Journal for Geometry and Graphics
Volume 25 (2021), No. 2, 243–251

Angles of Isosceles Tetrahedra

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Abstract. We give three new characteristics of an isosceles tetrahedron. These three characteristics are; (i) the sum of cosines of dihedral angles of a tetrahedron at each vertex is 1, (ii) the opposing dihedral angles of a tetrahedron are pairwise same, and (iii) all four solid angles of a tetrahedron are the same. It is known that "isosceles" implies (ii) and (iii), but we think the converse of these and (i) are new. The statement (iii) suggests that a solid angle may determine an isosceles tetrahedron uniquely up to a similarity. However, we give an example to show that this is not the case unless it is a regular tetrahedron. And finally, we obtain a trigonometric identity from an isosceles tetrahedron. We use a theorem on a spherical triangle.

Key Words: isosceles tetrahedron, equifacial tetrahedron, dihedral angle, solid angle

MSC 2020: 51M04

1 Introduction

We will convert a theorem on a spherical triangle to a theorem on a tetrahedron in order to obtain relations between an isosceles tetrahedron and (angles, dihedral angles and solid angles). Let us start with definitions.

Definition 1. Let $OABC$ be a tetrahedron. The segment joining two points O and A is denoted by OA, and its length by $\overline{OA}$. The *angle* between the two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is denoted by $\sphericalangle AOB$. The interior angle between the triangular faces OAB and OAC of a tetrahedron $OABC$ is called the *dihedral angle* at the edge of OA, and it is denoted by $\sphericalangle OA$. The *solid angle* $\sphericalangle O$ of the tetrahedron $OABC$ at O is definted as $\sphericalangle O = \sphericalangle OA + \sphericalangle OB + \sphericalangle OC - \pi$.

A solid angle is also called a *trihedral angle* in [1].

Definition 2. Let $OABC$ be a tetrahedron. Let S be the sphere of radius 1 centered at O. Let A', B', C' be the points on the sphere S that intersect the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, respectively. We join (A' and B'), (B' and C'), and (C' and A') by parts of great circles on S, and the

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result is said to be a *spherical triangle* $A'B'C'$. The segment of the great circle joining A' and B' is denoted by $\widehat{A'B'}$. The arc length of $\widehat{A'B'}$ is also denoted by $\widehat{A'B'}$. (See, for example, Section 6.5 of [4] for spherical geometry.)

Then next theorem is a result on a spherical triangle.

Theorem 1 (Spherical Laws of Cosines)**.** *(See Proposition 6.5.3 and Corollary 6.5.6 of [4].) Let $A'B'C'$ be the spherical triangle. Let α, β, and γ be its interior angles at A', B', and C', respectively. (Hence, α, β, and γ are the angle opposite to the sides $\widehat{B'C'}$, $\widehat{C'A'}$, and $\widehat{A'B'}$, respectively, and $0 \le \alpha, \beta, \gamma < \pi$.) Then*

$$(a) \quad \cos\alpha = \frac{\cos\widehat{B'C'} - \cos\widehat{A'B'}\cos\widehat{C'A'}}{\sin\widehat{A'B'}\sin\widehat{C'A'}}, \text{ and}$$

$$(b) \quad \cos\widehat{A'B'} = \frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

The relation between the tetrahedron $OABC$ and the spherical triangle $A'B'C'$ is the following:

Lemma 1. *We use the notations in Definitions 1 and 2 and Theorem 1. Then the dihedral angle $\sphericalangle OA$ is the interior angle of the spherical triangle $A'B'C'$ at A', i.e. $\sphericalangle OA = \alpha$. Similarly, $\sphericalangle OB = \beta$ and $\sphericalangle OC = \gamma$. The angle $\sphericalangle AOB$ is given by $\sphericalangle AOB = \widehat{A'B'}$ (measured in radian). Similarly, $\sphericalangle BOC = \widehat{B'C'}$ and $\sphericalangle COA = \widehat{C'A'}$.*

Proof. The interior angle α of the spherical triangle $A'B'C'$ at A' is the angle between the two tangent lines at A' to the great circles $\widehat{A'B'}$ and $\widehat{A'C'}$. But the angle between these two tangent lines is the dihedral angle $\sphericalangle OA$.

