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Dihedral Angles of 4-Ball Tetrahedra

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Abstract. A tetrahedron is a 4-ball tetrahedron if there are four externally tangent spheres centered at the vertices of the tetrahedron. It is known that a tetrahedron being a 4-ball tetrahedron is equivalent to (1) three pairs of the sum of opposing edge lengths are the same, and to (2) there is a sphere tangent to each edge of the tetrahedron. We will prove that a tetrahedron is a 4-ball tetrahedron if, and only if three pairs of the sums of opposing dihedral angles are the same.

Key Words: 4-ball tetrahedron, balloon tetrahedron, edge-additive tetrahedron, edge-tangent sphere, circumscriptable tetrahedron, edge-incentric tetrahedron, dihedral-angle-additive tetrahedron

MSC 2020: 51M15 (primary), 51M04

1 Introduction

A molecule of methane CH_4 has four spherical hydrogen (H) atoms, the centers of hydrogen atoms are bonded by way of a carbon (C) atom in the center, equidistant from each H atom. Each $H-C$ bond is 1.09 angstroms (1 angstrom $= 10^{-10}$ meters) between atom centers, and the bond angle defined by $H-C-H$ is $\cos^{-1}\left(-\frac{1}{3}\right) \approx 109.47°$. Each face of the tetrahedron defined by four H atoms form an equilateral triangle. The distance between the centers of H atoms is 1.78 angstroms.

Tetra phosphorus P_4 is the gaseous form of white phosphorus. White phosphorus is pyrophoric, is very dangerous, and is the main ingredient of napalm. When white phosphorus is polymerized, it becomes red phosphorus. Red phosphorus is much stable, and is used for the tip of the matches. A molecule of tetra phosphorus P_4 has four spherical P atoms bonded directly to each other, the centers of P atoms forming a regular tetrahedron. Unlike the methane molecule, the tetra phosphorus molecule has no central atom. The $P-P$ bond length is 2.25 angstroms.

We learned this chemistry from our chemist friend Dr. George Cabaniss recently, and this motivated us to investigate a tetrahedron formed by connecting the centers of four mutually tangent spheres, not necessarily of the same radius.

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Definition 1. A tetrahedron is a *4-ball tetrahedron* [6], or a *balloon tetrahedron* ([2, page 146]) if there are four externally tangent spheres centered at the vertices of the tetrahedron, and we say that these four mutually and externally tangent spheres *generate* a 4-ball tetrahedron.

It is possible to place three mutually and externally tangent circles centered at vertices of any triangle. However, it is not possible to place four mutually and externally tangent spheres centered at vertices of any tetrahedron.

Definition 2. We denote by AB the segment AB as well as its length. If a tetrahedron $ABCD$ has the property $AB + CD = AC + BD = AD + BC$, then the tetrahedron $ABCD$ is said to be *edge-additive*.

Definition 3. If a tetrahedron has a unique sphere tangent to all six edges, then the sphere is called an *edge-tangent sphere* of the tetrahedron, and the tetrahedron is said to be *circumscriptable* [1] or *edge-incentric* [4]. (In [4], a "3-intouch sphere" is used for a three-dimensional edge-tangent sphere since the author was considering higher dimensional spaces.)

The following basic result can be found in [1, Chapter IX, B].

Theorem 1. *Let $ABCD$ be a tetrahedron. The following statements are equivalent.*

1. *The tetrahedron $ABCD$ is a 4-ball tetrahedron.*
2. *The tetrahedron $ABCD$ is edge-additive.*
3. *The tetrahedron $ABCD$ has an edge-tangent sphere.*

There are many more basic theorems on a 4-ball tetrahedron. In order to make the narrative simpler and concise, we summarize them in the next Remarks. For more detail, please see [6] in addition to [1]. For more detail and for higher dimensional analog, please see [4].

