P_0Q_0 and V_0W_0 of the circle $\sigma_{P_0Q_0}$* (see Figure 6).

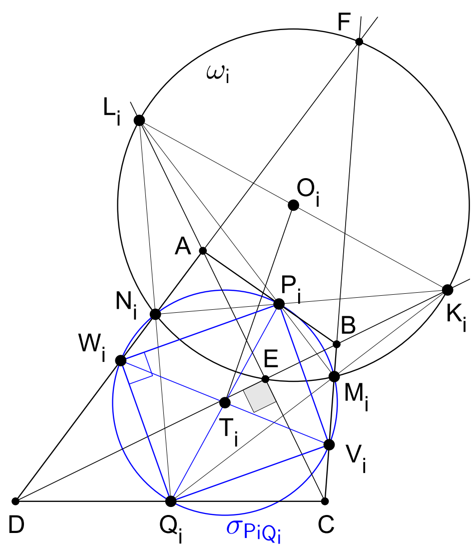

Figure 5

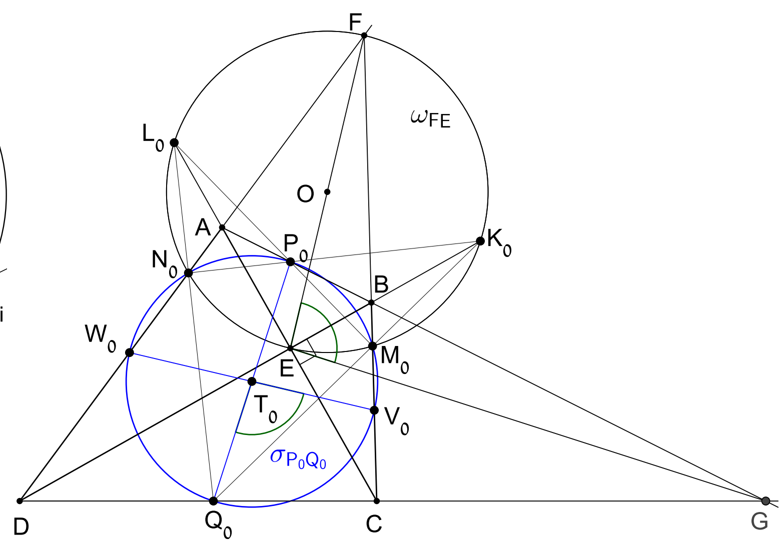

Figure 6

We now define precisely the concept of "a rectangle defined by a Pascal-points circle".

Definition 1. Let $PVQW$ be a rectangle inscribed in the orthodiagonal quadrilateral $ABCD$, where the vertices P and Q of the rectangle are Pascal points formed by the circle ω on the sides AB and CD of the quadrilateral. The vertices V and W of the rectangle are the points of intersection of the Pascal-points circle σ_{PQ} with the sides BC and AD. Then we call $PVQW$ *"the rectangle defined by the circle ω and the Pascal-points circle σ_{PQ}".*

Note: Under circumstances where it is clear which circle forms the Pascal points P and Q, we can call rectangle $PVQW$ *"the rectangle defined by the Pascal-points circle σ_{PQ}"*, for short.

Corollary 1. *For each orthodiagonal quadrilateral $ABCD$, there exists an infinite set $\{\omega_i\}$ of circles that form Pascal points P_i and Q_i. This set defines an infinite set $\{\sigma_{P_iQ_i}\}$ of Pascal-points circles, which further defines an infinite set $\{P_iV_iQ_iW_i\}$ of rectangles inscribed in the quadrilateral $ABCD$. We denote this set of rectangles by $\mathbb{M}_\odot$.*

The article contains five sections: In Section 2, we study the properties of rectangle set $\mathbb{M}_{\odot}$. In Section 3, we compare this rectangle set with another set of rectangles inscribed in the quadrilateral $ABCD$, whose sides are parallel to the diagonals of the quadrilateral (we denote it by $\mathbb{M}_{\|}$). In Section 4, we examine the case where the sides AB and CD are also not parallel ($AB \cap CD = G$). In this case, there is another infinite set of circles that pass through the points E and G and form Pascal points on the sides BC and AD (see [5]). We prove that the set of Pascal-points circles defined using this set defines precisely the same set, $\mathbb{M}_{\odot}$, of rectangles inscribed in the quadrilateral $ABCD$. Thereby we prove that $\mathbb{M}_{\odot}$ is the only set of rectangles defined by circles that form Pascal points and Pascal-points circles.

In Section 5, we use the set $\mathbb{M}_{\odot}$ to define a new concept, "two mutually-coordinated circles with respect to an orthogonal quadrilateral", and we give properties of the circles that meet this definition.

2. The properties of set of rectangles $\mathbb{M}_{\odot}$

Theorem 1. *For all rectangles from the set $\mathbb{M}_{\odot}$, the angle between the diagonals is the same and depends only on the quadrilateral $ABCD$, and not on the choice of the circle ω_i.*

Proof. Let ω_1 and ω_2 be two circles that form the Pascal points P_1, Q_1 and P_2, Q_2, respectively, on the sides AB and CD, and let $\sigma_{P_1Q_1}$ and $\sigma_{P_2Q_2}$ be the corresponding Pascal-points circles. The circle $\sigma_{P_1Q_1}$ intersects the sides BC and AD at points V_1 and W_1, respectively, in addition to the points M_1 and N_1 (see Figure 7). The circle $\sigma_{P_2Q_2}$ intersects the sides BC and AD at points V_2 and W_2, respectively, in addition to the points M_2 and N_2. In accordance with item (ii) in Property 2 above, the quadrilaterals $P_1V_1Q_1W_1$ and $P_2V_2Q_2W_2$ are rectangles.

We shall prove that the angle between the diagonals P_1Q_1 and V_1W_1 in the first rectangle equals the angle between the diagonals P_2Q_2 and V_2W_2 in the second rectangle. Through point F, which is common to both circle ω_1 and ω_2, we draw two tangents: FX is tangent to the circle ω_1, and FY is tangent to the circle ω_2. We denote by φ the acute angle between the tangents (angle XFY in Figure 7).

Now we shall prove that the angle between the diagonals P_1Q_1 and P_2Q_2 equals φ. First, we prove that for any circle ω that forms Pascal points P and Q there holds: The straight line PQ is perpendicular to the diameter KL of the circle ω (see Figure 8). Since both angle KML and angle KNL are equal to 90°, the segments KN and LM are altitudes to the sides QL and QK, respectively, in the triangle QKL. These altitudes intersect at the point P. It follows that the third altitude, which issues from the vertex Q to the side KL, is contained in the line QP. Thus, we obtain $PQ \perp KL$.

Therefore, in our case, $P_1Q_1 \perp K_1L_1$ and $P_2Q_2 \perp K_2L_2$ (see Figure 7). In the circle ω_1, the inscribed angle FK_1E is equal to the angle XFE between the tangent XF and the chord FE (see Figure 9). In a similar manner, in the circle ω_2, the angle FK_2E is equal to the angle YFE. Therefore there holds

$$\measuredangle FK_1E - \measuredangle FK_2E = \measuredangle XFE - \measuredangle YFE.$$

Since the angle FK_1E is an exterior angle of the triangle FK_1K_2, there holds

$$\measuredangle FK_1E - \measuredangle FK_2E = \measuredangle K_1FK_2.$$

On the other hand, $\measuredangle XFE - \measuredangle YFE = \measuredangle XFY = \varphi$, and therefore $\measuredangle K_1FK_2 = \varphi$.

