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On Pepe Trapezoids

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Abstract. In this article, we explore a special geometric figure, consisting of a trapezoid, the Pepe trapezoid, and an ellipse. The trapezoid is non-rectangular and has integer side lengths. The ellipse is non-circular, inscribed into the trapezoid, and has integer semi-axis lengths. We prove that there are infinitely many non-similar isosceles Pepe trapezoids, right-angled Pepe trapezoids and Pepe trapezoids in the form of parallelogram.

Key Words: trapezoid, ellipse.

MSC 2020: 51M04

1 Introduction

Pepe trapezoids were first described in a problem in Instagram Math Olympiad (IGMO) 2020 Round 2 [2]. A Pepe trapezoid is defined as a non-rectangular trapezoid which has

1. integral side lengths *and*
2. an inscribed ellipse (which is not a circle) with integral lengths of semi-major axis and semi-minor axis, such that the major axis or minor axis of the ellipse is perpendicular to the bases of the trapezoid.

The contestants were asked to prove the existence of infinitely many non-similar Pepe trapezoids in the problem. In this problem, a trapezoid (also referred to as a trapezium in English outside North America) is defined as a convex quadrilateral with at least one pair of parallel sides.

Instagram Math Olympiad is a free to enter online Mathematics Olympiad made by the Instagram Math Community [3]. The problem was proposed by @Pepemath from Instagram, and the name of this special type of trapezoid was given by the author of the problem.

In this article, we explore different types of Pepe trapezoids. We prove that there are infinitely many non-similar isosceles Pepe trapezoids, right-angled Pepe trapezoids and Pepe trapezoids in the form of parallelogram in Section 2, Section 3 and Section 4, respectively. We give a direction to future research in Section 5.

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2 Isosceles Pepe Trapezoids

In this section, we give parametrizations of isosceles Pepe trapezoids, and prove that there are infinitely many non-similar isosceles Pepe trapezoids.

Suppose an ellipse is inscribed in an isosceles trapezoid $ABCD$ with bases AB and CD ($CD > AB$). The lateral sides BC and DA are equal. We denote the lengths of the semi-axis of the ellipse that is perpendicular to the bases of the trapezoid and parallel to the bases of the trapezoid as l_1 and l_2, respectively. Let $AB = s$, $CD = t$, $BC = DA = u$, where s, t, u are positive and $t > s$ (Figure 1). These notations are used throughout the section. The isosceles trapezoid exists if $s + 2u > t$.

We first introduce two important lemmas. One of that states the relationships between the lengths of sides of an isosceles trapezoid and the lengths of semi-axes of its inscribed ellipse, if the inscribed ellipse has an axis, either major axis or minor axis, which is perpendicular to the bases of the trapezoid. The other lemma shows the existence of an inscribed ellipse with one of the axes perpendicular to the bases for any isosceles trapezoids.

Lemma 2.1. *If an ellipse is inscribed in an isosceles trapezoid, and an axis of the ellipse, either the major-axis or the minor-axis, is perpendicular to the bases of the trapezoid, then*

$$l_1 = \frac{\sqrt{4u^2 - (t-s)^2}}{4} \tag{1}$$

and

$$l_2 = \frac{\sqrt{st}}{2}. \tag{2}$$

Proof. By Pythagoras theorem, height of the trapezoid is $\sqrt{u^2 - (\frac{t-s}{2})^2} = \frac{\sqrt{4u^2-(t-s)^2}}{2}$. So $2l_1 = \frac{\sqrt{4u^2-(t-s)^2}}{2}$, rearranging gives (1).

Then apply an affine transformation which dilates/contracts the figure along the axis of the ellipse which is perpendicular to the bases with a scale of $\frac{l_2}{l_1}$. The ellipse is transformed to a circle with radius l_2. Suppose points A, B, C, D are transformed to A', B', C', D', respectively. After the transformation, $A'B' = s$, $C'D' = t$, $B'C' = D'A'$. Moreover, $A'B'C'D'$ is a tangential trapezoid, by Pitot's theorem, $A'B' + C'D' = B'C' + D'A'$. Hence, $B'C' = D'A' = \frac{s+t}{2}$. Consider the height of the transformed trapezoid, using Pythagoras theorem, we have $2l_2 = \sqrt{(\frac{s+t}{2})^2 - (\frac{t-s}{2})^2} = \sqrt{st}$. Rearranging gives (2). □

Lemma 2.2. *For any isosceles trapezoid, there exists an inscribed ellipse with one of the axes perpendicular to its bases.*

Proof. Consider the isosceles trapezoid $ABCD$, where AB and CD are its bases ($CD > AB$). $AB = s$, $CD = t$, $BC = DA = u$, and $t > s$. Name this trapezoid as T_1. From the proof of Lemma 2.1, we know that applying an affine transformation which dilates/contracts T_1 along a line perpendicular to its bases with a scale of $\frac{2\sqrt{st}}{\sqrt{4u^2-(t-s)^2}}$ transforms T_1 to an isosceles trapezoid $A'B'C'D'$, where $A'B'$ and $C'D'$ are its bases ($C'D' > A'B'$). $A'B' = s$, $C'D' = t$, $B'C' = D'A' = \frac{s+t}{2}$. Name this trapezoid as T_2.

