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Generalizations of Fagnano's Problem

Tran Quang Hung[1], Nguyen Thi Thuy Duong[2]

[1]*High School for Gifted Students, Vietnam National University, Vietnam*
tranquanghung@hus.edu.vn

[2]*Student at Concord College, Shropshire, United Kingdom*
nt.thuyduong3402@gmail.com

Abstract. We generalize Fagnano's famous problem of minimal inscribed perimeter by replacing the orthocenter with an arbitrary interior point P. By adding weights associated with P to Fagnano's inequality, we show that the new, generalized expression reaches minimum for the pedal triangle of P. We then further generalize our main theorem and derive some extensions by relating them to Fermat-Torricelli problem.

Key Words: Fagnano's inequality, generalized theorem, extremum problem.
MSC 2020: 51M04, 51M16

1 Introduction

In 1775, the Italian mathematician Giovanni Fagnano proposed his famous problem which can be restated as follows: find in the acute-angled triangle ABC the inscribed triangle with the smallest perimeter; see [1, 3, 4, 6] as well as [7, 9]. He then provided an analytic solution, showing that the triangle in question is ABC's orthic triangle.

Theorem 1 (Fagnano, 1775)**.** *Let ABC be an acute-angled triangle. Of all inscribed triangles of ABC, its orthic triangle has the smallest perimeter (See Figure 1).*

Ever since Fagnano, multiple other proofs have been discovered using a wide range of tools from geometry to physics. A classic example of a geometric proof is that by Lipot Fejér, using reflections and isosceles triangles; see [1, 3].

In this paper, similar to the approach taken in [2], we generalize Fagnano's problem by adding weights to Fagnano's inequality. We shall substitute orthocenter H with any point P, then add the inverse circumradii of triangles PBC, PCA, and PAB as weights.

Theorem 2 (Generalization of Fagnano's problem)**.** *Let P be an interior point of given triangle ABC. Denote by R_a, R_b, and R_c the circumradii of triangles PBC, PCA, and PAB*

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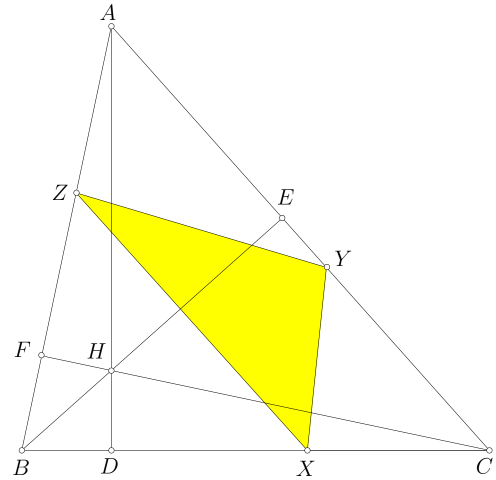

Figure 1: Orthic triangle DEF of triangle ABC and inscribed triangle XYZ.

respectively. Let X, Y, and Z be points on the lines BC, CA, and AB respectively. Then the value of the expression

$$\frac{YZ}{\mathrm{R}_a} + \frac{ZX}{\mathrm{R}_b} + \frac{XY}{\mathrm{R}_c} \tag{1}$$

attains minimum value if and only if XYZ is the pedal triangle of P with respect to triangle ABC.

The original Fagnano's problem can easily be spotted in the particular case where triangle ABC is acute-angled and P coincides with the orthocenter of triangle ABC. Then, R_a, R_b, and R_c are equal to R (circumradius of triangle ABC), and Equation (1) becomes $\frac{XY+YZ+ZX}{\mathrm{R}}$. According to Theorem 2, since R is a constant, the expression $\frac{XY+YZ+ZX}{\mathrm{R}}$ or $XY+YZ+ZX$ attains minimum value if and only if XYZ is the pedal triangle of orthocenter H with respect to triangle ABC.

In the last section of this paper, we will also present some extensions of the lemma and main theorem.

2 Proof of main theorem

We start with a lemma which helps integrate areas and later circumradii into the problem.