The angle $\sphericalangle AOB$ is the arc length $\widehat{A'B'}$ of the great circle by the defintion of the radian measurement. □

Hence, Lemma 1 allows us to rewrite Theorem 1 as follows:

Theorem 2. *Let $OABC$ be a tetrahedron. Then we have the following two equations.*

$$(a') \quad \cos\sphericalangle OA = \frac{\cos\sphericalangle BOC - \cos\sphericalangle COA\cos\sphericalangle AOB}{\sin\sphericalangle COA\sin\sphericalangle AOB}.$$

$$(b') \quad \cos\sphericalangle BOC = \frac{\cos\sphericalangle OA + \cos\sphericalangle OB\cos\sphericalangle OC}{\sin\sphericalangle OB\sin\sphericalangle OC}.$$

The next Lemma 2 may be the motivation for a solid angle in Definition 1.

Lemma 2. *(See Theorem 6.4.7 of [4].) The area of the spherical triangle $A'B'C'$ with interior angles α, β and γ (as in Definition 2 and Theorem 1) is given by $\alpha + \beta + \gamma - \pi$.*

Definition 3. A tetrahedron $ABCD$ is an *isosceles tetrahedron* if $AB = CD, AC = BD$, and $AD = BC$.

For basic information on an isosceles tetrahedron, see pages 94–102 in [1]. The next lemma is what we need most from [1].

Lemma 3. *A tetrahedron has four congruent triangle faces if, and only if, the tetrahedron is isosceles. (An isosceles tetrahedron is also called equifacial.)*

Proof. The faces of an isosceles tetrahedron are congruent by Theorem 293 of [1]. The converse is not difficult to prove, and we leave the proof of the converse to the readers. A stronger statement of this converse is given in Corollary 307 of [1]. That is, if the four faces of a tetrahedron have the equal area, then it must be isosceles. □

Now, we can state our result. Theorem 3 will give three characterizations of an isosceles tetrahedron in terms of dihedral angles and solid angles. Let us make the following four statements: (1) A tetrahedron is isosceles, (2) the sum of cosines of three dihedral angles of a tetrahedron at each vertex equals 1, (3) the opposing dihedral angles of a tetrahedron are equal, and (4) all four solid angles of a tetrahedron are equal. We will prove that these four statements are equivalent in Theorem 3. Exercises 14 and 15 in [1, Page 102] asks to prove the implications (1) $\implies$ (3) and (1) $\implies$ (4), and therefore, these implications are known. However, to the best of our knowledge, the converses (3) $\implies$ (1) and (4) $\implies$ (1), and the equivalence (1) $\iff$ (2) are new. For a comparison, it is interesting to note that a tetrahedron is isosceles if, and only if, the sum of three angles of a tetrahedron at each vertex is equal to π (See Lemma 5 below).

Corollary 1 will show that the sum of all six cosines of dihedral angles of an isosceles tetrahedron is 2. But we do not know if the converse of Corollary 1 is true (see Conjecture 1).

In Corollary 2, we will show that the largest solid angle of an isosceles tetrahedron is $2(\pi - 3\cos^{-1}\frac{1}{\sqrt{3}})$. The equivalence (1) $\iff$ (4) suggests that a solid angle may determine an isosceles tetrahedron uniquely up to a similarity. In general, this is shown to be false in Example 1. However, in Corollary 3, we will show that a tetrahedron has four equal solid angles $(\pi - 3\cos^{-1}\frac{1}{\sqrt{3}})$ if, and only if, it is a regular tetrahedron.

Finally, we will prove a trigonometric identity in Theorem 4 using the sum of the three angles at a vertex of an isosceles tetrahedron is π (Lemma 5).

To the best of our knowledge, Corollaries 1, 2 and 3, Example 1, and Theorem 4 are all new.

2 Isosceles Tetrahedra

Let us begin with a lemma on a triangle.

Lemma 4. *Let ABC be a triangle. If A, B, and C are the angles $\sphericalangle CAB$, $\sphericalangle ABC$, and $\sphericalangle BCA$, respectively, then we have*

$$(c) \quad \frac{\cos A - \cos B\cos C}{\sin B\sin C} + \frac{\cos B - \cos A\cos C}{\sin A\sin C} + \frac{\cos C - \cos A\cos B}{\sin A\sin B} = 1.$$

Proof. The identity $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$ is known and it is not difficult to prove it. (See [5], for example.) Since $A + B + C = \pi$, we have $\cos C = -\cos(A + B)$. So, we have

$$\begin{aligned}\frac{\cos C - \cos A\cos B}{\sin A\sin B} &= \frac{-\cos(A+B) - \cos A\cos B}{\sin A\sin B}\\ &= \frac{-\cos A\cos B + \sin A\sin B - \cos A\cos B}{\sin A\sin B} = 1 - 2\frac{\cos A\cos B}{\sin A\sin B} = 1 - 2\cot A\cot B.\end{aligned}$$