Apply an affine transformation which dilates/contracts T_2 along a line perpendicular to its bases with a scale of $\frac{\sqrt{4u^2-(t-s)^2}}{2\sqrt{st}}$ transforms T_2 to T_1. Since the sum of lengths of opposite

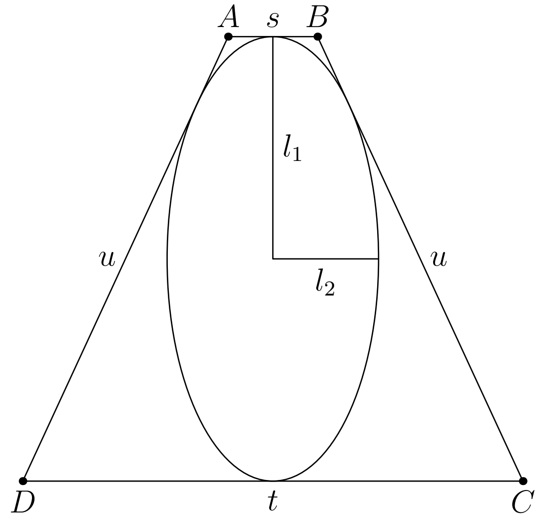

Figure 1: An ellipse inscribed in an isosceles trapezoid

sides of T_2 are equal, by Pitot's theorem, there exists a circle inscribed in T_2. After the transformation, the circle is transformed to an ellipse inscribed in T_1, with one of the axes perpendicular to the bases of T_1. Hence, for any isosceles trapezoid, there exists an inscribed ellipse with one of the axes perpendicular to its bases. □

We then give three different parametrizations of isosceles Pepe trapezoids. For the following trapezoids, the existence of an inscribed ellipse with one of the axes perpendicular to the bases is guaranteed by Lemma 2.2.

Theorem 2.1. *If $u = 18m^2 + 2n^2$, $s = 8mn$, $t = 32mn$, where m and n are positive integers such that $3m \neq n$ and $m \neq n$, then the isosceles trapezoid $ABCD$ is a Pepe trapezoid.*

Proof. $AB = s = 8mn$, $CD = t = 32mn$, $BC = DA = u = 18m^2 + 2n^2$ are obviously integers, hence the trapezoid has integral side lengths.

By Lemma 2.1, substituting $u = 18m^2 + 2n^2$, $s = 8mn$ and $t = 32mn$ into (1) and (2), respectively, gives $l_1 = |9m^2 - n^2|$, $l_2 = 8mn$, which are also integers. Also, it is easy to verify that if $m \neq n$, then $l_1 \neq l_2$, so the inscribed trapezoid is not a circle. Moreover, $s + 2u - t = 8mn + 2(18m^2 + 2n^2) - 32mn = 4(3m - n)^2 > 0$ since $3m \neq n$. Therefore, the isosceles trapezoid $ABCD$ exists and it is a Pepe trapezoid. □

Theorem 2.2. *If $u = 4m^2 + 4n^2$, $s = 2mn$, $t = 18mn$, where m and n are positive integers such that $m \neq n$, $m \neq 2n$ and $n \neq 2m$, then the isosceles trapezoid $ABCD$ is a Pepe trapezoid.*

Proof. Similarly, $AB = s = 2mn$, $CD = t = 18mn$, $BC = DA = u = 4m^2 + 4n^2$ are obviously integers. So the trapezoid has integral side lengths.

