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Solid Angle Sum of a Tetrahedron

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Abstract. J.W. Gaddum proved in 1952 that the solid angles sum of a tetrahedron is less than 2π by finding the bound to the sum of six angles between four vertical segments from an interior point to the faces of the tetrahedron. We will give a new proof of this result by embedding the tetrahedron into a parallelepiped. In addition, we will give the bound on the sum of the four solid angles of a right tetrahedron using direction angles, and prove that the sum of the four solid angles of an equifacial tetrahedron is at most that of a regular tetrahedron.

Key Words: solid angles of a tetrahedron, dihedral angles, direction angles, right tetrahedron, equifacial tetrahedron

MSC 2010: 51M16, 51M04

1. Introduction

The sum of the three angles of a triangle is π. However, it is surprising to note that the sum of the four solid angles of a tetrahedron is not fixed. It was only 1952 when Gaddum [4] proved that the sum of the four solid angles of a tetrahedron ω is less than 2π. He proves this by first drawing four perpendicular segments from an interior point to its faces, and adding six angles between them and called it R. Then he proves that $R \geq 3\pi$, and concludes that $\omega \leq 2\pi$ by observing $\omega = 8\pi - 2R$. Our approach in Theorem 1 is to embed a tetrahedron into a parallelepiped, and deduce that $\omega \leq 2\pi$ from the solid angle sum of the parallelepiped being 4π. In Corollary 1.1, we show that the sum of six dihedral angles of a tetrahedron must be between 2π and 3π.

Using direction angles, we will investigate the bound of the sum of four solid angles and the sum of dihedral angles of a right tetrahedron (trirectangular tetrahedron) in Theorem 2. From this result, we will prove that the sum of the four solid angles of an equaifacial tetrahedron is at most that of a regular tetrahedron (Theorem 3). From this, we obtain the equation $\pi - 2\cos^{-1}\frac{1}{\sqrt{3}} = \cos^{-1}\frac{1}{3}$.

For your interest, papers [1], [3], [6], [8], and [9] mostly deal with the measurements of dihedral angles and solid angles of tetrahedra. Among them, Abu-Saymeh and Hajja [1] proved that the solid angle $\angle(O)$ at O of a tetrahedron $OABC$ is given by $\cos\angle(O) =$

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$\frac{1}{3}[\cos\angle AOB + \cos\angle BOC + \cos\angle COA]$. F. Eriksson [3] proved that if $\vec{a} = \overline{OA}$, $\vec{b} = \overline{OB}$, $\vec{c} = \overline{OC}$ are three vectors in $\mathbb{R}^3$, then $\tan\frac{\angle(O)}{2} = \frac{\vec{a}\cdot\vec{b}\times\vec{c}}{1+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}}$. These results are interesting, but we did not use them in this paper. Lagarius, Richardson and Lindsey [7] dealt with an inequality comparing the solid angle at a vertex of a tetrahedron to the area of the opposite face (see Remark 1 below for more detail).

2. Sum of solid angles

Let us begin this section by explaining the notion of solid angles using spherical geometry.

Definition 1. Let A, B, and C be points on a unit sphere, not all on a great circle. A *spherical triangle* ABC is the triangle whose edges AB, BC, and AC are parts of great circles on the unit sphere. The *angle* at A of the spherical triangle ABC is the angle between the tangent lines to the great circles AB and AC at A. If points A, B, and C are points on a great circle, the angle at A is π.

The next lemma is known, and we leave the proof to the readers.

Lemma 1. *Let A, B, and C be points on a unit sphere. Let α, β, γ be the angles of the spherical triangle ABC at A, B, and C, respectively. Then the area of the spherical triangle is given by $\alpha+\beta+\gamma-\pi$.*

Definition 2. Let $OABC$ be a tetrahedron. Let A_1, B_1, and C_1 be points of intersections of the unit sphere centered at O with the rays OA, OB, and OC, respectively. The *solid angle at the vertex O of the tetrahedron $OABC$* is defined to be the area of the spherical triangle $A_1B_1C_1$, and we denote it by $\angle(O;ABC)$ or $\angle(O)$ when there is no confusion. Sometimes, a solid angle of a tetrahedron is also called a *trihedral angle*.

We can generalize the solid angle of a tetrahedron to the solid angle of a polyhedron. Our goal in this section is to prove $\angle(O)+\angle(A)+\angle(B)+\angle(C) < 2\pi$.

