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Tangential quadrilateral in isotropic plane

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Abstract. In this paper we prove some properties of tangential quadrilateral in an isotropic plane. We also determine a condition for tangential quadrilateral to be cyclic.

Key Words: isotropic plane, tangential quadrilateral

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1 Motivation

The concept of cyclic quadrangle in an isotropic plane was introduced and studied in [12], and its specilal case, harmonic quadrangle, was discussed in [5]. Non-tangential quadrilaterals were observed in [1] and [11]. On the other hand, tangential quadrilaterals have been somehow neglected. Motivated by that fact, in this paper we study the properties of tangential quadrilaterals in the isotropic plane.

In Euclidean geometry, tangential quadrilateral or circumscribed quadrilateral is a convex quadrilateral whose sides are tangent to a circle within the quadrilateral. The tangential quadrilateral $PQRS$ is characterized by the fact that the two pairs of opposite sides add up to the same total length i.e.

$$d(P,Q) + d(R,S) = d(Q,R) + d(S,P). \tag{1}$$

An ex-tangential quadrilateral or exscriptible quadrilateral is a convex quadrilateral where the extensions of its sides are tangent to a circle outside the quadrilateral. The convex quadrilateral $PQRS$ is ex-tangential quadrilateral if and only if the sum of two adjacent sides is equal to the sum of the other two sides. This is possible if

$$d(P,Q) + d(Q,R) = d(R,S) + d(S,P) \tag{2}$$

or

$$d(Q,P) + d(P,S) = d(Q,R) + d(R,S). \tag{3}$$

We will prove that in the isotropic plane a tangential quadrilateral is characterized with relation similar to (1) i.e.

$$d(P,Q) = -d(R,S), \quad d(Q,R) = -d(S,P). \tag{4}$$

It should be noticed that relations (2) and (3) always hold in the isotropic plane.

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2 Introduction

All metric quantities and notations related to the geometry of the isotropic plane can be found in [9]. Here we recall only same basic definitions and facts about such a plane. It is a real projective plane where the metric is induced by a real line f and a real point F incident with it. All lines through the *absolute point* F are called *isotropic lines*, and all points incident with the *absolute line* f are called *isotropic points*. Two lines are parallel if they pass through the same isotropic point, and two points are parallel if they lie on the same isotropic line. In the affine model of the isotropic plane where the coordinates of points are defined by $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$, the absolute line has the equation $x_0 = 0$ and the absolute point has the coordinates $(0, 0, 1)$. For two non-parallel points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ the *distance* is defined by $d(A, B) = x_B - x_A$, and for two parallel points $A = (x, y_A)$ and $B = (x, y_B)$ the *span* is defined by $s(A, B) = y_B - y_A$. Two non-parallel lines p and q given by the equations $y = k_p x + l_p$ and $y = k_q x + l_q$ form the *angle* defined by $\angle(p, q) = k_q - k_p$. The *midpoint* of points A and B is given by $\left(\frac{1}{2}(x_A + x_B), \frac{1}{2}(y_A + y_B)\right)$, while the *bisector* of lines p and q is given by the equation $y = \frac{1}{2}(k_p + k_q)x + \frac{1}{2}(l_p + l_q)$. A *circle* is defined as a conic touching the absolute line at the absolute point and therefore it has an equation of the form $y = ux^2 + vx + w$, $u, v, w \in \mathbb{R}$.

A figure consisting of four lines $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ and their six intersections is called a complete quadrilateral $\mathcal{ABCD}$. The pairs of points $P = \mathcal{A} \cap \mathcal{B}$, $R = \mathcal{C} \cap \mathcal{D}$; $M = \mathcal{A} \cap \mathcal{C}$, $N = \mathcal{B} \cap \mathcal{D}$ and $S = \mathcal{A} \cap \mathcal{D}$, $Q = \mathcal{B} \cap \mathcal{C}$ are called the opposite vertices of quadrilateral. There is unique conic touching lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and the absolute line of the isotropic plane. If that conic is a circle, $\mathcal{ABCD}$ is a tangential quadrilateral. If that conic is a parabola, $\mathcal{ABCD}$ is a non-tangential quadrilateral, [11].

Let a circle k with the equation

$$y = x^2 \tag{5}$$

be given. Let the lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be its tangents with respective contact points

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2), \quad D = (d, d^2). \tag{6}$$

Then lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ have equations

$$\begin{aligned} \mathcal{A} \ldots y &= 2ax - a^2, & \mathcal{B} \ldots y &= 2bx - b^2, \\ \mathcal{C} \ldots y &= 2cx - c^2, & \mathcal{D} \ldots y &= 2dx - d^2. \end{aligned} \tag{7}$$

The vertices of the quadrilateral $\mathcal{ABCD}$ are

$$\begin{aligned} P &= \mathcal{A} \cap \mathcal{B} = \left(\frac{a+b}{2}, ab\right), & R &= \mathcal{C} \cap \mathcal{D} = \left(\frac{c+d}{2}, cd\right), \\ Q &= \mathcal{B} \cap \mathcal{C} = \left(\frac{b+c}{2}, bc\right), & S &= \mathcal{D} \cap \mathcal{A} = \left(\frac{d+a}{2}, da\right), \\ M &= \mathcal{A} \cap \mathcal{C} = \left(\frac{a+c}{2}, ac\right), & N &= \mathcal{B} \cap \mathcal{D} = \left(\frac{b+d}{2}, bd\right). \end{aligned} \tag{8}$$

