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Recalling Thread Constructions of Quadrics*

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Abstract. The role of quadrics in Euclidean 3-space is similar to that of conics. Therefore, it is natural to ask for thread constructions of quadrics, as spatial analogues of the gardener's construction or GRAVES' construction of ellipses. The first solution given in 1882 by O. STAUDE is based on an ellipse e and its focal hyperbola h. A thread of given length, fixed with one end at a focal point of h, is passed behind the nearest branch of h and in front of e and finally attached to the vertex of the second branch of h. If the thread is stretched at a point P between, then P traces a patch of an ellipsoid $\mathcal{E}$ confocal with e and h. Later, STAUDE presented a second type of thread constructions where e and h are replaced by an ellipsoid $\mathcal{E}_0$ and a confocal hyperboloid H_0. Here, the thread follows at its ends the two branches of the line of curvature $\mathcal{E}_0 \cap \mathrm{H}_0$. We provide a synthetic approach to these constructions and discuss the case of paraboloids.

Key Words: quadric, thread construction, focal conics, confocal quadrics

MSC 2020: 51M04, 53A05, 53A17

1 Introduction

In 1882, Otto STAUDE [9] presented a thread construction for ellipsoids, based on a pair of focal conics e and h (Fig. 1). It was proposed as a spatial analogue of the gardener's construction and GRAVES's construction of ellipses (see, e.g., [4, Figs. 1.8 and 2.29]). Some years later, STAUDE [10] came up with a second version: Instead of the pair of focal conics, an ellipsoid $\mathcal{E}_0$ and a confocal hyperboloid H_0 are used. A thread which is stretched at the point P follows at its ends the two branches of the curve of intersection $\mathcal{E}_0 \cap \mathrm{H}_0$, which are lines of curvature for both quadrics. Then the thread continues along geodesics on $\mathcal{E}_0$ or H_0, while point P traces a portion of an ellipsoid $\mathcal{E}$ being confocal with $\mathcal{E}_0$ and H_0.

STAUDE's thread constructions of ellipsoids are subject of two models in SCHILLING's famous collection of mathematical models (listed in [8]), namely the models VII, no. 191 and 192 (see https://mathematical-models.org/index.php/models/view/345 and https:

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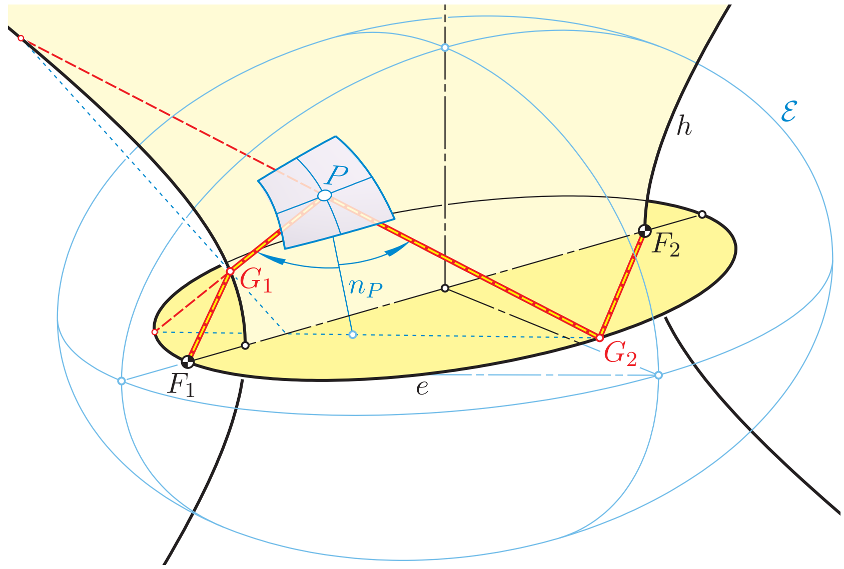

Figure 1: Staude's first thread construction of the ellipsoid $\mathcal{E}$ uses its focal conics e and h.

`//mathematical-models.org/index.php/models/view/279`, Digitales Archiv Mathematischer Modelle, TU Dresden). According to D. Hilbert, Staude's thread constructions of quadrics were one of the great mathematical results of the 19$^{\text{th}}$ century [1, p. 236].

We present a synthetic approach to these constructions, thus reducing the proof to uniqueness theorems for the solutions of first order differential equations. Moreover, we discuss the case of focal parabolas and, similar to the second version mentioned above, that of confocal paraboloids. For historical remarks, generalizations, and additional references see [2], [3, p. 11], [5, p. 19], [6, Sect. 3.3.5], or [9, Theorem 4.3]. Since the thread constructions result from properties of quadrics in a confocal family, we start recalling a few of them.

2 Confocal central quadrics

Let $\mathcal{E}$ be a triaxial ellipsoid with semiaxes a, b, and c. The one-parameter family of quadrics being *confocal* with $\mathcal{E}$ is given as

$$F(x,y,z;k) := \frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} + \frac{z^2}{c^2+k} - 1 = 0, \tag{1}$$

where $k \in \mathbb{R} \setminus \{-a^2, -b^2, -c^2\}$ serves as a parameter within the family. In the case $a > b > c > 0$, this family includes

$$\text{for} \left\{ \begin{array}{ll} -c^2 < k < \infty & \text{triaxial ellipsoids,} \\ -b^2 < k < -c^2 & \text{one-sheeted hyperboloids,} \\ -a^2 < k < -b^2 & \text{two-sheeted hyperboloids.} \end{array} \right. \tag{2}$$

Confocal quadrics intersect their common planes of symmetry along confocal conics. As limits for $k \to -c^2$ and $k \to -b^2$ we obtain 'flat' quadrics, i.e., the focal ellipse e and the focal hyperbola h, satisfying

$$e\colon\ \frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1,\ z = 0\,, \quad h\colon\ \frac{x^2}{a^2-b^2} - \frac{z^2}{b^2-c^2} = 1,\ y = 0\,. \tag{3}$$

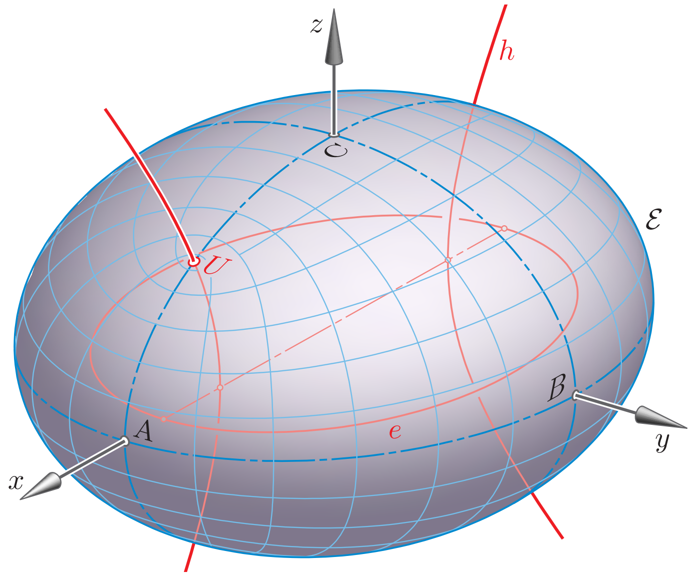

Figure 2: Ellipsoid $\mathcal{E}$ with lines of curvature, focal conics e, h, and an *umbilic point* U.