By Lemma 2.1, substituting $u = 4m^2 + 4n^2$, $s = 2mn$ and $t = 18mn$ into (1) and (2), respectively, gives $l_1 = 2|m^2 - n^2|$, $l_2 = 3mn$, which are also integers. In addition, it is easy to verify that if $m \neq 2n$ and $n \neq 2m$, then $l_1 \neq l_2$, and therefore the inscribed trapezoid is not a circle. In addition, $s + 2u - t = 2mn + 2(4m^2 + 4n^2) - 18mn = 8(m - n)^2 > 0$ since $m \neq n$. Hence, the isosceles trapezoid $ABCD$ exists and it is a Pepe trapezoid. □

Theorem 2.3. *If $u = 5(2^{2n} - 1)$, $s = 6$, $t = 3 \cdot 2^{2n+1}$, where n is a positive integer such that $n > 1$, then the isosceles trapezoid $ABCD$ is a Pepe trapezoid.*

Proof. $AB = s = 6$, $CD = 3 \cdot 2^{2n+1}$, $BC = DA = u = 5(2^{2n} - 1)$ are integers. So the trapezoid has integral side lengths.

By Lemma 2.1, substituting $u = 5(2^{2n} - 1)$, $s = 6$ and $t = 3 \cdot 2^{2n+1}$ into (1) and (2), respectively, gives $l_1 = 2(2^{2n} - 1)$, $l_2 = 3 \cdot 2^n$, which are also integers. We shall now show that the inscribed trapezoid is not a circle. Assume that it is a circle, then $l_1 = l_2$, then $2(2^{2n} - 1) = 3 \cdot 2^n$. The only solution of the equation is $n = 1$. Hence if $n > 1$, the inscribed ellipse is not a circle. Also, $s + 2u - t = 6 + 2[5(2^{2n} - 1)] - 3 \cdot 2^{2n+1} = 4(2^{2n} - 1) > 0$ if $n > 1$. Therefore, the isosceles trapezoid $ABCD$ exists and it is a Pepe trapezoid. □

By substituting different values of m and n in the parametrization given in Theorem 2.1 and 2.2, or different values of n in Theorem 2.3, we can get the following result.

Theorem 2.4. *There exists infinitely many non-similar isosceles Pepe trapezoids.*

3 Right-angled Pepe Trapezoids

In this section, we prove that there exists infinitely many non-similar right-angled Pepe trapezoids.

Suppose an ellipse is inscribed in an isosceles trapezoid $ABCD$ with bases AB and CD ($CD > AB$). $\angle ADC$ is a right-angle. We denote the lengths of the semi-axis of the ellipse that is perpendicular to the bases of the trapezoid and parallel to the bases of the trapezoid as l_1 and l_2, respectively. Let $AB = s$, $CD = t$, $DA = u$, $BC = v = \sqrt{u^2 + (t-s)^2}$ (Figure 2) where s, t, u are positive and $t > s$. These notations are used throughout the section. Note that under these constrains, this right-angled trapezoid always exists.

We first give a useful lemma which states the relationships between the lengths of sides of an right-angled trapezoid and the lengths of semi-axes of its inscribed ellipses, if the right-angled ellipses have an axis, either major axis or minor axis, which is perpendicular to the bases of the trapezoid. We then show the existence of an inscribed ellipse with one of the axes perpendicular to the bases for any right-angled trapezoids.

Lemma 3.1. *If an ellipse is inscribed in a right-angled trapezoid, and an axis of the ellipse, either the major-axis or the minor-axis, is perpendicular to the bases of the trapezoid, then*

$$l_1 = \frac{u}{2} \tag{3}$$

and

$$l_2 = \frac{st}{s+t}. \tag{4}$$

Proof. It is obvious that the length of axis that is perpendicular to the bases of the trapezoid is equal to height of the trapezoid, so $l_1 = \frac{u}{2}$.

Apply an affine transformation which dilates/contracts the figure along the axis of the ellipse which is perpendicular to the bases with a scale of $\frac{l_2}{l_1}$. The ellipse is transformed to a circle with radius l_2. Suppose points A, B, C, D are transformed to A', B', C', D', respectively. After the transformation, $A'B' = s$, $C'D' = t$, $D'A' = u'$, $B'C' = v'$. Note that $u' = 2l_2$. Moreover, $A'B'C'D'$ is a tangential trapezoid, by Pitot's theorem, $A'B' + C'D' = B'C' + D'A'$. Hence, $s + t = 2l_2 + \sqrt{(2l_2)^2 + (t-s)^2}$. Solving gives (4). □

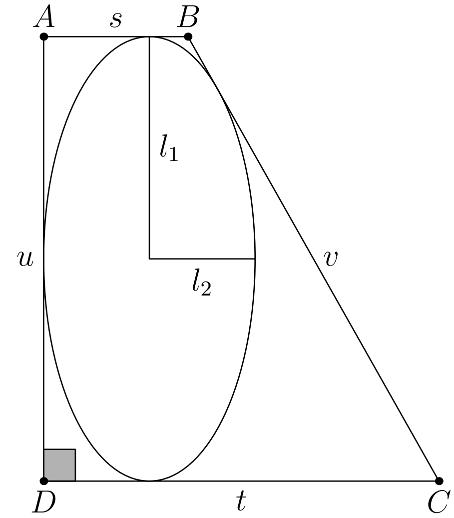

Figure 2: An ellipse inscribed in a right-angled trapezoid

Lemma 3.2. *For any right-angled trapezoid, there exists an inscribed ellipse with one of the axes perpendicular to its bases.*

