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Space Kinematics and Projective Differential Geometry Over the Ring of Dual Numbers*

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Abstract. We study an isomorphism between the group of rigid body displacements and the group of dual quaternions modulo the dual number multiplicative group from the viewpoint of differential geometry in a projective space over the dual numbers. Some seemingly weird phenomena in this space have lucid kinematic interpretations. An example is the existence of non-straight curves with a continuum of osculating tangents which correspond to motions in a cylinder group with osculating vertical Darboux motions. We also suggest geometrically meaningful ways to select osculating conics of a curve in this projective space and illustrate their corresponding motions. Furthermore, we investigate factorizability of these special motions and use the obtained results for the construction of overconstrained linkages.

Key Words: rational motion, motion polynomial, factorization, vertical Darboux motion, helical motion, osculating line, osculating conic, null cone motion, linkage

MSC 2020: 16S36 (primary), 53A20, 70B10

1 Introduction

The eight-dimensional real algebra $\mathbb{DH}$ of dual quaternions provides a well-known model for the group SE(3) of rigid body displacements. Dual quaternions with non-zero real norm represent elements of SE(3) and are uniquely determined up to real scalar multiples. In the projectivization $\mathbb{P}(\mathbb{DH}) \cong \mathbb{P}^7(\mathbb{R})$ they correspond to points of the Study quadric $\mathcal{S}$ minus an exceptional subspace E of dimension three [5].

The Study quadric model provides a rich geometric and algebraic environment for investigating questions of space kinematics. However, its "curved" nature poses serious problems

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in numerous applications. One way of getting around this is to consider dual quaternions modulo multiplication by *dual numbers* instead of just real numbers. The locus of the ensuing geometry is then not the set $\mathcal{S} \setminus E \subset \mathbb{P}^7(\mathbb{R})$ but the projective space $\mathbb{P}^3(\mathbb{D})$ of dimension three over the dual numbers (minus a low dimensional subset). It provides a *linear* model of space kinematics which is certainly a big advantage. However, it also comes with some rather counter-intuitive properties: The connecting straight line of two points is no longer unique and there exist curves with an osculating tangent in any of their points.

What seems rather strange from a traditional geometric viewpoint becomes much more natural in a kinematic interpretation where straight lines in $\mathbb{P}^3(\mathbb{D})$ correspond to vertical Darboux motions. Two poses may be interpolated by an infinity of vertical Darboux motions [10] and motions in cylinder groups, for example helical motions, admit osculating Darboux motions at any instance. We demonstrate and illustrate this in Section 3.

In Section 4 we present some results on osculating conics/motions. Generically, there exists a four dimensional set of osculating conics in every curve point. Among them we find the well-known Bennett motions [2, 3] but we also suggest another type of osculating conic with geometric significance. It allows an interpretation as osculating circle in an elliptic geometry. We further investigate factorizations of polynomial parametric representations of these motions and give a geometric criterion for factorizability. The obtained results are used to construct overconstrained linkages whose couplers perform such motions.

2 Preliminaries

A dual number is an element of the factor ring $\mathbb{D} \coloneqq \mathbb{R}[\varepsilon]/\langle\varepsilon^2\rangle$. It is uniquely represented by a linear polynomial $a + \varepsilon b$ in the indeterminate ε with coefficients a, $b \in \mathbb{R}$, the *primal* and *dual part,* respectively. Sum and product of two dual numbers as implied by this definition are

$$(a + \varepsilon b) + (c + \varepsilon d) = a + c + \varepsilon(b + d), \quad (a + \varepsilon b)(c + \varepsilon d) = ac + \varepsilon(ad + bc).$$

Multiplication obeys the rule $\varepsilon^2 = 0$. Provided $a \neq 0$, the multiplicative inverse of $a + \varepsilon b$ exists and is given by $(a + \varepsilon b)^{-1} = a^{-1} - \varepsilon b a^{-2}$. We denote the set of invertible dual numbers by $\mathbb{D}^\times$.

2.1 Projective Geometry over Dual Numbers

Similar to common projective geometry over the real or complex numbers, we can study projective geometry over the dual numbers. We focus on the projective space $\mathbb{P}^3(\mathbb{D})$ of dimension three over the dual numbers as this will be the relevant case for rigid body kinematics. The elements of $\mathbb{P}^3(\mathbb{D})$ are equivalence classes of elements of $\mathbb{D}^4 \setminus \{0\}$ where two vectors x and y are considered equivalent if there exists an invertible dual number $a + \varepsilon b$ such that $(a + \varepsilon b)x = y$. We denote equivalence classes by square brackets, i. e. as $[x]$ where $x \in \mathbb{D}^4$ or as $[x_0, x_1, x_2, x_3]$ where x_0, x_1, x_2, $x_3 \in \mathbb{D}$.

In spite of its formal similarity with $\mathbb{P}^3(\mathbb{R})$ or $\mathbb{P}^3(\mathbb{C})$, the space $\mathbb{P}^3(\mathbb{D})$ exhibits some rather unusual properties. Let us consider the connecting line of two points $[a]$ and $[b]$. For its definition we already have two choices. It can be considered as point set

$$\{[\alpha a + \beta b] \mid (\alpha, \beta) \in \mathbb{F}^2, (\alpha, \beta) \neq (0, 0)\} \tag{1}$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{D}$, respectively. We will reserve the word *straight line* for the case $\mathbb{F} = \mathbb{R}$. There are two reasons for this preference: Firstly, it seems to be the common notion in projective geometry over rings. Secondly, a straight line in this sense has real dimension one while it has real dimension two otherwise. With regard to kinematics, this means that a straight line describes an, again more common, one-parametric motion.

