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Rupert Properties of Polyhedra and the Generalised Nieuwland Constant

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Abstract. New results are discussed in terms of the Rupert property of polyhedra, which is about finding a hole (a straight tunnel) in a solid through which a congruent copy of the solid can pass. Recently it is proved in [7] that 8 of the 13 Archimedean solids have this property. In our paper we prove that the simplest Archimedean solid, the truncated tetrahedron is also of Rupert property. Moreover, we prove general results on the Nieuwland constant, a scaling factor between the passing and the original solids if a larger copy can also pass through. We also define a generalised Nieuwland constant for those solids not possessing this property and prove that this constant can be arbitrary small.

Key Words: Rupert property, Archimedean solids, Nieuwland constant

MSC 2010: 51M20 , 52B10

1. Introduction

More than 300 years ago Prince Rupert of Rhine with fellow mathematician John Wallis considered and solved the problem of finding a straight hole in a cube through which a congruent cube can pass [5]. Around 100 years later, Pieter Nieuwland proved that even a larger cube can pass through, and the maximum scale of the passing cube has been found to be $\frac{3\sqrt{2}}{4}$. In 1950 Schrek published a detailed overview of the problem of Rupert and the proof of Nieuwland [3]. Now the Rupert problem is still in the forefront of research with relevant new results in recent years.

At first we define the basic notions, based on [2].

Definition 1. By a *hole* we mean the intersection of the given solid and a generalised cylinder, where the intersection is entirely in the inner part of the solid, that is each generator line of the cylinder intersects the solid in a single line segment and generators cannot even be tangent lines of the surface of the solid.

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Definition 2. A solid is called *Rupert* (or having Rupert property), if there exists a hole in the solid in a way that a congruent solid can pass through this hole.

Definition 3. Suppose that the solid has Rupert property. If a larger copy of the solid can also pass through a proper hole, than the ratio of the scale of the maximally enlarged solid and the original one is called *Nieuwland constant.*

Based on the above mentioned fact, the Nieuwland constant of the cube is $\frac{3\sqrt{2}}{4}$. Here we note, that continuously enlarging a solid passing through the enlarging hole, sooner or later we reach the limit when the solid (and one of the generators of the hole) will touch the surface of the initial solid, which is not allowed by definition. Consequently, the exact Nieuwland constant is in fact an upper limit of the scale.

As it is described in [2], it is easy to see that the search for the appropriate hole in a convex solid is identical to the search for two different orthogonal parallel projections of the solid to two different planes in a way, that one projected shape can entirely fit into the inner part of the other projection. Based on this view it is also trivial that not all convex solids possess the Rupert property: for example all orthogonal projections of a sphere are congruent circles, none of which can fit entirely into the inner part of another.

Back to polyhedra, Scriba proved that, beside the cube, the tetrahedron and the octahedron also have this property [4]. Finally, in 2017 Jerrard, Wetzel and Yuan proved the Rupert property for the dodecahedron and the icosahedron, completing the discussion of Rupert property of Platonic solids in a positive manner [2]. In 2018 Huber et al. proved that the n-cube is also Rupert [1].

Recently Ying, Yuan and Zamfirescu studied Archimedean solids and proved that 8 of the 13 polyhedra possess the Rupert property (cuboctahedron, truncated octahedron, truncated cube, rhombicubooctahedron, truncated cuboctahedron, icosidodecahedron, truncated icosahedron and truncated dodecahedron) [7]. Moreover, the Nieuwland constants of these Platonic and Archimedean solids were also estimated [2, 7]. However, the existence of the Rupert hole of five of the Archimedean solids are still unsolved.

In this paper we extend the results of Ying, Yuan and Zamfirescu, by proving in Section 2, that the Archimedean truncated tetrahedron also has this property. In Section 3 we study and generalise the notion of Nieuwland constant for those solids having no Rupert property. If there is no hole in the solid where a congruent copy can pass through, we will consider the largest downscale copy (where the scaling factor is less than 1) for which one can find a hole. This factor will be called generalised Nieuwland constant. Moreover here we prove, that the Nieuwland constant can be arbitrary large and arbitrary small. The existence of a solid with arbitrary large Nieuwland constant is more or less trivial, thinking about an ellipsoid with properly selected axes $a >> b >> c$. Based on this idea here we prove that for any $k > 1$, $k \in \mathbb{R}$, there exists a symmetric convex polytope in $\mathbb{R}^n$ (actually an n-orthotope) with a Nieuwland constant larger than k. Furthermore, we prove that for any $0 < k \leq 1$, $k \in \mathbb{R}$, there exists a polyhedron with generalised Nieuwland constant smaller than k.