Since $d(P, Q) = \frac{c-a}{2}$, $d(Q, R) = \frac{d-b}{2}$, $d(R, S) = \frac{a-c}{2}$ and $d(S, P) = \frac{b-d}{2}$, it follows immediately $d(P, Q) = -d(R, S)$ and $d(Q, R) = -d(S, P)$. Similarly we get $d(M, P) = -d(N, R)$, $d(P, N) = -d(R, M)$ and also $d(M, S) = -d(N, Q)$, $d(S, N) = -d(Q, M)$.

Therefore, we can conclude that if $\mathcal{ABCD}$ is tangential quadrilateral and P, R; Q, S and M, N are the pairs of its opposite vertices, the following six equalities hold:

$$\begin{aligned} d(P,Q) &= -d(R,S), \qquad & d(Q,R) &= -d(S,P), \\ d(M,P) &= -d(N,R), \qquad & d(P,N) &= -d(R,M), \\ d(M,S) &= -d(N,Q), \qquad & d(S,N) &= -d(Q,M). \end{aligned} \tag{9}$$

Let us now prove the opposite. We will assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are tangent to a circle and that $\mathcal{D}$ is a line. We have to show that if (9) holds, then $\mathcal{D}$ is tangent to the same circle. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be tangent to the circle k with equation $y = x^2$. Thus, they have equations of the following form

$$\mathcal{A} \ldots y = 2ax - a^2, \quad \mathcal{B} \ldots y = 2bx - b^2, \quad \mathcal{C} \ldots y = 2cx - c^2$$

and

$$\mathcal{D} \ldots y = kx + l. \tag{10}$$

Now we have,

$$\begin{aligned} P &= \mathcal{A} \cap \mathcal{B} = \left(\frac{a+b}{2}, ab\right), \qquad & R &= \mathcal{C} \cap \mathcal{D} = \left(\frac{c^2+l}{2c-k}, \frac{c(kc+2l)}{2c-k}\right), \\ Q &= \mathcal{B} \cap \mathcal{C} = \left(\frac{b+c}{2}, bc\right), \qquad & S &= \mathcal{D} \cap \mathcal{A} = \left(\frac{a^2+l}{2a-k}, \frac{a(ka+2l)}{2a-k}\right), \\ M &= \mathcal{A} \cap \mathcal{C} = \left(\frac{a+c}{2}, ac\right), \qquad & N &= \mathcal{B} \cap \mathcal{D} = \left(\frac{b^2+l}{2b-k}, \frac{b(kb+2l)}{2b-k}\right). \end{aligned} \tag{11}$$

If, $d(P, Q) = -d(R, S)$ then $\frac{c-a}{2} = \frac{c^2+l}{2c-k} - \frac{a^2+l}{2a-k}$. That is true precisely when $(2c-k)(2a-k) = 2(2ac - k\,(a+c) - 2l)$ which holds if and only if $l = -\frac{k^2}{4}$ i.e. if $\mathcal{D}$ is tangent to k. If we used some of the other five equalities from (9), a similar calculation would lead to the same result. Thus, if one of the six equalities stated in (9) holds, the other five are also valid. Our observation gives a characterization of the tangential quadrilaterals:

Theorem 1. *The quadrilateral $\mathcal{ABCD}$ with the pairs of the opposite verices P, R; Q, S and M, N is a tangential quadrilateral if and only if*

$$\begin{aligned} d(P,Q) &= -d(R,S), \qquad & d(Q,R) &= -d(S,P), \\ d(M,P) &= -d(N,R), \qquad & d(P,N) &= -d(R,M), \\ d(M,S) &= -d(N,Q), \qquad & d(S,N) &= -d(Q,M). \end{aligned}$$

3 Some properties of tangential quadrilateral

In this section, we will prove several theorems dealing with the properties of the tangential quadrilateral $\mathcal{ABCD}$. In the proofs we use equations of the sides of $\mathcal{ABCD}$ given by (7) and coordinates of the vertices given by (8).

Theorem 2. *Let $\mathcal{ABCD}$ be a tangential quadrilateral with sides $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ touching a circle k at points A, B, C, D, respectively. Vertices $P = \mathcal{A} \cap \mathcal{B}$, $R = \mathcal{C} \cap \mathcal{D}$, $M = \mathcal{A} \cap \mathcal{C}$, $N = \mathcal{B} \cap \mathcal{D}$, $S = \mathcal{A} \cap \mathcal{D}$, $Q = \mathcal{B} \cap \mathcal{C}$ are parallel to the midpoints M_{AB}, M_{CD}, M_{AC}, M_{BD}, M_{AD}, M_{BC} of the line segments AB, CD, AC, BD, AD, BC, respectively.*