A first, possibly surprising, geometric property of $\mathbb{P}^3(\mathbb{D})$ refers to the connecting straight lines of two points. In contrast to geometry over the real numbers, it is no longer unique.

Proposition 2.1. *Let c and $d \in \mathbb{D}^4$ be two points such that c or d has at least one entry with non-zero primal part. Provided $[c]$ and $[d] \in \mathbb{P}^3(\mathbb{D})$ do not coincide, they have infinitely many connecting straight lines. If only one of the points c and d has an entry with non-zero primal part, the real dimension of the set of all connecting straight lines is equal to one, otherwise the dimension is two.*

Proof. We may parameterize any straight line connecting the given points by (1) where $a = \gamma c$, $b = \delta d$ and γ, $\delta \in \mathbb{D}^\times$. This gives four real parameters, the coefficients of γ and δ. But multiplying c and d simultaneously with the same invertible dual number yields identical lines. Thus, only two essential real parameters remain. If all entries of c or d have primal part zero, the dual part of γ or δ does not affect the product γc or δd, respectively. Thus we only have one essential parameter.

□

2.2 Space Kinematics

A *quaternion* is an element of the algebra $\mathbb{H}$ generated by basis elements 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ with generating relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ over the real numbers. A *dual* quaternion q is an element of the algebra $\mathbb{DH}$ with the same basis elements and generating relations but over the *dual numbers* $\mathbb{D}$. Thus, we may write $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ with q_0, q_1, q_2, $q_3 \in \mathbb{D}$ or, separating primal and dual parts, $q = p + \varepsilon d$ where $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $d = d_0 + d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ are elements of $\mathbb{H}$.

The conjugate dual quaternion is $q^\star = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k} = p^\star + \varepsilon d^\star$, the dual quaternion norm is $qq^\star$. In terms of (coefficients of) p and d it may be written as $qq^\star = pp^\star + \varepsilon(pd^\star + dp^\star) = p_0^2 + p_1^2 + p_2^2 + p_3^2 + 2\varepsilon(p_0d_0 + p_1d_1 + p_2d_2 + p_3d_3) \in \mathbb{D}$. A dual quaternion q is invertible if and only if its norm is invertible as a dual number. Its inverse is then given by $q^{-1} = (qq^\star)^{-1}q^\star$. The unit norm condition for dual quaternions reads as

$$pp^\star = 1, \quad pd^\star + dp^\star = 0.$$

Because the norm is multiplicative, the unit dual quaternions form a multiplicative group $\mathbb{DH}_0^\times$. We embed $\mathbb{R}^3$ into $\mathbb{DH}$ via $(x_1, x_2, x_3) \hookrightarrow 1 + \varepsilon(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})$ and define the action of $q = p + \varepsilon d \in \mathbb{DH}_0^\times$ on points of $\mathbb{R}^3$ in the usual way as

$$1 + \varepsilon(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) \mapsto 1 + \varepsilon(y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}) = (p - \varepsilon d)x(p^\star + \varepsilon d^\star) \tag{2}$$

[5]. This action provides us with a double cover of SE(3), the group of rigid body displacements, by the group $\mathbb{DH}_0^\times$.

A slight modification of (2) extends the action to points $[x_0, x_1, x_2, x_3]$ in the projective extension $\mathbb{P}^3(\mathbb{R})$ of $\mathbb{R}^3$:

$$[x_0 + \varepsilon(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})] \mapsto [y_0 + \varepsilon(y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k})] = [(p - \varepsilon d)x(p^\star + \varepsilon d^\star)]. \tag{3}$$

This gives an isomorphism between the group of dual quaternions of non-zero real norm modulo $\mathbb{R}^\times$, the real multiplicative group, and SE(3). The unit norm condition of (2) is replaced by the condition that the norm of q be real but non-zero:

$$pd^\star + dp^\star = 0, \quad pp^\star \neq 0. \tag{4}$$

This means that $[q] = [p + \varepsilon d]$ is a point on the so-called *Study quadric* $\mathcal{S}$ given by the quadratic form $pd^\star + dp^\star$, minus the *null cone* $\mathcal{N}$ given by the singular quadratic form $pp^\star$. The only real points of $\mathcal{N}$ are those of its three-dimensional vertex space E, given by $p = 0$. We call it the *exceptional generator.*

A crucial observation for this article is that even the real norm requirement (4) can be abandoned: As long as $qq^\star$ is invertible, (3) will describe a valid action on $\mathbb{P}^3(\mathbb{R})$ and provide a *homomorphism* from the group $\mathbb{DH}^\times$ of invertible dual quaternions modulo $\mathbb{R}^\times$ to SE(3) or even an *isomorphism* between $\mathbb{DH}^\times/\mathbb{D}^\times$ and SE(3).

Proposition 2.2. *The groups* $\mathbb{DH}^\times/\mathbb{D}^\times$ *and* SE(3) *are isomorphic via the action* (3).

Proof. It is easy to see that $\mathbb{DH}^\times$ is homomorphic to SE(3) via (3). In order to see that $\mathbb{DH}^\times/\mathbb{D}^\times$ is isomorphic, we have to show that dual multiples yield the same action and that identical action implies a dual factor.

