Journal for Geometry and Graphics
Volume 27 (2023), No. 1, 1–9

Volume of an n-dimensional Polyhedron: Revisited

Abhijit Bhattacharya[1], Kamlesh Kumar Dubey[2], Bikromadittya Mondal[1]

[1] *B. P. Poddar Institute of Management and Technology, Kolkata, India*
bhattaabhi14@gmail.com, bikmondal@gmail.com

[2] *Invertis University, Uttar Pradesh, India*
kkdubey4maths@gmail.com

Abstract. The paper presents a computational technique to determine the volume of an n-dimensional polyhedron. Initially, the volume is computed for an n-dimensional simplex which is used later to calculate the volume of an arbitrary polytope using the method of signed simplex decomposition. A recursive algorithm is used to compute the volume in n-dimensions. The proposed algorithm not only calculates the volume efficiently but also avoids complex calculations in higher dimensions.

Key Words: Cayley-Menger determinant, simplex, inradius, circumradius, face angles

MSC 2020: 51M04 (primary), 51M05

1 Introduction

The calculation of volume for a n-dimensional polyhedron is a very difficult problem. One can visualize the objects up-to three dimension but for the objects of dimensions more than three, it is quite challenging to analyze its geometry. The area of an arbitrary polygon can be determined by decomposing it into number of non overlapping triangles with common edges. Analogously, the volume of a convex polyhedron can also be determined by decomposing it into number of non overlapping tetrahedrons (3-simplex). The volume of a simplex can be determined by Cayley-Menger determinant but it cannot be used directly for an arbitrary polyhedron. Polyhedrons can be decomposed into simplices using signed simplex decomposition.

Cho [4] derived a three dimensional formula for volume of a tetrahedron in terms of six dihedral angles. Cho used the Cayley-Menger determinant to compute the volume of the tetrahedron inscribed into a sphere. Bhattacharyya and Pal [2] proposed a three dimensional volume formula for tetrahedron in terms face angles, inradii and circumradii of the faces of

ISSN 1433-8157/

the tetrahedron. In this article, authors derived two volume formulas for a tetrahedron using Caley-Menger determinant. They also assumed the tetrahedron circumscribed by a sphere. They used Sine law and the properties of triangles to compute the exact volume. Choudhury et al. [5] described the formula for volume of a pentahedron in terms of dihedral angles, face angles and the sides of the tetrahedrons to which the pentahedron is decomposed. Caley-Menger determinant is again used by the authors and some tensorial notation of volume for the pentahedron is also given in this article. Sabitov [9] described the volume invariance of a continuously deforming polyhedron through its dihedral angles. Dompierre et al. [6] discussed the technique of subdividing hexahedra, pyramids, prisms into tetrahedra. This technique is basically applied to use efficient algorithm for volume rendering in computer graphics. Buchholz and Smith [3] proposed a tensorial version of volume for a simplex in n dimensions. Newson [8] deduced a general formula for volume of a polyhedron with n number of faces. Károlyi and Lovász [7] determined the existence of a signed sum decomposition of a convex polytope in d-dimensional Euclidean space into number of simplices with its facets in general position.

In this work, we have focused to calculate the volume of a general polyhedron in n-dimensions. The organization of this paper is described as follows. Section 2 describes the methodology used to derive the volume. Section 3 consists the justification of the proposed method. Section 4 describes how signed simplex decomposition works. In Section 5, signed simplex decomposition is established by the help of an example. Section 6 describes the volume calculation for n-dimensional simplex using two algorithms. In Section 7, volume of n-dimensional polyhedron is described and finally we have summarized our work in Section 8.