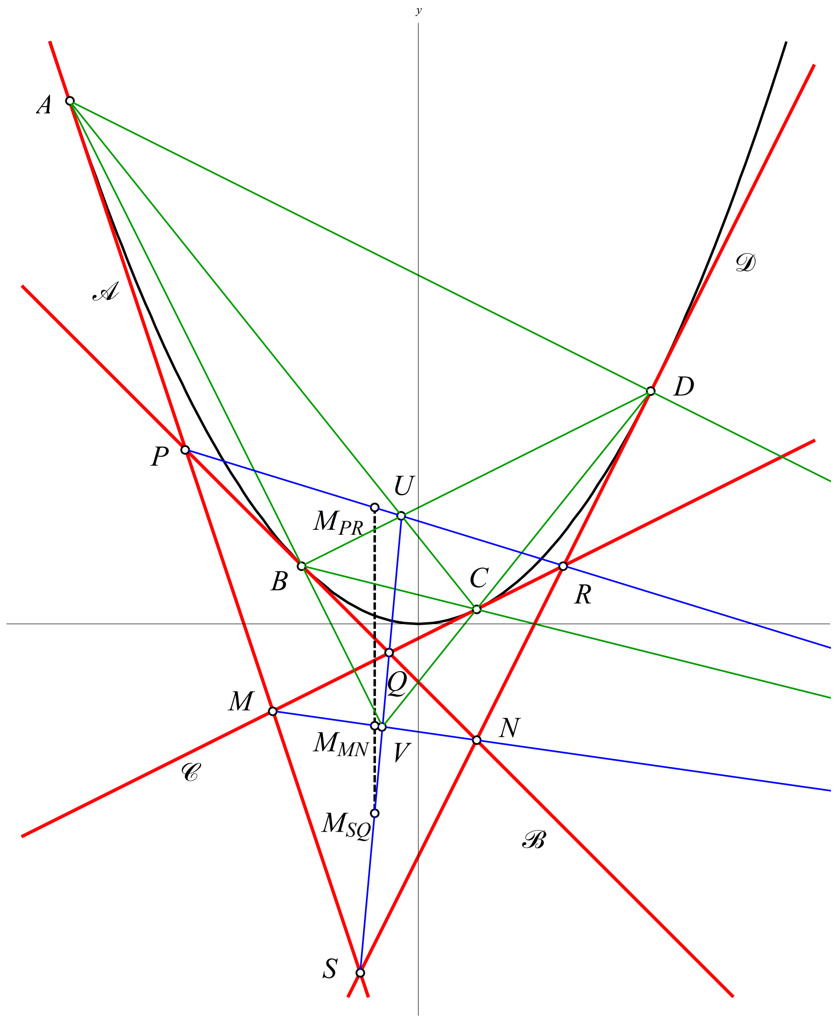

Figure 1: Tangential quadrilateral $\mathcal{ABCD}$ with vertices P, R, Q, S, M, N.

Proof. Let $\mathcal{ABCD}$ be a tangential quadrilateral with sides given by (7) and vertices given by (8). The point M_{AB} has coordinates $\left(\frac{a+b}{2}, \frac{a^2+b^2}{2}\right)$ and is parallel to the point $P = \left(\frac{a+b}{2}, ab\right)$. □

Theorem 3. *Let $\mathcal{ABCD}$ be a tangential quadrilateral. Midpoints M_{PR}, M_{SQ}, M_{MN} of the line segments formed by the pairs of opposite vertices are parallel points, Figure 1.*

Proof. Let $\mathcal{ABCD}$ be a tangential quadrilateral with vertices given by (8). The points M_{PR}, M_{SQ}, M_{MN} have coordinates

$$\left(\frac{a+b+c+d}{4}, \frac{ab+cd}{2}\right), \quad \left(\frac{a+b+c+d}{4}, \frac{bc+da}{2}\right), \quad \left(\frac{a+b+c+d}{4}, \frac{ac+bd}{2}\right), \tag{12}$$

respectively. They lie on the isotropic line with the equation $x = \frac{a+b+c+d}{4}$. □

The sides of the diagonal triangle of the tangential quadrilateral $\mathcal{ABCD}$ are given by

$$\begin{aligned} PR\ldots \quad y &= \frac{2(ab-cd)}{a+b-c-d}x - \frac{ab(c+d)-cd(a+b)}{a+b-c-d}, \\ SQ\ldots \quad y &= \frac{2(ad-bc)}{a-b-c+d}x - \frac{ad(b+c)-bc(a+d)}{a-b-c+d}, \\ MN\ldots \quad y &= \frac{2(ac-bd)}{a-b+c-d}x - \frac{ac(b+d)-bd(a+c)}{a-b+c-d}, \end{aligned} \tag{13}$$

and its vertices are

$$\begin{aligned} U = PR\cap SQ \ &= \Big(\frac{ac-bd}{a-b+c-d}, \frac{ac(b+d)-bd(a+c)}{a-b+c-d}\Big), \\ V = SQ\cap MN &= \Big(\frac{ab-cd}{a+b-c-d}, \frac{ab(c+d)-cd(a+b)}{a+b-c-d}\Big), \\ W = MN\cap PR &= \Big(\frac{ad-bc}{a-b-c+d}, \frac{ad(b+c)-bc(a+d)}{a-b-c+d}\Big). \end{aligned} \tag{14}$$