Similarly,

$$\frac{\cos A - \cos B\cos C}{\sin B\sin C} = 1 - 2\cot B\cot C \quad \text{and} \quad \frac{\cos B - \cos A\cos C}{\sin A\sin C} = 1 - 2\cot A\cot C.$$

Hence,

$$\frac{\cos A - \cos B\cos C}{\sin B\sin C} + \frac{\cos B - \cos A\cos C}{\sin A\sin C} + \frac{\cos C - \cos A\cos B}{\sin A\sin B}$$
$$= (1 - 2\cot A\cot B) + (1 - 2\cot B\cot C) + (1 - 2\cot A\cot C)$$
$$= 3 - 2(\cot A\cot B + \cot B\cot C + \cot A\cot C) = 3 - 2 = 1. \quad \square$$

Now, we are ready to state and prove our theorem.

Theorem 3. *Let $ABCD$ be a tetrahedron. The following statements are equivalent.*

1. *The tetrahedron $ABCD$ is isosceles.*
2. $\cos\sphericalangle AB + \cos\sphericalangle AC + \cos\sphericalangle AD = 1$,
   $\cos\sphericalangle AB + \cos\sphericalangle BC + \cos\sphericalangle BD = 1$
   $\cos\sphericalangle AC + \cos\sphericalangle BC + \cos\sphericalangle CD = 1$, *and*
   $\cos\sphericalangle AD + \cos\sphericalangle BD + \cos\sphericalangle CD = 1$.
   (This (2) is equivalent to

   $$\cos\sphericalangle AB + \cos\sphericalangle BC + \cos\sphericalangle CA = 1, \quad \cos\sphericalangle AB + \cos\sphericalangle BD + \cos\sphericalangle AD = 1,$$
   $$\cos\sphericalangle AC + \cos\sphericalangle CD + \cos\sphericalangle AD = 1, \quad \text{and} \quad \cos\sphericalangle BC + \cos\sphericalangle CD + \cos\sphericalangle BD = 1.$$

   See Remark 1 below.)
3. $\sphericalangle AB = \sphericalangle CD$, $\sphericalangle AC = \sphericalangle BD$, *and* $\sphericalangle AD = \sphericalangle BC$.
4. $\sphericalangle A = \sphericalangle B = \sphericalangle C = \sphericalangle D$.

Proof. Proof of (1) $\implies$ (2): Suppose the tetrahedron $ABCD$ is isosceles. Let $\alpha = \sphericalangle BDC$, $\beta = \sphericalangle ADC$, $\gamma = \sphericalangle ADB$. By (a'), we have

$$\cos\sphericalangle DA = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}, \quad \cos\sphericalangle DB = \frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma},$$
$$\text{and} \quad \cos\sphericalangle DC = \frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

Since all faces are congruent by Lemma 3, the angles α, β, γ are angles of a triangular face. In particular, we have $\alpha = \sphericalangle BAC$, $\beta = \sphericalangle ABC$, $\gamma = \sphericalangle ACB$ on the face ABC. Hence, by (c), we have $\cos\sphericalangle DA + \cos\sphericalangle DB + \cos\sphericalangle DC = 1$. Similarly, we have

$$\cos\sphericalangle AB + \cos\sphericalangle BC + \cos\sphericalangle BD = 1, \quad \cos\sphericalangle AC + \cos\sphericalangle BC + \cos\sphericalangle CD = 1,$$
$$\text{and} \quad \cos\sphericalangle AD + \cos\sphericalangle BD + \cos\sphericalangle CD = 1.$$

Proof of (2) $\implies$ (3): Suppose (2) holds. Let $s = \cos\sphericalangle AB$, $t = \cos\sphericalangle CD$, $u = \cos\sphericalangle AC$, $v = \cos\sphericalangle BD$, $w = \cos\sphericalangle AD$, and $x = \cos\sphericalangle BC$. Then the equations in (2) become $t+v+w = 1$, $t+u+x = 1$, $s+v+x = 1$, $s+u+w = 1$. From these, we have $v+w = u+x$, $v+x = u+w$ and $t+w = s+x$. From $v+w = u+x$ and $v+x = u+w$, we have $v+w = u+x$ and $v-w = u-x$. These two show that $v = u$ and $x = w$. Similarly, from $t+w = s+x$, we have $x = w$ and $t = s$. Hence, we have shown that $\cos\sphericalangle AB = \cos\sphericalangle CD$, $\cos\sphericalangle AC = \cos\sphericalangle BD$, and $\cos\sphericalangle AD = \cos\sphericalangle BC$. These prove that $\sphericalangle AB = \sphericalangle CD$, $\sphericalangle AC = \sphericalangle BD$, and $\sphericalangle AD = \sphericalangle BC$.