They form a pair of *focal conics*.[1]

The confocal family sends through each point $P = (\xi, \eta, \zeta)$ outside the coordinate planes, i.e., with $\xi\eta\zeta \neq 0$, exactly one ellipsoid, one one-sheeted hyperboloid, and one two-sheeted hyperboloid. The respective parameters (k_1, k_2, k_3) define the three *elliptic coordinates* of P, where

$$-a^2 < k_3 < -b^2 < k_2 < -c^2 < k_1\,. \tag{4}$$

For given Cartesian coordinates (ξ, η, ζ) of any point P, we obtain the elliptic coordinates by solving $F(\xi, \eta, \zeta; k) = 0$ in (1) for k. Conversely, if the tripel (k_1, k_2, k_3) of elliptic coordinates is given, then the Cartesian coordinates (ξ, η, ζ) of the corresponding points $P \in \mathcal{E}$ satisfy

$$\begin{gathered}\xi^2 = \frac{(a^2+k_1)(a^2+k_2)(a^2+k_3)}{(a^2-b^2)(a^2-c^2)}, \quad \eta^2 = \frac{(b^2+k_1)(b^2+k_2)(b^2+k_3)}{(b^2-c^2)(b^2-a^2)},\\ \zeta^2 = \frac{(c^2+k_1)(c^2+k_2)(c^2+k_3)}{(c^2-a^2)(c^2-b^2)}.\end{gathered} \tag{5}$$

There exist eight such points, symmetric w.r.t. the coordinate planes.

At each point P outside the coordinate planes, the surface normals

$$\mathbf{v}_i := \left(\frac{\xi}{a^2+k_i}, \frac{\eta}{b^2+k_i}, \frac{\zeta}{c^2+k_i}\right), \quad i = 1, 2, 3\,, \tag{6}$$

to the three quadrics through P are mutually orthogonal. Therefore, confocal quadrics form a triply orthogonal system of surfaces. Due to a classical theorem of Ch. Dupin, they intersect each other along lines of curvature. Fig. 2 shows the net of lines of curvature on a triaxial ellipsoid $\mathcal{E}$ with singularities at the umbilic points.

Lemma 1. *The tangent cones from any point P to the quadrics of a confocal family are confocal with the cones connecting P with the focal conics. Their common planes of symmetry are*

[1] The conics of a pair of focal conics lie in orthogonal planes and share the principal axis. The focal points of one conic coincide with vertices of the other (see Fig. 2 and, e.g., [4, Sect. 4.2]).

tangent to the quadrics passing through P. The tangent cones are coaxial cones of revolution if and only if P is a point of a focal conic.

For the definition of confocal quadratic cones see, e.g., [7, p. 284]. A proof of Lemma 1 can be found in [7, p. 286]

Given a confocal family, each line other than a generator of any contained ruled quadric contacts exactly two surfaces of the family, and the tangent planes at the corresponding points of contact are orthogonal. This results in the lemma below, which dates back to Jacobi (1839) and Chasles.

Lemma 2. *On each quadric $\mathcal{Q}$, the geodesics are curves with tangents contacting another fixed quadric $\mathcal{Q}'$ that is confocal with $\mathcal{Q}$ (see* [7, Fig. 7.7]*).*

For a proof, see [7, p. 291].

3 Thread constructions of central quadrics

3.1 Staude's first thread construction

Theorem 3. *Let e be an ellipse with the focal hyperbola h. Let F_1 denote a vertex of e and focal point of h and F_2 the focal point of e and vertex of h at a greater distance to F_1. A thread of a given length, fixed with one end at F_1, is passed behind the nearest branch of h and in front of e and finally attached to F_2 (see Fig. 1).*
If the thread is stretched at a point P such that it forms a spatial polygon with vertices F_1, $G_1 \in h$, P, $G_2 \in e$, and F_2, then P traces a patch of an ellipsoid $\mathcal{E}$ confocal with e and h.

The presented proof is based on two lemmas.

Lemma 4. *Let a thread with fixed endpoints F_1 and F_2 be stretched over a given curve c. Then, the corner-point $G \in c$ of the thread satisfies two conditions:*
(i) *The tangent t_G to c at G encloses congruent angles with the straight segments F_1G and F_2G, and*
(ii) *the normal plane to c at G either passes through both endpoints or separates F_1 and F_2. In the latter case, the lines $[F_1, G]$ and $[F_2, G]$ are generators of a cone of revolution with apex G and axis t_G (Fig. 3).*

Proof. When the thread has reached its equilibrium at $G \in c$, the stress along the thread induces two forces of equal quantity which act along the segments GF_1 and GF_2 and result in a force orthogonal to c. Therefore, the components of the two forces in direction of the tangent t_G must be opposite in order to compensate each other (see Fig. 3). This implies congruent angles between t_G and the two segments of the strengthened thread. □

It is noteworthy that G needs not be unique on c. If, for example, the curve c is an ellipse with focal points F_1 and F_2, then each point $G \in c$ satisfies the claimed equilibrium condition, since the sum of distances $\overline{GF_1}$ and $\overline{GF_2}$ is stationary. Other examples can be found below in Fig. 11 (where $c = o$) or in [4, p. 143, Fig. 4.17]).

Lemma 5. *Let a strengthened thread of a given length with one fixed endpoint F_1 be bent over a curve c while the second endpoint P traces a smooth curve p. Then, at each pose P, the curve p is orthogonal to the final segment of the thread.*

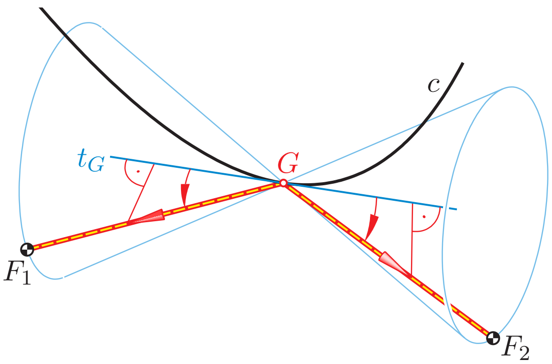

Figure 3: A thread with fixed endpoints F_1, F_2 and stretched over the curve c makes equal angles with the tangent t_G to c at the point G (Lemma 4).

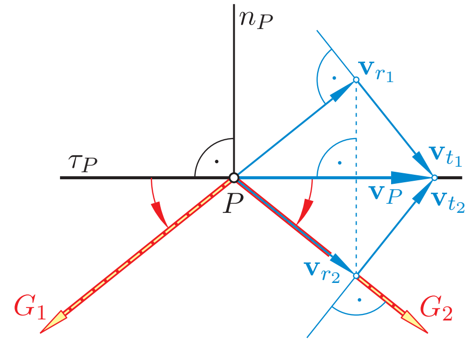

Figure 4: Decomposition of the velocity vector $\mathbf{v}_P$ at P, while the length of the strengthened thread $F_1G_1PG_2F_2$ is kept fixed.