2. The Rupert property of the truncated tetrahedron

In this section we prove that there is a hole in the Archimedean truncated tetrahedron through which a congruent truncated tetrahedron can pass. We will follow the idea described in [7], finding two appropriate orthogonal projections, one of which can entirely be placed into the

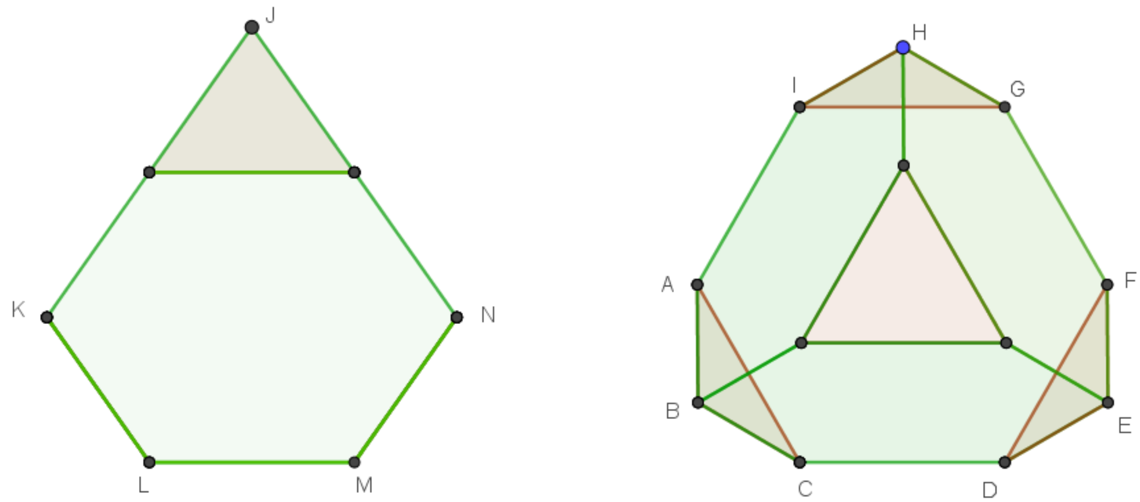

Figure 1: Two important orthogonal projections of the truncated tetrahedron

other one. We will pay special attention to the important restriction that all of the vertices and edges of the first projection must be in the inner part of the other projection.

Theorem 1. *The truncated tetrahedron has the Rupert property.*

Proof. To prove that the truncated tetrahedron is Rupert, let us consider two different initial orthogonal projections, seen in Figure 1. One of them (Figure 1, left) is a diamond shape image mapped onto a symmetry plane of the solid passing through the common edge LM of two hexagons. The other projection (Figure 1, right) is mapping the solid onto the plane of one of the hexagonal side, in our case onto the plane of hexagon ACDFGI. In both cases the horizontal diagonals of the mapped hexagons, KN, AF, and BE, and the horizontal sides of the hexagons, LM and CD, are parallel to the image planes, consequently they can be seen in real length. Since AF and BE are of equal lengths, for symmetry reasons the quadrilateral ABEF is a rectangle. In their current positions, the left diamond shape image is higher than the right one, that is, the distance between vertex J and edge LM is longer than the distance between vertex H and edge CD. Consequently, the left image cannot fit into the right one.