Lemma 2. *The sum of eight solid angle of a parallelepiped is 4π.*

The proof is left to the readers.

Theorem 1. *The sum of the four solid angles of a tetrahedron is less than 2π.*

Proof. Let $OABC$ be a tetrahedron. Let Ω be the parallelepiped defined by vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$. Let O', A', B', C' be diagonally opposite vertices of vertices O, A, B, C in Ω, respectively. Let $\omega = \angle(O)+\angle(A)+\angle(B)+\angle(C)$, the sum of four solid angles of the tetrahedron $OABC$.
Since the tetrahedra $OABC$ and $O'A'B'C'$ are congruent, the sum of four solid angles of $O'A'B'C'$ is also ω. Let λ be the sum of six solid angles of the octahedron $ABCA'B'C'$. Since the union of $OABC$, $O'A'B'C'$, and $ABCA'B'C'$ is the parallelepiped Ω, we have $2\omega+\lambda = 4\pi$ by Lemma 2. Since $\lambda > 0$, we have $\omega < 2\pi$. This proves the theorem.
Alternately, we divide Ω by the plane $ABA'B'$ into two equal polyhedra, $OAB'A'BC$ and $O'A'BAB'C'$, each having solid angle sum of 2π. Since the polyhedron $OAB'A'BC$ contains the tetrahedron $OABC$, the sum ω of the four solid angles of $OABC$ is less than 2π. This also proves the theorem. □

Definition 3. Let $OABC$ be a tetrahedron. The *dihedral angle at the edge* OA of the tetrahedron $OABC$ is the inner angle between the two faces OAB and OAC, and it is denoted by $(\overline{OA}; OABC)$, or $\overline{OA}$ for short if there is no confusion.

Lemma 3. *Let $OABC$ be a tetrahedron. Let A_1, B_1 and C_1 be points of intersections with the unit sphere centered at O and the rays OA, OB, and OC, respectively. Then the angle at A_1 of the spherical triangle $A_1B_1C_1$ is the same as the dihedral angle $\overline{OA}$ of the tetrahedron $OABC$.*

The proof is left to the readers.

The next lemma is about the relation between dihedral and solid angles.

Lemma 4. *Let $OABC$ be a tetrahedron. Then $\angle(O) = \overline{OA} + \overline{OB} + \overline{OC} - \pi$. In particular, the relation between the sum of solid angles and the sum of dihedral angles is given by*

$$\angle(O) + \angle(A) + \angle(B) + \angle(C) = 2 \cdot [\overline{OA} + \overline{OB} + \overline{OC} + \overline{AB} + \overline{BC} + \overline{CA} - 2\pi].$$

Proof. Combining Lemmas 1 and 3, we have $\angle(O) = \overline{OA} + \overline{OB} + \overline{OC} - \pi$. This implies that $\angle(A) = \overline{OA} + \overline{AB} + \overline{CA} - \pi$, $\angle(B) = \overline{AB} + \overline{OB} + \overline{BC} - \pi$, $\angle(C) = \overline{CA} + \overline{BC} + \overline{OC} - \pi$. Hence, we have $\angle(O) + \angle(A) + \angle(B) + \angle(C) = 2 \cdot [\overline{OA} + \overline{OB} + \overline{OC} + \overline{AB} + \overline{BC} + \overline{CA} - 2\pi]$. □

Corollary 1.1. *The sum of the six dihedral angles of a tetrahedron is between 2π and 3π.*

Proof. Let $OABC$ be a tetrahedron. By Theorem 1 and Lemma 4, we have $0 < 2\cdot[\overline{OA}+\overline{OB}+\overline{OC}+\overline{AB}+\overline{BC}+\overline{CA}-2\pi] < 2\pi$. This shows that $2\pi < [\overline{OA}+\overline{OB}+\overline{OC}+\overline{AB}+\overline{BC}+\overline{CA}] < 3\pi$. □

Remark 1. (1) As pointed out in [4], the bounds in Theorem 1 and Corollary 1.1 cannot be improved.

(2) If (ABC) is the area of the face ABC of the tetrahedron $OABC$, and if

$$\alpha = \frac{\angle(O)}{\angle(O) + \angle(A) + \angle(B) + \angle(C)} \quad \text{and} \quad \beta = \frac{(ABC)}{(ABC) + (OBC) + (OAB) + (OAC)},$$

then $\beta \geq \dfrac{1}{\csc(\pi\alpha) + 1} \geq \alpha$ according to [7].