Using the notation $q_\varepsilon := p - \varepsilon d$ for the ε-conjugate of $q = p + \varepsilon d$ we can write the right-hand side of (3) as $q_\varepsilon x q^\star$. Multiplying q with a dual number a yields $(aq)_\varepsilon x(aq)^\star = a_\varepsilon q_\varepsilon x a q^\star = (a_\varepsilon a) q_\varepsilon x q^\star$. Because $a_\varepsilon a$ equals the squared primal part of a, this does not change the action on $\mathbb{P}^3(\mathbb{R})$. Existence of a dual factor from identical action follows from equal dimension of $\mathbb{DH}^\times/\mathbb{D}^\times$ and SE(3) and the fact that these groups have only one connected component. □

Since all elements of $\mathbb{DH}^\times/\mathbb{D}^\times$ are points of $\mathbb{P}^3(\mathbb{D})$, it is natural to study space kinematics via the projective geometry of $\mathbb{P}^3(\mathbb{D})$. This point of view is not new. It played a role in [7] and [8]. From an old paper by C. Segre [11] we even infer that probably already E. Study and his disciples were aware of these connections in the first decades of the 20th century.

2.3 Straight Lines and Vertical Darboux Motions

Via the action (3), a curve in $\mathbb{P}^3(\mathbb{D})$ corresponds to a one-parametric rigid body motion. In particular, polynomial curves yield motions with polynomial trajectories in homogeneous coordinates, that is, *rational motions.* The simplest example of such motions comes from straight lines in $\mathbb{P}^3(\mathbb{D})$ which correspond to *vertical Darboux motions* [8, 9]. A vertical Darboux motion is the composition of a unit speed rotation about a fixed axis with a harmonic oscillation along the axis such that one full rotation corresponds to one oscillation period. Its trajectories are bounded rational curves of degree two (ellipses or straight-line segments). Rotations and translations are considered as special cases of vertical Darboux motions with zero or infinite oscillation amplitude, respectively.

We illustrate a vertical Darboux motion in Figure 1. This figure also helps us explain a generally useful concept: Motions obtained as composition of rotation around an axis and translation along the same axis have trajectories on a right circular cylinder. Any curve γ on such a cylinder can be used to completely specify the motion by adding a Cartesian frame consisting of cylinder normal, cylinder generator and horizontal cylinder tangent. Instead of the curve on the cylinder, we may equally well consider its image when developing the cylinder surface. In case of a vertical Darboux motion, γ is an ellipse. Its development is a sine curve which is scaled in direction of the developed cylinder generators in order to adapt to the oscillation's amplitude.

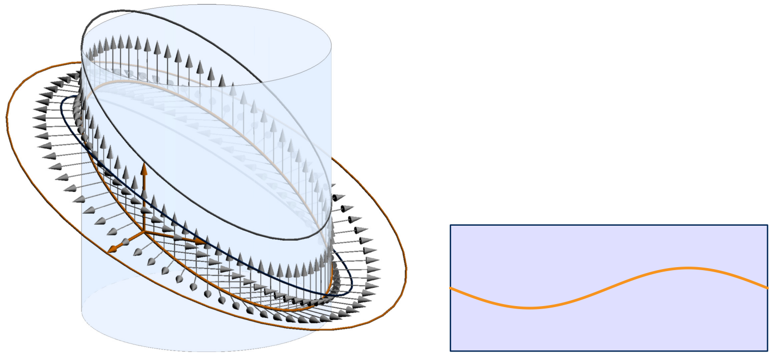

Figure 1: Vertical Darboux motion with some elliptic trajectories, right circular cylinder, and development.

3 Osculating Lines

In this section we demonstrate that a helical motion and a vertical Darboux motion can have second order contact at any parameter value. Since vertical Darboux motions correspond to straight lines in $\mathbb{P}^3(\mathbb{D})$ this amounts to saying that the curve corresponding to a helical motion has an osculating tangent at any point. This is a remarkable difference to classical differential geometry over the real numbers where this property characterizes straight lines. We also prove that an infinity of osculating lines not only exists for a helical motion but for any generic motion in a *cylinder group,* i. e., a group generated by rotations around and translations along a fixed axis.

A rotation around axis $\mathbf{k}$ with rotation angle ω is given by the dual quaternion $r = \cos\frac{\omega}{2} + \sin\frac{\omega}{2}\mathbf{k}$, a translation with oriented distance δ in direction of $\mathbf{k}$ is given by $t = 1 - \frac{1}{2}\varepsilon\delta\mathbf{k}$. Thus, a helical motion and a Darboux motion d with amplitude c are obtained by substituting $p\omega$ and $c\sin\omega$, respectively, for δ in the product rt:

$$\begin{aligned} h &= \cos\left(\tfrac{\omega}{2}\right) + \sin\left(\tfrac{\omega}{2}\right)\mathbf{k} + \frac{p}{2}\omega\varepsilon\left(\sin\left(\tfrac{\omega}{2}\right) - \cos\left(\tfrac{\omega}{2}\right)\mathbf{k}\right), \\ d &= \cos\left(\tfrac{\omega}{2}\right) + \sin\left(\tfrac{\omega}{2}\right)\mathbf{k} + \frac{c}{2}\sin\omega\varepsilon\left(\sin\left(\tfrac{\omega}{2}\right) - \cos\left(\tfrac{\omega}{2}\right)\mathbf{k}\right). \end{aligned} \tag{5}$$