Theorem 4. *Let $\mathcal{ABCD}$ be a tangential quadrilateral with sides $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ touching a circle k at the points A, B, C, D, respectively. The diagonal triangle of the quadrilateral $\mathcal{ABCD}$ coincide with the diagonal triangle of the contact quadrangle $ABCD$, Figure 1. Particularly,*

$$\begin{aligned} U = PR\cap SQ \quad &= AC\cap BD, \\ V = SQ\cap MN \quad &= AB\cap CD, \\ W = MN\cap PR \quad &= AD\cap BC. \end{aligned}$$

Proof. It follows from (6) that the lines AB and CD have equations $y=(a+b)x-ab$ and $y=(c+d)x-cd$, respectively. Therefore, $AC\cap BD = \left(\frac{ac-bd}{a-b+c-d}, \frac{ac(b+d)-bd(a+c)}{a-b+c-d}\right)$. The other two claims can be proved similarly. □

Theorem 5. *Let $\mathcal{ABCD}$ be a tangential quadrilateral with sides $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ touching a circle k at the points A, B, C, D, respectively, and the pairs of the opposite vertices $P=\mathcal{A}\cap\mathcal{B}$, $R=\mathcal{C}\cap\mathcal{D}$; $M=\mathcal{A}\cap\mathcal{C}$, $N=\mathcal{B}\cap\mathcal{D}$; $S=\mathcal{A}\cap\mathcal{D}$, $Q=\mathcal{B}\cap\mathcal{C}$. The following equalities are valid:*

$$\begin{aligned} \frac{d^2(B,D)}{d^2(A,C)} &= \frac{d(P,S)\cdot d(Q,R)}{d(S,R)\cdot d(P,Q)}, \\ \frac{d^2(C,D)}{d^2(A,B)} &= \frac{d(M,S)\cdot d(Q,N)}{d(S,N)\cdot d(M,Q)}, \\ \frac{d^2(B,C)}{d^2(A,D)} &= \frac{d(P,M)\cdot d(N,R)}{d(M,R)\cdot d(P,N)}. \end{aligned}$$

Proof. From (6) and (8) we get

$$\frac{d(P,S)\cdot d(Q,R)}{d(S,R)\cdot d(P,Q)} = \frac{\frac{d-b}{2}\cdot\frac{d-b}{2}}{\frac{c-a}{2}\cdot\frac{c-a}{2}} = \frac{(d-b)^2}{(c-a)^2} = \frac{d^2(B,D)}{d^2(A,C)}$$

The other two equalities can be proved similarly. □

Theorem 6. *Let $\mathcal{ABCD}$ be a tangential quadrilateral with sides $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ touching a circle k at the points A, B, C, D, respectively, and the pairs of the opposite vertices $P=\mathcal{A}\cap\mathcal{B}$,*

$R = \mathcal{C} \cap \mathcal{D}$; $M = \mathcal{A} \cap \mathcal{C}$, $N = \mathcal{B} \cap \mathcal{D}$; $S = \mathcal{A} \cap \mathcal{D}$, $Q = \mathcal{B} \cap \mathcal{C}$. The following equalities are valid:

$$\begin{aligned} d(A,P) &= d(P,B), & d(C,R) &= d(R,D),\\ d(A,S) &= d(S,D), & d(C,Q) &= d(Q,B),\\ d(A,M) &= d(M,C), & d(B,N) &= d(N,D). \end{aligned}$$

Proof. From (6) and (8) we get

$$d(A,P) = \frac{a+b}{2} - a = \frac{b-a}{2} = b - \frac{a+b}{2} = d(P,B).$$

The other five equalities can be proved similarly. □

Theorem 7. *Let $\mathcal{ABCD}$ be a tangential quadrilateral with sides $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ touching a circle k at the points A, B, C, D, respectively, and the pairs of the opposite vertices $P = \mathcal{A} \cap \mathcal{B}$, $R = \mathcal{C} \cap \mathcal{D}$; $M = \mathcal{A} \cap \mathcal{C}$, $N = \mathcal{B} \cap \mathcal{D}$; $S = \mathcal{A} \cap \mathcal{D}$, $Q = \mathcal{B} \cap \mathcal{C}$. Let $U = PR \cap SQ$, $V = SQ \cap MN$ and $W = MN \cap PR$ be its diagonal points. The following equalities are valid:*

$$\begin{aligned} \frac{d(P,U)}{d(U,R)} &= \frac{d(P,B)}{d(R,D)}, & \frac{d(S,U)}{d(U,Q)} &= \frac{d(S,D)}{d(Q,B)},\\ \frac{d(M,V)}{d(V,N)} &= \frac{d(M,C)}{d(N,D)}, & \frac{d(S,V)}{d(V,Q)} &= \frac{d(S,D)}{d(Q,C)},\\ \frac{d(M,W)}{d(W,N)} &= \frac{d(M,C)}{d(N,B)}, & \frac{d(P,W)}{d(W,R)} &= \frac{d(P,B)}{d(R,C)}. \end{aligned}$$

Proof. We will prove the first equality and other five can be proved similarly. From (6), (8) and (14) we get

$$\frac{d(P,U)}{d(U,R)} = \frac{2(ac-bd)-(a+b)(a-b+c-d)}{(c+d)(a-b+c-d)-2(ac-bd)} = \frac{(a-b)(c+d-a-b)}{(d-c)(a+b-c-d)} = \frac{b-a}{d-c} = \frac{d(P,B)}{d(R,D)}.$$

□

Theorems 4–7 are also valid in the Euclidean plane, [3], [10], [14].