Proof. With respect to F_1 as the origin of a coordinate frame, the curve c can be parametrized as $\mathbf{c}(t) = \lambda(t)\mathbf{e}_1(t)$ with $\|\mathbf{e}_1(t)\| = 1$ for t in some interval J. After being bent over c, the remaining segments of the thread form a ruled surface, and the position vector of the trajectory of P can be written as

$$\mathbf{f}(t) = \mathbf{c}(t) + (k - \lambda(t))\,\mathbf{e}_2(t) \text{ with } \|\mathbf{e}_2(t)\| = 1 \text{ and } k = \text{const.}$$

The angle condition claimed in Lemma 4 implies for all $t \in J$

$$\langle\,\dot{\mathbf{c}}, \mathbf{e}_2\rangle = \langle\,\dot{\mathbf{c}}, \mathbf{e}_1\rangle, \text{ and hence } \langle\dot{\lambda}\mathbf{e}_1 + \lambda\dot{\mathbf{e}}_1,\, \mathbf{e}_2\rangle = \langle\dot{\lambda}\mathbf{e}_1 + \lambda\dot{\mathbf{e}}_1,\, \mathbf{e}_1\rangle = \dot{\lambda}$$

if the dot indicates differentiation by t and $\langle\ ,\ \rangle$ denotes the standard dot product in $\mathbb{R}^3$. Consequently, we obtain

$$\langle\,\dot{\mathbf{f}}, \mathbf{e}_2\rangle = \langle\dot{\mathbf{c}} - \dot{\lambda}\mathbf{e}_2 + (k-\lambda)\,\dot{\mathbf{e}}_2,\ \mathbf{e}_2\rangle = \dot{\lambda} + \langle -\dot{\lambda}\mathbf{e}_2 + (k-\lambda)\,\dot{\mathbf{e}}_2,\ \mathbf{e}_2\rangle = \dot{\lambda} - \dot{\lambda} = 0.$$

This proves Lem. 5. □

Theorem 3. By virtue of Lemma 4, point $G_1 \in h$ is the apex of a cone of revolution which passes through F_1 and P and has the tangent t_{G_1} to h as its axis (Fig. 1). We learned in Lemma 1 that both conics e and h are the locus of apices of cones of revolution which pass through the other focal conic, and the axes of these cones are tangents to the conic. Therefore, since the segment G_1F_1 meets the focal ellipse e, the same must hold for the extension of the segment G_1P.

Lines through the point P meeting e and h are common generators of two confocal cones (Lemma 1). Thus, if there exists one transversal, then there are four that are mutually symmetric w.r.t. the tangent planes to the three confocal quadrics through P. A 180° rotation about the surface normal n_P of the ellipsoid $\mathcal{E}$ through P takes the line $[P, G_1]$ to a line $[P, G_2]$ which again meets the two focal conics e and h. The traces of the plane $[P, G_1, G_2]$ in the planes of e and h reveal (note Fig. 1) that, starting from P, the line $[P, G_2]$ meets first e and then h. Due to the properties of a pair of focal conics, the bent portion PG_2F_2 of the thread is in equilibrium because of Lemma 4.

Let point P move in such a way that the thread remains strengthened. We are going to prove that in this case the tangents to all possible trajectories of P are orthogonal to n_P.

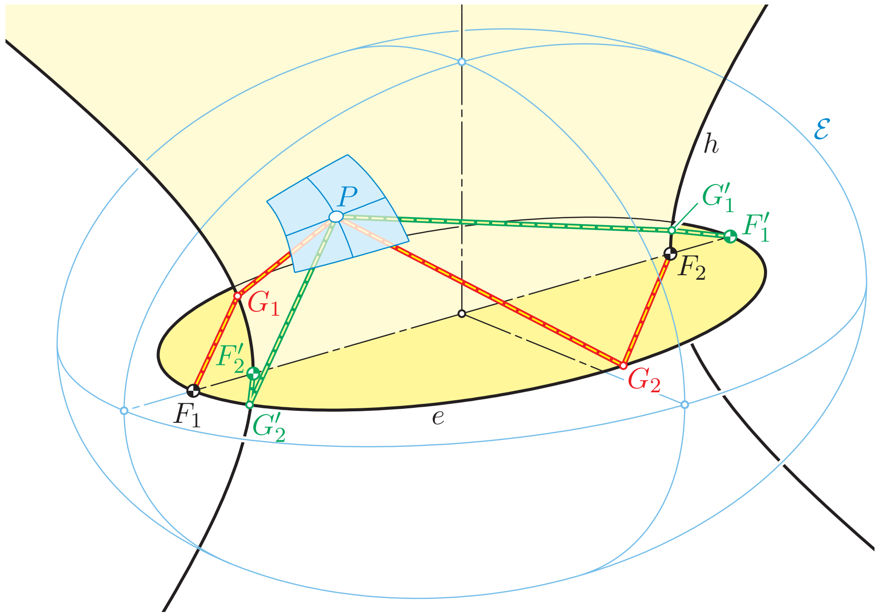

Figure 5: Two thread constructions of the same ellipsoid $\mathcal{E}$ using the focal conics e, h.

If the point P is fixed on the moving thread with the endpoint F_1, then, by Lemma 5, the tangent vector $\mathbf{v}_{t_1}$ of the point P is orthogonal to PG_1. Similarly, for the point P being fixed on the final portion of the thread, the velocity vector $\mathbf{v}_{t_2}$ would be orthogonal to PG_2. Because of the constant total length of the thread, the relative velocities of P with respect to the two parts of the thread must be equal; when the length of the initial part increases, that of the final part must decrease about the same rate, and vice versa. This implies for the vector $\mathbf{v}_P$ of the absolute velocity of P

$$\mathbf{v}_P = \mathbf{v}_{t_1} + \mathbf{v}_{r_1} = \mathbf{v}_{t_2} + \mathbf{v}_{r_2} \tag{7}$$

that the vectors $\mathbf{v}_{r_1}$ and $-\mathbf{v}_{r_2}$ are symmetric w.r.t. n_P. The orthogonal projection of the involved vectors into the plane $[P, G_1, G_2]$ reveals, as shown in Fig. 4, that $\mathbf{v}_P$ must be orthogonal to n_P.

Consequently, at all poses in some neighbourhood, the point P moves tangentially to the confocal ellipsoid through P. In other words, if in elliptic coordinates the parametrization of the requested trajectory of P is assumed as

$$\mathbf{k}(u,v) = (k_1(u,v), k_2(u,v), k_3(u,v)),$$

then the partial derivatives $\mathbf{k}_u$ and $\mathbf{k}_v$, which span the tangent plane, satisfy the conditions

$$\frac{\partial k_1}{\partial u} = \frac{\partial k_1}{\partial v} = 0.$$

This implies $k_1(u,v) = \text{const.}$ and confirms that the trajectory is a patch of an ellipsoid.