For $c = p$ the first two derivatives of h and d are equal,

$$\frac{\mathrm{d}h}{\mathrm{d}\omega}(0) = \frac{\mathrm{d}d}{\mathrm{d}\omega}(0) = \frac{1}{2}\mathbf{k} - \frac{1}{2}p\varepsilon\mathbf{k} \quad \text{and} \quad \frac{\mathrm{d}^2h}{\mathrm{d}\omega^2}(0) = \frac{\mathrm{d}^2d}{\mathrm{d}\omega^2}(0) = -\frac{1}{4} + \frac{1}{2}p\varepsilon,$$

while the third derivatives differ,

$$\frac{\mathrm{d}^3h}{\mathrm{d}\omega^3}(0) = -\frac{1}{8}\mathbf{k} + \frac{3}{8}p\varepsilon\mathbf{k} \neq -\frac{1}{8}\mathbf{k} + \frac{7}{8}p\varepsilon\mathbf{k} = \frac{\mathrm{d}^3d}{\mathrm{d}\omega^3}(0).$$

Thus, for $p = c$, the motions (5) have second order contact at $\omega = 0$. Since this parameter value has no particular meaning for a helical motion, we may state that *for any instance of a helical motion there exists a vertical Darboux motion with second order contact.*

Let us also verify that d is actually a straight line in $\mathbb{P}^3(\mathbb{D})$ by multiplying its parametric representation (5) with a suitable dual number valued function. Indeed, we have

$$\left(1 + p\varepsilon\cos^2\left(\tfrac{\omega}{2}\right)\right)d = \cos\left(\tfrac{\omega}{2}\right)(1 + p\varepsilon) + \sin\left(\tfrac{\omega}{2}\right)\mathbf{k}$$

which is a parametric representation of the straight line spanned by $1 + p\varepsilon$ and $\mathbf{k}$. Summarizing, we can thus state

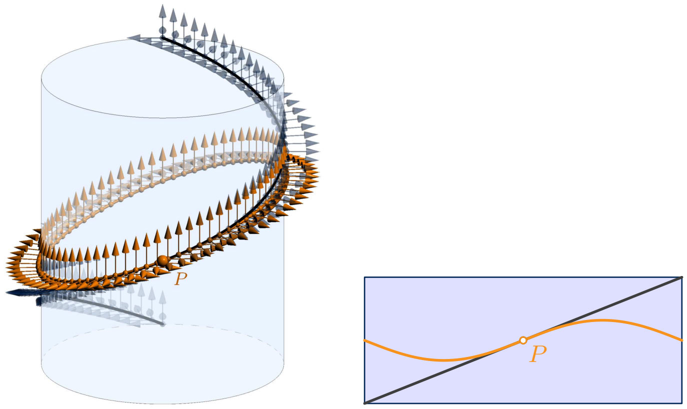

Figure 2: Geometric interpretation of osculating lines.

Theorem 3.1. *At any instance in time any helical motion, viewed as a curve in kinematic space $\mathbb{P}^3(\mathbb{D})$, has second order contact with a straight line. Yet, it is not a straight line itself.*

This seemingly strange behavior allows a clear geometric interpretation that also gives additional insight. Figure 2 displays helical motion and osculating Darboux motion via the cylinder model we discussed earlier. In the development, the helical motion corresponds to a straight line while the Darboux motion is a sine curve with this line as inflection tangent. Obviously, it is possible to determine uniquely a suitable sine function in every point. It gives rise to the unique osculating vertical Darboux motion in a point of the helical motion.

Helical motions are not the only curves in $\mathbb{P}^3(\mathbb{D})$ susceptible to second order approximation by straight lines in every point. An arbitrary motion in the cylinder group C corresponds to a curve in the development. There, an osculating sine function can be drawn in any sufficiently smooth point and gives rise to an osculating vertical Darboux motion. The possibility to do so is a direct consequence of the following lemma: Among all candidate sine functions there exist one with prescribed slope and curvature.

Lemma 3.2. *Given two real numbers k, $\varkappa \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that some point on the graph of the function $\varphi \mapsto a \sin\varphi$ has slope k and curvature $\varkappa$.*

Proof. The subgraphs corresponding to parameter intervals $[i\frac{\pi}{2}, (i+1)\frac{\pi}{2}]$ for $i \in \{0,1,2,3\}$ are congruent and, up to respective signs, have points of equal slope and curvature. Thus, we may restrict to the case $i = 0$, $k \geq 0$, $\varkappa \leq 0$ and search for $a > 0$. Slope and curvature are given by

$$k = a\cos\varphi \quad \text{and} \quad \varkappa = -\frac{a\sin\varphi}{(1 + a^2\cos^2\varphi)^{3/2}}$$

These identities can be written in the following way

$$k = a\cos\varphi \quad \text{and} \quad -\varkappa(1 + k^2)^{3/2} = a\sin\varphi.$$

This allows us to view a and φ as the polar coordinates of the point $(k, -\varkappa(1+k^2)^{3/2})$ which lies, due to our assumptions, in the first quadrant. As its polar coordinates are determined (and even unique if k and $\varkappa$ are not both zero), we can find $a \geq 0$ and $\varphi \in [0, \frac{\pi}{2}]$.

□

Corollary 3.3. *Any sufficiently smooth motion in a cylinder group has an osculating Darboux motion in any of its points (at any instance).*

4 Conic Sections

We now turn our attention to conic sections in $\mathbb{P}^3(\mathbb{D})$. We study them as rational curves of degree two. A parametric representation is simply a polynomial c of degree two in one indeterminate t that serves as a real parameter. We assume that c has no scalar polynomial factor of positive degree and also that its coefficients are linearly independent, as otherwise it would parametrize a straight line or a point. A conic parameterizes a rational motion with trajectories of degree at most four.