Using the notation from the previous theorem we can state:

Theorem 8. *Let $U_\mathcal{A}$, $U_\mathcal{B}$, $U_\mathcal{C}$, $U_\mathcal{D}$, $V_\mathcal{A}$, $V_\mathcal{B}$, $V_\mathcal{C}$, $V_\mathcal{D}$, $W_\mathcal{A}$, $W_\mathcal{B}$, $W_\mathcal{C}$, $W_\mathcal{D}$ be the points parallel to the diagonal points U, V, W, and lying on the $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, respectively. The following equalities are valid:*

$$\begin{aligned} \frac{1}{s(U,U_\mathcal{A})} + \frac{1}{s(U,U_\mathcal{C})} &= \frac{1}{s(U,U_\mathcal{B})} + \frac{1}{s(U,U_\mathcal{D})}\\ \frac{1}{s(V,V_\mathcal{A})} + \frac{1}{s(V,V_\mathcal{B})} &= \frac{1}{s(V,V_\mathcal{C})} + \frac{1}{s(V,V_\mathcal{D})}\\ \frac{1}{s(W,W_\mathcal{A})} + \frac{1}{s(W,W_\mathcal{D})} &= \frac{1}{s(W,W_\mathcal{B})} + \frac{1}{s(W,W_\mathcal{C})} \end{aligned}$$

where by s the span between two parallel points is denoted.

Proof. We will prove the first equality, and the other two can be proved similarly. According to (7) and (14) the intersection point $U_\mathcal{A}$ of the isotropic line through U and line $\mathcal{A}$ has coordinates $\left(\frac{ac-bd}{a-b+c-d}, 2a\frac{ac-bd}{a-b+c-d} - a^2\right)$. Therefore, $s(U,U_\mathcal{A}) = 2a\frac{ac-bd}{a-b+c-d} - a^2 - \frac{ac(b+d)-bd(a+c)}{a-b+c-d} =$

$-\frac{(a-b)(a-c)(a-d)}{a-b+c-d}$. Analogously we get spans $s(U,U_{\mathcal{B}})=\frac{(b-a)(b-c)(b-d)}{a-b+c-d}$, $s(U,U_{\mathcal{C}})=-\frac{(c-a)(c-b)(c-d)}{a-b+c-d}$ and $s(U,U_{\mathcal{D}})=\frac{(d-a)(d-b)(d-c)}{a-b+c-d}$. Thus,

$$\frac{1}{s(U,U_{\mathcal{A}})}+\frac{1}{s(U,U_{\mathcal{C}})}=\frac{(a-b+c-d)^2}{(a-b)(b-c)(a-d)(d-c)}=\frac{1}{s(U,U_{\mathcal{B}})}+\frac{1}{s(U,U_{\mathcal{D}})}. \qquad \square$$

In [6] the author presented the Euclidean version of Theorem 8 in which the isotropic lines through a diagonal point are substituted by the perpendicular lines to the sides of the quadrilateral.

Let us consider two of three sides of diagonal triangle of $\mathcal{ABCD}$, e.g. PR and SQ. If k_i, $i=1,\ldots,4$, with equations $y=E_ix^2+F_ix+G_i$ are circles touching two observed diagonals and sides $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, respectively, then we have: $E_1=-E_3$ and $E_2=-E_4$. Indeed, a short calculation in program *Wolfram Mathematica* results with

$$E_1=\frac{(a-c)(a-b+c-d)}{(a+b-c-d)(a-b-c+d)}=-E_3$$

and

$$E_2=\frac{(b-d)(a-b+c-d)}{(a+b-c-d)(a-b-c+d)}=-E_4.$$

The Euclidean version of this property can be found in [13], while the version of the property that follows can be found in [4].

Let us now consider one diagonal point of $\mathcal{ABCD}$, e.g. $U=PR\cup SQ$. If k_j, $j=1,\ldots,4$, with equations $y=I_jx^2+J_jx+K_j$ are circles passing through U and the pairs of vertices P,S; P,Q; R,Q; R,S, respectively, then we have: $I_1=-I_3$ and $I_2=-I_4$. Indeed, a short calculation in program *Wolfram Mathematica* results with

$$I_1=\frac{4(a-c)(a-b+c-d)}{(a+b-c-d)(a-b-c+d)}=-I_3$$

and

$$I_2=\frac{4(b-d)(a-b+c-d)}{(a+b-c-d)(a-b-c+d)}=-I_4.$$

In the Euclidean plane tangential quadrilateral $PQRS$ is also cyclic if and only if $d(P,A)\cdot d(R,C)=d(Q,C)\cdot d(S,A)$, [2]. In the isotropic plane the same condition leads to different result:

Theorem 9. *Let $\mathcal{ABCD}$ be a tangential quadrilateral with sides $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ touching a circle k at the points A, B, C, D, respectively, and the pairs of the opposite vertices $P=\mathcal{A}\cap\mathcal{B}$, $R=\mathcal{C}\cap\mathcal{D}$; $S=\mathcal{A}\cap\mathcal{D}$, $Q=\mathcal{B}\cap\mathcal{C}$. Equality*

$$|d(P,A)\cdot d(R,C)|=|d(Q,C)\cdot d(S,A)| \tag{15}$$

holds if and only if $ABCD$ is a harmonic quadrangle.