Throughout this section, we denote the distance between points A and B simply by AB (which is not misleading if we also denote the line passing through two points A and B as AB). The circumcircle of the triangle PQR will be denoted by (PQR).

Lemma 1. *Let P be an interior point of triangle ABC. Denote by S_a, S_b, and S_c the areas of triangles PBC, PCA, and PAB respectively. Let R be the second intersection of PA and (PBC), and M be an arbitrary point in this plane. Then*

$$S_a \cdot PA \cdot MA + S_b \cdot PB \cdot MB + S_c \cdot PC \cdot MC \geq S_a \cdot PA \cdot AR$$

and equality holds if and only if M coincides with P.

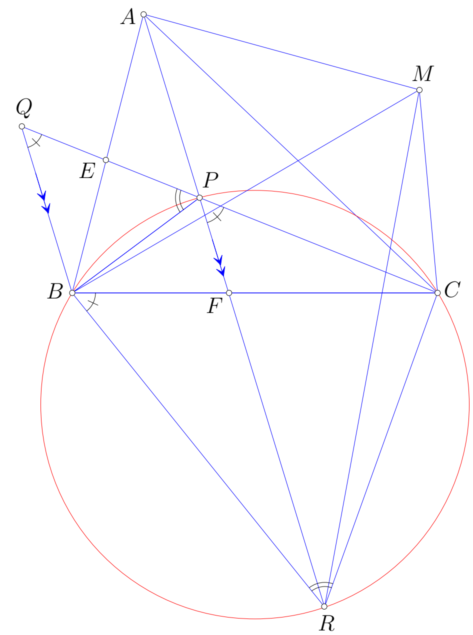

Figure 2: Proof of Lemma 1

Proof. (See Figure 2) Let E be the intersection of lines PC and AB, and F be the intersection of lines PA and BC. Point Q lies on PC such that BQ is parallel to AP. By the standard formula for the area of a triangle, we have

$$\frac{EA}{EB} = \frac{S_b}{S_a} \quad \text{and} \quad \frac{FB}{FC} = \frac{S_c}{S_b}.$$

Since triangles EBQ and EAP are similar, we have $\frac{PA}{BQ} = \frac{EA}{EB} = \frac{S_b}{S_a}$, giving

$$\frac{S_a \cdot PA}{BQ} = S_b. \tag{2}$$

Since triangles CBQ and CFP are similar, we have $\frac{PQ}{PC} = \frac{FB}{FC} = \frac{S_c}{S_b}$, so

$$\frac{S_c \cdot PC}{PQ} = S_b. \tag{3}$$

It follows from (2) and (3) that

$$\frac{S_a \cdot PA}{BQ} = \frac{S_b \cdot PB}{PB} = \frac{S_c \cdot PC}{PQ}. \tag{4}$$

As P is an inner point of triangle ABC, it is obvious that F is an inner point of segment BC. Therefore P and R are at different sides of line BC or the quadrilateral $PBCR$ is convex and cyclic. Using the Theorem of angles at circumference and that $\angle BPQ + \angle BPC = 180°$, we have

$$\angle BPQ = \angle BRC \tag{5}$$

and

$$\angle BQP = \angle RPC = \angle RBC. \tag{6}$$

By (5) and (6), triangles PBQ and RCB are similar, therefore

$$\frac{QB}{BC} = \frac{PB}{CR} = \frac{PQ}{RB}. \tag{7}$$

Combining with (4) gives

$$\frac{S_a \cdot PA}{BC} = \frac{S_b \cdot PB}{CR} = \frac{S_c \cdot PC}{RB}. \tag{8}$$

Using Ptolemy's inequality for point M and triangle RBC, we have

$$MR \cdot BC \leq MB \cdot RC + MC \cdot RB. \tag{9}$$

From (8) and (9), we obtain

$$S_a \cdot PA \cdot MR \leq S_b \cdot PB \cdot MB + S_c \cdot PC \cdot MC. \tag{10}$$