Proof of (3) $\implies$ (1): Suppose (3) holds. Let $\rho = \sphericalangle AB = \sphericalangle CD$, $\sigma = \sphericalangle AC = \sphericalangle BD$, and $\omega = \sphericalangle AD = \sphericalangle BC$. By equation (b'), we have

$$\frac{\cos\rho + \cos\sigma\cos\omega}{\sin\sigma\sin\omega} = \cos\sphericalangle ADB = \cos\sphericalangle ACB = \cos\sphericalangle CAD = \cos\sphericalangle CBD,$$
$$\frac{\cos\sigma + \cos\rho\cos\omega}{\sin\rho\sin\omega} = \cos\sphericalangle ADC = \cos\sphericalangle ABC = \cos\sphericalangle BAD = \cos\sphericalangle BCD,$$
$$\text{and}\quad \frac{\cos\omega + \cos\rho\cos\sigma}{\sin\rho\sin\sigma} = \cos\sphericalangle BDC = \cos\sphericalangle BAC = \cos\sphericalangle ABD = \cos\sphericalangle ACD.$$

Hence, we have

$$\sphericalangle ADB = \sphericalangle ACB = \sphericalangle CAD = \sphericalangle CBD,$$
$$\sphericalangle BAD = \sphericalangle ABC = \sphericalangle ADC = \sphericalangle BCD,$$
$$\text{and}\quad \sphericalangle ABD = \sphericalangle BAC = \sphericalangle ACD = \sphericalangle BDC.$$

These prove that triangles ABD, ABC, ACD, and BCD are congruent. Therefore, the tetrahedron $ABCD$ is isosceles by Lemma 3.

Proof of (3) $\implies$ (4): Suppose we have $\sphericalangle AD = \sphericalangle BC$, $\sphericalangle BD = \sphericalangle AC$, and $\sphericalangle CD = \sphericalangle AB$. This shows that $\sphericalangle A = \sphericalangle B = \sphericalangle C = \sphericalangle D$ by the definition of a solid angle.

Proof of (4) $\implies$ (3): Suppose $\sphericalangle A = \sphericalangle B = \sphericalangle C = \sphericalangle D$. We have

$$\begin{aligned}
&\text{(i)} && \sphericalangle AD + \sphericalangle BD + \sphericalangle CD = \sphericalangle D + \pi,\\
&\text{(ii)} && \sphericalangle AD + \sphericalangle AB + \sphericalangle AC = \sphericalangle A + \pi = \sphericalangle D + \pi,\\
&\text{(iii)} && \sphericalangle AB + \sphericalangle BD + \sphericalangle BC = \sphericalangle B + \pi = \sphericalangle D + \pi,\\
&\text{(iv)} && \sphericalangle AC + \sphericalangle BC + \sphericalangle CD = \sphericalangle C + \pi = \sphericalangle D + \pi
\end{aligned}$$

From these, we obtain a system of three homogeneous equations

$$\begin{aligned}
&\text{(ii)} - \text{(i)}: && (\sphericalangle AB - \sphericalangle CD) + (\sphericalangle AC - \sphericalangle BD) = 0,\\
&\text{(iii)} - \text{(i)}: && (\sphericalangle AB - \sphericalangle CD) + (\sphericalangle BC - \sphericalangle AD) = 0,\\
&\text{(iv)} - \text{(i)}: && (\sphericalangle AC - \sphericalangle BD) + (\sphericalangle BC - \sphericalangle AD) = 0,
\end{aligned}$$

in three variables $(\sphericalangle AB - \sphericalangle CD)$, $(\sphericalangle AC - \sphericalangle BD)$, and $(\sphericalangle BC - \sphericalangle AD)$. Therefore, we have $(\sphericalangle AB - \sphericalangle CD) = (\sphericalangle AC - \sphericalangle BD) = (\sphericalangle BC - \sphericalangle AD) = 0$, i.e., $\sphericalangle AD = \sphericalangle BC$, $\sphericalangle BD = \sphericalangle AC$, and $\sphericalangle CD = \sphericalangle AB$. □

We will use the next lemma in the proof of Theorem 4 in the next section. But since its statement is similar to Theorem 3, we include it here for a comparison.