3. Right tetrahedra

We will investigate the sum of four solid angles of a right tetrahedron.

Definition 4. A tetrahedron $OABC$ is a *right tetrahedron with right angle at O* if three faces OAB, OAC, and OBC are right triangles with all right angles at the vertex O. Since the dihedral angles $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ are $\frac{\pi}{2}$, we have $\angle(O) = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} - \pi = \frac{\pi}{2}$ by Lemmas 1 and 4. This is the motivation of calling $OABC$ a *right tetrahedron.* A right tetrahedron is called *trirectangular tetrahedron* in [2].

Lemma 5. *Let $P = (x, y, z)$, where $x, y, z > 0$, be an arbitrary point in the first octant such that $x^2 + y^2 + z^2 = 1$. Let Λ be the plane through P normal to the vector $\overrightarrow{OP} = \langle x, y, z\rangle$. Let O be the origin and let A, B, C be the x, y, and z intercepts of the plane Λ, respectively.*

(1) *Then* $\cos\overline{BC} = x$, $\cos\overline{CA} = y$, *and* $\cos\overline{AB} = z$.

(2) *Let α be the angle between $\overrightarrow{OP}$ and $\langle 1,0,0\rangle$, β the angle between $\overrightarrow{OP}$ and $\langle 0,1,0\rangle$, and γ the angle between $\overrightarrow{OP}$ and $\langle 0,0,1\rangle$. (The angles α,β,γ are the* ***direction angles*** *of the vector $\overrightarrow{OP}$.) Then $\alpha=\overline{BC}$, $\beta=\overline{CA}, \gamma=\overline{AB}$.*

(3) $3\cos^{-1}\frac{1}{\sqrt{3}} \le \alpha+\beta+\gamma<\pi$. *The equality holds only when $x=y=z=\frac{1}{\sqrt{3}}$.*

Proof. (1) and (2): Since $\overrightarrow{OP}$ and $\langle 0,0,1\rangle$ are normal vector to the planes ABC and OAB, respectively, and since γ is the angle between $\overrightarrow{OP}$ and $\langle 0,0,1\rangle$, the dot product gives us $\cos\overline{AB}=\overrightarrow{OP}\cdot\langle 0,0,1\rangle = z=\cos\gamma$. Similarly, we have $\cos\overline{BC}=x=\cos\alpha$, and $\cos\overline{CA}=y=\cos\beta$. Hence, $\alpha=\overline{BC}$, $\beta=\overline{CA}$, $\gamma=\overline{AB}$.

(3): Let $f(x,y,z)=\cos^{-1}x+\cos^{-1}y+\cos^{-1}z$, and $g(x,y,z)=x^2+y^2+z^2$. We want to minimize $f(x,y,z)$ subject to $g(x,y,z)=1,\ x,y,z>0$. We use Lagrange multipliers method. We have the gradients

$$\nabla f(x,y,z)=\left\langle -\frac{1}{\sqrt{1-x^2}},\ -\frac{1}{\sqrt{1-y^2}},\ -\frac{1}{\sqrt{1-z^2}}\right\rangle$$

and $\nabla g(x,y,z)=\langle 2x,2y,2z\rangle$. From the equation $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$, we have $-2\lambda=\frac{1}{x\sqrt{1-x^2}}=\frac{1}{y\sqrt{1-y^2}}=\frac{1}{z\sqrt{1-z^2}}$. This shows that the minimal value of $f(x,y,z)=\cos^{-1}x+\cos^{-1}y+\cos^{-1}z$ is achieved only when $x=y=z=\frac{1}{\sqrt{3}}$. We have $f(x,y,z)\ge 3\cos^{-1}\frac{1}{\sqrt{3}}$. By (1) and (2), we have $f(x,y,z)=\alpha+\beta+\gamma\ge 3\cos^{-1}\frac{1}{\sqrt{3}}$.