Remarks: Let S_A, S_B, S_C, S_D be the four spheres that generate a 4-ball tetrahedron $ABCD$ centered at A, B, C, D, respectively. Let E, F, G, H, I, J be the points of tangency of the generating spheres on the edges AB, BC, CA, DB, DC, DA, respectively. So, $S_A \cap S_B = \{E\}$, $S_B \cap S_C = \{F\}$, $S_C \cap S_D = \{G\}$, $S_D \cap S_B = \{H\}$, $S_D \cap S_C = \{I\}$, $S_D \cap S_A = \{J\}$. Thus, for example, $AE = AG = AJ$ is the radius of the sphere S_A. Let S be the edge-tangent sphere of a 4-ball tetrahedron $ABCD$. Then S meets edges AB BC, CA, DB, DC, DA at E, F, G, H, I, J,respectively. The intersection of the edge-tangent sphere S with the surface of the tetrahedron $ABCD$ are four incircles U_A, U_B, U_C, U_D on the faces BCD, ACD, ABD, and ABC, respectively. For each $X = E$, F, G, H, I, J, let Γ_X be the plane through the point X normal to the edge of the tetrahedron that contains X. Then for example, $\Gamma_E \cap \Gamma_F \cap \Gamma_G$ is a line normal to the face ABC, and intersection of this normal line with the face ABC, denoted by D', is the center of U_D. The intersection of the six planes $\Gamma_E \cap \Gamma_F \cap \Gamma_G \cap \Gamma_H \cap \Gamma_I \cap \Gamma_J$ is a point P, and this point P is the center of the edge-tangent sphere S. Thus, for example, E, F, G are the feet of the normal line from P to the edges AB, BC, and CA, respectively, and $D'E = D'F = D'G$ is the radius of U_D.

The center P of the edge-tangent sphere S can be inside as well as outside of the tetrahedron $ABCD$. The tetrahedron $ABCD$ in Figure 1 is a regular tetrahedron, so it is a 4-ball tetrahedron. Hence, the point P is its center, and is inside of the tetrahedron $ABCD$. The vertices of the tetrahedron $ABCD$ in Figure 2 are $B = (4, 0, 0)$, $C = (-2, 2\sqrt{3}, 0)$, $D = (-2, -2\sqrt{3}, 0)$, and $A = (0, 0, 1)$. Since the face BCD is an equilateral triangle, and since $AB = AC = AD$, $ABCD$ is a 4-ball tetrahedron by Theorem 1.2. The point F is the midpoint of BC. Since $BF = BE$, it can be shown that $E = \left(4 - 8\left(\frac{3}{17}\right)^{\frac{1}{2}}, 0, 16 - 32\left(\frac{3}{17}\right)^{\frac{1}{2}}\right)$. Thus, the point P is the intersection of Γ_E and the z-axis. From this, we can show that $P = (0, 0, 2\sqrt{51} - 16)$. Because $2\sqrt{51} - 16 < 0$, P is outside of the tetrahedron $ABCD$. The radius r of the edge-tangent sphere S is given by $r^2 = 464 - 64\sqrt{51}$.

We need two additional definitions.

Definition 4. The *dihedral angle* at the edge AB of a tetrahedron $ABCD$ is the inside angle between the triangular faces CAB and DAB, and it is denoted by $(C, \overline{AB}, D)$ or $\overline{AB}$ when there is no confusion.

Definition 5. A tetrahedron $ABCD$ is said to be *dihedral-angle-additive* if $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD} = \overline{AD} + \overline{BC}$.

Our main theorem is to prove that a tetrahderon is a 4-ball tetrahedron if, and only if it is dihedral-angle-additive (Theorem 2). Please contrast this to Theorem 1.2. As far as we can tell, this result seems new.

2 The Main Result

Theorem 2. *A tetrahedron is a 4-ball tetrahedron if, and only if it is dihedral-angle-additive.*

We divide the proof into two parts. We use the notations introduced in Remarks.