Lemma 5. *A tetrahedron $ABCD$ is isosceles if, and only if, the sum of three angles at each vertex equal π, i.e.,*

$$\sphericalangle BDC + \sphericalangle CDA + \sphericalangle ADB = \pi,$$
$$\sphericalangle ACB + \sphericalangle ACD + \sphericalangle BCD = \pi,$$
$$\sphericalangle ABC + \sphericalangle ABD + \sphericalangle CBD = \pi, \text{ and}$$
$$\sphericalangle BAC + \sphericalangle BAD + \sphericalangle CAD = \pi.$$

Proof. If a tetrahedron is isosceles, then the sum of angles at a vertex is π by Lemma 3. For the converse, see Problem 498 of [2]. □

Remark 1. Alternately, the implication (1) $\Longrightarrow$ (2) in Theorem 3 can be proven without (a') and (c) as follows: If we denote the area of a triangle ABC by $\mathcal{A}_{ABC}$, and if $ABCD$ is a tetrahedron, then we have $\mathcal{A}_{ABC} = \mathcal{A}_{ABD} \cos \sphericalangle AB + \mathcal{A}_{BCD} \cos \sphericalangle BC + \mathcal{A}_{ACD} \cos \sphericalangle AC$. We can see this equation by projecting triangular faces ABD, BCD, and ACD onto the face ABC. So, if the tetrahedron $ABCD$ is isosceles, then $\mathcal{A}_{ABC} = \mathcal{A}_{ABD} = \mathcal{A}_{BCD} = \mathcal{A}_{ACD}$ by Lemma 3. This proves that

(d) $\cos \sphericalangle AB + \cos \sphericalangle BC + \cos \sphericalangle CA = 1$. Similarly, we have
(e) $\cos \sphericalangle AB + \cos \sphericalangle BD + \cos \sphericalangle AD = 1$,
(f) $\cos \sphericalangle AC + \cos \sphericalangle CD + \cos \sphericalangle AD = 1$, and
(g) $\cos \sphericalangle BC + \cos \sphericalangle CD + \cos \sphericalangle BD = 1$.

Performing (d) + (e) + (f) − (g) gives us $\cos \sphericalangle AB + \cos \sphericalangle BC + \cos \sphericalangle CA = 1$. We can also confirm other three identities in (2). Similarly, the four equations in (2) imply (d)–(g).

Corollary 1. *An isosceles tetrahedron $ABCD$ has the property*

$$\cos \sphericalangle AB + \cos \sphericalangle AC + \cos \sphericalangle AD + \cos \sphericalangle BC + \cos \sphericalangle CD + \cos \sphericalangle BD = 2.$$

Proof. Adding all four equations in (2) of Theorem 3, we obtain $2(\cos \sphericalangle AB + \cos \sphericalangle AC + \cos \sphericalangle AD + \cos \sphericalangle BC + \cos \sphericalangle CD + \cos \sphericalangle BD) = 4$. This implies this corollary. □

We do not know much about the converse of Corollary 1 except to make the following conjecture.

Conjecture 1. If $ABCD$ is a tetrahedron, then $\cos \sphericalangle AB + \cos \sphericalangle AC + \cos \sphericalangle AD + \cos \sphericalangle BC + \cos \sphericalangle CD + \cos \sphericalangle BD \leq 2$. The equality holds if, and only if, the tetrahedron is isosceles.

Corollary 2. *The solid angle of an isosceles tetrahedron at a vertex is at most $2(\pi - 3\cos^{-1}\frac{1}{\sqrt{3}})$. The maximum value of a solid angle among all isosceles tetrahedra is attained only when an isosceles tetrahedron is regular.*

Proof. Let $ABCD$ be an isosceles tetrahedron. Then

$$\sphericalangle A + \sphericalangle B + \sphericalangle C + \sphericalangle D \leq 8\left(\pi - 3\cos^{-1}\frac{1}{\sqrt{3}}\right)$$

by Theorem 3(1) in [3], with the equality holding only when the tetrahedron is regular. Since $\sphericalangle A = \sphericalangle B = \sphericalangle C = \sphericalangle D$ by equation Theorem 3, we have this corollary. □

Definition 4. Two tetrahdra $ABCD$ and $A'B'C'D'$ are *similar* if

$$\frac{\overline{AB}}{\overline{A'B'}} = \frac{\overline{AC}}{\overline{A'C'}} = \frac{\overline{AD}}{\overline{A'D'}} = \frac{\overline{BC}}{\overline{B'C'}} = \frac{\overline{BD}}{\overline{B'D'}} = \frac{\overline{CD}}{\overline{C'D'}}.$$