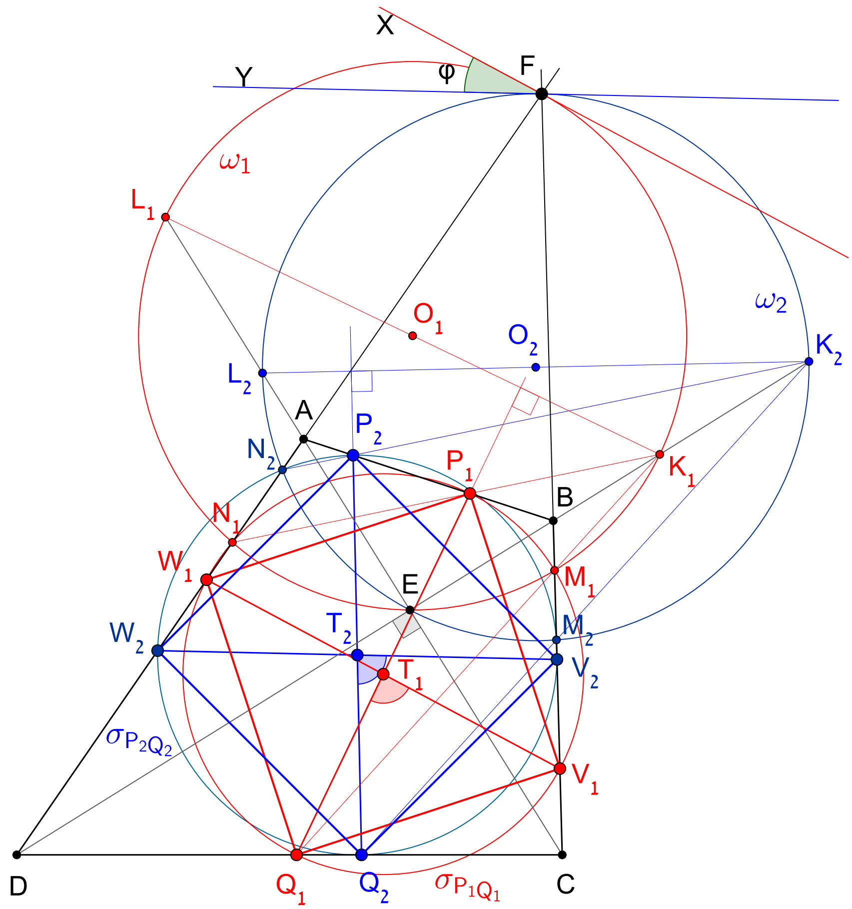

Figure 7

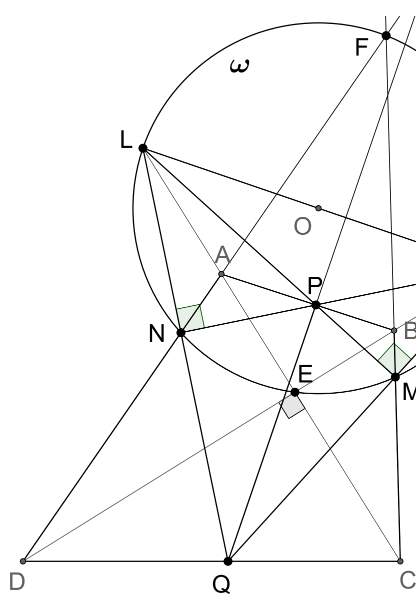

Figure 8

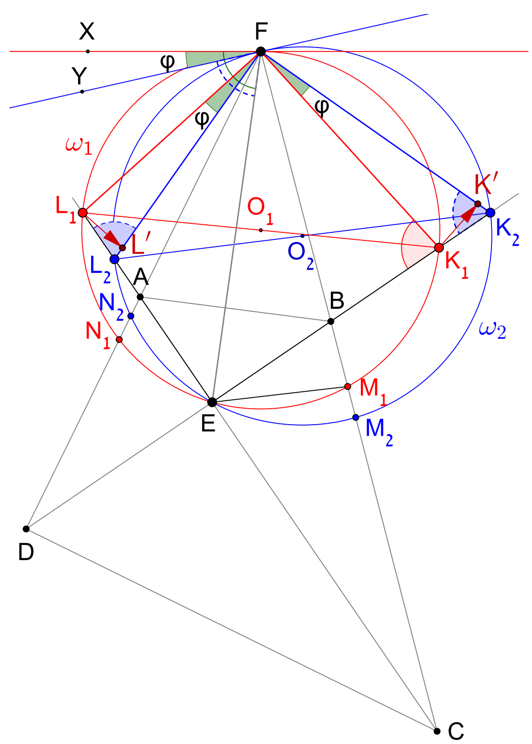

Figure 9

The quadrilaterals FK_1EL_1 and FK_2EL_2 are inscribed in circles ω_1 and ω_2, respectively. Therefore, the opposite angles of the quadrilaterals satisfy

$$\measuredangle FL_1E = 180^\circ - \measuredangle FK_1E \text{ and } \measuredangle FL_2E = 180^\circ - \measuredangle FK_2E.$$

Hence, for angle L_1FL_2, being an interior angle in triangle L_1FL_2, there holds

$$\begin{aligned}\measuredangle L_1FL_2 &= \measuredangle FL_2E - \measuredangle FL_1L_2 = (180^\circ - \measuredangle FK_2E) - (180^\circ - \measuredangle FK_1E)\\ &= \measuredangle FK_1E - \measuredangle FK_2E = \varphi.\end{aligned}$$

We obtained that $\measuredangle K_1FK_2 = \measuredangle L_1FL_2$. In addition, there holds $\measuredangle FK_2E = \measuredangle FL_2L_1$ (because the angle FL_2L_1 is the supplementary adjacent of $\measuredangle FL_2E$). Hence it follows that the triangles FK_2K_1 and FL_2L_1 are similar, and therefore $\frac{FL_1}{FL_2} = \frac{FK_1}{FK_2}$.

We now compose the following two geometric transformations (see Figure 9): The first transformation is the counterclockwise rotation by angle φ about point F (we denote this rotation by R_F^φ). The second transformation is the homothety with the center at point F and a factor of $k = \frac{FL_2}{FL_1}$ (we denote this homothety by H_F^k).

The rotation R_F^φ transforms the point L_1 to the point L', which belongs to the ray FL_2 and satisfies $FL_1 = FL'$. The rotation also transforms the point K_1 to the point K', which belongs to the ray FK_2 and satisfies $FK_1 = FK'$. Therefore the rotation transforms the line L_1K_1 into the line $L'K'$.

Rotations satisfy the following property: *"the angle between the original straight line and the image line equals the angle of rotation"*. Therefore, since $R_F^\varphi(L_1K_1) = L'K'$, it follows that $\left(\widehat{L_1K_1, L'K'}\right) = \varphi$.

The homothety H_F^k transforms the point L' to the point L'', which belongs to the ray FL_2 and satisfies $FL'' = FL' \cdot k = FL_1 \cdot \frac{FL_2}{FL_1} = FL_2$. In other words, L'' is the point L_2. Similarly, $H_F^k(K') = K_2$.

Thus we have obtained that homothety H_F^k transforms the line $L'K'$ into the line L_2K_2. From properties of homothety, it follows that the straight lines $L'K'$ and L_2K_2 are parallel. Therefore the angle between the lines L_1K_1 and L_2K_2 is equal to the angle between the straight lines L_1K_1 and $L'K'$. In other words, $\left(\widehat{L_1K_1, L_2K_2}\right) = \varphi$.

We have shown above that the lines P_1Q_1 and P_2Q_2 are respectively perpendicular to the straight lines L_1K_1 and L_2K_2. Therefore there holds that the angle between P_1Q_1 and P_2Q_2 equals φ. We obtained that the angle between the diagonals P_1Q_1 and P_2Q_2 of the rectangles $P_1V_1Q_1W_1$ and $P_2V_2Q_2W_2$ equals φ. Now we shall calculate the angle between diagonals V_1W_1 and V_2W_2.

First, we shall prove that the diagonal V_1W_1 is perpendicular to the straight line FO_1 (O_1 is the center of the circle ω_1). We use the method of complex numbers in the geometry of the plane. The principles of the method and formulas that we shall use appear, for example, in [8, pp. 154–181].

We choose a system of coordinates whose origin is at point O_1 (the center of circle ω_1), and the unit length equals O_1E. In this system ω_1 is the unit circle, and the equation of the unit circle is $z \cdot \overline{z} = 1$, where z and $\overline{z}$ are the complex coordinate and the conjugate of the coordinate of an arbitrary point Z that lies on circle ω_1.

Let e, f, k, l, m, and n be the complex coordinates of the points E, F, K_1, L_1, M_1, and N_1, respectively. These points lie on the unit circle, and therefore there holds:

$$\overline{e} = \frac{1}{e}, \quad \overline{f} = \frac{1}{f}, \quad \overline{k} = \frac{1}{k}, \quad \overline{l} = \frac{1}{l}, \quad \overline{m} = \frac{1}{m} \quad \text{and} \quad \overline{n} = \frac{1}{n}. \tag{1}$$

In this chosen system, one can express the coordinates v and w of points V_1 and W_1 (and their conjugates) using the coordinates f, m, and n of the points F, M_1, and N_1 (which belong to the unit circle) as follows (see the proof of Theorem 2 in [5]):

$$v = \frac{2mn - fm + fn}{m+n} \quad \text{and} \quad \overline{v} = \frac{2f - n + m}{f\,(m+n)},$$
$$w = \frac{2mn - fn + fm}{m+n} \quad \text{and} \quad \overline{w} = \frac{2f - m + n}{f\,(m+n)}.$$

The straight lines V_1W_1 and FO_1 are perpendicular if there holds

$$(w - v)\left(\overline{f} - \overline{o}\right) + (\overline{w} - \overline{v})\,(f - o) = 0. \tag{2}$$

For the origin O_1 holds $\overline{o} = o = 0$. Let us substitute the corresponding expressions for f, $\overline{f}$, v, $\overline{v}$, w, and $\overline{w}$ in the left-hand side of (2):

$$\left(\frac{2mn - fn + fm}{m+n} - \frac{2mn - fm + fn}{m+n}\right)\cdot\frac{1}{f} + \left(\frac{2f - m + n}{f\,(m+n)} - \frac{2f - n + m}{f\,(m+n)}\right)\cdot f$$
$$= \frac{-2fn + 2fm}{m+n}\cdot\frac{1}{f} + \frac{-2m + 2n}{f\,(m+n)}\cdot f = \frac{2\,(m-n)}{m+n} + \frac{2\,(n-m)}{m+n} = 0.$$

In other words, (2) is satisfied and therefore $V_1W_1 \perp FO_1$, as we set out to prove.
Note too, that the line FX is also perpendicular to the line FO_1 (because FX is tangent to the circle ω_1 at the point F, and FO_1 is a radius at the point F). Therefore the lines V_1W_1 and FX are parallel. Similarly we prove that the line V_2W_2 is parallel to the line FY, which is tangent to the circle ω_2 at the point F. Therefore the angle between the line V_1W_1 and the line V_2W_2 equals the angle between the tangents FX and FY, which is equal to φ.

We obtained that the angle between the two diagonals, V_1W_1 and V_2W_2, of the rectangles $P_1V_1Q_1W_1$ and $P_2V_2Q_2W_2$ equals φ. Let H be the point of intersection of the diagonals P_1Q_1 and P_2Q_2, and let U be the point of intersection of the diagonals V_1W_1 and V_2W_2 (see Figure 10). The counterclockwise rotation by angle φ about the point H transforms the line P_1Q_1 into the line P_2Q_2, and the counterclockwise rotation by angle φ about the point U transforms the line V_1W_1 into the line V_2W_2. Therefore there holds that the angle between the diagonals of the rectangle $P_1V_1Q_1W_1$ is equal to the angle between the diagonals of the rectangle $P_2V_2Q_2W_2$. □

The following corollary follows from Theorem 1 and item (ii) of Property 3 above:

Corollary 2. *Given the general data* (see above)*, then*

1. *in the case where the extensions of the sides AB and CD intersect at the point G, the angle $V_iT_iQ_i$ between the diagonals in every rectangle $P_iV_iQ_iW_i$ from the set $\mathbb{M}_\odot$ is equal to the angle FEG* (see Figure 11);
2. *in the case where the sides AB and CD are parallel (in other words, the quadrilateral $ABCD$ is a trapezoid), the angle $V_iT_iQ_i$ between the diagonals is equal to the angle FEJ, where $J \in BC$ and $EJ \parallel AB$* (see Figure 12).

Explanation: In the case where the sides AB and CD are parallel, we consider the point of their intersection, G, as a point at infinity. Therefore, every straight line that is parallel to AB and CD (and in particular, line EJ) passes through the point G. Therefore, in this case, it holds that $\measuredangle FEJ = \measuredangle V_iT_iQ_i$.