Conversely, if P remains on the ellipsoid $\mathcal{E}$, then $\mathbf{v}_P$ is orthogonal to n_P. This implies equal relative velocities $\|\mathbf{v}_{r_1}\| = \|\mathbf{v}_{r_2}\|$ in appropriate directions (see Fig. 4), and therefore, a constant length of the thread. □

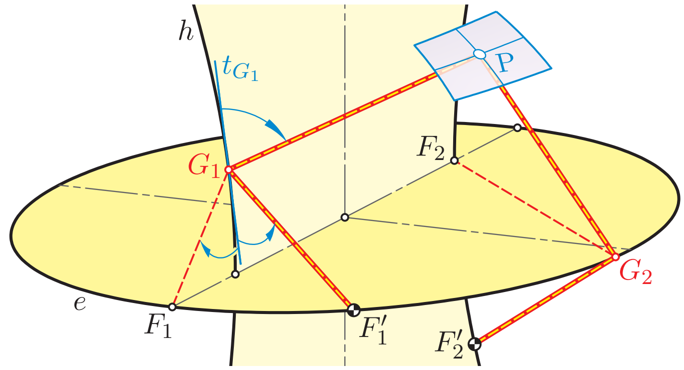

Figure 6: The fixed points F_1 and F_2 can be replaced by $F_1' \in e$ and $F_2' \in h$.

We find the total length L of the thread by specifying the point P at one of the vertices of the ellipsoid $\mathcal{E}$. This yields

$$L = 2a + a_e - a_h,$$

where a, a_e, and a_h are the respective principal semiaxes of the ellipsoid $\mathcal{E}$, the focal ellipse e, and the focal hyperbola h. It should be noted that W. Böhm [2] used Ivory's Theorem to prove that, for all poses of P, the sum of distances equals L.

Remark 1. (i) One cannot obtain the complete ellipsoid with the thread construction described in Theorem 3, since the thread, starting at F_1 and coming from behind, has to be bent around the hyperbola h. This does not work if point P lies behind the plane spanned by h. With regard to the other end of the thread, the point P cannot lie under the plane of the ellipse e. (ii) The same ellipsoid can be generated by using the remaining two common generators of the confocal cones which connect P with the pair of confocal conics (Fig. 5). The two strengthened threads could even be bound together at P by a small ring through which the two threads can glide independently from each other, while P remains on the quadric.

Corollary 6. *The thread construction of Theorem 3 for the triaxial ellipsoid $\mathcal{E}$ remains valid if the fixed endpoints F_1 and F_2 are replaced by two other sufficiently close points on the respective conics (Fig. 6). This variation affects only the total length L of the thread and the domain, that is traced by the point P on the ellipsoid $\mathcal{E}$.*

Proof. The condition stated in Lemma 4 remains valid when F_1 is replaced by a sufficiently close point $F_1' \in e$ (Fig. 6). On the other hand, since t_{G_1} encloses congruent angles with G_1F_1 and G_1F_1', while F_1 and F_1' lie on the same side of the normal plane to t_{G_1} at G_1, the difference of distances

$$d := \overline{G_1F_1'} - \overline{G_1F_1}$$

remains constant. Therefore, the difference d must be added to the total length L of the thread in order to keep P on the same ellipsoid. The same is valid for the replacement of the other fixed endpoint $F_2 \in h$ by a sufficiently close point $F_2' \in h$ (Fig. 6). □

3.2 Staude's second thread construction

In [10], Staude presented another thread construction, which is also documented as a historical model in [8, p. 139] (note also [6, Sect. 3.3.5]). It generalizes the version of Graves's construction on an ellipsoid $\mathcal{E}_0$, as displayed in [7, Fig. 7.14].

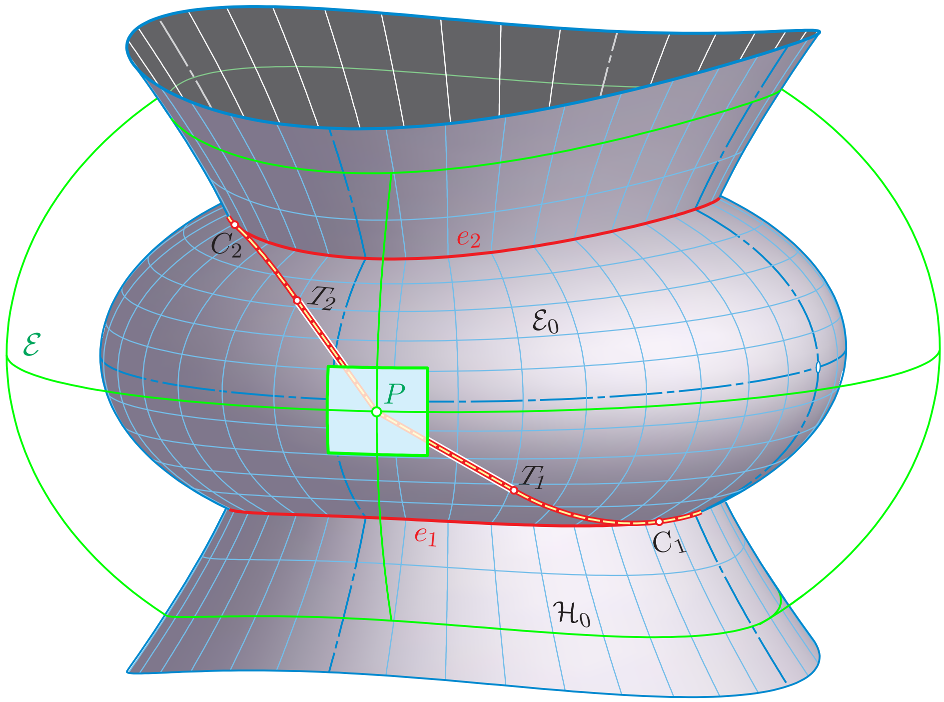

Figure 7: According to STAUDE's second thread construction, the two ends of the thread have to be attached to two antipodal lines of curvature e_1, e_2 of an ellipsoid $\mathcal{E}_0$, while the point P, which keeps the thread taut, moves on a confocal ellipsoid $\mathcal{E}$.

Theorem 7. *Let a thread of appropriate length with both ends be attached to a pair of antipodal lines of curvature e_1, e_2 of an ellipsoid $\mathcal{E}_0$ and kept taut so that it follows a geodesic crossing from e_1 to e_2. If we keep the ends as they are, but elongate this thread to a fixed length and keep it taut at a point P between the two curves e_1 and e_2, then P traces a patch of an ellipsoid $\mathcal{E}$ confocal with $\mathcal{E}_0$ (Fig. 7).*
Conversely, for a point P moving locally on $\mathcal{E}$, the length of the described taut thread connecting e_1 via P with e_2 remains fixed.

If the thread is sufficiently short, then it continues, from the two antipodal lines of curvature $e_1, e_2 \subset \mathcal{E}_0$ on, along geodesic arcs on $\mathcal{E}_0$ and furtheron along respective tangents. They meet at the point P, which keeps the thread taut (see Fig. 7). By virtue of Lemma 2, the two tangents $[P, T_1]$ and $[P, T_2]$ contact $\mathcal{E}_0$ and a confocal hyperboloid H_0, which is the second confocal quadric through e_1 and e_2. Due to Lemma 1, the two tangents are common to two confocal tangent cones with apex P, and consequently, in symmetric position w.r.t. the normal at P to one of the three confocal quadrics through P. In order to prove Theorem 7, we need a statement similar to Lemma 5.