Proof. Consider the right-angled trapezoid $ABCD$, with bases AB and CD ($CD > AB$). $\angle ADC$ is a right-angle. $AB = s$, $CD = t$, $DA = u$, $BC = v = \sqrt{u^2 + (t-s)^2}$. Name this trapezoid as T_1. The proof of Lemma 3.2 shows that applying an affine transformation which dilates/contracts T_1 along a line perpendicular to its bases with a scale of $\frac{2st}{u(s+t)}$ transforms T_1 to a right-angled trapezoid $A'B'C'D'$, where $A'B'$ and $C'D'$ are its bases ($C'D' > A'B'$). $\angle A'D'C'$ is a right-angle. $A'B' = s$, $C'D' = t$, $D'A' = \frac{2st}{s+t}$. $B'C' = \sqrt{(\frac{2st}{s+t})^2 + (t-s)^2} = \frac{s^2+t^2}{s+t}$. Name this trapezoid as T_2.

Apply an affine transformation which dilates/ contracts T_2 along a line perpendicular to its bases with a scale of $\frac{u(s+t)}{2st}$ transforms T_2 to T_1. Note that the sum of lengths of opposite sides of T_2 are equal. Hence, by Pitot's theorem, there exists a circle inscribed in T_2. After the transformation, the circle is transformed to an ellipse inscribed in T_1, with one of the axes perpendicular to the bases of T_1. Therefore for any right-angled trapezoid, there exists an inscribed ellipse with one of the axes perpendicular to its bases. □

Theorem 3.1. *There exists infinitely many non-similar right-angled Pepe trapezoids.*

Proof. Note that it suffices to prove that there exists infinitely many non-similar non-rectangular trapezoids that have rational side lengths, and an ellipse with rational lengths of the semi-major axis and semi-minor axis can be inscribed in the trapezoid such that the major axis or minor axis of the ellipse is perpendicular to the trapezoid. This is because we can always scale up the trapezoid by a factor to turn the rational lengths to integral lengths.

By Lemma 3.2, there exists an ellipse inscribed in the trapezoid with one of the axes perpendicular to its bases. By Lemma 3.1, if u, s, t are rational, then l_1 and l_2 are rational. Hence, we just have to prove that there exists infinitely many non-similar right-angled trapezoids with rational side lengths, i.e. infinitely many rational solutions which satisfy the equation $(t-s)^2 + u^2 = v^2$.

Choose any rational t and s, where $t > s$. Then choose rational m and n (where $m > n$) such that $t - s = 2mn$. Put $u = m^2 - n^2$, $v = m^2 + n^2$. Then s, t, u, v are rational numbers

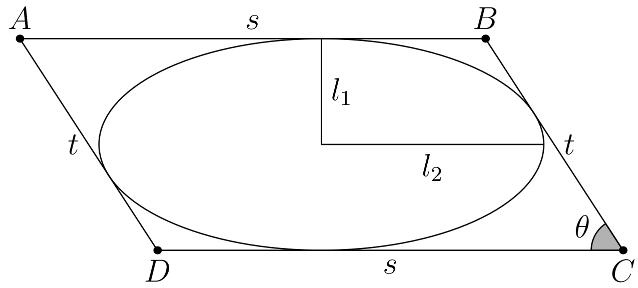

Figure 3: An ellipse inscribed in a parallelogram

that satisfy the equation. Note that this is the well-known solution for Pythagoras equation. Hence, by putting in different rational s, t, m and n, such that $\frac{st}{s+t}$ and $\frac{u}{2}$ are not equal, and then scaling up the trapezoid appropriately, we can generate infinitely many non-similar Pepe trapezoids. □

4 Pepe Trapezoids in the Form of Parallelograms

In this section, we prove that there exist infinitely many Pepe trapezoids in the form of parallelograms. Note that by definition, parallelograms are special types of trapezoids.

Suppose an ellipse is inscribed in a parallelogram $ABCD$ with bases AB and CD ($AB = CD$, AB and CD are parallel). Also, $BC = AD$, BC and AD are parallel. Let $AB = CD = s$, $BC = AD = t$, where s and t are positive. $\angle BCD = \theta$, where $0 < \theta < \pi$. We denote the lengths of the semi-axis of the ellipse that is perpendicular to the bases of the trapezoid and parallel to the bases of the trapezoid (AB and CD are considered as bases of the trapezoid) as l_1 and l_2, respectively (Figure 3). These notations are used throughout the section.