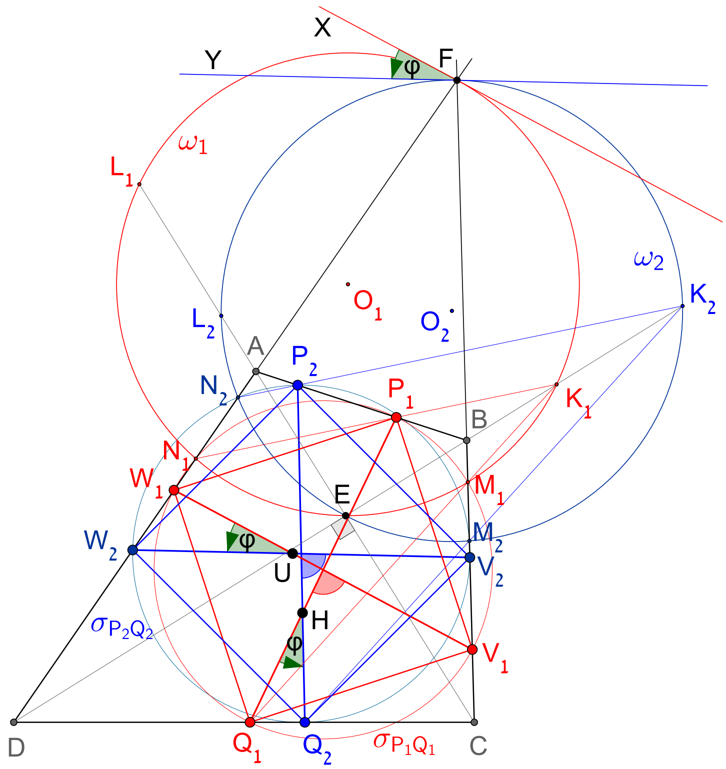

Figure 10

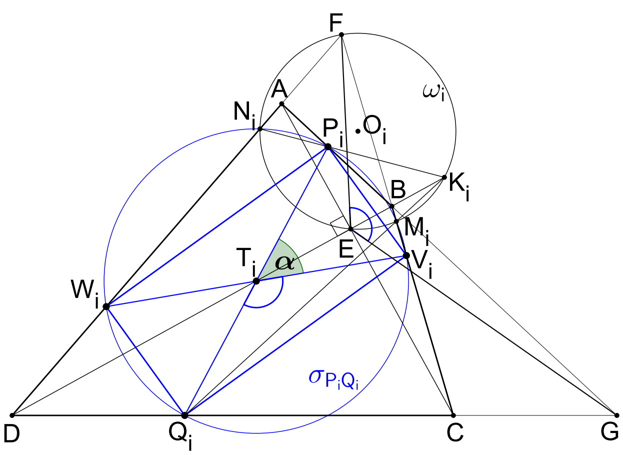

Figure 11

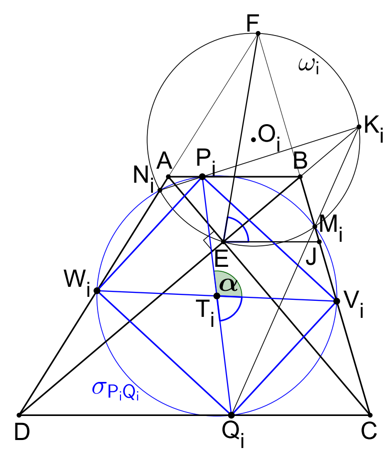

Figure 12

Let $P_iV_iQ_iW_i$ be an arbitrary rectangle from the set $\mathbb{M}_\odot$. We denote the angle $\measuredangle P_iT_iV_i$ between the diagonals of the rectangle by α (see Figures 11 and 12).

The ratio $\frac{P_iV_i}{V_iQ_i}$ between adjacent sides equals $\tan \frac{1}{2}\alpha$. Therefore, for every two rectangles, $P_1V_1Q_1W_1$ and $P_2V_2Q_2W_2$, from the set $\mathbb{M}_\odot$ there holds that $\frac{P_1V_1}{V_1Q_1} = \frac{P_2V_2}{V_2Q_2}$. In other words, every two rectangles from the set $\mathbb{M}_\odot$ are similar. Thus, we may conclude the following

Corollary 3. *Every two rectangles from the set* $\mathbb{M}_\odot$ *are similar.*

From Corollary 2 follows that if the segments EF and EG are perpendicular ($\measuredangle FEG = 90°$), then in each rectangle from the set $\mathbb{M}_\odot$ one of the angles between the diagonals equals 90°. In other words, in this case all rectangles from the set $\mathbb{M}_{\odot}$ are squares. This leads to the following corollary:

Corollary 4. *Let $ABCD$ be an orthodiagonal quadrilateral in which the diagonals intersect at the point E and the extensions of the sides BC and AD intersect at the point F. Then each of the following two conditions is sufficient for the set $\mathbb{M}_{\odot}$ to be a set of squares inscribed in the quadrilateral $ABCD$:*

1. *The extensions of the sides AB and CD intersect at the point G, and the angle FEG is equal to* 90° (see Figure 13).
2. *$ABCD$ is an isosceles trapezoid.*

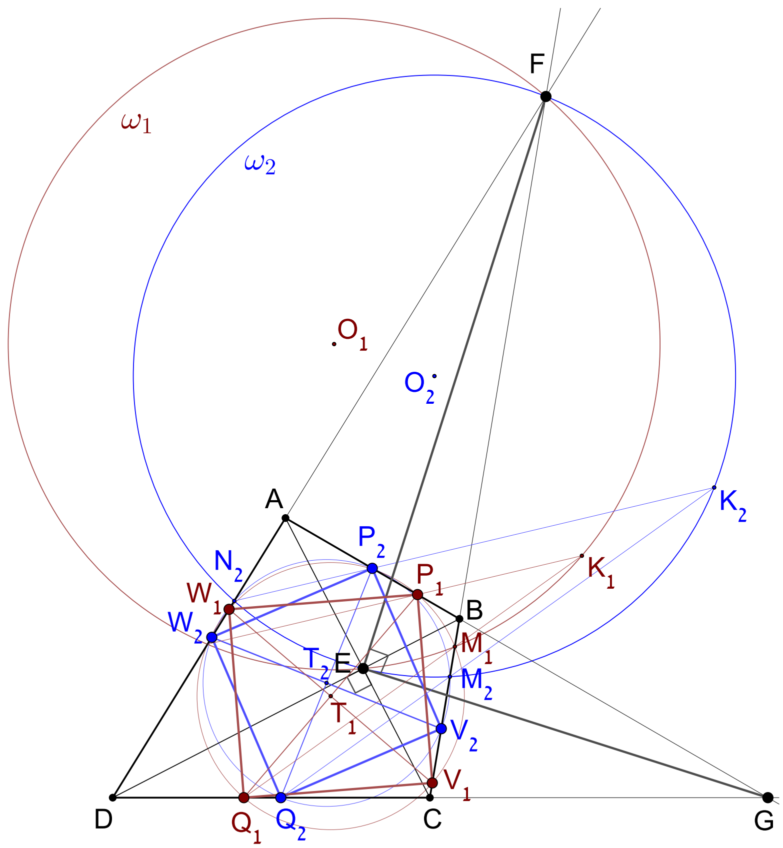

Figure 13

Item 2 of Corollary 4 follows from item 2 of Corollary 2 and from the fact that in the isosceles trapezoid the straight line EF is perpendicular to the bases AB and CD, and therefore also perpendicular to the line EJ, which is parallel to the bases.

Theorem 2. *Given the general data* (see above)*, then among all the rectangles of the set $\mathbb{M}_{\odot}$, the one with the minimal perimeter and area is the rectangle defined by the circle ω_{EF} and the Pascal-points circle $\sigma_{P_0Q_0}$.*

Proof. Let ω_i be an arbitrary circle that forms the Pascal points P_i and Q_i. We first prove that the length of the segment P_iQ_i depends only on the length of the diameter of circle ω_i and the angle of viewing of the diameter K_iL_i from the point Q_i (see Figure 14).

Let the angle of viewing of the segment K_iL_i from point Q_i be $\measuredangle K_iQ_iL_i = \beta$, and let $\measuredangle L_iQ_iZ = \gamma$. Therefore $\measuredangle K_iQ_iZ = \beta - \gamma$, $\measuredangle N_iK_iL_i = \gamma$, and $\measuredangle K_iL_iM_i = \beta - \gamma$. In the right-angled triangles $K_iL_iN_i$ and $P_iQ_iN_i$ holds

$$N_iL_i = K_iL_i \cdot \sin\gamma \quad \text{and} \quad Q_iN_i = P_iQ_i \cdot \cos\gamma.$$

Hence, we obtain for the segment Q_iL_i: $Q_iL_i = P_iQ_i \cdot \cos\gamma + K_iL_i \cdot \sin\gamma$.