Addition of $S_a \cdot PA \cdot MA$ to both sides of (10) gives

$$S_a \cdot PA \cdot (MR + MA) \leq S_a \cdot PA \cdot MA + S_b \cdot PB \cdot MB + S_c \cdot PC \cdot MC. \tag{11}$$

Using the triangle inequality, we have

$$S_a \cdot PA \cdot MA + S_b \cdot PB \cdot MB + S_c \cdot PC \cdot MC \geq S_a \cdot PA \cdot AR. \tag{12}$$

The right hand side of the inequality (12) is a constant, and equality holds if and only if A, R, and M are collinear and quadrilateral $MBRC$ is cyclic. In other words, equality holds if and only if M coincides with P. Thus, the value of the expression

$$S_a \cdot PA \cdot MA + S_b \cdot PB \cdot MB + S_c \cdot PC \cdot MC$$

attains minimum value if and only if M coincides with P. □

We now present a synthetic proof of Theorem 2 by using Lemma 1 and Miquel's Theorem [5].

Proof of Theorem 2. (See Figure 2) Since X, Y, and Z lie on the lines BC, CA, and AB respectively, using Miquel's theorem [5], circles (AYZ), (BZX), and (CXY) have a common point M.

Denote by d_a the diameter length of (AYZ) and R the radius of (ABC). By the law of sines, we have

$$YZ = d_a \cdot \sin A = d_a \cdot \frac{BC}{2\mathrm{R}}. \tag{13}$$

Hence,

$$\frac{YZ}{\mathrm{R}_a} = \frac{d_a \cdot \frac{BC}{2\mathrm{R}}}{\frac{PB \cdot PC \cdot BC}{4S_a}} = \frac{2d_a \cdot S_a}{\mathrm{R} \cdot PB \cdot PC}. \tag{14}$$

Since MA is a chord of (AYZ),

$$d_a \geq MA \tag{15}$$

and equality occurs iff AM is diameter of (AYZ), in other words Y and Z are orthogonal projections of M on sides CA and AB respectively. From (14) and (15), we get

$$\frac{YZ}{\mathrm{R}_a} \geq \frac{2S_a \cdot MA \cdot PA}{\mathrm{R} \cdot PA \cdot PB \cdot PC}. \tag{16}$$

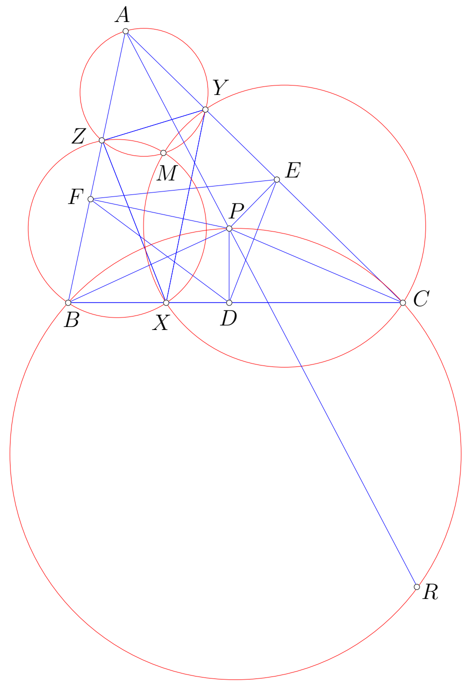

Figure 3: Proof of Theorem 2

Similarly, we get the same inequalities

$$\frac{ZX}{\mathrm{R}_b} \geq \frac{2S_b \cdot MB \cdot PB}{\mathrm{R} \cdot PA \cdot PB \cdot PC} \tag{17}$$

and

$$\frac{XY}{\mathrm{R}_c} \geq \frac{2S_c \cdot MC \cdot PC}{\mathrm{R} \cdot PA \cdot PB \cdot PC}. \tag{18}$$