In other words, two tetrahedra $ABCD$ and $A'B'C'D'$ are similar if, and only if, $\triangle ABC \approx \triangle A'B'C'$, $\triangle ABD \approx \triangle A'B'D'$, $\triangle ACD \approx \triangle A'C'D'$, and $\triangle BCD \approx \triangle B'C'D'$. (Here, by $\triangle ABC \approx \triangle A'B'C'$, we mean the triangles ABC and $A'B'C'$ are similar.) Hence, by Theorem 2 and by the definition of a solid angle, two similar tetrahedra have the same solid angles at each corresponding vertex. On the other hand, having the same solid angles at each corresponding vertex *does not* imply that the two tetrahedra are similar as we will show in the next example.

Example 1. Equation (4) in Theorem 3 suggests that a solid angle may uniquely determine an isosceles tetrahedron up to a similarity. However, we will show the existence of two *non-similar* isosceles tetrahedra $ABCD$ and $A'B'C'D'$ such that their solid angles are equal. That is, two non-similar isosceles tetrahedra can have the equal solid angle.

We will construct two one-parameter families of tetrahedra $ABCD = T(x)$ and $A'B'C'D' = U(t)$ to show this.

Let $x \geq 1$. Let $A = (x, 1, 1)$, $B = (-x, -1, 1)$, $C = (x, -1, -1)$, $D = (-x, 1, -1)$. Then $AB = CD = AD = BC = 2\sqrt{x^2+1}$ and $AC = BD = 2\sqrt{2}$. Let us denote the one parameter family of isosceles tetrahedra $ABCD$ by $T(x)$ for each $x \geq 1$. The vectors $\vec{l} = \langle 1, -x, x\rangle$, $\vec{m} = \langle -1, x, x\rangle$, $\vec{n} = \langle 1, x, -x\rangle$ are normal to the three faces ABC, ABD, and ACD, respectively. These imply that

$$\cos\sphericalangle AB = -\frac{\vec{l}\cdot\vec{m}}{|\vec{l}||\vec{m}|} = \frac{1}{2x^2+1}, \quad \cos\sphericalangle AC = -\frac{\vec{l}\cdot\vec{n}}{|\vec{l}||\vec{n}|} = \frac{2x^2-1}{2x^2+1},$$
$$\cos\sphericalangle AD = -\frac{\vec{m}\cdot\vec{n}}{|\vec{m}||\vec{n}|} = \frac{1}{2x^2+1}.$$

Let

$$f(x) = 2\cos^{-1}\frac{1}{2x^2+1} + \cos^{-1}\frac{2x^2-1}{2x^2+1} - \pi.$$

Then, the function $f(x)$ assigns the solid angle to the tetrahedra $T(x)$ at the vertex $A = A(x)$. Hence, it is the solid angle at each vertex of the isosceles tetrahedron $T(x)$ for each $x \geq 1$ by Theorem 3.

Let $t \geq 1$. Let $A' = (t, 2t, 1)$, $B' = (-t, -2t, 1)$, $C' = (t, -2t, -1)$, $D' = (-t, 2t, -1)$. Then $A'B' = C'D' = 2t\sqrt{5}$, $A'D' = B'C' = 2\sqrt{t^2+1}$, and $A'C' = B'D' = 2\sqrt{4t^2+1}$. Let $U(t)$ denote the one-parameter family of isosceles tetrahedra $A'B'C'D'$ for each $t \geq 1$. Then, the vectors $\vec{p} = \langle 2, -1, 2t\rangle$, $\vec{q} = \langle -2, 1, 2t\rangle$, $\vec{r} = \langle 2, 1, -2t\rangle$ are normal to the three faces $A'B'C'$, $A'B'D'$, and $A'C'D'$, respectively. From these, we have

$$\cos\sphericalangle A'B' = -\frac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|} = \frac{-4t^2+5}{4t^2+5}, \quad \cos\sphericalangle A'C' = -\frac{\vec{p}\cdot\vec{r}}{|\vec{p}||\vec{r}|} = \frac{-4t^2-3}{4t^2+5},$$
$$\cos\sphericalangle A'D' = -\frac{\vec{q}\cdot\vec{r}}{|\vec{q}||\vec{r}|} = \frac{-4t^2+3}{4t^2+5}.$$

Let

$$g(t) = \cos^{-1}\frac{-4t^2+5}{4t^2+5} + \cos^{-1}\frac{4t^2-3}{4t^2+5} + \cos^{-1}\frac{4t^2+3}{4t^2+5} - \pi.$$

Then, the function $g(t)$ assigns the solid angle to the tetrahedra $U(t)$ at the vertex $A' = A'(t)$, and hence, it is the solid angle at each vertex of the isosceles tetrahedron $U(t)$ for each $t \geq 1$.