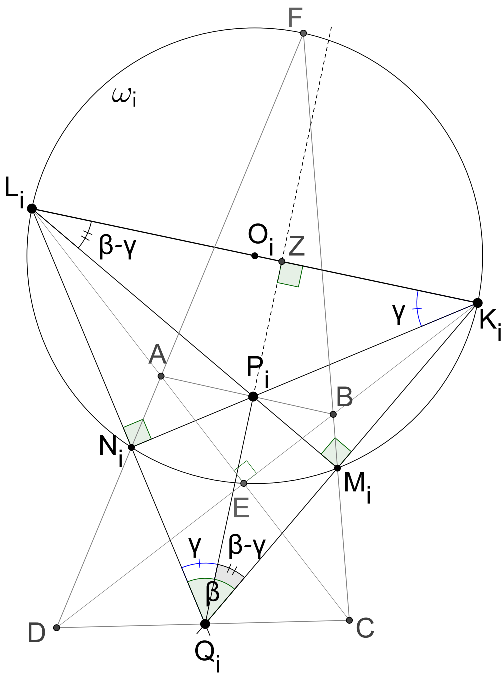

Figure 14

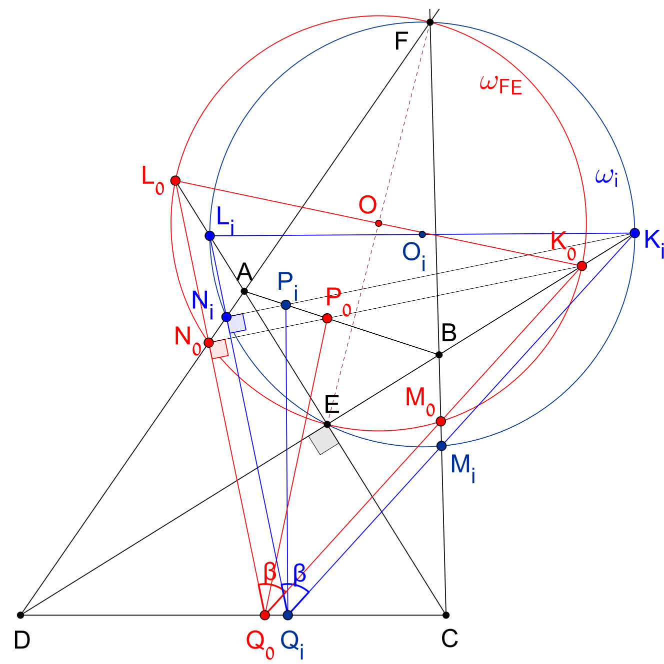

Figure 15

In a similar manner, in the right-angled triangles $K_iL_iM_i$ and $P_iQ_iM_i$ holds

$$M_iK_i = K_iL_i \cdot \sin(\beta - \gamma) \quad \text{and} \quad Q_iM_i = P_iQ_i \cdot \cos(\beta - \gamma),$$

and hence $Q_iK_i = P_iQ_i \cdot \cos(\beta - \gamma) + K_iL_i \cdot \sin(\beta - \gamma)$.
Q_iK_i and Q_iL_i are two secants of the circle ω_i that issue from the point Q_i. Therefore, there holds $Q_iN_i \cdot Q_iL_i = Q_iM_i \cdot Q_iK_i$.
We substitute the expressions for the four segments in this equality, to obtain

$$\begin{aligned} &P_iQ_i \cdot \cos\gamma \cdot (P_iQ_i \cdot \cos\gamma + K_iL_i \cdot \sin\gamma) \\ =\ &P_iQ_i \cos(\beta - \gamma) \cdot (P_iQ_i \cdot \cos(\beta - \gamma) + K_iL_i \cdot \sin(\beta - \gamma)). \end{aligned}$$

From this, by using known trigonometric identities, we obtain, in stages,

$$\begin{aligned} P_iQ_i \left(\cos^2\gamma - \cos^2(\beta - \gamma)\right) &= K_iL_i \left(\sin^2(\beta - \gamma) - \sin^2\gamma\right), \\ P_iQ_i \left(\frac{1 + \cos 2\gamma}{2} - \frac{1 + \cos(2\beta - 2\gamma)}{2}\right) &= K_iL_i \left(\frac{\sin(2\beta - 2\gamma)}{2} - \frac{\sin 2\gamma}{2}\right), \\ P_iQ_i \left(\cos 2\gamma - \cos(2\beta - 2\gamma)\right) &= K_iL_i \left(\sin(2\beta - 2\gamma) - \sin 2\gamma\right), \\ P_iQ_i \left(-2\sin\beta \sin(2\gamma - \beta)\right) &= K_iL_i \left(2\sin(\beta - 2\gamma)\cos\beta\right), \\ P_iQ_i \sin\beta = K_iL_i \cos\beta &\Rightarrow P_iQ_i = K_iL_i \cot\beta, \end{aligned}$$

and finally

$$P_iQ_i = K_iL_i \cot \measuredangle K_iQ_iL_i. \tag{3}$$

Formula (3) is satisfied for every circle ω_i that forms Pascal points on the sides of an orthodiagonal quadrilateral, and in particular, it is satisfied for the circle ω_{EF}, whose diameter is the segment EF (see Figure 15). In other words, there holds that $P_0Q_0 = K_0L_0 \cot \measuredangle K_0Q_0L_0$.

Let us now compare the lengths of the diameters K_iL_i and K_0L_0. The sides of the angles $\measuredangle K_0Q_0L_0$ and $\measuredangle K_iQ_iL_i$ are parallel in the same direction, therefore $\measuredangle K_iQ_iL_i = \measuredangle K_0Q_0L_0$.

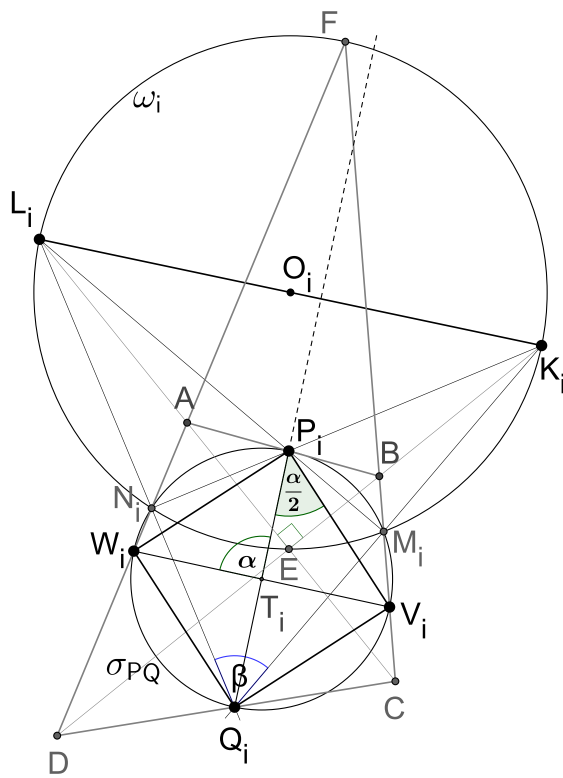

Figure 16

In the circle ω_{EF}, there holds $K_0L_0 = EF$ (equal diameters). In the circle ω_i, the segment FE is a chord that is not a diameter, therefore $K_iL_i > EF$. Therefore $K_iL_i > K_0L_0$, meaning that there exists a number $\lambda > 1$, for which there holds $K_iL_i = \lambda \cdot K_0L_0$. Hence,

$$P_iQ_i = K_iL_i \cot \measuredangle K_iQ_iL_i = \lambda \cdot K_0L_0 \cot \measuredangle K_0Q_0L_0 = \lambda \cdot P_0Q_0.$$

In other words:

$$P_iQ_i = \lambda \cdot P_0Q_0. \tag{4}$$

Recall that, in accordance with Theorem 1, the following holds for the orthodiagonal quadrilateral $ABCD$: In all rectangles defined by Pascal-points circles, the angle between the diagonals is a fixed value (we denote it by α). In addition, in every rectangle, it holds that the angle between the large side and the diagonal is equal to one half of the acute angle between the diagonals, that is to say, $\frac{\alpha}{2}$.

Therefore, for a rectangle $P_iV_iQ_iW_i$, defined by the circles ω_i and $\sigma_{P_iQ_i}$ (see Figure 16), there holds: the angle between the diagonal P_iQ_i and the large side P_iV_i equals $\frac{\alpha}{2}$. Therefore we obtain for the right-angled triangle $P_iQ_iV_i$ and the sides P_iV_i and V_iQ_i of the rectangle

$$P_iV_i = P_iQ_i \cos\frac{\alpha}{2} \quad \text{and} \quad V_iQ_i = P_iQ_i \sin\frac{\alpha}{2}. \tag{5}$$

In particular, for two adjacent sides of the rectangle $P_0V_0Q_0W_0$, defined by the circle ω_{EF} and the Pascal-points circle $\sigma_{P_0Q_0}$, there holds

$$P_0V_0 = P_0Q_0 \cos\frac{\alpha}{2} \quad \text{and} \quad V_0Q_0 = P_0Q_0 \sin\frac{\alpha}{2}. \tag{6}$$

Let us now consider the perimeter of the rectangle $P_iV_iQ_iW_i$. From the formulas (4), (5), and (6) follows

$$\begin{aligned} P_{P_iV_iQ_iW_i} &= 2P_iV_i + 2V_iQ_i = 2P_iQ_i\cos\frac{\alpha}{2} + 2P_iQ_i\sin\frac{\alpha}{2} \\ &= \lambda\left(2P_0Q_0\cos\frac{\alpha}{2} + 2P_0Q_0\sin\frac{\alpha}{2}\right) = \lambda\left(2P_0V_0 + 2V_0Q_0\right) = \lambda P_{P_0V_0Q_0W_0}. \end{aligned}$$

Thus, we have obtained that $P_{P_iV_iQ_iW_i} = \lambda P_{P_0V_0Q_0W_0}$ when $\lambda > 1$ and consequently $P_{P_iV_iQ_iW_i} > P_{P_0V_0Q_0W_0}$. Let us compare the areas of the rectangles $P_0V_0Q_0W_0$ and $P_iV_iQ_iW_i$.

$$\begin{aligned} S_{P_iV_iQ_iW_i} &= \frac{1}{2}P_iQ_i \cdot V_iW_i \sin\alpha = \frac{1}{2}\left(P_iQ_i\right)^2 \sin\alpha = \frac{1}{2}\lambda^2\left(P_0Q_0\right)^2\sin\alpha \\ &= \frac{1}{2}\lambda^2 P_0Q_0 \cdot V_0W_0 \sin\alpha = \lambda^2 S_{P_0V_0Q_0W_0}. \end{aligned}$$

Therefore $S_{P_iV_iQ_iW_i} > S_{P_0V_0Q_0W_0}$. □

Theorem 3. *Given the general data* (see above). *Then, among all the rectangles of the set* $\mathbb{M}_\odot$, *the only one whose sides are parallel to the diagonals of the quadrilateral* $ABCD$ *is the rectangle defined by the circle* ω_{EF} *and the Pascal-points circle* $\sigma_{P_0Q_0}$.