However, rotating the polyhedron around the edge CD, whilst keeping the image plane identical, the height of the right image can be increased. With edge length $\sqrt{8}$ of the polyhedron, it is easy to calculate that the height of the left image is 6, but the maximum height of the right image of the rotating polyhedron can be increased up to $\sqrt{38}$. This maximum happens when the spatial segment between vertex H and midpoint of edge CD, along which the distance of vertex H and edge CD is measured, will be parallel to the image plane. During this rotation the length of the horizontal diagonals and horizontal edges does not change. Therefore, after this rotation the left image can fit into the modified (vertically streched) right image in a way, that edge LM is parallel but slightly above edge CD, vertex K is on edge AB, and vertex N is on edge EF. Due to the difference in height, vertex J is still below vertex H, inside the modified right image. By a sufficiently small rotation around the centroid of the diamond shape image, the vertices L, M and J will still be inside the modified right image, whilst vertices K and N will leave the sides of the rectangle ABEF and will also be inner points of the modified right image. The final position of these two projections can be seen in Figure 2 with enlarged images of the surroundings of vertices of the diamond shape image to show that each vertex is inside the larger image. And this was to be proved. □

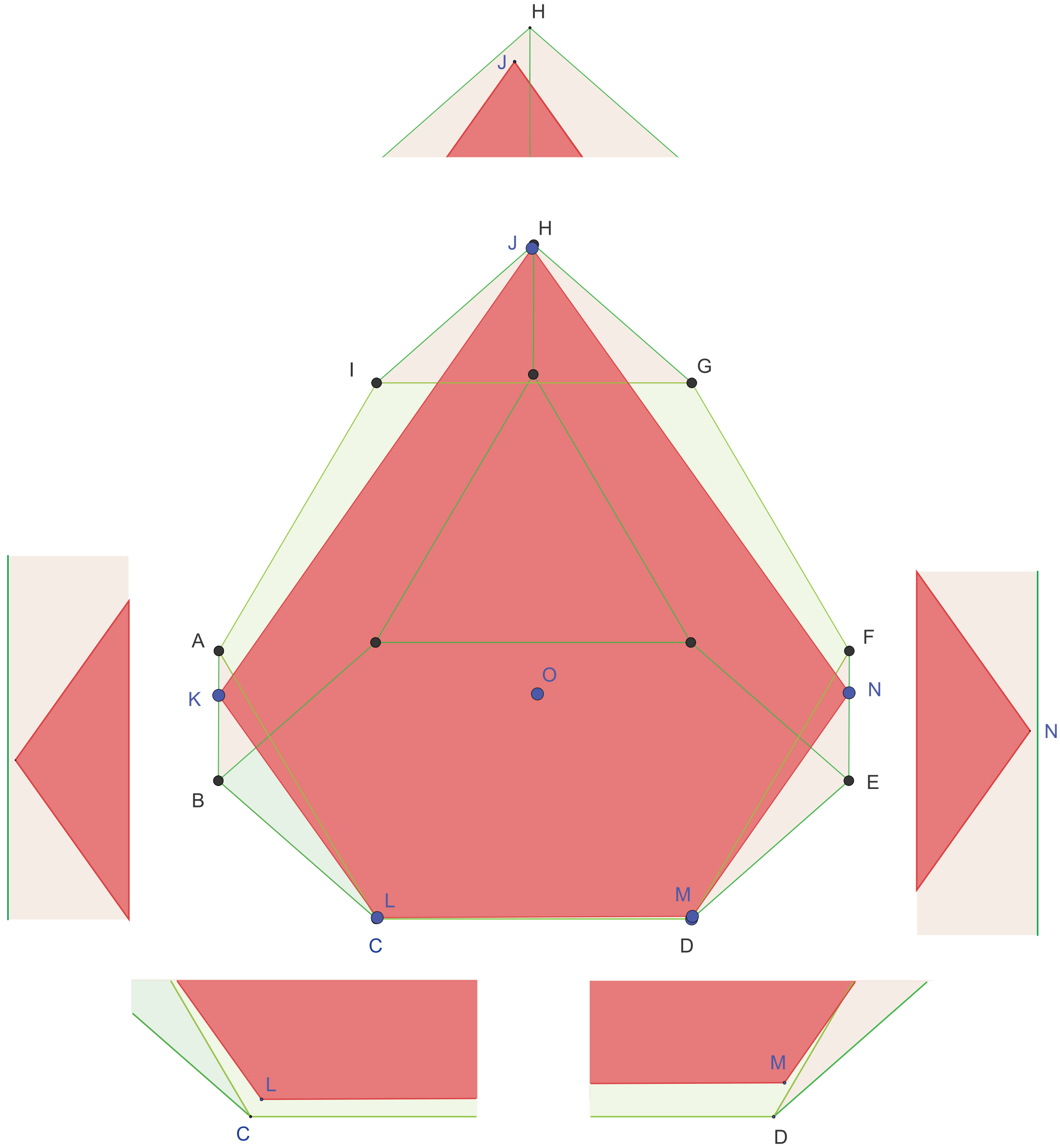

Figure 2: The Rupert property of the truncated tetrahedron with scaled images of the vertices. Note that the figure does not show symmetry due to the final (sufficiently small) rotation around centroid O