Proof of Theorem 2, Part 1: Suppose $ABCD$ is a 4-sphere tetrahedron. We will show that it is dihedral-angle-additive. Let D', C', B', A' be the centers of the incircles U_D, U_C, U_B, U_A of triangular faces ABC, ABD, ACD, BCD, respectively. Recall P is the center of the edge-tangent sphere S, and E, F, G are the feet of normal lines from P to the edge AB, BC, CA, respectively.

Consider the tetrahedron $ABCP$. The planes $\Gamma_E = PD'E$, $\Gamma_F = PD'F$, $\Gamma_G = PD'G$ are normal to the edges AB, BC, CA, respectively. So, we have $(P, \overline{AB}, C) = \angle PED'$, $(P, \overline{BC}, A) = \angle PFD'$, and $(P, \overline{CA}, B) = \angle PGD'$. $D'E = D'F = D'G$ is the radius of the incircle U_D. The segment PD' is shared by the triangles PED', PFD', PGD'. Hence, the triangles PED', PFD', PGD' are congruent. So, we have $\angle PED' = \angle PFD' = \angle PGD'$. This shows that $(P, \overline{AB}, C) = (P, \overline{BC}, A) = (P, \overline{CA}, B)$. Like Figures 1 and 2, P and D can be on the same side of the plane ABC, or P and D can be on opposite sides of the plane ABC. Let $\theta(D') = (P, \overline{AB}, C) = (P, \overline{BC}, A) = (P, \overline{CA}, B)$ if P and D are on the same side of the plane ABC, and $\theta(D') = -(P, \overline{BC}, A) = -(P, \overline{CA}, B)$ if P and D are on opposite sides of the plane ABC.

Similarly, we have

$$(P, \overline{AB}, D) = (P, \overline{BD}, A) = (P, \overline{DA}, B), \quad (P, \overline{AC}, D) = (P, \overline{CD}, A) = (P, \overline{DA}, C),$$
$$\text{and} \quad (P, \overline{CB}, D) = (P, \overline{BD}, C) = (P, \overline{DC}, B).$$

Let $\theta(C') = (P, \overline{AB}, D) = (P, \overline{BD}, A) = (P, \overline{DA}, B)$ if P and C are on the same side of the plane ABD, and let $\theta(C') = -(P, \overline{AC}, D) = -(P, \overline{BD}, A) = -(P, \overline{DA}, B)$ if P and C are on opposite sides of the plane ABD.

Let $\theta(B') = (P, \overline{AC}, D) = (P, \overline{CD}, A) = (P, \overline{DA}, C)$ if P and B are on the same side of the plane ACD, and let $\theta(B') = -(P, \overline{AC}, D) = -(P, \overline{CD}, A) = -(P, \overline{DA}, C)$ if P and B are on opposite sides of the plane ACD.

Let $\theta(A') = (P, \overline{CB}, D) = (P, \overline{BD}, C) = (P, \overline{DC}, B)$ if P and A are on the same side of the plane BCD, and let $\theta(A') = -(P, \overline{CB}, D) = -(P, \overline{BD}, C) = -(P, \overline{DC}, B)$ if P and A are on opposite sides of the plane BCD. These cases are portrayed in Figures 1 and 2.

Then, we have

$$\begin{aligned} \overline{AB} &= (C, \overline{AB}, D) = (P, \overline{AB}, C) + (P, \overline{AB}, D) = \theta(D') + \theta(C'), \\ \text{and} \quad \overline{CD} &= (A, \overline{CD}, B) = (P, \overline{CD}, A) + (P, \overline{CD}, B) = \theta(B') + \theta(A'). \end{aligned}$$

So $\overline{AB} + \overline{CD} = \theta(A') + \theta(B') + \theta(C') + \theta(D')$. Similarly, we can show that $\overline{AC} + \overline{BD} = \theta(A') + \theta(B') + \theta(C') + \theta(D') = \overline{AD} + \overline{BC}$. This proves that $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD} = \overline{AD} + \overline{BC}$. Therefore, the 4-ball tetrahedron $ABCD$ is dihedral-angle-additive. □

The proof of the converse is Part 2, and it is long. We need the following lemmas.