Proof. Let ω_i be an arbitrary circle, and let $P_iV_iQ_iW_i$ be a rectangle defined by the circles ω_i and $\sigma_{P_iQ_i}$. Let us check which additional conditions will guarantee that the sides of the rectangle $P_iV_iQ_iW_i$ will be parallel to the diagonals of the quadrilateral $ABCD$.
In order to prove that the sides of the rectangle $P_iV_iQ_iW_i$ are parallel to the diagonals of the quadrilateral $ABCD$, it is sufficient to prove, for example, that $P_iV_i \parallel AC$. The satisfaction of this property is equivalent to the satisfaction of the proportion

$$\frac{AP_i}{P_iB} = \frac{CV_i}{V_iB}. \tag{7}$$

In order to prove the proportion (7), we return to the method of complex numbers in the geometry of the plane. We choose a system of coordinates in which ω_i is the unit circle, in other words, the origin is at the center O_i of the circle ω_i and the length unit is equal to the radius O_iE. Using the complex coordinates of the points A, P_i, B, C, and V_i, one can write the proportion (7) as follows:

$$\frac{p-a}{b-p} = \frac{v-c}{b-v}. \tag{8}$$

We express the two sides of the equation (8), using the complex coordinates of the points E, F, K_i, L_i, M_i, and N_i that belong to the unit circle ω_i.
Since the ratios $\frac{AP_i}{P_iB}$ and $\frac{CV_i}{V_iB}$ are real numbers, it therefore holds that

$$\frac{p-a}{b-p} = \overline{\left(\frac{p-a}{b-p}\right)} = \frac{\overline{p}-\overline{a}}{\overline{b}-\overline{p}} \quad \text{and} \quad \frac{v-c}{b-v} = \overline{\left(\frac{v-c}{b-v}\right)} = \frac{\overline{v}-\overline{c}}{\overline{b}-\overline{v}}.$$

In order to obtain expressions for the numbers $\overline{a}$, $\overline{b}$, $\overline{c}$, and $\overline{p}$, we shall make use of the following property:

Let $U(u)$, $X(x)$, $Y(y)$, and $Z(z)$ be four points on the unit circle, and let $S(s)$ be the point of intersection of the straight lines UX and YZ. Then, for the conjugate $\overline{s}$ of the complex coordinate s holds:

$$\overline{s} = \frac{u+x-y-z}{ux-yz}. \tag{9}$$

Using (9), one can express $\overline{a}$, $\overline{b}$, $\overline{c}$, and $\overline{p}$ as follows: $A = FN \cap EL$ and therefore $\overline{a} = \frac{f+n-e-l}{fn-el}$; $B = FM \cap EK$ and therefore $\overline{b} = \frac{f+m-e-k}{fm-ek}$; $C = FM \cap EL$ and therefore $\overline{c} = \frac{f+m-e-l}{fm-el}$. $P = KN \cap LM$, and in addition K_iL_i is a diameter and therefore, in this case, $k = -l$, and therefore the expression for $\overline{p}$ is $\overline{p} = \frac{l+m-k-n}{lm-kn} = \frac{2l+m-n}{l(m+n)}$. In the proof of Theorem 1, the expression we used for $\overline{v}$ was $\overline{v} = \frac{2f-n+m}{f(m+n)}$.

If we substitute the expressions for $\overline{p}$, $\overline{a}$, and $\overline{b}$ in the left-hand side of the equation (8), we obtain

$$\frac{p-a}{b-p} = \frac{\overline{p}-\overline{a}}{\overline{b}-\overline{p}} = \frac{\frac{2l+m-n}{lm+ln} - \frac{f+n-e-l}{fn-el}}{\frac{f+m-e+l}{fm+el} - \frac{2l+m-n}{lm+ln}}.$$

This can be transformed into the following form:

$$\frac{(fln+fmn+2eln-2el^2-lmn-fn^2-ln^2+l^2m+l^2n-flm)(fm+el)}{(-efm+l^2m+l^2n+fln-flm+lmn+fmn-flm-elm-2el^2)(fn-el)}.$$

After factoring the parentheses in the numerator and the denominator, we obtain

$$\frac{(l-n)(fn-fm-2el+lm+ln)(fm+el)}{(l+m)(-2el-fm+lm+fn+ln)(fn-el)},$$

and finally, after cancellations,

$$\frac{p-a}{b-p} = \frac{(l-n)(fm+el)}{(l+m)(fn-el)}.$$

If we substitute the expressions for $\overline{b}$, $\overline{c}$, and $\overline{v}$ in the right-hand side of the equality (8), we obtain

$$\frac{v-c}{b-v} = \frac{\overline{v}-\overline{c}}{\overline{b}-\overline{v}} = \frac{\frac{2f+m-n}{fm+fn} - \frac{f+m-e-l}{fm-el}}{\frac{f+m-e+l}{fm+el} - \frac{2f+m-n}{fm+fn}}$$
$$= \frac{(f^2m-2fmn-2fel-elm+eln+fem+flm-f^2n+fen+fln)(fm+el)}{(-f^2m-fem+flm+f^2n+2fmn-fen+fln-2fel-elm+eln)(fm-el)}.$$

We substitute the obtained expressions in equality (8), cancel the factor $fm+el$ and obtain

$$\frac{l-n}{(l+m)(fn-el)}$$
$$= \frac{f^2m-2fmn-2fel-elm+eln+fem+flm-f^2n+fen+fln}{(-f^2m-fem+flm+f^2n+2fmn-fen+fln-2fel-elm+eln)(fm-el)}.$$

After cross-multiplication, collection of terms on the left-hand side, and collection of similar terms, we obtain the following equality:

$$f^3lm^2 - f^3ln^2 + f^2l^2n^2 - f^2l^2m^2 + e^2l^2m^2 - e^2l^2n^2 - fe^2lm^2 + fe^2ln^2 = 0.$$

The left-hand side of the last equality can be factored as follows:

$$\left(f^3 - f^2l + e^2l - fe^2\right) l \left(m^2 - n^2\right) = 0.$$

In the unit circle ω_i the chord M_iN_i is not a diameter, therefore $m \neq \pm n$; in addition $l \neq 0$. Thus, we obtain

$$f^3 - f^2l + e^2l - fe^2 = 0 \implies \left(f^2 - e^2\right)(f - l) = 0. \tag{10}$$

Finally, since $f \neq l$ and $f \neq e$, it follows that (10) will be satisfied only when the condition $f = -e$ holds. In other words, it will only hold in the case where the points E and F are the ends of a diameter in the unit circle ω_i.
Hence, it follows that the equalities (7) and (8) are satisfied only for the circle ω_{EF}. □

3. Comparison of the set of rectangles $\mathbb{M}_\odot$ and $\mathbb{M}_\parallel$

3.1. Properties of the set $\mathbb{M}_\parallel$

Let $ABCD$ be an orthodiagonal quadrilateral. It is easy to prove that for each point P, that is an interior point of side AB, there exists a rectangle $PXZY$, where $X \in BC$, $Y \in AD$, $Z \in CD$, and the sides PX and PY are respectively parallel to the diagonals AC and BD of the quadrilateral (see Figure 17). Therefore, there exists a set of rectangles inscribed in the quadrilateral $ABCD$ whose sides are parallel to the diagonals of the quadrilateral. We denote this set by $\mathbb{M}_\parallel$.

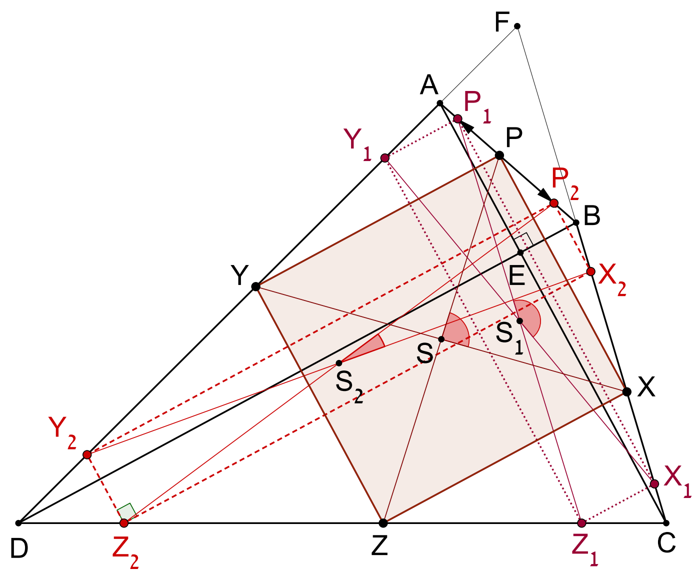

Figure 17

We denote by S the point of intersection of the diagonals of rectangle $PXZY$. For the angle PSX between the diagonals there holds that

$$\tan\left(\frac{1}{2}\measuredangle PSX\right) = \frac{PX}{PY}.$$

In other words, the magnitude of the angle depends on the location of point P on the side AB. Namely, when the point $P = P_1$ approaches point A (see Figure 17), the length of the side P_1X_1 of the rectangle $P_1X_1Z_1Y_1$ increases and approaches the length of the diagonal AC, and the length of the side P_1Y_1 decreases and approaches 0. In this case, the magnitude of the angle $P_1S_1X_1$ tends to 180°. When the point $P = P_2$ approaches point B (see Figure 17), then the length of the side P_2Y_2 of the rectangle $P_2X_2Z_2Y_2$ increases and approaches the length of the diagonal BD, and the length of the side P_2X_2 decreases and approaches 0. In this case, the magnitude of the angle $P_2S_2X_2$ tends to 0°.

To summarize, when point P moves from point A to point B on the segment AB, the angle PSX increases monotonically and scans all the values between 0° and 180°. In particular, there is a location of the point P for which $\measuredangle PSX = 90°$. At this point, the rectangle $PXZY$ is a square. Therefore, the set of rectangles $\mathbb{M}_{\|}$ includes a single rectangle that is a square.

It is easy to see that the area function of the rectangle $PXZY$ is a continuous function of the location of the point P on the segment AB. When P approaches the endpoint A or B, the area $PXZY$ approaches zero. Therefore the maximum area is obtained when P is at some interior point of the segment AB.

Let us prove that in the set $\mathbb{M}_{\|}$, there is a rectangle with a maximal area. Let $\frac{AP}{AB} = t$, therefore $\frac{BP}{AB} = 1 - t$. From the similarity of triangles $\triangle APY \sim \triangle ABD$ follows that

$$\frac{PY}{BD} = t \implies PY = t\,BD.$$

From the similarity of the triangles $\triangle BPX \sim \triangle BAC$ follows that

$$\frac{PX}{AC} = 1 - t \implies PX = (1-t)AC.$$

Therefore, we obtain for the area of rectangle $PXZY$

$$S_{PXZY} = PY \cdot PX = t\,BD \cdot (1-t)AC = (t - t^2)BD \cdot AC = 2(t - t^2)S_{ABCD}.$$

The function $S(t) = 2(t - t^2)S_{ABCD}$ has a maximum at the point $t = \frac{1}{2}$. Therefore for P located at the middle of the segment AB, the rectangle $PXZY$ has a maximal area.