An upper limit of the Nieuwland constant can easily be calculated by the ratio of the original height of the diamond shape image and the maximal height of the right image, which is $\frac{\sqrt{38}}{6}$.

3. The generalised Nieuwland constant

So far the Nieuwland constant has been defined and calculated only for those cases when a specific polyhedron has had the Rupert property. Instead of considering the solid at first, let us introduce a different approach, and consider the problem from the viewpoint of the Nieuwland constant: if an arbitrary positive real number is given, can we find a proper polyhedron with this number or larger as Nieuwland constant? Moreover, what happens, if a solid has no Rupert property? Can we find a hole through which a smaller copy of the solid can pass? How small can be the largest hole? These question will be discussed in this section. The first question will be answered in $\mathbb{R}^n$ using the approach of [1].

Theorem 2. *For any $k > 1$, $k \in \mathbb{R}$, there exists an n-dimensional convex polytope with Nieuwland constant larger than k.*

Proof. Consider a rectangular n-dimensional cuboid (an n-orthotope) with mutually orthogonal edges of length $a_1 = 1$, $a_2 = k+1$, $\ldots$, $a_n = (k+1)^{(n-1)}$. Consider the hyperplane determined by edges a_2, a_3, $\ldots$, a_n, onto which the orthogonal projection of the polytope is a $(n-1)$-dimensional cuboid with edge lengths $k+1$, $\ldots$, $(k+1)^{(n-1)}$.

Now consider the hyperplane determined by edges $a_1, \ldots, a_{n-1}$, onto which the orthogonal projection of the original polytope is another $(n-1)$-dimensional cuboid with edge lengths 1, $k+1$, $\ldots$, $(k+1)^{(n-2)}$. The scale of this latter polytope by a factor $k + \frac{1}{k}$ yields a $(n-1)$-dimensional cuboid with edge lengths $k + \frac{1}{k}$, $(k+1)(k+\frac{1}{k})$, $\ldots$, $(k+1)^{(n-2)}(k+\frac{1}{k})$. This enlarged polytope still fits into the projection on the hyperplane of $a_2, a_3, \ldots, a_n$ if the corresponding edges are parallel, respectively, since $k + \frac{1}{k} < k+1$ and $(k+1)^{(m)}(k+\frac{1}{k}) < (k+1)^{(m+1)}$ for any $m = 1, \ldots, n-2$ and $k > 1$. Consequently the Nieuwland constant of this rectangular polytope is at least $k + \frac{1}{k}$. □

So our answer is affirmative if $k > 1$. For other positive values, however, we have to reconsider the concept of Nieuwland. The constant is originally defined for solids with Rupert property. But even if a solid has no Rupert property, there can be a smaller copy of this solid which can pass through a proper hole. This is to be defined as a generalisation of the original concept.

Definition 4. The scale factor $0 < k \leq 1$ is called *generalised Nieuwland constant* of a given solid, if there is a copy of this solid downscaled by k, which can pass through a proper hole in the original solid, but there is no larger copy with this property.

The case $k = 1$ is enabled, since the generalised Nieuwland constant is, as well as the original concept of the constant, an upper limit. For instance, the generalised Nieuwland constant of the sphere is 1, because scaling the sphere by a factor arbitrarily close to 1 (but less than 1) the scaled sphere can obviously pass through a centrally located circular hole in the original sphere.

It is evident, that for any solid with no Rupert property a generalised Nieuwland constant can be assigned. Given a solid, there exists a secant line close enough to the surface of the solid intersecting the solid in one single chord segment. Considering a right circular cylinder with this chord as rotational axis and with sufficiently small radius, the intersection of the cylinder and the solid can function as a hole through which an appropriately downscaled solid (the bounding sphere of which is of smaller radius than the cylinder) can pass.