Proof. Condition (15) can be fulfilled in two different ways. The equality $d(P,A)\cdot d(R,C)=d(Q,C)\cdot d(S,A)$ is not possible since $a\neq c$ and $b\neq d$. On the other hand, condition $d(P,A)\cdot d(R,C)=-d(Q,C)\cdot d(S,A)$ is fulfilled if and only if $2(ac+bd)=(a+c)(b+d)$ which is according to [5] true precisely when $ABCD$ is a harmonic quadrangle, i.e. $M=\mathcal{A}\cap\mathcal{C}$ lies on BD and $N=\mathcal{B}\cap\mathcal{D}$ lies on AC. $\square$

One more characterization of bicentric quadrilateral $PQRS$ in the Euclidean plane is given in [8] by: $\frac{d(P,R)}{d(Q,S)}=\frac{d(P,B)+d(R,D)}{d(Q,C)+d(S,A)}$, but this condition does not hold in the isotropic plane since the given equality leads to the contradiction $b=d$.

4 Condition for tangential quadrilateral to be cyclic

Theorem 10. *The pairs of the opposite vertices P, R; M, N; S, Q of a tangential quadrilateral $\mathcal{ABCD}$ form three quadrangles $PRMN$, $PRQS$ and $QSMN$. The following statements hold*

- *$PRMN$ is cyclic if and only if $d(B,A) = d(D,C)$, $d(C,A) = d(D,B)$.*
- *$PRQS$ is cyclic if and only if $d(C,B) = d(D,A)$, $d(A,B) = d(D,C)$.*
- *$QSMN$ is cyclic if and only if $d(A,C) = d(D,B)$, $d(B,C) = d(D,A)$.*

Proof. We will prove the first statement. The other two can be proved similarly. The circle PRM has the equation

$$y = \frac{4(a-c)}{a+b-c-d}x^2 - \frac{2(a^2-bc-c^2+ad)}{a+b-c-d}x + \frac{a^2c-ac^2-bc^2+a^2d}{a+b-c-d},$$

while the circle PRN has the equation

$$y = \frac{4(b-d)}{a+b-c-d}x^2 - \frac{2(b^2+bc-ad-d^2)}{a+b-c-d}x + \frac{b^2c+b^2d-ad^2-bd^2}{a+b-c-d}.$$

They coincide precisely when the following three conditions are fulfilled:
(i) $a - c = b - d$
(ii) $a^2 - bc - c^2 + ad = b^2 + bc - ad - d^2$
(iii) $a^2c - ac^2 - bc^2 + a^2d = b^2c + b^2d - ad^2 - bd^2$.
Condition (i) is equivalent to the condition $a + d = b + c$, and condition (ii) is equivalent to $(a + d)^2 = (b + c)^2$. Condition (iii) is equivalent to the condition $(a - b)(a + b)(c + d) = (c - d)(c + d)(a + b)$. Therefore, if condition (i) is fulfilled the other two are fulfilled as well. Now we have $d(B,A) = a - b = c - d = d(D,C)$ and $d(C,A) = a - c = b - d = d(D,B)$. □

If a, b, c, d are real numbers chosen such that $a < b < c < d$, then only quadrangle $PRMN$ can be a cyclic quadrangle, Figure 2. That case is described in the following theorem.

Theorem 11. *Let the pairs of the opposite vertices of a tangential quadrilateral $\mathcal{ABCD}$ form a cyclic quadrangle $PRMN$ with circumscribed circle $\mathcal{K}$. The following statements hold*

- *The diagonal point W is an isotropic point, i.e. the lines AD, BC, PR and MN are parallel lines.*
- *The joint line of the intersection points of the circles k and $\mathcal{K}$ passes through W.*
- *The midpoint M_{SQ} lies on $\mathcal{K}$.*
- *The midpoints M_{PR}, M_{MN} coincide with the diagonal points U, V respectively.*
- *The bisectors of the pairs of sides of $\mathcal{ABCD}$ at vertices P, R, M and N are concurrent in the midpoint M_{SQ}.*
- *The bisectors of the pairs of sides of $\mathcal{ABCD}$ at vertices S and Q pass through the isotropic point W.*