Lemma 1. *Let $ABCD$ be a tetrahedron. Then we have the following equation.*

$$\cos \overline{AD} = \frac{\cos(\angle BDC) - \cos(\angle CDA)\cos(\angle ADB)}{\sin(\angle CDA)\sin(\angle ADB)}. \qquad \textit{See [5, page 940].}$$

(For your information,

$$\cos(\angle ABC) = \frac{\cos \overline{BD} + \cos \overline{AB} \cos \overline{BC}}{\sin \overline{AB} \sin \overline{BC}}.$$

See [3, page 731].)

Lemma 2. *Let m be a line in a plane Γ. Let P be a point not on Γ. Let O be the point on Γ such that PO is normal to Γ, and let X be the point on the line m such that PX is normal to m. Then the line OX is normal to m.*

Proof. Since the plane OPX is normal to the line m, the line OX is normal to m. □

Lemma 3 below is the key to prove Part 2 of the proof of Theorem 2.

Lemma 3. *Let $ABCD$ be a tetrahedron.*

1. *Let X, Y, Z be points on the edges AD, BD, CD, respectively, such that $DX = DY = DZ$ in length. Let $\Gamma_X, \Gamma_Y, \Gamma_Z$ be planes through X, Y, Z normal to the edges AD, BD, CD, respectively. Let Q be the intersection of these three planes Γ_X, Γ_Y, and Γ_Z. Then Q is a point such that $(Q, \overline{AD}, B) = (Q, \overline{BD}, A)$, $(Q, \overline{BD}, C) = (Q, \overline{CD}, B)$ and $(Q, \overline{CD}, A) = (Q, \overline{AD}, C)$. (See Figure 3.) (Note that Q and C may be on the opposite sides of the plane ABD. See Example 1 below.)*
2. *If P is a point on the ray DQ different from D, then we have $(P, \overline{AD}, B) = (P, \overline{BD}, A)$, $(P, \overline{BD}, C) = (P, \overline{CD}, B)$ and $(P, \overline{CD}, A) = (P, \overline{AD}, C)$.*
3. *Suppose P is a point such that $(P, \overline{AD}, B) = (P, \overline{BD}, A)$, $(P, \overline{BD}, C) = (P, \overline{CD}, B)$, and $(P, \overline{CD}, A) = (P, \overline{AD}, C)$. Let X, Y, Z be the feet of the line through P normal to the line AD, BD, CD respectively. Then $DX = DY = DZ$.*

Proof. 1. Note that $\angle DXQ = \angle DYQ = \frac{\pi}{2}$. Since $DX = DY$ and the edge DQ is shared by triangles $\triangle DQX$ and $\triangle DYQ$, we have $\triangle DQX \equiv \triangle DYQ$ (this means triangles DQX and DYQ are congruent). So $QX = QY$ so that $\triangle QXY$ is isosceles. Also, $\triangle DXY$ is isosceles. Thus, $\angle QXY = \angle QYX$ and $\angle DXY = \angle DYX$. As we said earlier, we have $\angle DXQ = \angle DYQ$. Now, we apply Lemma 1 to the dihedral angles $(Q, \overline{XD}, B)$ and $(Q, \overline{YD}, A)$ of the tetrahedron $DQYX$, and we have

$$\begin{aligned}\cos(Q, \overline{XD}, B) &= \frac{\cos(\angle QXY) - \cos(\angle DXQ)\cos(\angle DXY)}{\sin(\angle DXQ)\sin(\angle DXY)} \\ &= \frac{\cos(\angle QYX) - \cos(\angle DYQ)\cos(\angle DYX)}{\sin(\angle DYQ)\sin(\angle DYX)} \\ &= \cos(Q, \overline{YD}, A).\end{aligned}$$

Note that $(Q, \overline{AD}, B) = (Q, \overline{XD}, B)$ and $(Q, \overline{YD}, A) = (Q, \overline{BD}, A)$. This proves that $(Q, \overline{AD}, B) = (Q, \overline{BD}, A)$. Similarly, we can prove that $(Q, \overline{BD}, C) = (Q, \overline{CD}, B)$, and $(Q, \overline{CD}, A) = (Q, \overline{AD}, C)$.