In line with the general philosophy of this article we should consider a polynomial c up to multiplication with a dual number valued function. However, here it is sufficient to consider only dual number multiples, that is, we consider only polynomial representations of minimal degree. In projective differential geometry over the real numbers, a generic smooth space curve admits a two parametric set of osculating conics in a generic point. In projective geometry over the dual numbers, a further degree of freedom is added:

For the case of interpolating conics for three finitely separated points $[c_0]$, $[c_1]$, $[c_2] \in \mathbb{P}^3(\mathbb{D})$, this is easy to see. An interpolating conic may be parameterized as $[c(t)]$ where

$$c(t) = c_0 + (c_1 - c_0 - c_2)t + c_2 t^2.$$

The points $[c_0]$, $[c_1]$, $[c_2]$ correspond to parameter values $t = 0$, $t = 1$, and $t = \infty$, respectively. Obviously, different dual number multiples of c_0, c_1 and c_2 yield different conics, unless the dual factor is the same for all three points. We may use this freedom to have the dual factor 1 for c_1 whence a general parametric representation for interpolating conics can be written as

$$c(t) = \gamma_0 c_0 + (c_1 - \gamma_0 c_0 - \gamma_2 c_2)t + \gamma_2 c_2 t^2 \tag{6}$$

where γ_0 and γ_2 are invertible dual numbers. This gives the claimed four degrees of freedom.

In view of Section 3 it is natural to ask for space curves that admit a conic with even higher order contact in every point. We will not pursue this question any further at this point. Instead, we present two examples of osculating conics in this set with a special meaning for space kinematics. In Section 4.3 we discuss their factorization in the sense of [4] and use it for the construction of linkages.

4.1 Bennett Motions

The *Bennett motion* is a well-known example of a quartic space motion whose kinematic image in the "classical" sense is a conic section on the Study quadric $\mathcal{S}$ and which is determined by three general finitely separated or infinitesimally neighboring points in the Study quadric. In fact, we may simply define it as a regular conic in the Study quadric that does not intersect the exceptional generator E [2, 3]. In our context, we can re-derive the motion from the following observation:

Lemma 4.1. *Given an invertible dual quaternion p there exists an invertible dual number a such that ap has real norm. The dual number a is determined up to a real multiple.*

Proof. Write $p = p' + \varepsilon p''$ and $a = a' + \varepsilon a''$ with quaternions p', p'' and real numbers a', a''. The dual part of the norm of ap then reads as $a'^2(p'p''^\star + p''p'^\star) + 2a'a''p'p'^\star$. Both, $p'p''^\star + p''p'^\star$ and $p'p'^\star$ are real numbers and the latter is different from zero (because p is invertible). We may divide by a' (because we want to find an invertible dual number a) so that ultimately

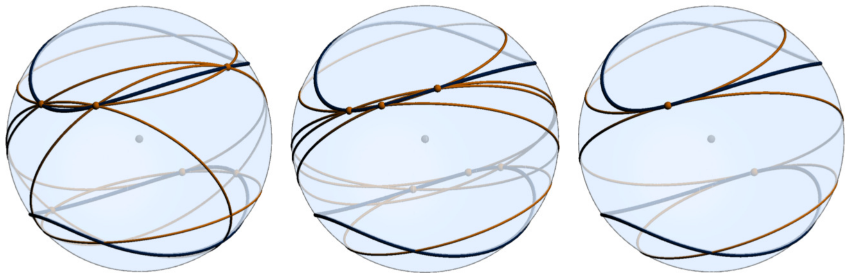

Figure 3: Circles through three points (left and middle) and osculating circle in elliptic geometry (right).

$a = a' + \varepsilon a''$ is determined, up to a real multiple, by one non-vanishing homegeneous linear equation. A (non-trivial) solution with $a' = 0$ is not possible because p is invertible whence $p'p'^{\star} \neq 0$. □

Returning to (6), we may assume $[c_0]$, $[c_1]$, $[c_2] \in \mathcal{S}$ as otherwise we can multiply with suitable dual numbers by Lemma 4.1. Now we are still free to multiply c_0, c_1, and c_2 with real numbers and it is well-known (c. f. for example [2]) that this freedom is enough to ensure that $[c(t)]$ lies on the Study quadric $\mathcal{S}$.

Bennett motions are rational motions with entirely circular trajectories of degree four. They appear as coupler motions of *Bennett linkages*, that is, spatial four-bar linkages with exceptional mobility [2]. An example for a Bennett motion can be found later in Figure 4.

4.2 Motions Based on Osculating Circles of Elliptic Geometry

An important object for the kinematic geometry in $\mathbb{P}^3(\mathbb{D})$ is the null cone $\mathcal{N}$. It consists of points represented by non-invertible dual quaternions, a property that is preserved under coordinate changes and thus makes $\mathcal{N}$ a *geometric invariant.* With this in mind, it is natural to look for osculating conics in special position with respect to $\mathcal{N}$. For a general parametric representation of the shape (6) it is possible to determine the dual factors γ_0, $\gamma_2 \in \mathbb{D}$ in such a way that the conic parameterized by $[c(t)]$ is *tangent to $\mathcal{N}$ in two points.* In fact, if we only consider *real* factors, this amounts to determining a circle through three points in the real elliptic plane with absolute conic $\mathcal{N} \cap \varphi$ where φ is the conic's plane. For three finitely separated points this problem has four solutions as can be seen in the spherical model of elliptic geometry (Figure 3). But this property does not translate to three infinitesimally neighboring points as in the limit three of the four circles converge to the curve tangent so that the osculating circle is unique. This is also visualized in Figure 3.