On the boundary of $\{(x,y,z): x^2+y^2+z^2=1,\ x,y,z>0\}$, i.e., when at least one of x,y,z is 0, we have $f(x,y,z)=\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$. This shows that $f(x,y,z)=\alpha+\beta+\gamma<\pi$.
□

Theorem 2. *Let $OABC$ be a right tetrahedron with right angle at O. Then*

(1) $3\cos^{-1}\frac{1}{\sqrt{3}}\le\overline{AB}+\overline{BC}+\overline{CA}<\pi$.
The equality on the left side holds only when ABC is an equilateral triangle.

(2) $6\cos^{-1}\frac{1}{\sqrt{3}}-\frac{3}{2}\pi\le\angle(A)+\angle(B)+\angle(C)<\frac{1}{2}\pi$.
The equality on the left side holds only when ABC is an equilateral triangle.

(3) $6\cos^{-1}\frac{1}{\sqrt{3}}-\pi\le\angle(O)+\angle(A)+\angle(B)+\angle(C)<\pi$.
The equality on the left side holds only when ABC is an equilateral triangle.

Proof. (1): This is by Lemma 5, parts (2) and (3).
(2): Since $\overline{OA}=\overline{OB}=\overline{OC}=\frac{1}{2}\pi$, we have
$\angle(A)=\overline{OA}+\overline{AB}+\overline{AC}-\pi=\overline{AB}+\overline{AC}-\frac{1}{2}\pi$, $\angle(B)=\overline{AB}+\overline{BC}-\frac{1}{2}\pi$, and $\angle(C)=\overline{AB}+\overline{BC}-\frac{1}{2}\pi$. So $\angle(A)+\angle(B)+\angle(C)=2[\overline{AB}+\overline{BC}+\overline{AC}]-\frac{3}{2}\pi$, or $\overline{AB}+\overline{BC}+\overline{AC}=\frac{1}{2}\left[\frac{3}{2}\pi+\angle(A)+\angle(B)+\angle(C)\right]$. Hence, from (1), we have $3\cos^{-1}\frac{1}{\sqrt{3}}\le\frac{1}{2}\left[\frac{3}{2}\pi+\angle(A)+\angle(B)+\angle(C)\right]<\pi$ or $6\cos^{-1}\frac{1}{\sqrt{3}}-\frac{3}{2}\pi\le\angle(A)+\angle(B)+\angle(C)<\frac{1}{2}\pi$.
(3): This is from part (2) and by noting that $\angle(O)=\frac{1}{2}\pi$. □

4. Equifacial tetrahedra

We will label a tetrahedron by $ABCD$ rather than $OABC$ in this section because of the symmetrical nature of an equifacial tetrahedron $ABCD$.

Definition 5. A tetrahedron $ABCD$ is *equifacial* if $AB = CD$, $AC = BD$, and $AD = BC$. So four triangular faces of an equifacial tetrahedron are congruent and $\overline{AB} = \overline{CD}$, $\overline{AC} = \overline{BD}$, $\overline{AD} = \overline{BC}$ (see Exercise 15 on page 102 in [2]). An equifacial tetrahedron is also called an *isosceles tetrahedron.*

The next lemma is Lemma 1 of [5], so we omit its proof.

Lemma 6. *If $ABCD$ is an equifacial tetrahedron, then there is a rectangular box Ω that contains an equifacial tetrahedron $ABCD$ so that the six edges of the tetrahedron $ABCD$ are diagonals of six faces of Ω (see Figure 1. We say the rectangular box Ω is said to* diagonally embed the tetrahedron *$ABCD$.) Conversely, the diagonally embedded tetrahedron in a rectangular box is equifacial.*

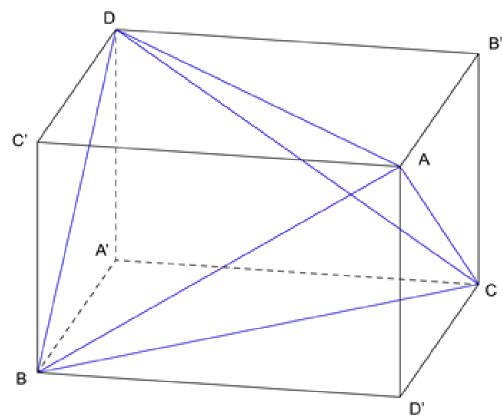

Figure 1: Ω is diagonally embedding the tetrahedron $ABCD$.