2. Since P is on the ray DQ different from D, P is on the half plane QDA on the side of the line DA as Q, so that $(Q, \overline{AD}, B) = (P, \overline{AD}, B)$. Similarly, P is on the half plane QDB on the side of the line DB as Q, so that $(Q, \overline{BD}, A) = (P, \overline{BD}, A)$. By 1., we have $(Q, \overline{AD}, B) = (Q, \overline{BD}, A)$. Hence, $(P, \overline{AD}, B) = (P, \overline{BD}, A)$. Similarly, we can prove $(P, \overline{BD}, C) = (P, \overline{CD}, B)$, and $(P, \overline{CD}, A) = (P, \overline{AD}, C)$.
3. Let C' be the foot of the line through P normal to the plane DAB. Then $\angle PXC' = (P, \overline{AD}, B) = (P, \overline{BD}, A) = \angle PYC'$. Also, $\angle PC'X = \angle PC'Y$ and segment PC' is shared by both $\triangle PXC'$ and $\triangle PYC'$. Thus, $\triangle PXC' \equiv \triangle PYC'$ so that $XC' = YC'$. Since PE is normal to the plane ADB, and since PX is normal to AD, XC' and AD are normal by Lemma 2. So, we have $\angle DXC' = \frac{\pi}{2}$. Similarly, $\angle DYC' = \frac{\pi}{2}$. Hence, we have $\triangle DXC' \equiv \triangle DYC'$. This proves that $DX = DY$. Similarly, we can show that $DX = DZ$. □

Example 1. In Lemma 3.1, let $A = X = (1, 0, 0)$, $B = Y = (0, 1, 0)$, $C = Z = (0, 0, 1)$, and $D = (0, 0, 0)$. Then $Q = (1, 1, 1)$, and Q and C are on the same side of the plane ABD. And it appears that for any tetrahedron $ABCD$, the point Q defined in Lemma 3.1, Q and C are on the same side of the plane ABD, but this is not the case.

Let $A = X = (1, 0, 0)$, $B = Y = (0, 1, 0)$, $C = Z = \left(\frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{1}{2}\right)$, and $D = (0, 0, 0)$. Then it can be shown that $Q = \left(1, 1, -\frac{\sqrt{3}-1}{2}\right)$. Since the z-coordinate of Q is negative, Q and C are on opposite sides of the ABD plane (=xy-plane).

Proof of Theorem 2, Part 2: Suppose a tetrahedron $ABCD$ is dihedral-angle-additive, i.e., $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD} = \overline{AD} + \overline{BC}$. We will prove that the tetrahedron $ABCD$ is a 4-ball tetrahedron.

Let us denote the point Q in Lemma 3.1 by Q_D. Then $(Q_D, \overline{AD}, B) = (Q_D, \overline{BD}, A)$, $(Q_D, \overline{BD}, C) = (Q_D, \overline{CD}, B)$, and $(Q_D, \overline{CD}, A) = (Q_D, \overline{AD}, C)$.

a. If Q_D and C are on the same side of the plane ABD, let $x = (Q_D, \overline{AD}, B) = (Q_D, \overline{BD}, A)$. If Q_D and A are on opposite sides of the plane ABD, let $x = -(Q_D, \overline{AD}, B) = -(Q_D, \overline{BD}, A)$. (See Figure 3.)
b. If Q_D and A are on the same side of the plane BCD, let $y = (Q_D, \overline{BD}, C) = (Q_D, \overline{CD}, B)$. If Q_D and A are on opposite sides of the plane BCD let $y = -(Q_D, \overline{BD}, C) = -(Q_D, \overline{CD}, B)$.
c. If Q_D and B are on the same side of the plane ACD, let $z = (Q_D, \overline{CD}, A) = (Q_D, \overline{AD}, C)$. If Q_D and B are on opposite sides of the plane ACD, let $z = -(Q_D, \overline{CD}, A) = -(Q_D, \overline{AD}, C)$.