2 Method Outlines

In this method, it is assumed that the simplex is circumscribed by a sphere of radius R, i.e. the simplex is well centered. The method initiates with a general expression for edge of the simplex in terms of inradii or circumradii and face angles of the faces of the simplex. Using this, a new expression defining a relation between the inradii or circumradii corresponding to the faces of the simplex has been obtained. Finally the volume of the simplex is derived in terms of inradii or circumradii and face angles of the faces of the simplex using the obtained expressions in the expansion of Caley-Menger determinant. For a non simplex polytope, it is decomposed into finite number of simplices using signed simplex decomposition and taking the algebraic sum over the individual volumes to obtain the original volume of the polytope.

2.1 Signed Simplex Decomposition

This is a computation technique which can be used to subdivide polyhedrons into component tetrahedral (3-simplex). This technique is widely used in computer graphics to write efficient algorithms for volume rendering. This technique helps us to decompose an arbitrary n-dimensional convex polyhedron into n-dimensional simplices whose volumes are already determined by the proposed algorithm. The algebraic sum over the computed volumes of the component simplices will give the desired volume of the polyhedron.

3 Justification of the Methodology

The elements of Caley-Menger determinant are the distances between the points of the simplex. As the dimension increases, it is very difficult to explain the results analytically due to huge number of calculations and also a significant amount of error can be associated with it. So we use a different technique to compute the volume which enables us to define the edges of simplex in terms of inradii or circumradii and face angles of its faces. The proposed methodology follows a computational approach to calculate the volume in higher dimensions which not only reduces the complex calculations but also produces a computer generated solution.

4 Volume of the Polyhedron Using Signed Simplex Decomposition

Let us consider a polyhedron with n vertices and m edges is circumscribed by a sphere of radius R. Let the polyhedron be subdivided into p number of tetrahedrons in such a way that all the p tetrahedrons are non overlapping and covering all the edges of the polyhedron. Now it has been observed that the number of non overlapping decomposed tetrahedrons may not be unique and depends on the mechanism of decomposition. e.g. a polyhedron with nine edges can be decomposed in two or three non overlapping tetrahedrons as a tetrahedron have six edges and obviously only one tetrahedron is not sufficient to cover all the nine edges. Now, if we cut the polyhedron using a plane then it is obvious that the number of non overlapping tetrahedrons covering all the edges of polyhedron to which it is decomposed depends on how the plane cut the polyhedron.

Now, applying the technique of signed simplex decomposition on the results described by Bhattacharyya and Pal [2], we can propose two generalized formula for volume of the given polyhedron, one is in terms of inradii and another is in terms of circumradii along with the face angles of the faces of the component tetrahedrons. The volume of the polyhedron is determined by adding the volumes of its component tetrahedrons. The results are as follows:

4.1 Result 1

Let r_i^t, $i \in \{0,1,2,3\}$, $t \in \{1,2,3,\ldots,p\}$ be the inradii and θ_{ij}^t, $i,j \in \{0,1,2,3\}$, $i \neq j$, $t \in \{1,2,3\ldots,p\}$ be the face angles of the faces of p component tetrahedrons of the polyhedron. The volume of the polyhedron is given by

$$V = \sum_{t=1}^{p} \frac{{r_1^t}^2 {r_3^t}^2}{24R{D_t}^2}\sqrt{(-M_t)} \tag{1}$$

where

$$\begin{aligned}
M_t &= A_t^4 + B_t^4 + C_t^4 - 2A_t^2B_t^2 - 2B_t^2C_t^2 - 2C_t^2A_t^2,\\
A_t &= \tfrac{1}{8}\sin\theta_{32}^t \sin\theta_{21}^t \sin\theta_{10}^t,\\
B_t &= \tfrac{1}{8}\sin\theta_{31}^t \sin\theta_{20}^t \sin\theta_{12}^t,\\
C_t &= \tfrac{1}{8}\sin\theta_{30}^t \sin\theta_{21}^t \sin\theta_{12}^t,\\
D_t &= \sin\tfrac{\theta_{32}^t}{2}\sin\tfrac{\theta_{30}^t}{2}\sin\tfrac{\theta_{31}^t}{2}\sin\tfrac{\theta_{21}^t}{2}\sin\tfrac{\theta_{10}^t}{2}\sin\tfrac{\theta_{13}^t}{2}\sin\tfrac{\theta_{12}^t}{2}\cos\tfrac{\theta_{21}^t}{2}
\end{aligned}$$

and R is the radius of the sphere circumscribing the polyhedron.