To summarize, in the set $\mathbb{M}_{\|}$, the rectangle with the maximal area is the one whose vertices are the midpoints of the sides of the quadrilateral $ABCD$.

3.2. Comparing the sets $\mathbb{M}_{\odot}$ and $\mathbb{M}_{\|}$

In every orthodiagonal quadrilateral $ABCD$ there are two sets of inscribed rectangles: the set $\mathbb{M}_{\odot}$ — rectangles defined by Pascal-points circles, and the set $\mathbb{M}_{\|}$ — rectangles whose sides are parallel to the diagonals AC and BD. The sets $\mathbb{M}_{\odot}$ and $\mathbb{M}_{\|}$ contain different rectangles. The only rectangle that belongs to both set $\mathbb{M}_{\odot}$ and set $\mathbb{M}_{\|}$ is the rectangle $P_0V_0Q_0W_0$, which is defined by the circle ω_{EF} and the Pascal-points circle $\sigma_{P_0Q_0}$.

Table 1 shows a comparison between the properties of the rectangles from each of the sets $\mathbb{M}_{\odot}$ and $\mathbb{M}_{\|}$:

Corollary 5. *Let $ABCD$ be an orthodiagonal quadrilateral in which E is the point of intersection of the diagonals, and F is the point of intersection of the rays CB and DA. Every interior point P of the side AB is a vertex of two rectangles inscribed in the quadrilateral.*

Table 1: Comparison of the rectangles in $\mathbb{M}_\odot$ and $\mathbb{M}_\|$

Property	*In* $\mathbb{M}_\|$	*In* $\mathbb{M}_\odot$
The sides of the rectangle are parallel to the diagonals of the quadrilateral.	In every rectangle, the sides are parallel to the diagonals of the quadrilateral $ABCD$.	There is but a single rectangle whose sides are parallel to the diagonals of the quadrilateral $ABCD$. This is the rectangle defined by the circles ω_{EF} and $\sigma_{P_0Q_0}$.
The corresponding sides of two rectangles are parallel.	The corresponding sides of every two rectangles are parallel.	There are no two rectangles whose corresponding sides are parallel.
The value of the angle between the diagonals of the rectangle.	The angle varies, and gets all the values in the range (0°,180°).	The angle is fixed for all rectangles, and is determined by the quadrilateral $ABCD$.
There exists a rectangle that is a square.	There exists a single rectangle that is a square.	If $\measuredangle FEG \neq 90°$, then there are no squares in the set $\mathbb{M}_\odot$. If $\measuredangle FEG = 90°$, then all rectangles in the set are squares.
Similarity of rectangles	There is an infinite number of pairs of similar rectangles.	Every two rectangles in the set are similar.
Rectangle with maximal or minimal area	There exists a rectangle with maximal area. This is the rectangle whose vertices are the midpoints of the sides of the quadrilateral $ABCD$.	There exists a rectangle with minimal area. This is the rectangle defined by the circles ω_{EF} and $\sigma_{P_0Q_0}$.

The first rectangle is a rectangle whose sides are parallel to the diagonals of the quadrilateral, and the second rectangle is a rectangle defined by the Pascal-points circle σ_{PQ} (see Figure 18). *The point P_0 (the Pascal point formed by the circle whose diameter is the segment EF) is the only point on the side AB for which the two rectangles coincide.*

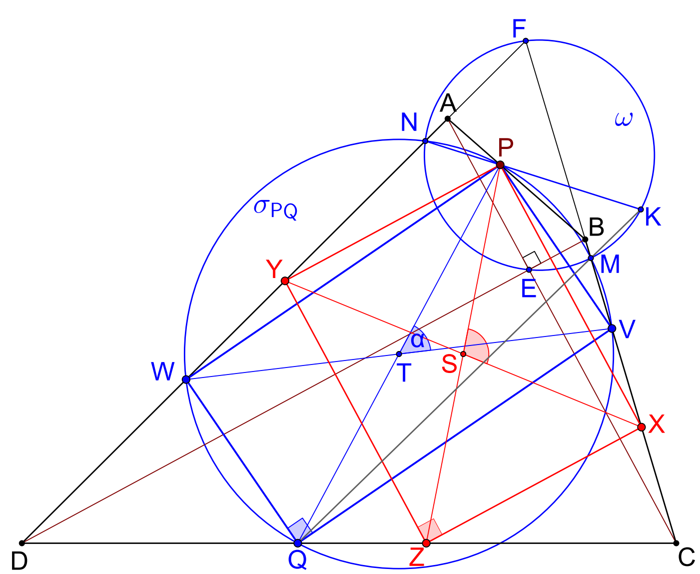

Figure 18

We have seen above, that for rectangles of the set $\mathbb{M}_{\|}$ the angle PSX between the diagonals of the rectangle (the angle for which B is an interior point) increases monotonically and obtains values between 0° and 180°. Therefore, the following corollary holds:

Corollary 6. *Let $ABCD$ be an orthodiagonal quadrilateral; $\mathbb{M}_{\odot}$ is the set of rectangles inscribed in $ABCD$ and defined by Pascal-points circles; α is the fixed value of the angles $P_iT_iV_i$ between the diagonals of the rectangles of the set $\mathbb{M}_{\odot}$.*
Then the point P_0 is the only point on the side AB which is a vertex of a rectangle inscribed in the quadrilateral $ABCD$ for which the following two conditions hold simultaneously:

(i) *the sides of the rectangle are parallel to the diagonals of the quadrilateral, and*

(ii) *the angle between the diagonals of the rectangle (the angle for which B is an interior point) equals α.*

4. Uniqueness of the set $\mathbb{M}_{\odot}$ as a set of rectangles defined by circles that form Pascal points and Pascal-points circles

Let G be the point of intersection of the sides AB and CD. For circles ψ_j, that pass through the points E and G and form the Pascal points P_j and Q_j on the sides BC and AD, the following properties are satisfied:

Corollary 7. *Below are properties of circles ψ_j, which are the result of Property 3* (above) *and of the Theorems 1 and 3:*

Property (a): *For every orthodiagonal quadrilateral $ABCD$, there is an infinite set $\{\psi_j\}$ of circles that pass through the points E and G and form Pascal points P_j and Q_j on the sides BC and AD. This set defines the infinite set $\{\sigma_{P_jQ_j}\}$ of Pascal-points circles. This set of Pascal-points circles defines an infinite set $\{P_jV_jQ_jW_j\}$ of rectangles inscribed in the quadrilateral $ABCD$.*

Property (b): *Among all rectangles in set $\{P_jV_jQ_jW_j\}$, the only rectangle whose sides are parallel to the diagonals of the quadrilateral $ABCD$ is the one defined by both of the following circles: by the circle ψ_{EG}, whose diameter is the segment EG, and by the Pascal-points circle $\sigma_{P_{0'}Q_{0'}}$, in which $P_{0'}$ and $Q_{0'}$ are Pascal points formed by ψ_{EG}* (see Figure 19).

Property (c): *The angle FEG between the diameters EF and EG of the circles ω_{EF} and ψ_{EG} equals the angle $V_{0'}T_{0'}Q_{0'}$ between the diameters $P_{0'}Q_{0'}$ and $V_{0'}W_{0'}$ of the circle $\sigma_{P_{0'}Q_{0'}}$* (see Figure 19).

Property (d): *For every rectangle of the set $\{P_jV_jQ_jW_j\}$, the value of the angle between the diagonals is constant and depends only on the quadrilateral $ABCD$ and not on the choice of the circle ψ_j.*

It is therefore natural to ask what relation exists between the rectangle $P_0V_0Q_0W_0$, defined using the circles ω_{EF} and $\sigma_{P_0Q_0}$, and the rectangle $P_{0'}V_{0'}Q_{0'}W_{0'}$, defined using the circles ψ_{EG} and $\sigma_{P_{0'}Q_{0'}}$. A more general question is: What is the relation between the set of rectangles $\mathbb{M}_{\odot}$ and the set of rectangles $\{P_jV_jQ_jW_j\}$ defined using the circles ψ_j and $\sigma_{P_jQ_j}$?

In the following Theorem 4 we answer the first question, in Theorem 5 the general one.

Theorem 4. *In addition to item 2 in the general data, it is given that $P_0V_0Q_0W_0$ is the rectangle defined by the circles ω_{EF} and $\sigma_{P_0Q_0}$, and that ψ_{EG} is a circle whose diameter is the segment EG, where G is the point of intersection of the extensions of the sides AB and CD.*

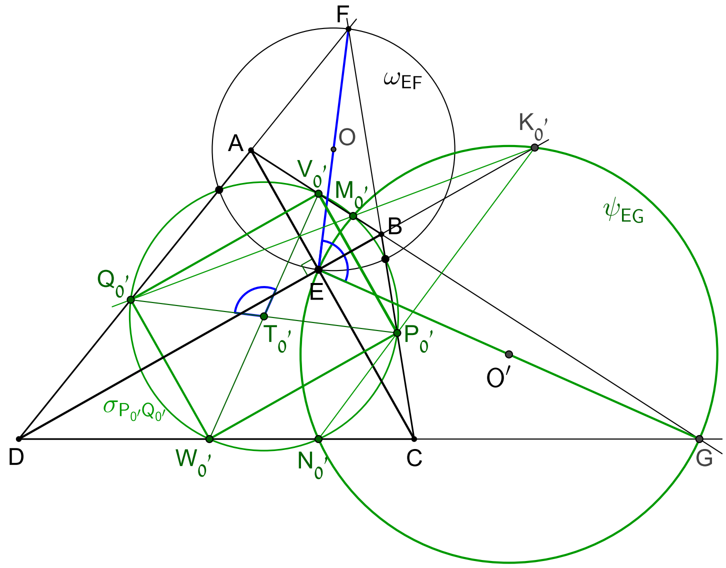

Figure 19

Then the rectangle defined by the circles ψ_{EG} and $\sigma_{P_{0'}Q_{0'}}$ coincides with the rectangle $P_0V_0Q_0W_0$. In particular, the Pascal points formed by the circle ψ_{EG} on the sides BC and AD coincide with the points V_0 and W_0.