Hence, we have

1. $\overline{AD} = x + z, \overline{BD} = x + y, \overline{CD} = y + z$.
Similarly, by Lemma 3.1, there is a point Q_A such that

$$a = \pm(Q_A, \overline{AD}, B) = \pm(Q_A, \overline{AB}, D), b = \pm(Q_A, \overline{AB}, C) = \pm(Q_A, \overline{AC}, B),$$
$$\text{and} \quad c = \pm(Q_A, \overline{AC}, D) = \pm(Q_A, \overline{AD}, C),$$

where the signs "±" are determined in a similar way as in a.–c. Then we have
2. $\overline{AD} = a + c, \overline{AB} = a + b, \overline{AC} = b + c$. (See Figure 4.)
Again, there is a point Q_B such that

$$s = \pm(Q_B, \overline{BD}, C) = \pm(Q_B, \overline{BC}, D), \quad t = \pm(Q_B, \overline{BC}, A) = \pm(Q_B, \overline{AB}, C),$$
$$\text{and} \quad u = \pm(Q_B, \overline{AB}, D) = \pm(Q_B, \overline{BD}, A).$$

Hence,
3. $\overline{BD} = s + u$, $\overline{BC} = s + t$, $\overline{AB} = t + u$ (See Figure 5.)
Again, there is a point Q_C such that

$$p = \pm(Q_C, \overline{CD}, A) = \pm(Q_C, \overline{AC}, D), \quad q = \pm(Q_C, \overline{AC}, B) = \pm(Q_C, \overline{BC}, A),$$
$$\text{and} \quad r = \pm(Q_C, \overline{BC}, D) = \pm(Q_C, \overline{CD}, B).$$

Hence,
4. $\overline{CD} = p + r, \overline{AC} = p + q, \overline{BC} = q + r$. (See Figure 6).
From 1. through 4., we have $\overline{AD} = x+z = a+c$, $\overline{BD} = x+y = s+u$, $\overline{CD} = y+z = p+r$, $\overline{AB} = a + b = t + u$, $\overline{BC} = s + t = q + r$, $\overline{AC} = b + c = p + q$.
Since $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD}$, we have $(a + b) + (y + z) = (b + c) + (x + y)$, and $(t + u) + (p + r) = (p + q) + (s + u)$. They simplify to $a - c = x - z$, and $t - s = q - r$. But $\overline{AD} = a + c = x + z$, and $\overline{BC} = s + t = q + r$. This proves that
5. $a = x, c = z, t = q$, and $s = r$.
Similarly, since $\overline{AB} + \overline{CD} = \overline{AD} + \overline{BC}$, we have $(t+u)+(y+z) = (x+z)+(s+t)$, and $(a + b) + (p + r) = (a + c) + (q + r)$. They simplify to $u - s = x - y$, and $b - c = q - p$. But $\overline{BD} = x + y = s + u$ and $\overline{AC} = b + c = p + q$. This proves that
6. $u = x, s = y, b = q$, and $c = p$.
Again, since $\overline{AC} + \overline{BD} = \overline{AD} + \overline{BC}$, we have $(b + c) + (s + u) = (a + c) + (s + t)$, and $(p + q) + (x + y) = (x + z) + (q + r)$. They simplify to $b - a = t - u$, and $p - r = z - y$. But $\overline{AB} = a + b = t + u$ and $\overline{CD} = y + z = p + r$. This proves that
7. $b = t, a = u, p = z$, and $y = r$.
From 5. through 7., we have
8. $a = x = u, c = z = p, b = t = q$, and $s = r = y$.
9. The planes AQ_AD and AQ_DD are the same since $a = x$. We denote these common planes by Ω_{AD}.
10. The planes $BQ_BD = BQ_DD \coloneqq \Omega_{BD}$ since $x = u$.
11. The planes $CQ_CD = CQ_DD \coloneqq \Omega_{CD}$ since $z = p$.
12. The planes $AQ_AB = AQ_BB \coloneqq \Omega_{AB}$ since $b = t$.
13. The planes $BQ_BC = BQ_CC \coloneqq \Omega_{BC}$ since $t = q$.
14. The planes $AQ_AC = AQ_CC \coloneqq \Omega_{AC}$ since $b = q$.
15. $\Omega_{AD} \cap \Omega_{BD} \cap \Omega_{CD} = DQ_D$ because of 9., 10., and 11.
16. $\Omega_{AD} \cap \Omega_{AB} \cap \Omega_{AC} = AQ_A$ because of 9., 12., and 14.
17. $\Omega_{BD} \cap \Omega_{AB} \cap \Omega_{BC} = BQ_B$ because of (10), (12), and (13).