In lack of a better name, we refer to the motions in question as *quadratic null cone motions.* The four-dimensional set of osculating conics contains a two-dimensional set of these motions. Their generic trajectories are rational of degree four, not circular in general but tangent to the plane at infinity in two points. Figure 4 displays a null cone motion and a Bennett motion that osculate at one pose which is drawn a little larger.

4.3 Factorization of Bennett Motions and Null Cone Motions

It is well known that a general conic section on the Study quadric (a Bennett motion) occurs as the coupler motion of a closed spatial four-bar linkage (Bennett linkage) [2, 3]. One

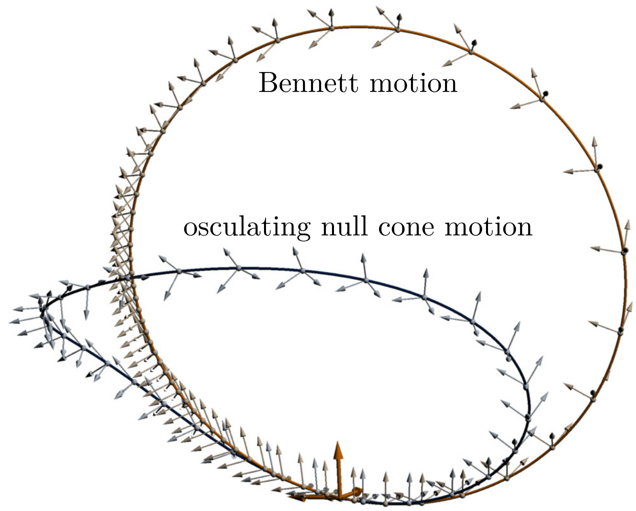

Figure 4: A Bennet motion and an osculating null cone motion.

way to find the corresponding linkage to a given Bennett motion is to compute different factorizations of a parametric representation into linear polynomials over the ring of dual quaternions. Each factor parametrizes a straight line on the Study quadric and therefore, generically, a rotation [4]. It can be realized mechanically via a revolute joint. Combining the joints of two factorizations yields the spatial four-bar linkage. For special conic sections on the Study quadric, factorization and construction of linkages can be more involved. It is possible that revolute joints have to be replaced by prismatic (translation) joints [3, 4]. It may also be the case that only one or infinitely many factorizations exist.

The case of only one factorization is briefly mentioned in [12]. This happens precisely if the Bennett motion is also a null cone motion. Since essentially everything is known about the factorization of Bennett motions, we now have a closer look at factorizability of null cone motions. We also discuss how to construct a linkage to generate this motion, provided factorizations exist. In $\mathbb{P}^3(\mathbb{D})$, straight lines no longer correspond to rotations but to vertical Darboux motions with rotations and translations as special cases. Since a vertical Darboux motion is a composition of a rotation and a harmonic oscillation along the same axis, it can be realized by a cylindrical joint which allows an independent rotation around and translation along a fixed axis. When constructing linkages one will have to ensure that the cylindrical joints do not introduce unwanted degrees of freedom.

Let $c(t) = c_0 + (c_1 - c_0 - c_2)t + c_2 t^2 = p(t) + \varepsilon d(t)$ be a conic section which is tangent to $\mathcal{N}$. We will assume that $c_2 = 1$ which can be achieved by a change of coordinates. Since c is tangent to the null cone in two points, we know that the norm polynomial's primal part $pp^\star$ is a real polynomial with two roots of multiplicity two.

If the two roots coincide, i.e. $pp^\star$ has one root with multiplicity four, we deduce that this root is actually real. But then there exists a real parameter value t_0 at which the norm of p is zero whence $p(t_0) = 0$. This implies that p is the square of a linear polynomial $s \in \mathbb{R}[t]$. The underlying motion is the translation along a quadratic curve with one point at infinity, that is, a parabola or a quadratically parametrized half-line. A factorization necessarily is of the form $c = (s + \varepsilon d_1)(s + \varepsilon d_2)$. But in this case s is a factor of c and therefore c does not parametrize a conic but a line and it cannot be a quadratic translation. Therefore we assume for the remainder of this section that the two roots of $pp^\star$ are distinct.

Theorem 4.2. *If $c = p+\varepsilon d$ parametrizes a quadratic null-cone motion, a factorization exists precisely if $pp^\star$ has two distinct roots t_0, $t_1 \in \mathbb{C}$ of multiplicity two and the points $[c(t_0)]$, $[c(t_1)]$ lie on the Study quadric.*

Proof. We distinguish between three cases:
(a) p does not have a real polynomial factor,
(b) p is an irreducible quadratic real polynomial, or
(c) p is the product of two distinct linear real polynomials.
The seemingly missing case of just one linear real factor is not compatible with the assumption of c parametrizing a null cone motion. It can only happen in Case (a) that the points $[c(t_0)]$, $[c(t_1)]$ don't lie on the Study quadric since both (b) and (c) ensure that the dual part of $cc^\star$ has a real polynomial factor with roots t_0, t_1. Thus it suffices to show the existence of factorizations in Case (b) and Case (c).

Let us consider Case (a) at first. Here, the two roots t_0, t_1 are complex conjugates and we have $pp^\star = s^2$ for the irreducible quadratic polynomial $s = (t-t_0)(t-t_1) \in \mathbb{R}[t]$ (recall that c is assumed to be monic). An obviously necessary condition for c to admit a factorization into linear polynomials is that $cc^\star = s^2 + \varepsilon(pd^\star + dp^\star)$ is a product of two quadratic polynomials with dual coefficients, that is $cc^\star = (s+\varepsilon\lambda_1)(s+\varepsilon\lambda_2)$ for some λ_1, $\lambda_2 \in \mathbb{R}[t]$. This implies $pd^\star + dp^\star = s(\lambda_1+\lambda_2)$ so that s is a factor of $cc^\star$ and the two points $[c(t_0)]$, $[c(t_1)]$ of tangency of the conic c and the nullcone also lie on the Study quadric. For the converse statement, we appeal to [4, Lemma 3] which ensures existence of a factorization.