Similarly, we can write the other formula as

4.2 Result 2

Let R_i^t, $i \in \{0,1,2,3\}$, $t \in \{1,2,3,\ldots,p\}$ be the circumradii corresponding to the faces and θ_{ij}^t, $i,j \in \{0,1,2,3\}$, $i \neq j$, $t \in \{1,2,3,\ldots,p\}$ be the face angles of the faces of p component tetrahedrons of the polyhedron. The volume of the polyhedron is given by

$$V = \sum_{t=1}^{p} \frac{2{R_1^t}^2 {R_3^t}^2}{3R} \sqrt{(-N_t)} \qquad (2)$$

where

$$\begin{aligned} N_t &= A_t^4 + B_t^4 + C_t^4 - 2A_t^2 B_t^2 - 2B_t^2 C_t^2 - 2C_t^2 A_t^2, \\ A_t &= \sin\theta_{32}^t \sin\theta_{10}^t, \\ B_t &= \frac{\sin\theta_{31}^t \sin\theta_{20}^t \sin\theta_{12}^t}{\sin\theta_{21}^t}, \\ C_t &= \sin\theta_{30}^t \sin\theta_{12}^t \end{aligned}$$

and R is the radius of the sphere circumscribing the polyhedron.

5 Illustration of Signed Simplex Decomposition

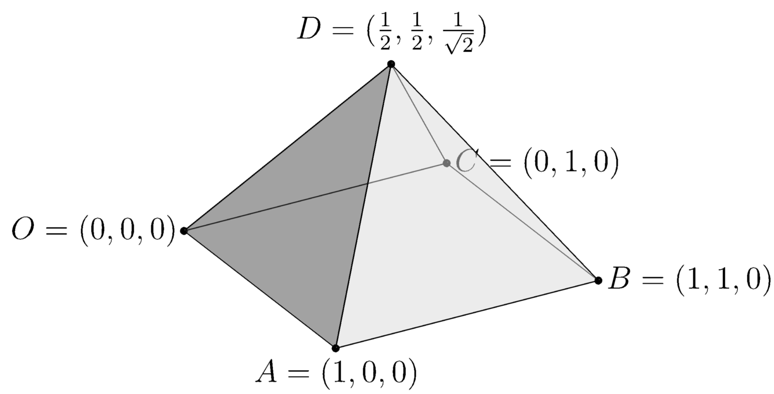

Figure 1: A squared base polyhedron having edge length 1.

Let us consider a polyhedron $OABCD$ (Figure 1) having squared base $OABC$ with the point O is at the origin. Let the lengths of the edges of the polyhedron be 1. Let us assume a vertical plane decomposing the polyhedron diagonally (AC) into two identical tetrahedrons $OACD$ and $ABCD$ keeping two co-planner triangles in base. The coordinates of A, B, C and D are $(1,0,0)$, $(1,1,0)$, $(0,1,0)$ and $(1/2, 1/2, 1/\sqrt{2})$ respectively. The component tetrahedrons are $OACD$ and $ACBD$. It is assumed that the polyhedron is circumscribed by a sphere of radius R. Let the equation of the sphere circumscribing the polyhedron be

$$x^2 + y^2 + z^2 - 2gx - 2fy - 2hz + c = 0. \qquad (3)$$

Since the sphere passes through the origin, hence we have $c = 0$. Again the sphere passes through the points A, B and D respectively. So Equation (3) gives $g = -\frac{1}{2}$, $f = -\frac{1}{2}$ and $h = 0$ respectively. Using the values of f, g, h and c, we get the radius of the sphere

