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Lie Superalgebras and Lie Supergroups, I

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1. Graded Algebras

Let K denote the base field. K is assumed to be of characteristic zero. In the examples K is the field of real or complex numbers.

Let $\mathbb{Z}$ denote the additive group of the integers. A $\mathbb{Z}$*-graded linear space* is a K-linear space V and a family of subspaces V_k, $(k = 0, \pm 1, \pm 2, \ldots)$ such that

$$V = \bigoplus_{k \in \mathbb{Z}} V_k.$$

A $\mathbb{Z}$*-graded associative algebra* is an associative K-algebra $\mathcal{A}$, which is $\mathbb{Z}$-graded as a K-linear space

$$\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k,$$

such that

$$\mathcal{A}_k \mathcal{A}_l \subseteq \mathcal{A}_{k+\ell}, \quad (k, \, l \in \mathbb{Z}).$$

The elements of $\mathcal{A}_k$ are called *homogeneous of degree* k.

Example 1$^\diamond$. Let $V = \bigoplus_{k \in \mathbb{Z}} V_k$ be a $\mathbb{Z}$-graded linear space. A linear operator $A \colon V_k \longrightarrow V_{k+l}$ $(k \in \mathbb{Z})$ is called *homogeneous of degree* ℓ. If $L_\ell(V)$ denotes the space of linear operators of degree ℓ, then

$$L'(V) = \bigoplus_{k \in \mathbb{Z}} L_k(V)$$

is a $\mathbb{Z}$ -graded associative algebra with unit. If V is finite dimensional we have

$$L(V) = \bigoplus_{k \in \mathbb{Z}} L_k(V).$$

Assume V to be finite dimensional and $V_k = \{0\}$, $(k = -1, \pm 2, \pm 3, \ldots)$, $V = V_0 \oplus V_1$.

Then the following relations hold

$$\begin{aligned} L_0(V) &= L(V_0) \oplus L(V_1), \\ L_{-1}(V) &= L(V_1, V_0), \\ L_1(V) &= L(V_0, V_1), \text{ and} \\ L_k(V) &= \{0\}, \quad (k = \pm 2, \pm 3, \ldots). \end{aligned}$$

Chosing a basis of homogeneous elements in V we may represent the elements of $L_0(V)$ by diagonal block matrices

$$(1) \qquad \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}.$$

The elements of $L_{-1}(V)$ and $L_1(V)$ become represented by the block matrices

$$(2) \qquad \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ B_0 & 0 \end{pmatrix}, \quad \text{respectively.}$$

Example 2$^\diamond$. Let $\Lambda(n) = \Lambda(y_1, \ldots, y_n)$ denote the exterior or GRASSMANN algebra with n generators, which are assumed to be homogeneous of degree 1 . Put

$$\begin{aligned}
\Lambda_0 &= K, \\
\Lambda_1 &= \operatorname{span}\{y_1, \ldots, y_n\}, \\
\Lambda_k &= \operatorname{span}\{y_{i_1} \cdots y_{i_k} : \quad 1 \le i_1 < \cdots < i_k \le n\}, \quad (k = 2, \ldots, n-1), \\
\Lambda_n &= \operatorname{span}\{y_1 \cdots y_n\}, \\
\Lambda_\ell &= \{0\}, \quad \text{if } \ell = -1, -2, \ldots \text{ or } \ell = n+1, n+2, \ldots,
\end{aligned}$$

then we have

$$\Lambda(n) = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \oplus \Lambda_n,$$

and $\Lambda(n)$ becomes a $\mathbb{Z}$-graded associative algebra with unit. The $\mathbb{Z}$-graded algebra $\Lambda(n)$ is (*graded*) *commutative*, i.e., for homogeneous elements $a_k \in \Lambda_k$, $a_\ell \in \Lambda_\ell$ the following equation holds:

$$a_\ell a_k = (-1)^{k\ell} a_k a_\ell.$$

Example 3$^\diamond$. Let $\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k$ be a $\mathbb{Z}$-graded associative algebra with unit. We define a new product by

$$[a_k, a_\ell] = a_k a_\ell - (-1)^{k\ell} a_\ell a_k, \quad a_k \in \mathcal{A}_k.$$

With respect to that multiplication $\mathcal{A}$ becomes a $\mathbb{Z}$-graded LIE algebra

$$\mathcal{A}_L = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k.$$

We have

$$(3) \qquad [a_\ell, a_k] = -(-1)^{k\ell} [a_k, a_\ell],$$

and the modified JACOBI identity

$$(4) \qquad (-1)^{km}[a_k, [a_\ell, a_m]] + (-1)^{\ell k}[a_\ell [a_m, a_k]] + (-1)^{m\ell}[a_m [a_k, a_\ell]] = 0.$$

A $\mathbb{Z}$-graded LIE algebra is a $\mathbb{Z}$ -graded linear space $\mathcal{L} = \bigoplus_{k\in\mathbb{Z}} \mathcal{L}_k$ with a bracket multiplication compatible with the grading $[\mathcal{L}_k, \mathcal{L}_\ell] \subseteq \mathcal{L}_{k+\ell}$ and satisfying (3) and (4) for homogeneous elements.

Let $\mathbb{Z}_2 = \mathbb{Z}/(2) = \{\overline{0}, \overline{1}\}$ denote the additive group of two elements. A $\mathbb{Z}_2$-graded linear space is a K-linear space V with two distinguished subspaces $V_{\overline{0}}$ and $V_{\overline{1}}$ such that $V = V_{\overline{0}} \oplus V_{\overline{1}}$ holds. The elements of $V_{\overline{0}}$ are called *even* , those of $V_{\overline{1}}$ are called *odd*. A homogeneous element is either even or odd.