Lemma 8. *Let one end of a strengthened thread of given length be attached to the line of curvature e, which is the intersection of an ellipsoid $\mathcal{E}_0$ with a confocal hyperboloid H_0. Suppose that, in each pose, the thread is a C^1-composition of three arcs. It begins along the curved edge e, continues from a point $C \in e$ on along a geodesic c of $\mathcal{E}_0$ until a point T. Finally, there is a straight segment on the tangent to c at T. Then, in which way ever the second endpoint P of the thread moves smoothly in space, its trajectory p is orthogonal to the straight segment TP.*

Proof. Any point Q which is attached to the thread between the points C and T runs on an orthogonal trajectory of the geodesic c. There is a local parametrization $\mathbf{x}(u,v)$, $(u,v) \in I \times J$, of $\mathcal{E}_0$ with geodesics tangent to c as u-lines and its orthogonal trajectories as v-lines. By virtue of a theorem by Gauss, u can be assumed as common arc length along the geodesics. This implies for the partial derivatives

$$\langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0, \quad \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1, \quad \langle \mathbf{x}_u, \mathbf{x}_{uu} \rangle = \langle \mathbf{x}_u, \mathbf{x}_{uv} \rangle = \langle \mathbf{x}_v, \mathbf{x}_{uu} \rangle = 0$$

for all $(u,v) \in I \times J$. The equations $\langle \mathbf{x}_u, \mathbf{x}_{uu} \rangle = \langle \mathbf{x}_v, \mathbf{x}_{uu} \rangle = 0$ confirm that the osculating planes of the u-lines are orthogonal to the tangent plane.

Now we distinguish between two cases:

(i) If, under the motion of the thread's endpoint P in space, the point C and the geodesic c remain fixed, i.e., T runs along c, then the point P traces an involute, which is an orthogonal trajectory of the generators on the tangent surface of c and also a line of curvature on the developable.

(ii) Otherwise, the v-coordinate of T varies. Let T trace the curve $\mathbf{p}(t)$ on $\mathcal{E}_0$ given by $u = u(t)$ and $v = t$ for $t \in J$. If the point P is supposed to be attached to the thread, then we obtain for its path the parametrization

$$\mathbf{p}(t) = \mathbf{x}\,(u(t),\, t) + (k - u(t))\,\mathbf{x}_u(t) \quad \text{with} \quad k = \text{const.}$$

From

$$\dot{\mathbf{p}}(t) := \frac{\mathrm{d}\,\mathbf{p}(t)}{\mathrm{d}t} = \dot{u}\,\mathbf{x}_u + \mathbf{x}_v - \dot{u}\,\mathbf{x}_u + (k-u)(\dot{u}\,\mathbf{x}_{uu} + \mathbf{x}_{uv})$$

follows the stated orthogonality, since

$$\langle \mathbf{x}_u,\, \dot{\mathbf{p}} \rangle = \langle \mathbf{x}_u,\, \mathbf{x}_v \rangle + (k-u)\langle \mathbf{x}_u,\, \dot{u}\,\mathbf{x}_{uu} \rangle + (k-u)\langle \mathbf{x}_u,\, \mathbf{x}_{uv} \rangle = 0.$$

The same holds when we replace the ellipsoid $\mathcal{E}_0$ by the hyperboloid H_0. □

Theorem 7. Based on Lemma 8, the proof is similar to that of Theorem 3. With respect to the part of the thread attached to the line of curvature e_i, $i \in \{1,2\}$, a point P which is fixed on the moving thread has a tangent vector $\mathbf{v}_{t_i}$ orthogonal to the segment PT_i. If, additionally, the point P is moving relative to the thread with velocity vector $\mathbf{v}_{r_i}$ in direction of PT_i, we obtain the vector of absolute velocity of P as a sum of two orthogonal components. This holds for $i = 1, 2$ and yields (7).

When the total length of the thread remains constant, the relative velocities $\|\mathbf{v}_{r_1}\|$ and $\|\mathbf{v}_{r_2}\|$ must be equal. This implies, as depicted in Fig. 4, that $\mathbf{v}_P$ is orthogonal to the interior angle bisector of $\angle T_1 P T_2$ and tangent to the confocal ellipsoid $\mathcal{E}$ passing through P.

Conversely, if P remains on the ellipsoid $\mathcal{E}$, then we obtain equal relative velocities in appropriate directions, and therefore, a constant length of the thread. □

Remark 2. After extending PT_1 and PT_2 from their tangency points with $\mathcal{E}_0$ to their second intersection with the ellipsoid $\mathcal{E}$, we obtain two adjacent sides of a billiard in $\mathcal{E}$. More about billiards in ellipsoids can be found in [11, p. 62 ff.].

4 Paraboloids

All quadrics which are confocal with a given paraboloid can be represented as

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} - 2z - k = 0 \quad \text{for} \quad k \in \mathbb{R} \setminus \{-a^2, -b^2\}. \tag{8}$$

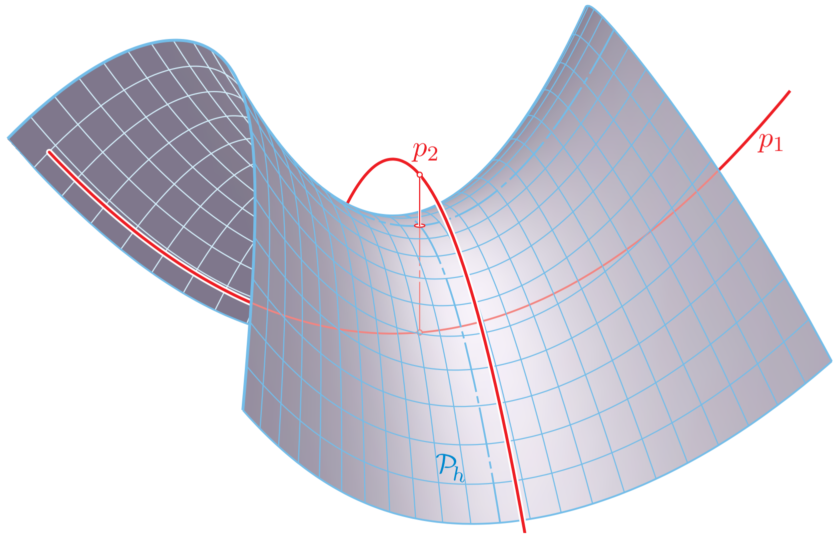

Figure 8: Hyperbolic paraboloid $\mathcal{P}_h$ with lines of curvature and focal parabolas p_1, p_2.

In the case $a > b > 0$, this one-parameter family contains

$$\text{for} \begin{cases} -b^2 < k < \infty & \text{elliptic paraboloids,} \\ -a^2 < k < -b^2 & \text{hyperbolic paraboloids,} \\ -\infty < k < -a^2 & \text{elliptic paraboloids.} \end{cases} \tag{9}$$

For each k, the vertex of the corresponding paraboloid has the coordinates $(0, 0, -k/2)$. The point $(0, 0, b^2/2)$ is the common focal point of the principal sections in the plane $x = 0$, and $(0, 0, a^2/2)$ is the focus for $y = 0$.