$R = \sqrt{g^2+f^2+h^2-c} = \sqrt{\frac{1}{2}}$. The four faces of the tetrahedron $OACD$ are OAC, OAD, OCD and ACD respectively. The twelve face angles are

$$\theta^1_{01} = \frac{\pi}{2}, \quad \theta^1_{02} = \frac{\pi}{4}, \quad \theta^1_{03} = \frac{\pi}{4}, \quad \theta^1_{10} = \frac{\pi}{3}, \quad \theta^1_{12} = \frac{\pi}{3}, \quad \theta^1_{13} = \frac{\pi}{3},$$
$$\theta^1_{20} = \frac{\pi}{3}, \quad \theta^1_{21} = \frac{\pi}{3}, \quad \theta^1_{23} = \frac{\pi}{3}, \quad \theta^1_{30} = \frac{\pi}{2}, \quad \theta^1_{31} = \frac{\pi}{4}, \quad \theta^1_{32} = \frac{\pi}{4}$$

corresponding to the triangular faces OAC, OAD, OCD, ACD respectively. Similarly the face angles of the corresponding faces of the tetrahedron $ABCD$ are

$$\theta^2_{01} = \frac{\pi}{2}, \quad \theta^2_{02} = \frac{\pi}{4}, \quad \theta^2_{03} = \frac{\pi}{4}, \quad \theta^2_{10} = \frac{\pi}{3}, \quad \theta^2_{12} = \frac{\pi}{3}, \quad \theta^2_{13} = \frac{\pi}{3},$$
$$\theta^2_{20} = \frac{\pi}{3}, \quad \theta^2_{21} = \frac{\pi}{3}, \quad \theta^2_{23} = \frac{\pi}{3}, \quad \theta^2_{30} = \frac{\pi}{2}, \quad \theta^2_{31} = \frac{\pi}{4}, \quad \theta^2_{32} = \frac{\pi}{4}.$$

5.1 Volume Computation Using the Formula Described in Section 4.1

Let r^1_0, r^1_1, r^1_2, and r^1_3 be the inradii of the faces OAC, OAD, OCD, and ACD respectively. Now we have to calculate the values of r^1_1 and r^1_3. Here r^1_1 is the radius of the inscribed circle of the triangular face OAD and r^1_3 be the radius of the inscribed circle of the triangular face ADC. It is obvious from Figure 1 that the face OAD is an equilateral triangle with the side length 1. Hence using geometry, we get $r^1_1 = \frac{1}{2\sqrt{3}}$. Similarly we can find the value of inradii of the face ACD i.e. $r^1_3 = \frac{2-\sqrt{2}}{2}$.

Now using the above data from the Figure 1, we can compute the values of

$$\begin{aligned}
A_1 &= \tfrac{1}{8}\sin\theta^1_{32}\sin\theta^1_{21}\sin\theta^1_{10} = \tfrac{3}{32\sqrt{2}},\\
B_1 &= \tfrac{1}{8}\sin\theta^1_{31}\sin\theta^1_{20}\sin\theta^1_{12} = \tfrac{3}{32\sqrt{2}},\\
C_1 &= \tfrac{1}{8}\sin\theta^1_{30}\sin\theta^1_{21}\sin\theta^1_{12} = \tfrac{3}{32},\\
D_1 &= \sin\tfrac{\theta^1_{32}}{2}\sin\tfrac{\theta^1_{30}}{2}\sin\tfrac{\theta^1_{31}}{2}\sin\tfrac{\theta^1_{21}}{2}\sin\tfrac{\theta^1_{10}}{2}\sin\tfrac{\theta^1_{13}}{2}\sin\tfrac{\theta^1_{12}}{2}\cos\tfrac{\theta^1_{21}}{2} = \tfrac{(\sqrt{2}-1)\sqrt{3}}{128},\\
M_1 &= A_1^4+B_1^4+C_1^4-2A_1^2B_1^2-2B_1^2C_1^2-2C_1^2A_1^2 = -\tfrac{3^4}{32^4}
\end{aligned}$$