With the same reason for equality occurs in (15), the equalities of (16), (17), and (18) occur iff XYZ is the pedal triangle of M. Adding (16), (17), and (18), and using Lemma 1 gives

$$\begin{aligned} \frac{YZ}{\mathrm{R}_a} + \frac{ZX}{\mathrm{R}_b} + \frac{XY}{\mathrm{R}_c} &\geq 2\frac{S_a \cdot PA \cdot MA + S_b \cdot PB \cdot MB + S_c \cdot PC \cdot MC}{\mathrm{R} \cdot PA \cdot PB \cdot PC} \\ &\geq \frac{2S_a \cdot PA \cdot AR}{\mathrm{R} \cdot PA \cdot PB \cdot PC} \end{aligned} \tag{19}$$

where R is the second intersection of PA with (PBC). Furthermore,

$$\frac{2S_a \cdot PA \cdot AR}{\mathrm{R} \cdot PA \cdot PB \cdot PC} = \frac{2S_a \cdot AR}{\mathrm{R} \cdot PB \cdot PC} \tag{20}$$

which is a constant. Thus, the right hand side of inequality (19) is a constant, and the equality in (19) holds if and only if M coincides with P, combining with the equalities of (16), (17), and (18) occur iff XYZ is the pedal triangle of M. We deduce that the equality of (19) occurs iff X, Y, and Z coincide with D, E, and F respectively, where DEF is the pedal triangle of P with respect to triangle ABC.

. Therefore, the value of the expression (1) reaches minimum value if and only if XYZ is the pedal triangle of P with respect to triangle ABC. □

3 Some extensions and consequences

Here, we will present some extensions and applications of Lemma 1 and the main theorem.

First, we can regard Lemma 1 as a generalization of Fermat-Torricelli problem [8, 10]. Indeed, let us consider a triangle with angles not exceeding 120° and the Fermat point F. It is conspicuous that $\angle BFC = \angle CFA = \angle AFB = 120°$, and thus the areas of triangles FBC, FCA, and FAB are directly proportional to FA, FB, and FC respectively. If we introduce Lemma 1 here, with $P = F$, then

$$\frac{S_a}{FA} = \frac{S_b}{FB} = \frac{S_c}{FC} = k. \tag{21}$$

As k is a constant, we are left with the problem of finding the minimum value of

$$MA + MB + MC. \tag{22}$$

Equality is reached when M coincides with F. Furthermore, Lemma 1 can be generalized with powers as follows:

Theorem 3. *Let P be a point in the interior of ABC. Denote by S_a, S_b, and S_c the areas of triangles PBC, PCA, and PAB respectively. Let p be a real number no less than 1, and M be an arbitrary point in this plane. Then the value of the expression*

$$S_a \cdot PA^{2-p} \cdot MA^p + S_b \cdot PB^{2-p} \cdot MB^p + S_c \cdot PC^{2-p} \cdot MC^p$$

is minimal if and only if M coincides with P.

Proof. Case $p = 1$, we obtain Lemma 1.

Case $p > 1$. Let R be the intersection of PA and (PBC). Since $p > 1$, let $q = \frac{p}{p-1}$, q is a positive real number and $\frac{1}{p} + \frac{1}{q} = 1$. Holder's inequality transforms this into

$$\begin{aligned}
&\left(S_a \cdot PA^{2-p} \cdot MA^p + S_b \cdot PB^{2-p} \cdot MB^p + S_c \cdot PC^{2-p} \cdot MC^p\right)^{\frac{1}{p}} \\
&= \frac{\left(\sum \left(S_a^{\frac{1}{p}} \cdot PA^{\frac{2-p}{p}} \cdot MA\right)^p\right)^{\frac{1}{p}} \cdot \left(\sum \left(S_a^{\frac{1}{q}} \cdot PA^{\frac{2}{q}}\right)^q\right)^{\frac{1}{q}}}{\left(\sum \left(S_a^{\frac{1}{q}} \cdot PA^{\frac{2}{q}}\right)^q\right)^{\frac{1}{q}}} \\
&\geq \frac{\sum S_a^{\frac{1}{p}+\frac{1}{q}} \cdot PA^{\frac{2-p}{p}+\frac{2}{q}} \cdot MA}{(\sum S_a \cdot PA^2)^{\frac{1}{q}}}.
\end{aligned} \tag{23}$$