It is not hard to prove that most of the properties of confocal central quadrics, in particular those reported in Lemmas 1 and 2 are also valid for confocal paraboloids.

In the family (8) of confocal paraboloids, the limits for $k \to -a^2$ or $k \to -b^2$ define the pair of *focal parabolas*

$$p_1\colon\ \frac{y^2}{a^2 - b^2} + 2z - a^2 = 0,\ x = 0\,, \quad p_2\colon\ \frac{x^2}{a^2 - b^2} - 2z + b^2 = 0,\ y = 0 \tag{10}$$

within the family of confocal quadrics (Fig. 8). The vertex of each focal parabola coincides with the focal point of the other parabola. Therefore, this pair is the same as shown in [4, Fig. 4.15]: each parabola is the locus of apices of cones of revolution passing through the other parabola.

4.1 First thread construction fails for parabolas

Now we discuss thread constructions for paraboloids. An analogue of the first construction according to Theorem 3 should be based on the two focal parabolas p_1 and p_2. By virtue of Lem. 4, we need to find lines which meet both parabolas simultaneously.

From Lemma 1 follows that through each point P outside the planes of symmetry there pass four lines $t_0, \dots, t_3$ meeting both focal parabolas p_1 and p_2. One of them, say t_0, is parallel to the common axis of the parabolas. Therefore, the remaining transversals t_1, t_2, t_3 can be obtained respectively by reflecting t_0 in the tangent planes of the three confocal paraboloids

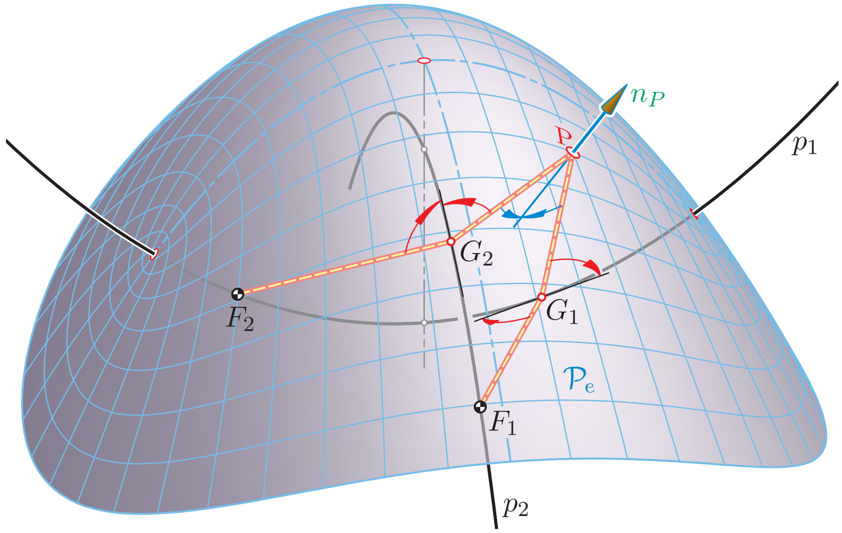

Figure 9: A thread construction based on a pair (p_1, p_2) of focal parabolas fails for the elliptic paraboloid $\mathcal{P}_e$, since Lemma 4,(ii) is not satisfied at the point G_2.

through P, the elliptic paraboloid $\mathcal{P}_1$ $(k = k_1)$, the hyperbolic paraboloid $\mathcal{P}_2$ $(k = k_2)$, and the elliptic paraboloid $\mathcal{P}_3$ $(k = k_3)$, where (k_1, k_2, k_3) are the elliptic coordinates of P with $k_3 < -a^2 < k_2 < -b^2 < k_1$. The normal lines n_i, $i = 1, 2, 3$, to the three paraboloids at P are the common axes of symmetry of the cones connecting P with the two focal parabolas p_1 and p_2.

Theorem 9. *The first thread construction fails for paraboloids. In the case of an elliptic paraboloid $\mathcal{P}_e$ ($k = k_1$ or $k = k_3$, see Fig. 9), the strengthened thread $F_1G_1PG_2F_2$ with $F_2, G_1 \in p_1$ and $F_1, G_2 \in p_2$ does not satisfy the second condition of Lemma 4, i.e., the normal plane of p_2 at G_2 does not separate the two adjacent segments G_2F_2 and G_2P.*
At a hyperbolic paraboloid (see Fig. 11), the surface normal n_P to $\mathcal{P}_h$ at P is the exterior angle bisector of $\angle G_1PG_2$.

Proof. We focus on the plane $\sigma\colon y = 0$ containing the focal parabola p_2. The images of points and lines under the orthogonal projection into σ are called front views and indicated by a prime. Hence, the point $P = (\xi, \eta, \zeta)$ has the front view $P' = (\xi, 0, \zeta) \in \sigma$. We assume that P is outside of the symmetry planes of the confocal parabolas; because of the symmetries of paraboloids, we may confine us to the case $\xi, \eta > 0$. The other focal parabola in the plane $x = 0$ appears in the front view as a half-line p_1' (Fig. 10).
The common transversals $t_0, \dots, t_3$ from P to p_1 and p_2 intersect σ at the points $T_0, \dots, T_3 \in p_2$, where T_0 is the ideal point of the z-axis. The diagonal triangle $N_1N_2N_3$ of the quadrangle $T_0 \dots T_3$ consists of the trace points of the surface normals n_1, n_2, n_3 to the paraboloids $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ through P. Therefore for each $i \in \{1, 2, 3\}$, the points $T_i \in p_2$ and N_i in σ have the same x-coordinate (Fig. 10). The meeting points S_i of t_i with the focal parabola p_1 in the plane $x = 0$ have their front views S_i' on the connections $[P', T_i]$, where P' is the orthocenter of the triangle $N_1N_2N_3$.
For each $i \in \{1, 2, 3\}$, the surface normal n_i at P to the paraboloid $\mathcal{P}_i$ with the elliptic coordinate $k = k_i$ has the direction of the vector

$$\mathbf{n}_i = \left(\frac{\xi}{a^2 + k_i}, \frac{\eta}{b^2 + k_i}, -1 \right)$$