Next, let us consider Case (b) where p is an irreducible quadratic real polynomial, that is $c = p + \varepsilon d$ with $d \in \mathbb{H}[t]$. A suitable parameter transformation allows us to assume $p = t^2+1$. In this case, c describes a translation along a bounded quadratic curve (an ellipse or line segment). After a change of coordinates we can assume that all trajectories lie in planes orthogonal to the third coordinate axis and their major axes are parallel to the first coordinate axis. Thus, the parametrization is of the form $c = t^2+1+\varepsilon(\gamma_1 t+\gamma_0+b\mathbf{j}t+a\mathbf{i})$, where $a \geq b \geq 0$, $a \neq 0$ are the lengths of their semi-axes and γ_1, γ_0 are arbitrary real numbers. As in [6], we solve the equation $c = F_1F_2$ for arbitrary linear factors F_1, $F_2 \in \mathbb{DH}[t]$. Since the primal part of c is a real polynomial, we get that the primal parts of F_1 and F_2 are conjugates of each other. Let us write

$$\begin{aligned} F_1 &= t + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} + \varepsilon(u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}),\\ F_2 &= t - p_1\mathbf{i} - p_2\mathbf{j} - p_3\mathbf{k} + \varepsilon(v_0 + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}).\end{aligned}$$

Comparing coefficients in $c = F_1F_2$, we obtain a system of nine algebraic equations:

$$\begin{gathered} u_0+v_0=\gamma_1, \quad u_1+v_1=0, \quad u_2+v_2=b, \quad u_3+v_3=0, \quad p_1^2+p_2^2+p_3^2=1,\\ p_1(u_1-v_1)+p_2(u_2-v_2)+p_3(u_3-v_3)=\gamma_0, \quad p_1(v_0-u_0)+p_2(u_3+v_3)-p_3(u_2+v_2)=a,\\ -p_1(u_3+v_3)+p_2(v_0-u_0)+p_3(u_1+v_1)=0, \quad p_1(u_2+v_2)-p_2(u_1+v_1)+p_3(v_0-u_0)=0.\end{gathered}$$

For $a > b$, this system has two two parametric families of solutions:

$$\begin{gathered} p_1 = \pm\frac{\sqrt{a^2-b^2}}{a}, \quad p_2 = 0, \quad p_3 = -\frac{b}{a}, \quad u_2 = -v_2 + b, \quad u_3 = -v_3\\ u_0 = \frac{\gamma_1}{2} \mp \frac{\sqrt{a^2-b^2}}{2}, \quad u_1 = -v_1 = \pm\frac{a\gamma_0 - 2bv_3}{2\sqrt{a^2-b^2}}, \quad v_0 = \frac{\gamma_1}{2} \pm \frac{\sqrt{a^2-b^2}}{2}.\end{gathered} \tag{7}$$

If $a = b > 0$, which is the case if and only if c is a circular translation, there is only one family of solutions, namely

$$p_1 = p_2 = 0, \quad p_3 = -1, \quad u_0 = v_0 = \frac{\gamma_1}{2}, \quad u_1 = -v_1, \quad u_2 = -v_2 + b, \quad u_3 = -v_3 = -\frac{\gamma_0}{2}.$$

In case of $\gamma_0 = \gamma_1 = 0$, these solutions correspond to the ones found in [6] for quadratic motion polynomials.

In the final Case (c) the polynomial p is the product of two distinct linear real polynomials s_1, $s_2 \in \mathbb{R}[t]$. Let us find the reduced polynomial $\widetilde{c}$ which parametrizes the same motion and fulfills the Study condition. Following the proof of Lemma 4.1 we can find the polynomial $a = s_1 s_2(2s_1 s_2 - \varepsilon(d + d^\star)) \in \mathbb{D}[t]$ with dual number coefficients such that ac has a real norm polynomial. This product has the real polynomial factor $(s_1 s_2)^2$, so after reducing ac and dividing off the leading coefficient we end up with $\widetilde{c} \coloneqq (1 - \varepsilon(d + d^\star)/(2s_1 s_2))c$. The norm polynomial of $\widetilde{c}$ equals $s_1^2 s_2^2$. Thus we can apply Lemma 3 of [4] to infer existence of a factorization $\widetilde{c} = F_1 F_2$ such that $F_1 F_1^\star = s_1^2$ and $F_2 F_2^\star = s_2^2$. As $\widetilde{c}$ is the product of c with a rational dual function, we can multiply $\widetilde{c}$ with the inverse of this function to obtain c, i. e. $c = (1+\varepsilon(d+d^\star)/(2s_1 s_2))F_1 F_2$. Let us use partial fraction decomposition for the dual part of the first factor such that $(1+\varepsilon(d+d^\star)/(2s_1 s_2)) = (1+\varepsilon\lambda_1/s_1)(1+\varepsilon\lambda_2/s_2)$ for some λ_1, $\lambda_2 \in \mathbb{R}$. This now yields a factorization of c as we obtain $c = (1 + \varepsilon\lambda_1/s_1)F_1(1 + \varepsilon\lambda_2/s_2)F_2$. □

As we have seen in the proof, polynomials which fulfill Case (b) parametrize bounded quadratic translations. In Case (c) they parametrize quadratic translations along a curve which intersects the plane at infinity for two real parameters. Thus, these polynomials parametrize translations along a hyperbola.