As we know that the two component tetrahedrons are identical, we can easily find the values $r^2_1 = \frac{1}{2\sqrt{3}}$ and $r^2_3 = \frac{2-\sqrt{2}}{2}$ and also

$$\begin{aligned}
A_2 &= \tfrac{1}{8}\sin\theta^2_{32}\sin\theta^2_{21}\sin\theta^2_{10} = \tfrac{3}{32\sqrt{2}},\\
B_2 &= \tfrac{1}{8}\sin\theta^2_{31}\sin\theta^2_{20}\sin\theta^2_{12} = \tfrac{3}{32\sqrt{2}},\\
C_2 &= \tfrac{1}{8}\sin\theta^2_{30}\sin\theta^2_{21}\sin\theta^2_{12} = \tfrac{3}{32},\\
D_2 &= \sin\tfrac{\theta^2_{32}}{2}\sin\tfrac{\theta^2_{30}}{2}\sin\tfrac{\theta^2_{31}}{2}\sin\tfrac{\theta^2_{21}}{2}\sin\tfrac{\theta^2_{10}}{2}\sin\tfrac{\theta^2_{13}}{2}\sin\tfrac{\theta^2_{12}}{2}\cos\tfrac{\theta^2_{21}}{2} = \tfrac{(\sqrt{2}-1)\sqrt{3}}{128},\\
M_2 &= A_2^4+B_2^4+C_2^4-2A_2^2B_2^2-2B_2^2C_2^2-2C_2^2A_2^2 = -\tfrac{3^4}{32^4}
\end{aligned}$$

Therefore using the result of Theorem 1. of section 3.1, the volume V of the polyhedron can be given by

$$V = \frac{{r^1_1}^2 {r^1_3}^2}{24RD_1^2}\sqrt{(-M_1)} + \frac{{r^2_1}^2 {r^2_3}^2}{24RD_2^2}\sqrt{(-M_2)} \tag{4}$$

Now substituting the above values in Equation (4), we get the volume of the polyhedron $V = 0.23570226039$.

5.2 Volume Computation Using the Formula Described in Section 4.2

Let R_0^1, R_1^1, R_2^1, and R_3^1 be the circumradii of the faces OAC, OAD, OCD, and ACD respectively. Using the basic geometry we can calculate the value of $R_1^1 = \frac{1}{\sqrt{3}}$ and $R_3^1 = \frac{1}{\sqrt{2}}$. Now from the data from Figure 1, in Section 5, we can calculate

$$\begin{aligned}
&A_1 = \sin\theta_{32}^1 \sin\theta_{10}^1 = \tfrac{\sqrt{3}}{2\sqrt{2}},\\
&B_1 = \tfrac{\sin\theta_{31}^1 \sin\theta_{20}^1 \sin\theta_{12}^1}{\sin\theta_{21}^1} = \tfrac{\sqrt{3}}{2\sqrt{2}},\\
&C_1 = \sin\theta_{30}^1 \sin\theta_{12}^1 = \tfrac{\sqrt{3}}{2},\\
&N_1 = A_1^4 + B_1^4 + C_1^4 - 2A_1^2B_1^2 - 2B_1^2C_1^2 - 2C_1^2A_1^2 = -\tfrac{36}{64}
\end{aligned}$$