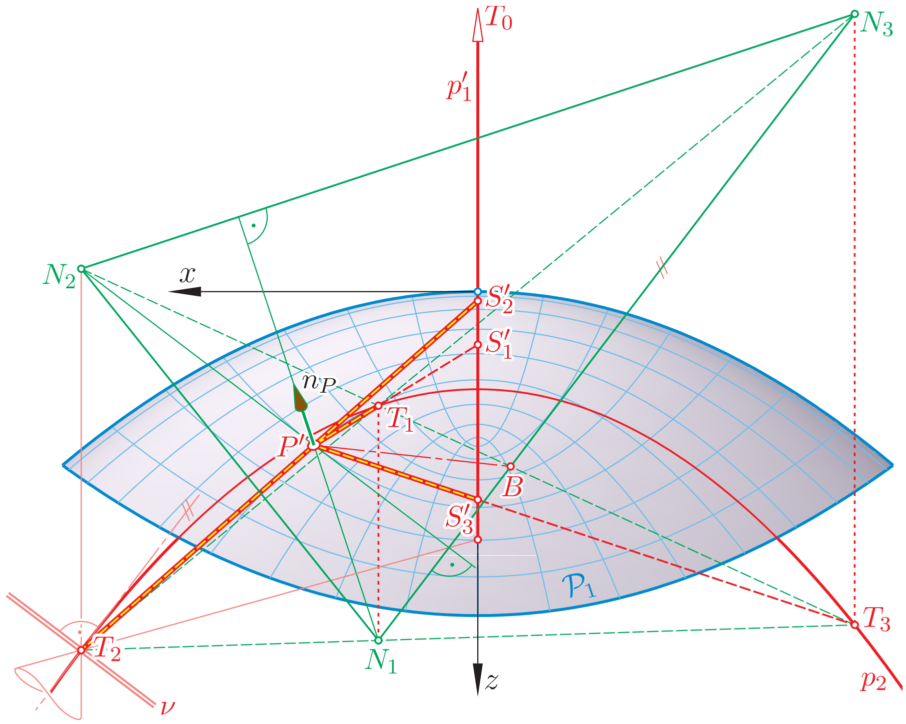

Figure 10: The transversals t_1, t_2, t_3 through P to the pair of focal parabolas meet p_1 at S_1, S_2, S_3 and p_2 at T_1, T_2, T_3, respectively. The points N_1, N_2, N_3 in the plane $x = 0$ of p_2 are the trace points of the surface normals at P to the three paraboloids $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$.

and intersects the plane $\sigma\colon y = 0$ at

$$N_i = \left(\xi\,\frac{a^2 - b^2}{a^2 + k_i},\, 0,\, \zeta + b^2 + k_i\right), \quad k_3 < -a^2 < k_2 < -b^2 < k_1.$$

This confirms that for each i there is an affine transformation $P' \mapsto N_i$ (see [7, p. 70]). We obtain the common x-coordinate of N_i and T_i from the x-coordinate of P' by multiplication with a constant factor λ_i, where

$$0 < \lambda_1 = \frac{a^2 - b^2}{a^2 + k_1} < 1, \quad \lambda_2 = \frac{a^2 - b^2}{a^2 + k_2} > 1, \quad \lambda_3 = \frac{a^2 - b^2}{a^2 + k_3} < 0.$$

Therefore, T_3 has a negative x-coordinate, while that of T_1 lies between 0 and ξ and that of T_2 is $> \xi$ (note Fig. 10).
For a thread construction of the elliptic paraboloid $\mathcal{P}_1$ through P, the middle part, i.e., the two segments terminated by P, must end on two different focal parabolas and span a plane through the surface normal n_1 of $\mathcal{P}_1$. Moreover, it is forbidden that the segment with endpoint on p_i hits the other parabola in between, and the meeting point with p_i must be finite. Consequently, the middle part for $\mathcal{P}_1$ is T_2PS_3, where T_2 is left of P' (Fig. 10). The segment $T_2S_2 \subset t_2$ as well as the final segment connecting T_2 with any point $F_2 \in p_1$ lies on the cone of revolution connecting T_2 with p_1. However, there is no point $F_2 \in p_1$ which is separated from P by the normal plane ν to p_2 at T_2. This follows since in the plane σ the normal lines of p_2 meet the z-axis at points with z-coordinates bigger than that of the focus of p_2 and vertex of p_1 (note [4, p. 145]). Thus, the strengthened thread for the elliptic

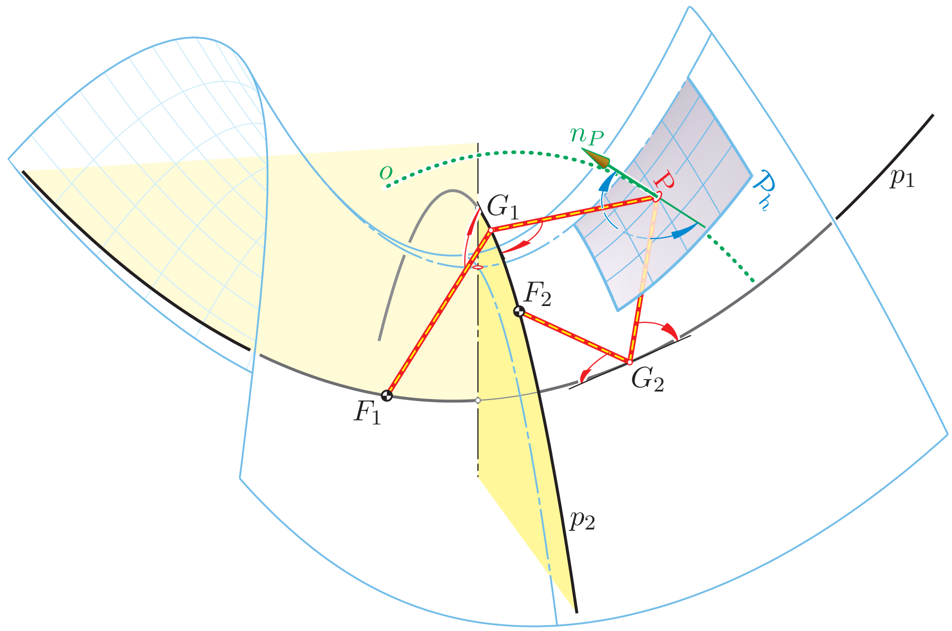

Figure 11: The difference of the threads' lengths $(\overline{PG_1}+\overline{G_1F_1})-(\overline{PG_2}+\overline{G_2F_2})$ remains constant while point P moves locally on the hyperbolic paraboloid $\mathcal{P}_h$ (Corollary 10). For P running on the orthogonal trajectory o (dotted green line) of the confocal hyperbolic paraboloids, the total length of the strenghtened thread $F_1G_1PG_2F_2$ with fixed endpoints $F_1 \in p_1$ and $F_2 \in p_2$ remains constant (Remark 3).

paraboloid $\mathcal{P}_1$ cannot satisfy the condition Lemma 4,(ii) at the point T_2. In the notation used in Figs. 1, 5, and 6, T_2 stands for the point G_2 sliding on p_2 (note Fig. 9).

The case of the elliptic paraboloid $\mathcal{P}_3$ can be reduced to the previous case, because the isometry

$$(x,y,z)\mapsto(x',y',z')=(y,x,-z) \quad \text{together with} \quad k':=-k-a^2-b^2$$

exchanges the elliptic paraboloids for $k<-a^2$ with those for $k'>-b^2$.