It is however not evident, if for any arbitrarily small k there exists a solid with Nieuwland constant smaller than k. Since the Wetzel-conjecture states that every convex polyhedron has the Rupert-property [2], we will try to find concave polyhedra to prove this statement.

Theorem 3. *For any real number $0 < k < 1$ there exists a polyhedron, such that its generalised Nieuwland constant is smaller than k.*

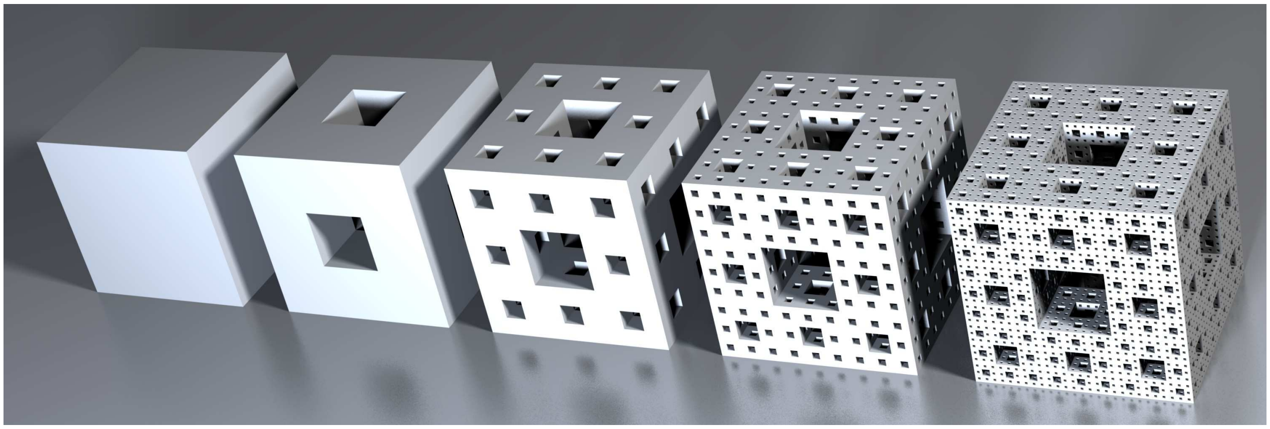

Figure 3: The unit cube, and iterations of the Menger-sponge (source: [6])

Proof. Consider the first iteration of the Menger-sponge in the unit cube (see the second polyhedron in Figure 3). The generalised Nieuwland constant of this solid cannot be greater than $\frac{1}{2}$, since the edges are trisected in constructing the Menger-sponge. Therefore a cube scaled by $\frac{1}{2}$ cannot be placed in a way that it is entirely covered by the original polyhedron. Consequently, it is impossible to find a hole for the polyhedron downscaled by $\frac{1}{2}$.

However, along the original edge of the unit cube there is an $\frac{1}{3} \times \frac{1}{3} \times 1$ cuboid. In this cuboid one can easily find a hole for a Menger-sponge downscaled by $\frac{1}{3}$. Thus, the generalised Nieuwland constant of the first iteration of the Menger-sponge is between $\frac{1}{2}$ and $\frac{1}{3}$. Analogously, the generalised Nieuwland constant of the n^{th} iteration of the Menger-sponge is between $\dfrac{1}{2^n}$ and $\dfrac{1}{3^n}$. Since the series $\frac{1}{2^n}$ converges to 0, when n tends to infinity, for any $0 < k < 1$ there exists an integer number n, for which $\frac{1}{2^n} < k$. Consequently, for any $0 < k < 1$ there exists an iteration of the Menger-sponge, the generalised Nieuwland constant of which is smaller than k, and this was to be proved. $\square$

Here we note that there are also many solids for which the generalised Nieuwland constant is between $\frac{1}{2} < k < 1$.

4. Conclusion and future work

We have proved that the truncated tetrahedron has the Rupert property and introduced the generalised Nieuwland constant also for those polyhedra not having the Rupert property. However, there are many problems still unsolved: to prove that the remaining Archimedean solids have the Rupert property and if every convex polyhedron has this property. Although we provided polyhedra with generalised Nieuwland constant smaller or larger than a predefined k, it would be interesting to show a polyhedron with generalised Nieuwland constant equal to a predefined k.

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