Similarly for another identical component, we have $R_1^2 = \frac{1}{\sqrt{3}}$ and $R_3^2 = \frac{1}{\sqrt{2}}$ and also

$$\begin{aligned}
&A_2 = \sin\theta_{32}^2 \sin\theta_{10}^2 = \tfrac{\sqrt{3}}{2\sqrt{2}},\\
&B_2 = \tfrac{\sin\theta_{31}^2 \sin\theta_{20}^2 \sin\theta_{12}^2}{\sin\theta_{21}^2} = \tfrac{\sqrt{3}}{2\sqrt{2}},\\
&C_2 = \sin\theta_{30}^2 \sin\theta_{12}^2 = \tfrac{\sqrt{3}}{2},\\
&N_2 = A_2^4 + B_2^4 + C_2^4 - 2A_2^2B_2^2 - 2B_2^2C_2^2 - 2C_2^2A_2^2 = -\tfrac{36}{64}
\end{aligned}$$

Therefore using the result of Theorem 2. of Section 4.1, the volume V of the polyhedron can be given by

$$V = \frac{2}{3}\frac{{R_1^1}^2 {R_3^1}^2}{3R}\sqrt{(-N_1)} + \frac{2}{3}\frac{{R_1^2}^2 {R_3^2}^2}{3R}\sqrt{(-N_2)} \tag{5}$$

Now substituting the above values in Equation (5), we get the volume of the polyhedron $V = 0.23570226039$.

6 Volume of n-Dimensional Simplex

In this section, our aim is to extend the above result to n-dimensions. The volume of n-dimensional simplex is given by the formula

$$V_n^2(S) = \frac{(-1)^{n+1}}{2^n (n!)^2}\det(M') \tag{6}$$

where, $\det(M')$ can be written as

$$\begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 & 1\\
1 & 0 & a_{01}^2 & \cdots & a_{0,n-2}^2 & a_{0,n-1}^2 & a_{0n}^2\\
1 & a_{10}^2 & 0 & \cdots & a_{1,n-2}^2 & a_{1,n-1}^2 & a_{1n}^2\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
1 & a_{n-2,0}^2 & a_{n-2,1}^2 & \cdots & 0 & a_{n-2,n-1}^2 & a_{n-2,n}^2\\
1 & a_{n-1,0}^2 & a_{n-1,1}^2 & \cdots & a_{n-1,n-2}^2 & 0 & a_{n-1,n}^2\\
1 & a_{n,0}^2 & a_{n,1}^2 & \cdots & a_{n,n-2}^2 & a_{n,n-1}^2 & 0
\end{vmatrix}$$

From Section 9.7 of Berger [1], the circumradius R can be given by

$$R^2 = -\frac{1}{2}\frac{\det(M'')}{\det(M')} \tag{7}$$

where $\det(M'')$ is given by

$$\begin{vmatrix} 0 & a_{01}^2 & \cdots & a_{0,n-1}^2 & a_{0n}^2 \\ a_{10}^2 & 0 & \cdots & a_{1,n-1}^2 & a_{1n}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,0}^2 & a_{n-1,1}^2 & \cdots & 0 & a_{n-1,n}^2 \\ a_{n,0}^2 & a_{n,1}^2 & \cdots & a_{n,n-1}^2 & 0 \end{vmatrix}$$

Now substituting the value of $\det(M')$ from (7) in (6), we have

$$V_n^2(S) = -\frac{(-1)^{n+1}}{2^{n+1}(n!)^2 R^2} \det(M'') \tag{8}$$

In the proposed methodology, it has been observed that the volume of the tetrahedron is computed by expanding the Cayley-Menger determinant. Now it is obvious that the order of the determinant increases with the increase of the dimension. It is not feasible to expand an $(n \times n)$ determinant, so we suggest to use a recursive algorithm to compute the volume in n-dimension.