Proof. Let m be a real number such that $a+d = b+c = m$, i.e. $d = m-a$, $c = m-b$. From (13) we get

$$PR \ldots \quad y = mx + ab - \frac{m(a+b)}{2},$$

$$MN \ldots \quad y = mx - ab + \frac{m(a+b)-m^2}{2}.$$

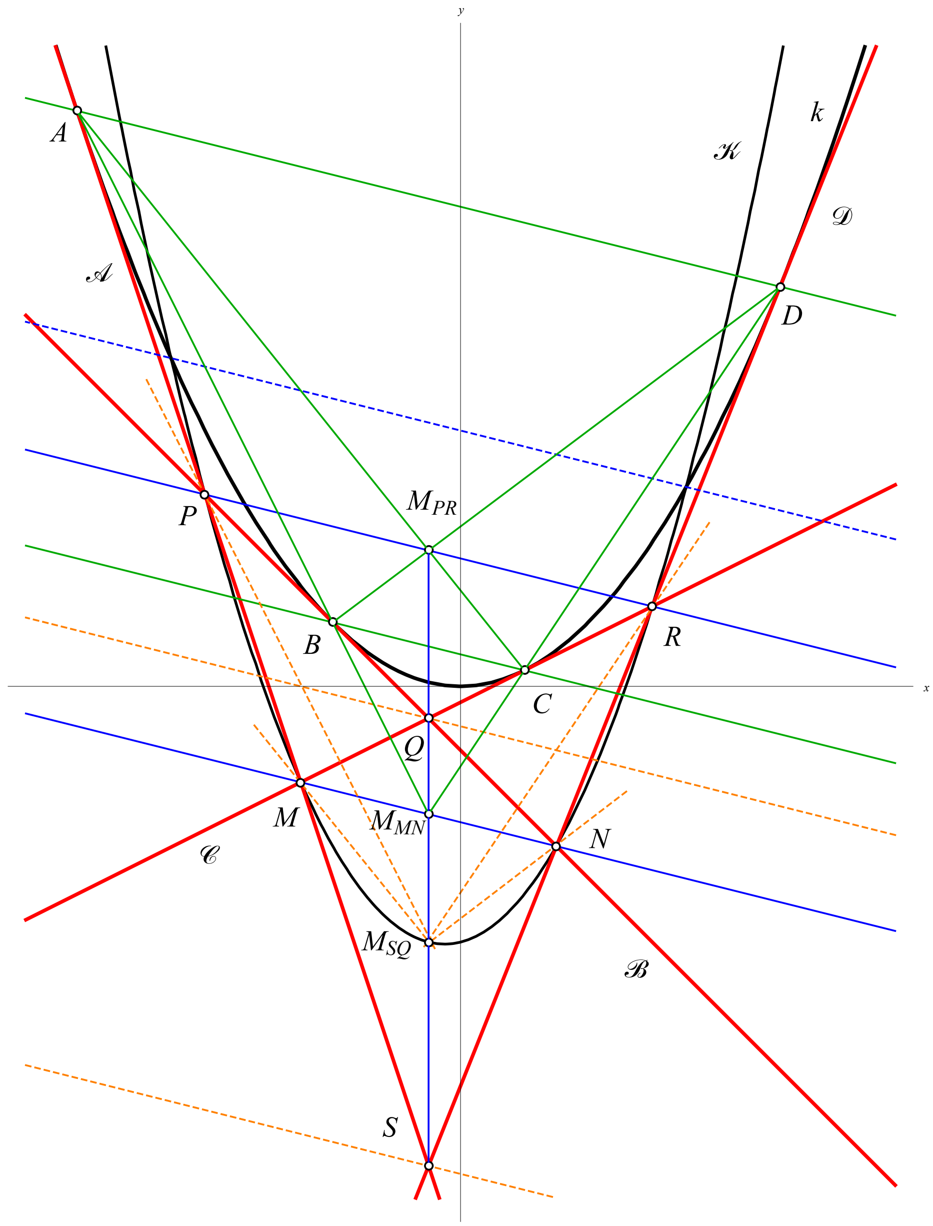

Figure 2: Tangential quadrilateral $\mathcal{ABCD}$ with cyclic quadrangle $PRMN$.

Thus, PR and MN are the parallel lines and W is an isotropic point. The circle $\mathcal{K}$ has the equation

$$y = 2x^2 - mx + \frac{m(a+b) - a^2 - b^2}{2}. \tag{16}$$

The intersection points of the circles k and $\mathcal{K}$ have coordinates

$$\left(\frac{m \pm \sqrt{(a+b-m)^2 + (a-b)^2}}{2}, \frac{a^2 + b^2 + m^2 - m(a+b) \pm m\sqrt{(a+b-m)^2 + (a-b)^2}}{2}\right)$$

and their joint line has the equation

$$y = mx + \frac{a^2 + b^2 - m(a+b)}{2}$$

being therefore parallel to PR and MN. The coordinates of the point $M_{SQ} = \left(\frac{m}{2}, \frac{m(a+b)-a^2-b^2}{2}\right)$ obviously satisfy (16). Therefore, M_{SQ} lies on $\mathcal{K}$. From (12) and (14) we get $M_{PR} = \left(\frac{m}{2}, ab + \frac{m^2-m(a+b)}{2}\right) = U$ and $M_{MN} = \left(\frac{m}{2}, -ab + \frac{m(a+b)}{2}\right) = V$.