According to Definition 4, two similar tetrahedra must have similar triangular faces. Note that for a fixed $t > 1$, $U(t)$ is not similar to $T(x)$ for any $x > 1$ since the faces of $T(x)$ are isosceles triangles for all $x > 1$ while the faces of the tetrahedron $U(t)$ are never isosceles triangles for any $t > 1$. Therefore, $T(x)$ and $U(t)$ are not similar. Since $T(1)$ is a regular tetrahedron, we have that $f(1) = 3\cos^{-1}\frac{1}{3} - \pi$ (see Corollary 2 and Remark 2, below). So, $g(1) < f(1)$ by Corollary 2. (Note that $g(1) = 2\cos^{-1}\frac{1}{9} + \cos^{-1}\frac{7}{9} - \pi \approx 0.4569$ and $f(1) \approx 0.5512$.) Since $\lim_{x\to\infty} f(x) = 0$ and $\lim_{t\to\infty} g(t) = 0$, there is an $x > 1$ such that $f(x) = g(t)$ for each $t > 1$ by the continuity of f. Therefore, we have shown that a solid angle does not uniquely determine an isosceles tetrahedron up to a similarity. This also shows that having the same solid angles at each corresponding vertex *does not* imply that the two tetrahedra are similar.

Even though two non-similar tetrahedra can have equal solid angles, the regular tetrahedron is an exception.

Corollary 3. *A tetrahedron has four equal solid angles equal to* $2\left(\pi - 3\cos^{-1}\frac{1}{\sqrt{3}}\right)$ *if, and only if, it is a regular tetrahedron.*

Proof. The proof is obtained by equation (4) of Theorem 3 and Corollary 2. □

Remark 2. Note that the dihedral angle of a regular tetrahedron can be represented by $\pi - 2\cos^{-1}\frac{1}{\sqrt{3}} = 2\tan^{-1}\frac{1}{\sqrt{2}} = \cos^{-1}\frac{1}{3}$. Hence, we have $2(\pi - 2\cos^{-1}\frac{1}{\sqrt{3}}) = 6\tan^{-1}\frac{1}{\sqrt{2}} - \pi = 3\cos^{-1}\frac{1}{3} - \pi$.

3 A Trigonometric Identity

The next theorem resembles the identity $\tan^{-1}\alpha + \tan^{-1}\frac{1}{\alpha} = \frac{\pi}{2}$ for any $\alpha > 0$. There are many known trigonometric identities. But to the best of our knowledge, we think Theorem 4 is new. We prove this theorem using Lemma 5 on an isosceles tetrahedron rather than a triangle.

Theorem 4. *For any* α, β, $\gamma > 0$, *we have*

$$\cos^{-1}\frac{\alpha}{\sqrt{(\alpha+\beta)(\alpha+\gamma)}} + \cos^{-1}\frac{\beta}{\sqrt{(\alpha+\beta)(\beta+\gamma)}} + \cos^{-1}\frac{\gamma}{\sqrt{(\alpha+\gamma)(\beta+\gamma)}} = \pi.$$

Proof. Let $a, b, c > 0$. Let $D = (a, b, c)$, $A = (a, -b, -c)$, $B = (-a, b, -c)$, $C = (-a, -b, c)$. Then $ABCD$ is an isosceles tetrahedron, and $\overrightarrow{DA} = -2\langle 0, b, c\rangle$, $\overrightarrow{DB} = -2\langle a, 0, c\rangle$, $\overrightarrow{DC} = -2\langle a, b, 0\rangle$. So $\overrightarrow{DA} \times \overrightarrow{DB} = 4\,\langle 0, b, c\rangle \times \langle a, 0, c\rangle = 4\,\langle bc, ca, -ab\rangle$. Thus, a normal vector to the plane DAB is $\vec{u} = \langle bc, ca, -ab\rangle$. Similarly, normal vectors to the planes DBC and DCA are $\vec{v} = \langle -bc, ca, ab\rangle$ and $\vec{w} = \langle bc, -ca, ab\rangle$, respectively. Note that these three normal vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are pointing outward of the tetrahedron $ABCD$. Hence,

$$\cos\sphericalangle DC = -\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{-a^2b^2 + b^2c^2 + c^2a^2}{a^2b^2 + b^2c^2 + c^2a^2}.$$