Furthermore, using Lemma 1 we have

$$\frac{\sum S_a^{\frac{1}{p}+\frac{1}{q}} \cdot PA^{\frac{2-p}{p}+\frac{2}{q}} \cdot MA}{(\sum S_a \cdot PA^2)^{\frac{1}{q}}} = \frac{\sum S_a \cdot PA \cdot MA}{(\sum S_a \cdot PA^2)^{\frac{1}{q}}} \geq \frac{S_a \cdot PA \cdot RA}{(\sum S_a \cdot PA^2)^{\frac{1}{q}}}. \tag{24}$$

From (23) and (24), we can now observe that

$$S_a \cdot PA^{2-p} \cdot MA^p + S_b \cdot PB^{2-p} \cdot MB^p + S_c \cdot PC^{2-p} \cdot MC^p \geq \left(\frac{S_a \cdot PA \cdot RA}{(\sum S_a \cdot PA^2)^{\frac{p-1}{p}}}\right)^p \tag{25}$$

or

$$S_a \cdot PA^{2-p} \cdot MA^p + S_b \cdot PB^{2-p} \cdot MB^p + S_c \cdot PC^{2-p} \cdot MC^p \geq \frac{(S_a \cdot PA \cdot RA)^p}{(\sum S_a \cdot PA^2)^{p-1}}. \tag{26}$$

From the conditions for equality of Holder's inequality and Lemma 1, it is not a challenge to realize that the equality in (26) is attained when and only when M coincides with P. □

Theorem 3 can also be used to obtain a result as follows:

Consequence 1. *For real number $p \geq 1$, suppose triangle ABC contains a point P which satisfies*

$$\frac{S_a}{PA^{p-2}} = \frac{S_b}{PB^{p-2}} = \frac{S_c}{PC^{p-2}} \tag{27}$$

where S_a, S_b, and S_c denote the areas of triangles PBC, PCA, and PAB respectively. Let M be an arbitrary point in this plane, then the expression

$$MA^p + MB^p + MC^p$$

attains minimum value when and only when M coincides with P.

This result can be considered a generalization for Fermat-Torricelli problem with powers. Another power generalization can be derived for Theorem 2

Theorem 4 (Generalization of Theorem 2 with powers)**.** *Let P be an interior point of a given triangle ABC. Denote by R_a, R_b, and R_c the circumradii of triangles PBC, PCA, and PAB respectively. Let p be a real number no less than 1. Let X, Y, and Z be points on the lines BC, CA, and AB respectively. Then, the value of the expression*

$$\frac{YZ^p}{\mathrm{R}_a \cdot (BC \cdot PA)^{p-1}} + \frac{ZX^p}{\mathrm{R}_b \cdot (CA \cdot PB)^{p-1}} + \frac{XY^p}{\mathrm{R}_c \cdot (AB \cdot PC)^{p-1}} \tag{28}$$

is minimal if and only if XYZ is the pedal triangle of P with respect to triangle ABC.

Proof. Recalling $p \geq 1$ and inequality (16) gives us

$$\left(\frac{YZ}{\mathrm{R}_a}\right)^p \geq \left(\frac{2S_a \cdot MA \cdot PA}{\mathrm{R} \cdot PA \cdot PB \cdot PC}\right)^p. \tag{29}$$

Thus

$$\frac{YZ^p}{\mathrm{R}_a \cdot (BC \cdot PA)^{p-1}} \geq \left(\frac{2S_a \cdot MA \cdot PA}{\mathrm{R} \cdot PA \cdot PB \cdot PC}\right)^p \cdot \frac{\mathrm{R}_a^{p-1}}{(BC \cdot PA)^{p-1}}. \tag{30}$$