Algorithm 1 Volume calculation using inradii and face angles

Input $A[][]$, $r[]$, $\theta[][]$, R, n

$A[0][1] \leftarrow \frac{r[2]\cos\frac{\theta[2][3]}{2}}{\sin\frac{\theta[2][0]}{2}\sin\frac{\theta[2][1]}{2}}$, $A[0][2] \leftarrow \frac{r[1]\cos\frac{\theta[1][3]}{2}}{\sin\frac{\theta[1][0]}{2}\sin\frac{\theta[1][2]}{2}}$, $A[0][3] \leftarrow \frac{r[1]\cos\frac{\theta[1][2]}{2}}{\sin\frac{\theta[1][0]}{2}\sin\frac{\theta[1][3]}{2}}$,

$A[1][2] \leftarrow \frac{r[0]\cos\frac{\theta[0][3]}{2}}{\sin\frac{\theta[0][1]}{2}\sin\frac{\theta[0][2]}{2}}$, $A[1][3] \leftarrow \frac{r[2]\cos\frac{\theta[2][0]}{2}}{\sin\frac{\theta[2][1]}{2}\sin\frac{\theta[2][3]}{2}}$, $A[2][3] \leftarrow \frac{r[0]\cos\frac{\theta[0][1]}{2}}{\sin\frac{\theta[0][2]}{2}\sin\frac{\theta[0][3]}{2}}$

function VOLINRAD(A, n)
  if $n = 3$ **then**
    return $(-\frac{1}{576R^2}\det(A, 4))$ ▷ Determinant of 4×4 matrix.
  else
    return $(-\frac{(-1)^{n+1}}{4n^2R^2\,\text{VOLINRAD}(A,n-1)}\det(A, n+1))$
    ▷ Determinant of $(n+1) \times (n+1)$ matrix.
  end if
end function

Algorithm 2 Volume calculation using circumradii and face angles

Input $A[][]$, $r[]$, $\theta[][]$, R, n

$A[0][1] \leftarrow 2R[2]\sin\theta[2][3]$, $A[0][2] \leftarrow 2R[1]\sin\theta[1][3]$, $A[0][3] \leftarrow 2R[1]\sin\theta[1][2]$

$A[1][2] \leftarrow 2R[0]\sin\theta[0][3]$, $A[1][3] \leftarrow 2R[2]\sin\theta[2][0]$, $A[2][3] \leftarrow 2R[0]\sin\theta[0][1]$

function VOLCIRCUMRAD(A, n)
  if $n = 3$ **then**
    return $-\frac{1}{576R^2}\det(A, 4)$ ▷ Determinant of 4×4 matrix.
  else
    return $-\frac{(-1)^{n+1}}{4n^2R^2\,\text{VOLCIRCUMRAD}(A,n-1)}\det(A, n+1)$
    ▷ Determinant of $(n+1) \times (n+1)$ matrix.
  end if
end function

In the proposed algorithm, we calculate the volume of the polyhedron in two ways. In the first one, we define a recursive function VOLINRAD which takes a two dimensional array A and the dimension n as the arguments. The array stores values of edges of the simplex which are defined in terms inradii and face angles of the faces of the simplex. Two more arrays are used to store inradii and face angles respectively. Another recursive function $\det(A, n)$ is used to calculate $(n \times n)$ determinants. The recursive function VOLINRAD terminates when the dimension reaches to $n = 3$.

In the second method the process is same but the inputs to the array are the edges in terms of circumradii and the face angles of the faces of the simplex. The method volcircumrad takes two arguments, one is a two dimensional array A and another is the dimension n.

7 Volume of n-Dimensional Polyhedron

In Section 5, we have already calculated the volume of a general polyhedron in three dimensions. The result is also demonstrated using a suitable example. The volume calculation for a polyhedron in n-dimensions is much more complicated. Now for a polyhedron which is a simplex, we derived the volume formula in the previous section. For a polyhedron which is not a simplex, we can not use the methodology directly described in the previous section. Signed simplex decomposition is a technique by which a convex polyhedron can be decomposed into number of component simplices. Thus a n-dimensional polyhedron can also be decomposed into number of component n-simplices. Now taking the algebraic sum over the volume of such n-simplices, we can calculate the volume of the proposed polyhedron.