Similary,

$$\cos\sphericalangle DA = -\frac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\|\vec{w}\|} = \frac{a^2b^2 - b^2c^2 + c^2a^2}{a^2b^2 + b^2c^2 + c^2a^2} \quad \text{and}$$
$$\cos\sphericalangle DB = -\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{a^2b^2 + b^2c^2 - c^2a^2}{a^2b^2 + b^2c^2 + c^2a^2}.$$

Also, we have

$$\sin^2\sphericalangle DC = 1 - \left(\frac{-a^2b^2 + b^2c^2 + c^2a^2}{a^2b^2 + b^2c^2 + c^2a^2}\right)^2 = \frac{4a^2b^2c^2(a^2+b^2)}{(a^2b^2 + b^2c^2 + c^2a^2)^2}.$$

Thus, we have

$$\sin\sphericalangle DC = \frac{2abc\sqrt{a^2+b^2}}{a^2b^2 + b^2c^2 + c^2a^2}.$$

Similary,

$$\sin \sphericalangle DA = \frac{2abc\sqrt{b^2+c^2}}{a^2b^2+b^2c^2+c^2a^2}, \quad \text{and} \quad \sin \sphericalangle DB = \frac{2abc\sqrt{a^2+c^2}}{a^2b^2+b^2c^2+c^2a^2}.$$

By (a'), we have

$$\begin{aligned}\cos \sphericalangle BDC &= \frac{\cos \sphericalangle DA + \cos \sphericalangle DB \cos \sphericalangle DC}{\sin \sphericalangle DB \sin \sphericalangle DC} \\ &= \frac{(a^2b^2+b^2c^2+c^2a^2)(a^2b^2-b^2c^2+c^2a^2)+(a^2b^2+b^2c^2-c^2a^2)(-a^2b^2+b^2c^2+c^2a^2)}{(2abc\sqrt{a^2+c^2})(2abc\sqrt{a^2+b^2})} \\ &= \frac{a^2}{\sqrt{(a^2+c^2)(a^2+b^2)}}.\end{aligned}$$

Let $\alpha = a^2$, $\beta = b^2$, $\gamma = c^2$. Hence,

$$\sphericalangle BDC = \cos^{-1}\frac{a^2}{\sqrt{(a^2+c^2)(a^2+b^2)}} = \cos^{-1}\frac{\alpha}{\sqrt{(\alpha+\beta)(\alpha+\gamma)}}.$$

Similarly,

$$\begin{aligned}\sphericalangle CDA &= \cos^{-1}\frac{b^2}{\sqrt{(a^2+b^2)(b^2+c^2)}} = \cos^{-1}\frac{\beta}{\sqrt{(\alpha+\beta)(\beta+\gamma)}} \quad \text{and} \\ \sphericalangle ADB &= \cos^{-1}\frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}} = \cos^{-1}\frac{\gamma}{\sqrt{(\alpha+\gamma)(\beta+\gamma)}}.\end{aligned}$$

Since the tetrahedron $ABCD$ is isosceles, we have that $\sphericalangle BDC + \sphericalangle CDA + \sphericalangle ADB = \pi$ by Lemma 5. This implies

$$\cos^{-1}\frac{\alpha}{\sqrt{(\alpha+\beta)(\alpha+\gamma)}} + \cos^{-1}\frac{\beta}{\sqrt{(\alpha+\beta)(\beta+\gamma)}} + \cos^{-1}\frac{\gamma}{\sqrt{(\alpha+\gamma)(\beta+\gamma)}} = \pi. \qquad \square$$

References

[1] N. Altshiller-Court: *Modern Pure Solid Geometry.* The Macmillan Co., New York, 1935.

[2] E. J. Barbeau, E. Barbeau, M. S. Klamkin, W. O. Moser, W. Moser, et al.: *Five Hundred Mathematical Challenges.* The Mathematical Association of America, 1995.

[3] H. Katsuura: *Solid Angle Sum of a Tetrahedron.* J. Geom. Graph. **24**(1), 29–34, 2020.

[4] A. N. Pressley: *Elementary Differential Geometry.* Springer Science & Business Media, 2010.

Internet Sources

[5] S. Mohammad: *Proof of* $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$, 2015. `https://www.askiitians.com/forums/Trigonometry/proof-of-cotacotb-cotbcotc-cotccota-1_97408.htm`.

Received October 26, 2021; final form January 31, 2022.