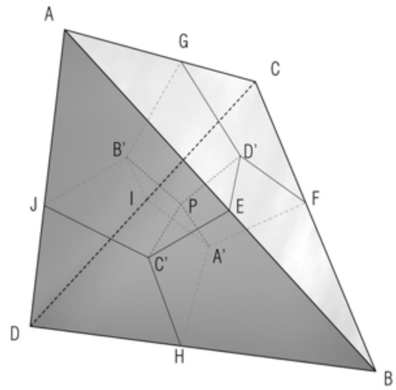


Figure 1

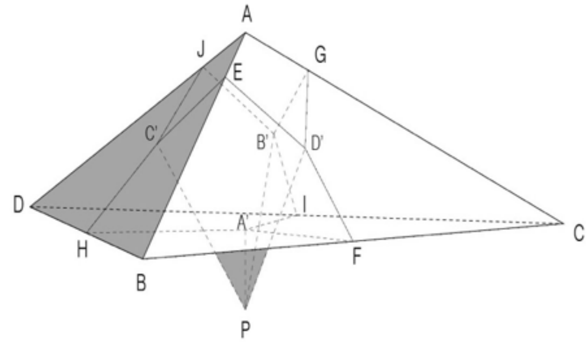
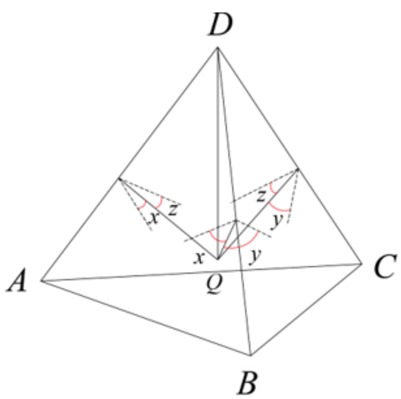
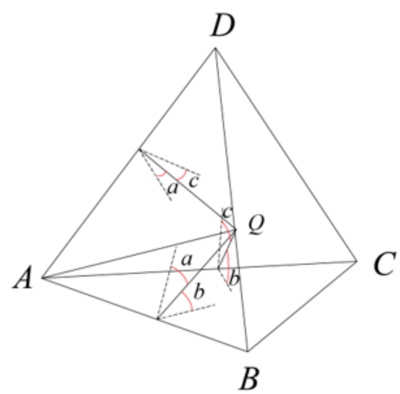