For obtaining a thread construction of type 1 for the hyperbolic paraboloid $\mathcal{P}_2$, the middle part of the strengthened thread consists of the segments T_1P and PS_3 in the plane with the trace $[T_1,T_3]$ through N_2 in σ (Fig. 10). Obviously, the segment PN_2 lies on the exterior angle bisector of $\angle T_1PS_3$, since N_2 is outside of T_1T_3, while the interior angle bisector meets the trace $[T_1,T_3]$ at the fourth harmonic conjugate B of N_2 w.r.t. T_1 and T_3. This implies that, for P varying locally on $\mathcal{P}_2$, the difference of thread lengths F_1T_1P and F_2S_3P remains constant, wherever the fixed end points $F_1\in p_1$ and $F_2\in p_2$ are specified (compare with Fig. 6). Conversely, a constant difference implies a tangent vector of P along the interior angle bisecting plane of $\angle T_1PS_3$, which contacts $\mathcal{P}_2$ (Fig. 11, where G_1 stands for T_1, G_2 for S_3, and $\mathcal{P}_h$ for $\mathcal{P}_2$). □

Corollary 10. *Let P_0 be a point of a hyperbolic paraboloid $\mathcal{P}_h$ with focal parabolas p_1 and p_2. If two strengthed threads $F_iG_iP_0$, $i=1,2$, are fixed at any $F_i\in p_i$ and passed around the other parabola p_j $(j\neq i)$, at the point G_i, then for all points $P\in\mathcal{P}_h$ sufficiently close to P_0 the difference of lengths of the two threads, i.e.,*

$$(\overline{PG_1}+\overline{G_1F_1})-(\overline{PG_2}+\overline{G_2F_2})$$

remains constant (Fig. 11). Conversely, if this difference of lengths is kept constant, then the point P moves on $\mathcal{P}_h$.

Remark 3. If point P varies on the orthogonal trajectory o of the confocal hyperbolic parabolas (note Fig. 11), then the interior angle bisector $[P, B]$ (Fig. 10) of $\angle T_1PS_3$ is orthogonal to o, which yields a constant length of the strengthened thread. However, the thread of fixed length does not define a constrained motion of P but admits two degrees of freedom; at P the tangent plane to the trajectory is orthogonal to $[P, B]$. Notice that o is a line of curvature on confocal elliptic paraboloids.

4.2 Second thread construction for paraboloids

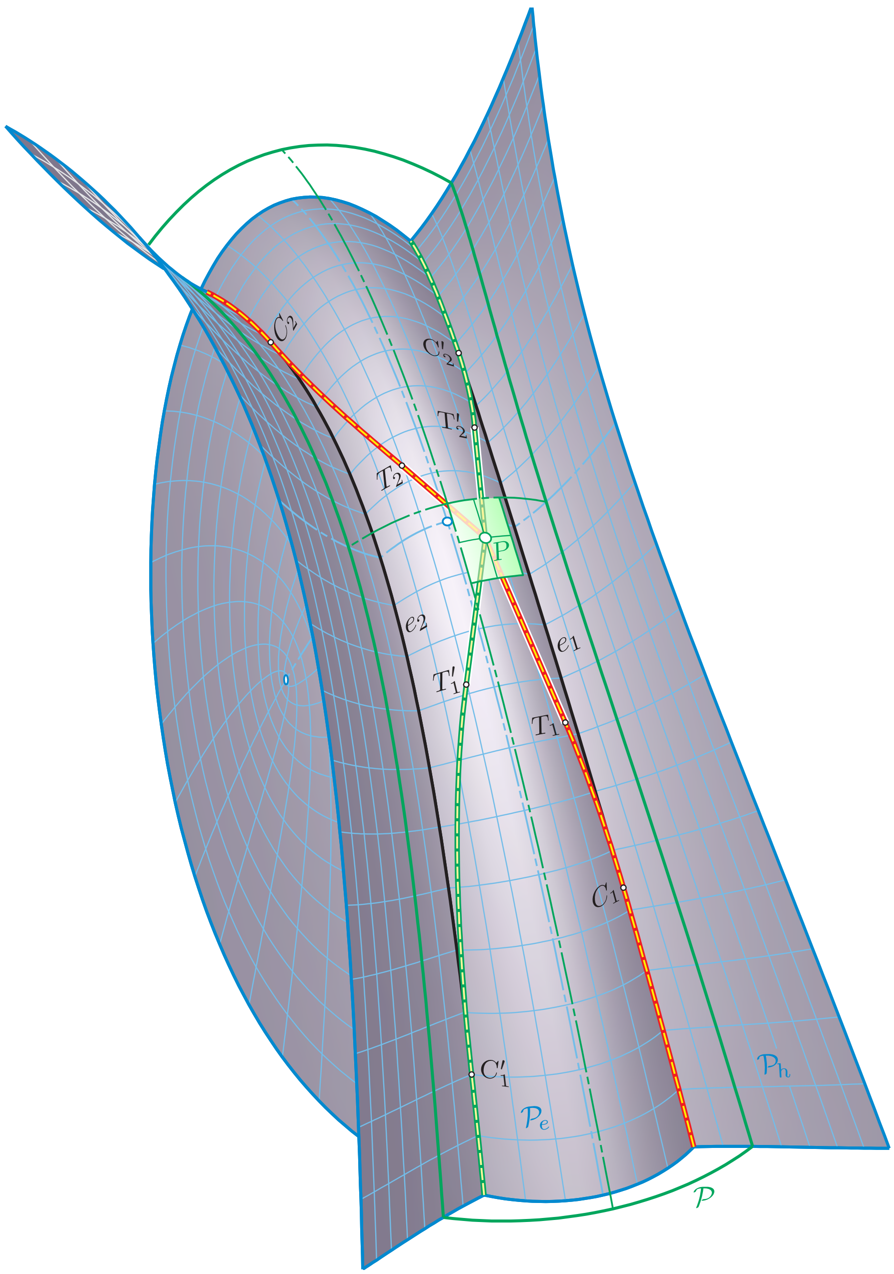

Figure 12: Thread construction of the elliptic paraboloid $\mathcal{P}$.

On the other hand, Staude's second thread construction remains valid for paraboloids. Fig. 12 shows a thread of fixed length with both ends attached to the two connected components e_1, e_2 of the line of curvature that is shared by the confocal paraboloids $\mathcal{P}_e$ and $\mathcal{P}_h$. If this thread is strengthened at the point P, then P is movable on an elliptic paraboloid $\mathcal{P}$ that is confocal with $\mathcal{P}_e$ and $\mathcal{P}_h$. The proof is the same as that for Theorem 7 when, for $i = 1, 2$, the points C_i on the thread denote the endpoints of the subarcs along the lines of curvature, while T_iP are the straight segments tangent to the geodesics at T_i.

If conversely point P moves locally on the elliptic paraboloid $\mathcal{P}$, then the thread remains strengthened, because n_P is the interior angle bisector of $\angle T_1PT_2$. There are even two possibilities for this thread (Fig. 12) since the four common tangents from P to $\mathcal{P}_e$ and $\mathcal{P}_h$ consist of two pairs of lines which are symmetric w.r.t. the surface normal n_P of $\mathcal{P}$ at P (note Lemma 1). We summarize:

Theorem 11. *Staude's second thread construction, as explained in Theorem 7 for triaxial ellipsoids, works similar for elliptic paraboloids, when the two ends of the thread are attached to different components of the intersection curve between confocal elliptic and hyperbolic paraboloids (Fig. 12).*

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