We give a parametrization of Pepe trapezoids in the form of parallelograms.

Theorem 4.1. *If $s = 8n$, $t = 2(n^2+3)$ and $\theta = \sin^{-1}(\frac{n^2-3}{n^2+3})$, where n is a positive integer such that $n > 1$ and $n \neq 3$, then parallelogram $ABCD$ is a Pepe trapezoid.*

Proof. We shall prove by introducing a Cartesian coordinate system. In order to satisfy the conditions in the theorem, let the coordinates of point A, B, C, D be $(-2(2+\sqrt{3})n, n^2-3)$, $(2(2-\sqrt{3})n, n^2-3)$, $(2(2+\sqrt{3})n, -(n^2-3))$, $(-2(2-\sqrt{3})n, -(n^2-3))$ respectively.

The equations of line AB, BC, CD and DA are $y = n^2-3$, $\frac{1}{4n}x + \frac{\sqrt{3}}{2(n^2-3)}y = 1$, $y = -(n^2-3)$, $\frac{1}{4n}x + \frac{\sqrt{3}}{2(n^2-3)}y = -1$ respectively. Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = 2n$ and $b = n^2-3$, then line AB, BC, CD and DA touch the ellipse at one point only. So the ellipse is inscribed in the parallelogram.

$s = 8n$, $t = 2(n^2+3)$, $l_1 = a = 2n$, $l_2 = b = n^2-3$ are all integers. Also, note that if $n \neq 3$, then $l_1 \neq l_2$. So the parallelogram $ABCD$ is a Pepe trapezoid. □

By substituting different values of n in the parametrization given in Theorem 4.1, we have the following result.

Theorem 4.2. *There exists infinitely many Pepe trapezoids in the form of parallelograms.*

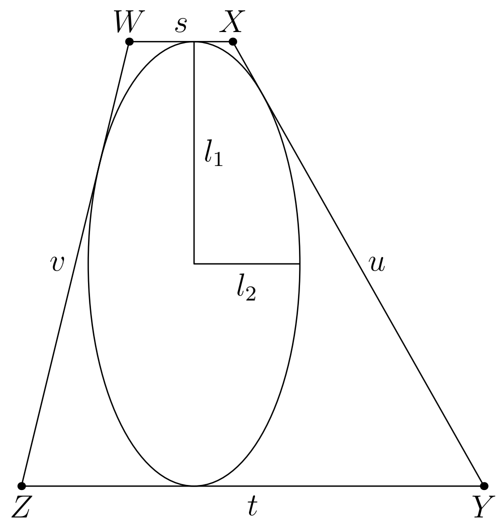

Figure 4: An ellipse inscribed in an arbitary trapezoid

5 Opportunities of Future Research

After the investigations on isosceles Pepe trapezoids, right-angled Pepe trapezoids and Pepe trapezoids in the form of parallelograms, it is natural for us to ask the following question.

Problem. Are there any parametrizations that give Pepe trapezoids that are not isosceles Pepe trapezoids, right-angled Pepe trapezoids or Pepe trapezoids in the form of parallelograms?

One of the approach of tackling this open question is to consider the following theorem proposed by the author in [1].

Theorem 5.1. *Consider an ellipse inscribed in trapezoid $WXYZ$ with bases WX and YZ ($YZ > WX$). Denote the lengths of the semi-axis of the ellipse that is perpendicular to the bases of the trapezoid and parallel to the bases of the trapezoid as l_1 and l_2, respectively. Let the lengths of WX, XY, YZ, ZW be s, v, t, u, respectively (Figure 4). Let $w = t - s$, $z = \frac{u+v+w}{2}$, $\alpha = \frac{u^2-v^2+w^2}{2w}$, $\beta = \frac{v^2-u^2+w^2}{2w}$, $\gamma = s + t$, $\delta = \frac{\alpha+\beta+\gamma}{2}$. If an axis of the ellipse, either the major-axis or the minor-axis, is perpendicular to the bases of the trapezoid, then*

$$l_1 = \frac{\sqrt{z(z-u)(z-v)(z-w)}}{w}. \tag{5}$$

and

$$l_2 = \frac{\sqrt{-\delta(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}}{\gamma}. \tag{6}$$

Solving the Diophantine equations (5) and (6) such that s, t, u, v, l_1 and l_2 are positive integers will allow us to construct a Pepe trapezoid.

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