Figure 2



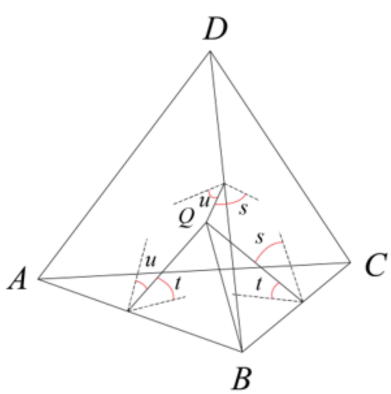
$Q = Q_D$

Figure 3



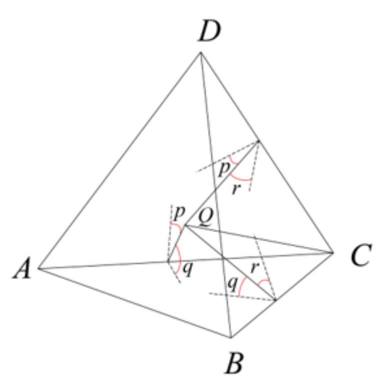
$Q = Q_A$

Figure 4



$Q = Q_B$

Figure 5



$Q = Q_C$

Figure 6

18. $\Omega_{CD} \cap \Omega_{BC} \cap \Omega_{AC} = CQ_C$ because of (11), (13), and (14).
 At the face ABD, we have $\Omega_{AD} \cap \Omega_{AB} \cap \Omega_{BD} \neq \emptyset$. That is, the intersection of the three planes Ω_{AD}, Ω_{AB}, Ω_{BD} is a point, say $\Omega_{AD} \cap \Omega_{AB} \cap \Omega_{BD} = \{P\}$. Hence, (15), (16), and (17) imply that $\{P\} = DQ_D \cap AQ_A \cap BQ_B$. So we have
19. $DQ_D \cap AQ_A \cap BQ_B \neq \emptyset$.
 At the face BCD, we also have $\Omega_{BD} \cap \Omega_{BC} \cap \Omega_{CD} \neq \emptyset$. This, together with (15), (17), (18) implies
20. $DQ_D \cap BQ_B \cap CQ_C \neq \emptyset$.

Since $DQ_D \cap BQ_B$ is on the left side of both (19) and (20), we have $DQ_D \cap AQ_A \cap BQ_B \cap CQ_C \neq \emptyset$. Therefore, the lines DQ_D, AQ_A, BQ_B, CQ_C concur at the point P, and $\overline{AD} = x + z$, $\overline{BD} = x + y$, $\overline{CD} = y + z$, $\overline{AB} = x + b$, $\overline{BC} = y + b$, $\overline{AC} = z + b$. Hence, by Lemma 3.2, we have

$$(P, \overline{AD}, B) = (P, \overline{BD}, A), (P, \overline{BD}, C) = (P, \overline{CD}, B),$$
$$\text{and} \quad (P, \overline{CD}, A) = (P, \overline{AD}, C), \quad (P, \overline{BD}, C) = (P, \overline{BC}, D), \quad (P, \overline{BC}, A) = (P, \overline{AB}, C),$$
$$\text{and} \quad (P, \overline{AB}, D) = (P, \overline{BD}, A), \quad \text{and} \quad (P, \overline{CD}, A) = (P, \overline{AC}, D), \quad (P, \overline{AC}, B) = (P, \overline{BC}, A),$$
$$\text{and} \quad (P, \overline{BC}, D) = (P, \overline{CD}, B).$$

Let E, F, G, H, I, J be the feet of the lines through P normal to the edge AB, BC, CA, BD, CD, AD, respectively, as in Figures 1 or 2. Then by Lemma 3.3, we have $r_A := AE = AG = AJ$, $r_B := BE = BF = BH$, $r_C := CF = CG = CI$, $r_D := DG = DI = DJ$. This proves that the tetrahedron $ABCD$ is generated by four-sphere of radii r_A, r_B, r_C, r_D, centered at A, B, C, D, respectively.

Therefore, this proves Theorem 2. □

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