Let b_P, b_R, b_M, b_N, b_Q, b_S denote the bisectors of the pair of lines $\mathcal{A}$, $\mathcal{B}$; $\mathcal{C}$, $\mathcal{D}$; $\mathcal{A}$, $\mathcal{C}$; $\mathcal{B}$, $\mathcal{D}$; $\mathcal{B}$, $\mathcal{C}$; $\mathcal{A}$, $\mathcal{D}$, respectively. Using (7) and $d = m - a$, $c = m - b$, after some short calculations we get

$$
\begin{aligned}
b_P \dots \quad & y = (a+b)x - \frac{a^2+b^2}{2} \\
b_R \dots \quad & y = (2m-a-b)x - m^2 + m(a+b) - \frac{a^2+b^2}{2} \\
b_M \dots \quad & y = (a-b+m)x + bm - \frac{a^2+b^2+m^2}{2} \\
b_N \dots \quad & y = (b-a+m)x + am - \frac{a^2+b^2+m^2}{2} \\
b_Q \dots \quad & y = mx - b^2 + bm - \frac{m^2}{2} \\
b_S \dots \quad & y = mx - a^2 + am - \frac{m^2}{2}.
\end{aligned}
$$

The coordinates of the point $M_{SQ} = \left(\frac{m}{2}, \frac{m(a+b)-a^2-b^2}{2}\right)$ satisfy the first four equations, which proves that b_P, b_R, b_M, b_N are concurrent. The last two equations represent the parallel lines passing through the isotropic point W. □

5 On centroids and Nagel points of tangential quadrilateral

In [7] the Nagel line of a circumscriptible quadrilateral, the line containing the Nagel point, centroid and incenter of the quadrilateral, in the Euclidean plane was studied. In this section we will study the analogous points in the isotropic plane. It is well known that in the isotropic plane the center of any circle coincide with the absolute point.

The pairs of the opposite vertices P, R; M, N; S, Q of a tangential quadrilateral $\mathcal{ABCD}$ form three quadrangles $PRMN$, $PRQS$ and $QSMN$. Let us denote their centroids by G_1, G_2, G_3, and their Nagel points by N_1, N_2 and N_3, respectively.

According to (8) the coordinates of this centroids are given by

$$
\begin{aligned}
G_1 &= \left(\frac{a+b+c+d}{4}, \frac{ab+cd+ac+bd}{4}\right) \\
G_2 &= \left(\frac{a+b+c+d}{4}, \frac{ab+cd+bc+da}{4}\right) \\
G_3 &= \left(\frac{a+b+c+d}{4}, \frac{bc+da+ac+bd}{4}\right)
\end{aligned}
$$

while the centroid G of the contact quadrangle $ABCD$ has coordinates

$$G = \left(\frac{a+b+c+d}{4}, \frac{a^2+b^2+c^2+d^2}{4}\right).$$

They obviously lie on the isotropic line $x = \frac{a+b+c+d}{4}$ and we can conclude:

Theorem 12. *The centroids of the three quadrangles $PRMN$, $PRQS$ and $QSMN$ formed by the pairs of the opposite vertices of a tangential quadrilateral $\mathcal{ABCD}$ and the centroid of the contact quadrangle $ABCD$ are parallel points.*

Let us notice that in the case when $PRMN$ is a cyclic quadrangle, then G_1 lies on the circle k. Indeed, $G_1 = \left(\frac{a+b+c+d}{4}, \frac{ab+cd+ac+bd}{4}\right) = \left(\frac{2m}{4}, \frac{(a+d)(b+c)}{4}\right) = \left(\frac{m}{2}, \frac{m^2}{4}\right)$.

We will now consider one of the three quadrangles, e.g. $PRMN$. Let A_1 be the isotomic conjugate of A with respect to the segment PM and D_1 be the isotomic conjugate of D with respect to the segment RN. The points B_1 and D_1 are defined in the same way. The point A_1 has coordinates $\left(\frac{b+c}{2}, a(b+c-a)\right)$ since $\frac{b+c}{2}+a = \frac{a+b}{2}+\frac{a+c}{2}$ and $a(b+c-a)+a^2 = ab+ac$. Similarly we get $D_1 = \left(\frac{b+c}{2}, d(b+c-d)\right)$, $C_1 = \left(\frac{a+d}{2}, c(a+d-c)\right)$ and $B_1 = \left(\frac{a+d}{2}, b(a+d-b)\right)$. The lines A_1D_1 and B_1C_1 are isotropic lines passing through Q and S, respectively, and intersecting at the absolute point of the isotropic plane. Thus, N_1 coincide with the absolute point. The same result holds for the other two quadrangles $PRQS$ and $QSMN$ and their Nagel points N_2 and N_3.

At the end let us notice that in the case when $PRMN$ is a cyclic quadrangle, then A_1 and D_1 coincide with Q, while B_1 and C_1 coincide with S. Indeed, $A_1 = \left(\frac{m}{2}, ad\right) = D_1 = Q$ and $B_1 = \left(\frac{m}{2}, bc\right) = C_1 = S$, where $m = a+d = b+c$.

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