7.1 Analysis

On the basis of observations by Károlyi Lovász [7], a polygon can be decomposed into its component triangles if the edges of the triangles lie on the same line as that of the polygon. The possible generalization of this observation for a convex polytope P in d-dimensional euclidean space is that the polytope can be decomposed into component simplices if its facets are parallel to P. Such decomposition is also possible even if the faces of the polytope are in general position. In this situation the decomposition is performed by allowing an auxiliary hyperplane to which the facets of the polytope are parallel. Let us consider a convex m-polytope P. Let us define a set of hyperplanes K_P determined by the m-1 facets of P. If the normals of K_P are in general position such that the simplices P_1, P_2, …, $P_n \subset R^m$ with ϵ_1, ϵ_2, …, ϵ_n having signs $\{+1, -1\}$ then we can write

$$P = \epsilon_1 P_1 + \epsilon_2 P_2 + \cdots + \epsilon_n P_n$$

In Section 6, we already computed the volume for n-dimensional simplex in terms of inradii, circumradii and face angles of the faces of the simplex. As we have decomposed the polytope into the number of simplices in n-dimension, we can separately calculate the volumes of each of the component simplices of the polytope and take the summation to get the volume of the required polytope.

In light of above discussion, we can derive the volume(V_P) of n-dimensional polyhedron as

$$V_P = \epsilon_1 V_1 + \epsilon_2 V_2 + \cdots + \epsilon_n V_n$$

where, V_1, V_2, …, V_n be the volumes of component simplices respectively.

8 Conclusion

In this article, the volume of a polyhedron is computed using the technique of singed simplex decomposition. According to this technique the polyhedron is decomposed into component tetrahedrons and the volume is calculated by taking the algebraic sum of the volumes of the tetrahedrons. Two formulas have been proposed by the authors in which the first one is in terms of inradii, face angles of the faces of the component tetrahedrons and the radius of the sphere circumscribing the polyhedron whereas the second formula is computed the volume in terms of circumradii, face angles of the faces of the component tetrahedrons along with the radius of the sphere circumscribing it. A squared base pyramid with all the edges of unit length is taken as an example to verify the result. An algorithm is designed to compute the volume of an n-dimensional simplex. Finally the volume of a n-dimensional polyhedron is described. This method may be extended to evaluate the volume of a general polyhedron in non euclidean spaces.

References

[1] M. Berger: *Notation and background.* In *Geometry I*, 1–3. Springer, 1987.

[2] A. Bhattacharyya and B. Pal: *Volume of a tetrahedron revisited.* Demonstr. Math. **47**(1), 253–257, 2014. https://doi.org/10.2478/dema-2014-0019.

[3] R. H. Buchholz and M. J. Smith: *Volume of an n dimensional simplex* 1996.

[4] Y. Cho: *Volume of a tetrahedron in terms of dihedral angles and circumradius.* Appl. Math. Lett. **13**(4), 45–47, 2000. https://doi.org/10.1016/S0893-9659(99)00206-2.

[5] S. D. Choudhury, S. Dey, and A. Bhattacharyya: *Volume of a Pentahedron Revisited.* Palest. J. Math. **8**(2), 2019.

[6] J. Dompierre, P. Labbé, M.-G. Vallet, and R. Camarero: *How to Subdivide Pyramids, Prisms, and Hexahedra into Tetrahedra.* IMR **99**, 195, 1999.

[7] G. Károlyi and L. Lovász: *Decomposition of convex polytopes into simplices.* Preprint, 1991.

[8] H. Newson: *On the volume of a polyhedron.* Ann. of Math. (2) **1**(1/4), 108–110, 1899. doi: https://doi.org/10.2307/1967277.

[9] I. K. Sabitov: *The volume as a metric invariant of polyhedra.* Discrete Comput. Geom. **20**(4), 405–425, 1998. doi: https://doi.org/10.1007/PL00009393.

Received November 3, 2022; final form March 3, 2023.