Simultaneously, we have

$$\begin{aligned} S_a^{p-1} \cdot PA^{2(p-1)} &= \left(\frac{BC \cdot PB \cdot PC}{4\mathrm{R}_a}\right)^{p-1} \cdot PA^{(p-1)} \cdot PA^{(p-1)} \\ &= \frac{(BC \cdot PA)^{p-1}}{(4\mathrm{R}_a)^{p-1}} \cdot (PA \cdot PB \cdot PC)^{p-1}, \end{aligned} \tag{31}$$

or

$$\frac{\mathrm{R}_a^{p-1}}{(BC \cdot PA)^{p-1}} = \left(\frac{PA \cdot PB \cdot PC}{4}\right)^{p-1} \cdot S_a^{1-p} \cdot PA^{2-2p}. \tag{32}$$

From (30) and (32), it follows that

$$\frac{YZ^p}{\mathrm{R}_a \cdot (BC \cdot PA)^{p-1}} \geq \frac{2^{2-p}}{PA \cdot PB \cdot PC \cdot \mathrm{R}^p} \cdot S_a \cdot PA^{2-p} \cdot MA^p. \tag{33}$$

Now using (26), we obtain

$$\begin{aligned} \sum \frac{YZ^p}{\mathrm{R}_a \cdot (BC \cdot PA)^{p-1}} &\geq \frac{2^{2-p}}{PA \cdot PB \cdot PC \cdot \mathrm{R}^p} \cdot \left(\sum S_a \cdot PA^{2-p} \cdot MA^p\right) \\ &\geq \frac{2^{2-p}}{PA \cdot PB \cdot PC \cdot \mathrm{R}^p} \cdot \frac{(S_a \cdot PA \cdot RA)^p}{(\sum S_a \cdot PA^2)^{p-1}} \end{aligned} \tag{34}$$

where R is the intersection of PA and (PBC). Equality is reached when and only when X, Y, and Z are projections of P on BC, CA, and AB respectively. □

In Theorem 4, if we let $X'Y'Z'$ be the pedal triangle of P in ABC, we notice that

$$\frac{Y'Z'}{BC \cdot PA} = \frac{Z'X'}{CA \cdot PB} = \frac{X'Y'}{AB \cdot PC}$$

which gives rise to the following result

Consequence 2. *Let P be an arbitrary interior point of triangle ABC, and $X'Y'Z'$ its pedal triangle with respect to ABC. Denote by R_a, R_b, and R_c the circumradii of triangles PBC, PCA, and PAB respectively. X, Y, and Z are points on the lines BC, CA, and AB. For real numbers $p \geq 1$, the value of the expression*

$$\frac{YZ^p}{\mathrm{R}_a \cdot Y'Z'^{p-1}} + \frac{ZX^p}{\mathrm{R}_b \cdot Z'X'^{p-1}} + \frac{XY^p}{\mathrm{R}_c \cdot X'Y'^{p-1}} \tag{35}$$

attains a minimum value if and only if $X = X'$, $Y = Y'$, and $Z = Z'$.

From Consequence 2, let triangle ABC be acute and P coincides with its orthocenter then $\mathrm{R}_a = \mathrm{R}_b = \mathrm{R}_c$, we obtain the following consequence

Consequence 3 (Generalization of Fagnano's problem with powers)**.** *Let ABC be a triangle and $X'Y'Z'$ its orthic triangle. Let X, Y, and Z be the points on the lines BC, CA, and AB respectively. For real numbers $p \geq 1$, the value of the expression*

$$\frac{YZ^p}{Y'Z'^{p-1}} + \frac{ZX^p}{Z'X'^{p-1}} + \frac{XY^p}{X'Y'^{p-1}} \tag{36}$$

attains a minimum value if and only if $X = X'$, $Y = Y'$, and $Z = Z'$.

4 Conclusion

By adding powers and specific weights to Fermat-Torricelli and Fagnano's problems, we found the generalized problem of which they are both special cases. We viewed Lemma 1 as a way to add weights to Fermat-Torricelli problem, thereby generalizing it and Fagnano's problem. We also give a general direction using powers for these two well-known geometric extremal problems.

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