Remark 4.3. The proof of Theorem 4.2 shows that all bounded quadratic translations admit a factorization. But only circular translations which fulfill the Study condition can be decomposed into two rotations, that is, they have factors which satisfy the Study condition. This confirms results of [6].

The proof of Theorem 4.2 together with Lemma 4.1 shows that in Case (b) and Case (c) there exist infinitely many quadratic polynomials which parametrize the same motion. In Case (b) choosing different representations allows us to decompose the given motion into linear polynomials in infinitely many different ways. In Case (c) however, the linear factors obtained from different representations only differ by dual rational factors. While the factorizations are different in algebraic sense, the underlying kinematic decompositions are all the same. The obtained factors parametrize translations and therefore can be realized by prismatic joints. As all quadratic translations have trajectories parallel to a plane, they can always be realized by coupling two prismatic joints with axes parallel to this plane but non-parallel to each other. Therefore, factorization does not allow for the construction of an overconstrained linkage performing motions of type (c).

For Case (a) and (b) however we have seen in the proof of Theorem 4.2 that we can find (at least) two different factorizations. For the construction of overconstrained linkages we need factorizations where one linear factor has a real norm polynomial, hence parametrizes a rotation. The other factors are linear polynomials with non-real norm polynomial and therefore parametrize vertical Darboux motions which can be realized by cylindrical joints. Using these types of factorization we can construct four-bar linkages consisting of two revolute and two cylindrical joints which are able to perform the given motion. The degree of freedom according to the formula of Chebyshev-Grübler-Kutzbach [1, Chapter 5] equals 0, thus the linkages are overconstrained.

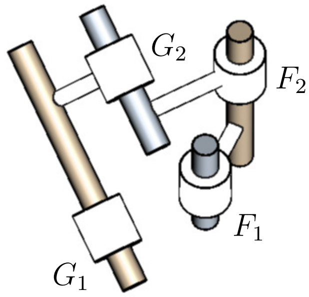

Figure 5: Spatial four-bar linkage with revolute joints F_1, G_2 and cylindrical joints F_2, G_1 whose coupler performs an elliptic translation.

As a linkage corresponding to a motion of Case (a) is described in [12], we focus on the construction of a linkage corresponding to Case (b). Let $c = t^2+1+\varepsilon(\gamma_1 t+\gamma_0+b\mathbf{j}t+a\mathbf{i})$ be an elliptic translation with $a > b \geq 0$. Note that varying γ_0 and γ_1 yield different representations of the same underlying motion. The linear factors obtained in the proof of Theorem 4.2 have the norm polynomials $t^2 + 1 + \varepsilon(t(\gamma_1 \pm \sqrt{a^2 - b^2}) + \gamma_0)$. Thus, we can choose $\gamma_0 = 0$ and $\gamma_1 = \mp\sqrt{a^2 - b^2}$ such that either the first, or the second factor has real norm.

All factors of one family of factorizations parametrize motions around parallel axes. Choosing two different factorizations from the same family therefore would yield a four-bar linkage with two revolute and two cylindrical joints with parallel axes. Such linkages however have in general two degrees of freedom. Therefore we need to choose one factorization from each family. The two different choices of γ_1 determine, which joints are the revolute joints.

Example 4.4. Let us consider a translation along an ellipse with semi-major axis of length $a = 2$ and semi-minor axis of length $b = 1$. For $\gamma_1 = \sqrt{a^2 - b^2}$ we have $c = t^2+1+\varepsilon(\sqrt{3}t+\mathbf{j}t+2\mathbf{i})$. Choosing $v_2 = 0$, $v_3 = 1/\sqrt{3}$ in Equation (7) we obtain the factorizations $c = F_1F_2 = G_1G_2$ with

$$F_1 = t + \frac{\sqrt{3}\mathbf{i} - \mathbf{k}}{2} - \varepsilon\frac{\mathbf{i} - 3\mathbf{j} + \sqrt{3}\mathbf{k}}{3}, \qquad F_2 = t - \frac{\sqrt{3}\mathbf{i} - \mathbf{k}}{2} + \varepsilon\frac{3\sqrt{3} + \mathbf{i} + \sqrt{3}\mathbf{k}}{3},$$

$$G_1 = t - \frac{\sqrt{3}\mathbf{i} + \mathbf{k}}{2} + \varepsilon\frac{3\sqrt{3} + \mathbf{i} + 3\mathbf{j} - \sqrt{3}\mathbf{k}}{3}, \qquad G_2 = t + \frac{\sqrt{3}\mathbf{i} + \mathbf{k}}{2} + \varepsilon\frac{-\mathbf{i} + \sqrt{3}\mathbf{k}}{3}.$$

The axes of the motions parametrized by F_1 and F_2 as well as G_1 and G_2, respectively, are parallel. The angle between the non-parallel axes is $\pi/3$. The distance between the parallel axes is 1 while the distance between the other axes is 4/3. The obtained linkage is depicted in Figure 5. It can be shown that it has two operation modes, both are elliptic translations.

5 Conclusion

We have related space kinematics to the geometry of the projective space $\mathbb{P}^3(\mathbb{D})$ over the ring of dual numbers. This interpretation seems well suited for kinematic visualization of certain differential geometric aspects and it also provides the proper mathematical framework for the systematic study of osculating motions. We presented results for ordinary and osculating tangents and some ideas about osculating conics and their factorizability. Factorization without the Study condition opens additional possibilities for the construction of linkages with cylindrical joints.

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