Proof. From Property (b) in Corollary 7 follows that the sides of the rectangle $P_{0'}V_{0'}Q_{0'}W_{0'}$ are parallel to the diagonals of the quadrilateral $ABCD$. From item (ii) in Property 3, above, and Property (c) of Corollary 7 follows that $\measuredangle V_0T_0Q_0 = \measuredangle FEG = \measuredangle V_{0'}T_{0'}Q_{0'}$. The angles $P_0T_0V_0$ and $P_{0'}T_{0'}V_{0'}$ are, respectively, the supplementary adjacent angles to the equal angles $V_0T_0Q_0$ and $V_{0'}T_{0'}Q_{0'}$. Therefore there also holds $\measuredangle P_0T_0V_0 = \measuredangle P_{0'}T_{0'}V_{0'}$.

Let α be the fixed value of the angles $P_iT_iV_i$ between the diagonals of the rectangles of the set $\mathbb{M}_{\odot}$, and, in particular, $\measuredangle P_0T_0V_0 = \alpha$. Therefore, angle $P_{0'}T_{0'}V_{0'}$ is also equal to α. We also note that point B is an interior point of the angle $P_{0'}T_{0'}V_{0'}$.

To summarize, we have obtained that the sides of the rectangle $P_{0'}V_{0'}Q_{0'}W_{0'}$ are parallel to the diagonals of the quadrilateral $ABCD$, that the angle $P_{0'}T_{0'}V_{0'}$ between the diagonals of the rectangle equals α, and that the point $V_{0'}$ belongs to the side AB.

According to Corollary 6, point P_0 is the only point on the side AB that is a vertex of a rectangle that satisfies at the same time: (i) the sides of the rectangle are parallel to the diagonals of the quadrilateral, and (ii) the angle between the diagonals of the rectangle (the angle for which B is an interior point) equals α. Therefore, it must hold that the points P_0 and $V_{0'}$ coincide, and therefore the points $P_{0'}$, $W_{0'}$ and $Q_{0'}$ coincide with the points V_0, Q_0 and W_0, respectively. □

Theorem 5. *In addition to item 1 in the general data, it is given that the extensions of the sides AB and CD intersect at the point G.*
Then for every circle ω_i, that passes through the points E and F and defines a rectangle $P_iV_iQ_iW_i$, there is a circle ψ_j, that passes through the points E and G and defines a rectangle $P_jV_jQ_jW_j$ that coincides with the rectangle $P_iV_iQ_iW_i$ (see Figure 20).

Proof. Let ω_{EF} be a circle whose diameter is the segment EF, that forms Pascal points P_0 and Q_0 on the sides AB and CD, respectively. $\sigma_{P_0Q_0}$ is a Pascal-points circle, $P_0V_0Q_0W_0$ is

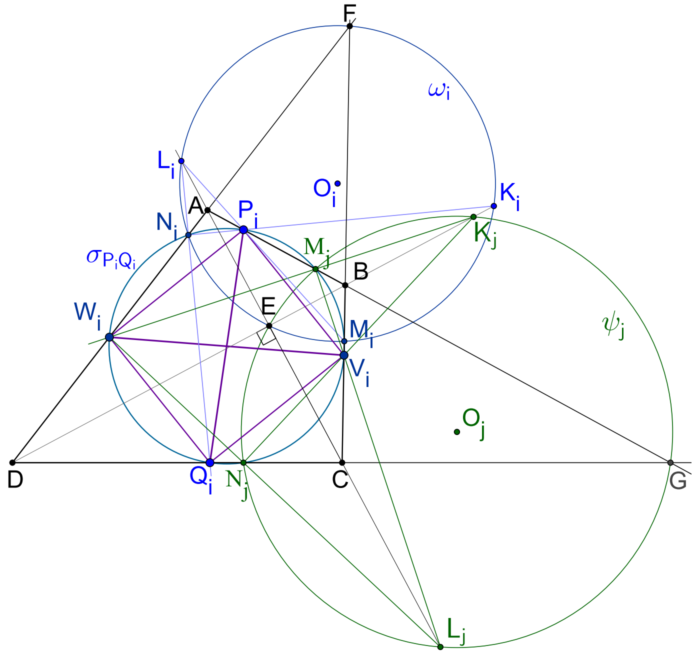

Figure 20

the rectangle defined by the circles ω_{EF} and $\sigma_{P_0Q_0}$, and ψ_{EG} is a circle whose diameter is the segment EG. Also, let $\measuredangle EFO_i = \varphi$, where O_i is the center of the circle ω_i.
Without loss of generality, we can assume that the counterclockwise rotation by the angle φ about point F (we denote this by R_F^φ) transforms the straight line FE into the line FO_i (see Figure 21). We rotate ray GE counterclockwise about point G by the angle φ and obtain the ray GZ. We denote by O_j the point of intersection of the ray GZ and the midperpendicular to segment EG.
The rotation R_F^φ transforms the tangent to the circle ω_{EF} at point F into the tangent to the circle ω_i. As we saw at the end of the proof of Theorem 1, in this case there holds:

(i) The rotation R_H^φ transforms the line P_0Q_0 into the line P_iQ_i, where H is the point of intersection of the straight lines P_0Q_0 and P_iQ_i (see Figure 21);

(ii) the rotation R_U^φ transforms line V_0W_0 (which passes through the diagonal of the rectangle $P_0V_0Q_0W_0$) into the line V_iW_i, (which passes through the diagonal of rectangle $P_iV_iQ_iW_i$), where U is the point of intersection of the straight lines V_0W_0 and V_iW_i.

Let ψ_j be a circle whose center is at point O_j and whose radius is O_jE; let P_j and Q_j be Pascal points formed by ψ_j on the sides BC and AD; and let $P_jV_jQ_jW_j$ be the rectangle defined by the circles ψ_j and $\sigma_{P_jQ_j}$ (see Figure 22).
In order to show that the rectangles $P_iV_iQ_iW_i$ and $P_jV_jQ_jW_j$ coincide, it is enough to show that the Pascal points P_j and Q_j respectively coincide with the points V_i and W_i or, alternatively, to prove that the points V_j and W_j respectively coincide with the Pascal points P_i and Q_i. (For example, in Figure 21, we see that the segment K_jN_j intersects the side BC at the point V_i and that the ray K_jM_j intersects the side AD at the point W_i.) In both cases, it is assured that the Pascal-points circles $\sigma_{P_iQ_i}$ and $\sigma_{P_jQ_j}$ intersect, and therefore the rectangles $P_iV_iQ_iW_i$ and $P_jV_jQ_jW_j$ coincide.
Consider now the counterclockwise rotation by angle φ about the point G (we denote this by R_G^φ). This rotation transforms the line GE (which contains the diameter of the circle ψ_{EG}) into the straight line GO_i (which contains the radius of the circle ψ_j) (see Figure 22).

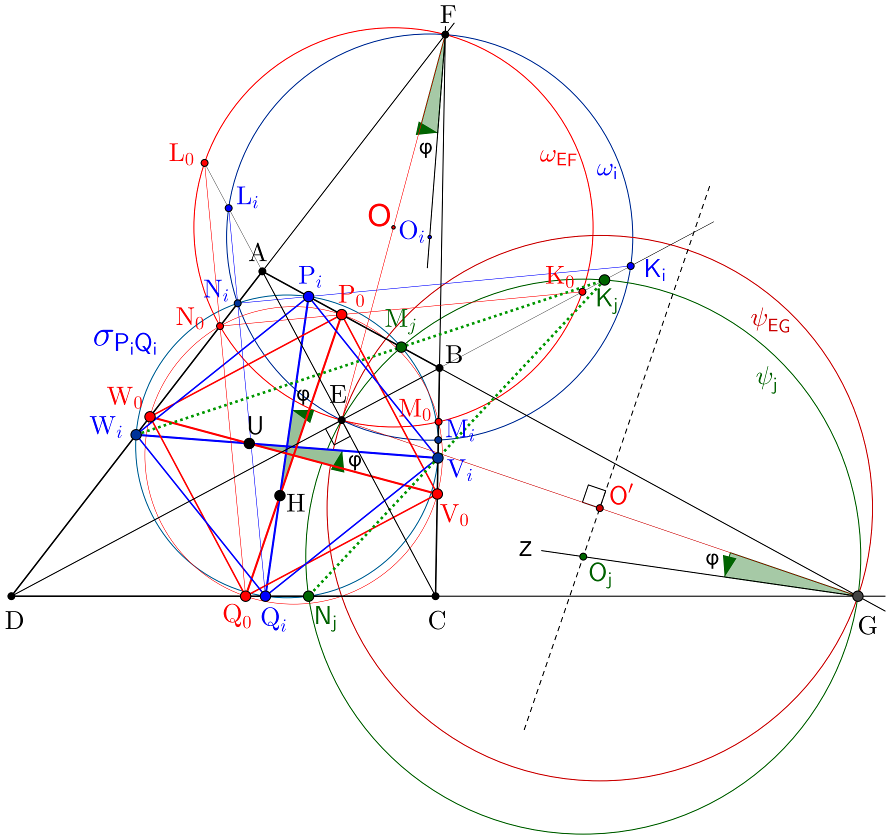

Figure 21

Therefore, the tangent to the circle ψ_{EG} at point G is transformed into the tangent to the circle ψ_j. It thus follows that for the rotation R_G^{φ} claims similar to (i) and (ii) of the rotation R_F^{φ} above will also hold true. In other words:

(iii) The rotation $R_{U'}^{\varphi}$ transforms the line V_0W_0 into the line P_jQ_j; recall that $P_{0'}Q_{0'} = V_0W_0$ is the straight line of Pascal points formed by the circle ψ_{EG}, where U' is the point of intersection of the straight lines V_0W_0 and P_jQ_j (see Figure 22);

(iv) the rotation $R_{H'}^{\varphi}$ transforms the line P_0Q_0 into the line V_jW_j, which contains the second diagonal of the rectangle $P_jV_jQ_jW_j$, where H' is the point of intersection of the straight lines P_0Q_0 and V_jW_j.