Theorem 3. *Let $ABCD$ be an equifacial tetrahedron.*

(1) $0 < \angle(A) + \angle(B) + \angle(C) + \angle(D) \leq 8\left(\pi - 3\cos^{-1}\frac{1}{\sqrt{3}}\right)$. *Equality on the right holds only when $ABCD$ is a regular tetrahedron.*

(2) $2\pi < [\overline{AD} + \overline{AB} + \overline{AC} + \overline{BC} + \overline{DC} + \overline{DB}] \leq 6\left(\pi - 2\cos^{-1}\frac{1}{\sqrt{3}}\right)$. *Equality on the right holds only when $ABCD$ is a regular tetrahedron.*

(3) $\pi - 2\cos^{-1}\frac{1}{\sqrt{3}} = \cos^{-1}\frac{1}{3}$. *So the right side of (2) can be replaced by* $6\cos^{-1}\frac{1}{3}$.

Proof. (1) By Lemma 6, we assume that $ABCD$ is embedded in a rectangular box Ω. Let A', B', C', D' be the diagonally opposite points of A, B, C, D in Ω. Let $\sigma = \angle(A'; BCD) + \angle(B; A'CD) + \angle(C; A'BD) + \angle(D; A'BC)$, the sum of four solid angle of the right tetrahedron $A'BCD$. Since the right tetrahedra, $AB'CD$, $ABC'D$, and $ABCD'$ are congruent to $A'BCD$, the sum of solid angles for each of these is σ. Let $\omega = \angle(A) + \angle(B) + \angle(C) + \angle(D)$, the solid angle sum of the equifacial tetrahedron $ABCD$. Since the solid angle sum of the rectangular box Ω is 4π, we have $4\pi = 4\sigma + \omega$ or $\sigma = \pi - \frac{1}{4}\omega$. Substituting it to $6\cos^{-1}\frac{1}{3} - \pi \leq \sigma < \pi$ in Theorem 2(3), we have $0 < \omega \leq 8\left(\pi - 3\cos^{-1}\frac{1}{\sqrt{3}}\right)$. The equality holds when BCD is an equilateral triangle. But when BCD is an equilateral triangle, the tetrahedron $ABCD$ is a regular tetrahedron.

Proof of (2): By Lemma 4, we have

$$\angle(A) + \angle(B) + \angle(C) + \angle(D) = 2 \cdot [\overline{OA} + \overline{OB} + \overline{OC} + \overline{AB} + \overline{BC} + \overline{CA} - 2\pi].$$

By the part (1), we have

$$0 < 2 \cdot [\overline{OA} + \overline{OB} + \overline{OC} + \overline{AB} + \overline{BC} + \overline{CA} - 2\pi] < 8\Big(\pi - 3\cos^{-1}\frac{1}{\sqrt{3}}\Big),$$

or

$$2\pi < [\overline{OA}+\overline{OB}+\overline{OC}+\overline{AB}+\overline{BC}+\overline{CA}-2\pi] < 4\Big(\pi-3\cos^{-1}\frac{1}{\sqrt{3}}\Big)+2\pi = 6\Big(\pi-2\cos^{-1}\frac{1}{\sqrt{3}}\Big).$$

The equality on the right holds only when $ABCD$ is a regular tetrahedron.
Proof of (3): The dihedral angle of a regular tetrahedron is $\cos^{-1}\frac{1}{3}$. From the part (2), the sum of the six dihedral angles of a regular tetrahedron is $6\big(\pi - 2\cos^{-1}\frac{1}{\sqrt{3}}\big)$. Hence, we have $\pi - 2\cos^{-1}\frac{1}{\sqrt{3}} = \cos^{-1}\frac{1}{3}$. □

Remark 2. Let $A = (a, 1, 1)$, $B = (a, -1, -1)$, $C = (-a, 1, -1)$, and $D = (-a, -1, 1)$. Then $ABCD$ is an equifacial tetrahedron. We can show that

$$\cos\overline{AD} = \frac{1}{1+2a^2}, \quad \cos\overline{AB} = \frac{-1+2a^2}{1+2a^2}, \quad \text{and} \quad \cos\overline{AC} = \frac{1}{1+2a^2}.$$

By letting $a \to \infty$, we have $\overline{AD} = \overline{BC} \to \frac{\pi}{2}$, $\overline{AB} = \overline{CD} \to 0$, and $\overline{AC} = \overline{BD} \to \frac{\pi}{2}$. This shows that, in Theorem 3(2), the sum of the six dihedral angles can be made as close to 2π as possible.

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