From Properties (i) and (iv) follows that the lines P_iQ_i and V_jW_j form equal corresponding angles with the line P_0Q_0 (see Figure 23). Therefore $P_iQ_i \parallel V_jW_j$. Similarly, from Properties (ii) and (iii) follows that the straight lines V_iW_i and P_jQ_j form equal corresponding angles with the line V_0W_0 (see Figure 24). Therefore, $V_iW_i \parallel P_jQ_j$.

Now we shall prove that the segments P_iQ_i and V_jW_j lie on a single straight line, and that the segments V_iW_i and P_jQ_j also lie on a single straight line. We check each of the possible cases:

1. We assume that the segments P_iQ_i and V_jW_j are of equal length and lie on two different parallel straight lines. In this case, the quadrilateral $V_jP_iQ_iW_j$ is a parallelogram. Therefore, the segments P_iV_j and Q_iW_j are parallel.
   Therefore, since $P_i, V_j \in AB$ and $Q_i, W_j \in CD$, we obtain that the straight lines AB and CD are parallel. This is contrary to the datum that states that the straight lines AB and

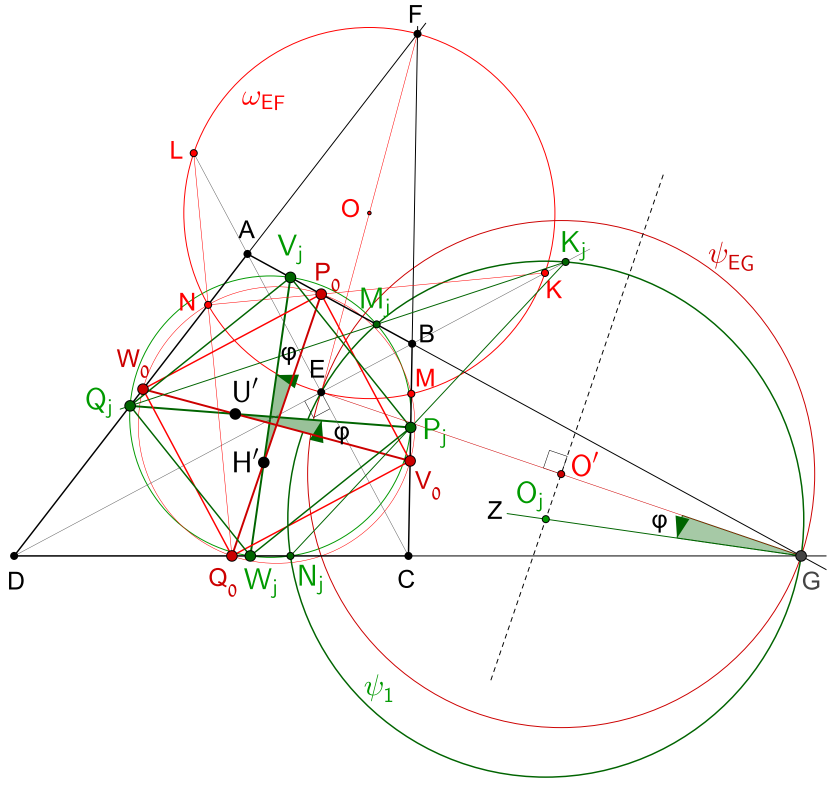

Figure 22

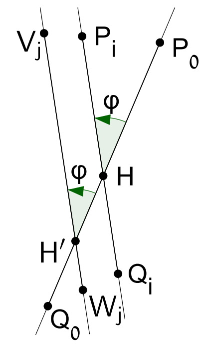

Figure 23

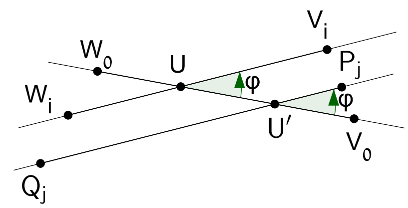

Figure 24

CD intersect (at point G).

2. In a similar manner, we can prove that it is impossible that the segments V_iW_i and P_jQ_j lie on two different parallel straight lines together with $V_iW_i = P_jQ_j$.
3. We assume that the segments P_iQ_i and V_jW_j lie on different parallel straight lines and $P_iQ_i \neq V_jW_j$, and also that the segments V_iW_i and P_jQ_j lie on different parallel straight lines and $V_iW_i \neq P_jQ_j$.
   Without losing generality, we assume that $V_jW_j > P_iQ_i$ and $P_jQ_j > V_iW_i$, as described

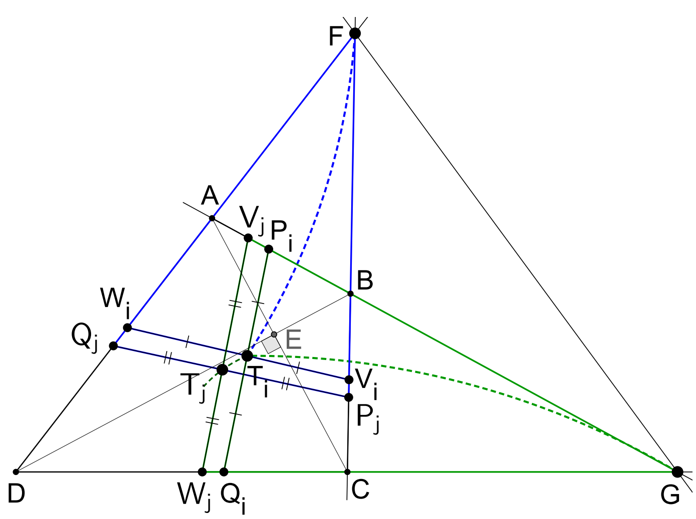

Figure 25

in Figure 25. We denote by T_i the point of intersection of the diagonals of the rectangle $P_iV_iQ_iW_i$, and by T_j the point of intersection of the diagonals of the rectangle $P_jV_jQ_jW_j$. In the triangle GV_jW_j there holds: the segment P_iQ_i (whose ends lie on sides GV_j and GW_j) is parallel to the third side V_jW_j. Therefore, the straight line T_jT_i, which passes through the midpoints of the segments P_iQ_i and V_jW_j, also passes through the vertex G. Similarly, in the triangle FP_jQ_j there holds: the segment V_iW_i (whose ends lie on sides FP_j and FQ_j) is parallel to the side P_jQ_j. Therefore, the straight line T_jT_i, which passes through the midpoints of the segments P_jQ_j and V_iW_i, also passes through the vertex F. Hence, it follows that the points F and G belong to the straight line T_jT_i, or equivalently, points T_i and T_j belong to the straight line FG. On the other hand, the points T_i and T_j are interior points of the plane angle ABC, the angle whose boundaries are the rays BA and BC.

The points F and G belong to rays BF and BG, or, in other words, to the sides of the angle FBG, which is vertically opposite to the angle ABC. Therefore, the straight line FG and the plane angle ABC do not intersect, and in particular, the points T_i and T_j do not belong to the straight line FG. We obtained a contradiction concerning the location of points T_i and T_j relative to straight line FG.

It has now been proven that the cases (1) – (3) are impossible. Therefore, it must necessarily hold true that the pair of segments P_iQ_i and V_jW_j lies on a single straight line or the pair of segments V_iW_i and P_jQ_j lies on a single straight line.

In the case that the segments V_iW_i and P_jQ_j lie on a single straight line, there holds for the ends of these segments $V_i, P_j \in BC$ and $W_i, Q_j \in AD$. In this case, the points P_j and V_i coincide and the points Q_j and W_i coincide and the theorem holds.

In the case that the segments P_iQ_i and V_jW_j lie on a single straight line, there holds for the ends of these segments $P_i, V_j \in AB$ and $Q_i, W_j \in CD$. Thus, the points V_j and W_j coincide respectively with the Pascal points P_i and Q_i and, in this case as well, the theorem holds. □

The following corollary follows from the Theorems 1, 4 and 5, and from Corollary 7.

Corollary 8. *The set of circles $\{\omega_i\}$ that form Pascal points on the sides AB and CD, and the set of circles $\{\psi_j\}$ that form Pascal points on the sides BC and AD define the same set $\mathbb{M}_\odot$ of rectangles inscribed in the orthodiagonal quadrilateral $ABCD$.*

5. Circles mutually coordinated relative to a quadrilateral

Definition 2. Let $ABCD$ be an orthodiagonal quadrilateral in which the diagonals intersect at the point E, the extensions of opposite sides BC and AD intersect at the point F, and the extension of opposite sides AB and CD intersect at the point G; ω_i is a circle that passes through the points E and F and forms Pascal points P_i and Q_i on the sides AB and CD, respectively; and ψ_j is a circle that passes through the points E and G, and forms Pascal points P_j and Q_j on the sides BC and AD, respectively.

Then any pair of circles (ω_i, ψ_j), for which the quadrilateral $P_iP_jQ_iQ_j$ is a rectangle, is called a *"pair of circles mutually coordinated relative to the quadrilateral $ABCD$"*.

For example, the circles ω_{EF} and ψ_{EG} are mutually coordinated relative to the quadrilateral $ABCD$.

Based on this definition, one can restate Theorem 5 as follows:

Corollary 9. *For any circle ω_i that forms Pascal points on the sides AB and CD, there exists a circle ψ_j that forms Pascal points on the sides BC and AD and which is coordinated with ω_i relative to the quadrilateral $ABCD$.*

For any pair of circles (ω_i, ψ_j) that are mutually coordinated relative to the quadrilateral $ABCD$, the following properties hold:

Property (i): *In addition to the general data, let ψ_{EG} be a circle whose diameter is EG, and let O, O', O_i, and O_j be the centers of circles ω_{EF}, ψ_{EG}, ω_i, and ψ_j, respectively. Then the angles OFO_i and $O'GO_j$ are oriented in the same direction and are equal in magnitude.*

Property (ii): *The circles ω_i and ψ_j define the same Pascal-points circle σ (using two different pairs of Pascal points). Each pair of the circles, ω_i and ψ_j, intersects the two opposite sides of the quadrilateral at the same points through which also passes the Pascal-points circle σ defined by them.*

For example, in the Figures 20 and 21, where the circles ω_i and ψ_j are coordinated with respect to the quadrilateral $ABCD$, it can be seen that ω_i intersects the sides BC and AD at points M_i and N_i, that the circle ψ_j intersects the sides AB and CD at points M_j and N_j, and that the circle $\sigma_{P_iQ_i}$ passes through the four points M_i, N_i, M_j, and N_j.

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