Every $\mathbb{Z}$-graded linear space admits a canonical $\mathbb{Z}_2$-gradation by $V_{\overline{0}} = \bigoplus_{k\in\mathbb{Z}} V_{2k}$, $V_{\overline{1}} = \bigoplus_{k\in\mathbb{Z}} V_{2k+1}$.

A $\mathbb{Z}_2$-graded associative algebra or an associative *superalgebra* is a K-algebra, which is $\mathbb{Z}_2$-graded as a linear space

$$\mathcal{A} = \mathcal{A}_{\overline{0}} \oplus \mathcal{A}_{\overline{1}},$$

such that the multiplication satisfies

$$\begin{aligned} &\mathcal{A}_{\overline{0}}\mathcal{A}_{\overline{0}} \subseteq \mathcal{A}_{\overline{0}}, \quad \mathcal{A}_{\overline{1}}\mathcal{A}_{\overline{1}} \subseteq \mathcal{A}_{\overline{0}}, \\ &\mathcal{A}_{\overline{0}}\mathcal{A}_{\overline{1}} \subseteq \mathcal{A}_{\overline{1}}, \quad \mathcal{A}_{\overline{1}}\mathcal{A}_{\overline{0}} \subseteq \mathcal{A}_{\overline{1}}. \end{aligned}$$

Every $\mathbb{Z}$-graded algebra $\mathcal{A}$ admits a canonical Z_2-gradation by

$$\mathcal{A}_{\overline{0}} = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_{2k}, \quad \mathcal{A}_{\overline{1}} = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_{2k+1}.$$

Example 4$^\diamond$. Let $V = V_{\overline{0}} \oplus V_{\overline{1}}$ denote a $\mathbb{Z}_2$-graded linear space. A linear operator A on V is called *paritiy preserving* or *even*, if it satisfies $A\colon V_{\overline{0}} \to V_{\overline{0}}$ and $A\colon V_{\overline{1}} \to V_{\overline{1}}$. The operator A is called *parity reversing* or *odd* if $A\colon V_{\overline{0}} \to V_{\overline{1}}$ and $A\colon V_{\overline{1}} \to V_{\overline{0}}$. By $L_{\overline{0}}(V)$ and $L_{\overline{1}}(V)$ we denote the linear space of even and odd linear operators on V, respectively.

$$L'(V) = L_{\overline{0}}(V) \oplus L_{\overline{1}}(V)$$

is a $\mathbb{Z}_2$-graded associative algebra with unit. If V is finite dimensional then

$$L(V) = L_{\overline{0}}(V) \oplus L_{\overline{1}}(V)$$

holds and

$$\begin{aligned} &L_{\overline{0}}(V) \cong L(V_{\overline{0}}) \oplus L(V_{\overline{1}}), \\ &\qquad\qquad L_{\overline{1}} \cong L(V_{\overline{0}}, V_{\overline{1}}) \oplus L(V_{\overline{1}}, V_{\overline{0}}). \end{aligned}$$

Choosing a basis of homogeneous elements in V, the elements of $L_{\overline{0}}(V)$ are represented by diagonal block matrices as in (1), while the elements of $L_{\overline{1}}(V)$ are represented by block matrices of the following type

$$\begin{pmatrix} 0 & B_1 \\ B_0 & 0 \end{pmatrix}.$$

Example 5$^\diamond$. The GRASSMANN algebra $\Lambda(n)$ admits a canonical $\mathbb{Z}_2$-grading with respect to a chosen system of generators $\Lambda(n) = \Lambda_{\overline{0}} \oplus \Lambda_{\overline{1}}$ with $\Lambda_{\overline{0}} = \Lambda_0 \oplus \Lambda_2 \oplus \cdots$, $\Lambda_{\overline{1}} = \Lambda_1 \oplus \Lambda_3 \oplus \cdots$ Now $\Lambda(n)$ is a $\mathbb{Z}_2$-graded associative and (graded) commutative algebra with unit or, equivalently, an associative, commutative superalgebra with unit.

Example 6$^\diamond$. Let $\mathcal{A} = \mathcal{A}_{\overline{0}} \oplus \mathcal{A}_{\overline{1}}$ denote a $\mathbb{Z}_2$-graded associative algebra with unit. We define brackets by $[a,b] = ab - (-1)^{|a||b|}ba$ for homogeneous elements $a, b \in \mathcal{A}$. Define the parity $|a|$ as follows:

$$|a| = \begin{cases} 0 & \text{if } a \in \mathcal{A}_{\overline{0}}, \\ 1 & \text{if } a \in \mathcal{A}_{\overline{1}}. \end{cases}$$

With respect to the brackets, $\mathcal{A}$ becomes a $\mathbb{Z}_2$-graded LIE algebra or,equivalently, a LIE superalgebra $\mathcal{A}_L = \mathcal{A}_{\overline{0}} \oplus \mathcal{A}_{\overline{1}}$. For homogeneous elements $a, b, c \in \mathcal{A}$ the following equations hold:

$$[b,a] = -(-1)^{|a||b|}[a,b], \tag{4}$$

$$(-1)^{|a||c|}[a,[b,c]] + (-1)^{|b||a|}[b,[c,a]] + (-1)^{|c||b|}[c,[a,b]] = 0 \tag{5}$$

(the modified JACOBI equation). A $\mathbb{Z}_2$-graded LIE algebra or LIE superalgebra is a $\mathbb{Z}_2$-graded linear space $\mathcal{L} = \mathcal{L}_{\overline{0}} \oplus \mathcal{L}_{\overline{1}}$ with a bracket multiplication $[\cdot,\cdot]$ compatible with the gradation

$$\begin{aligned} [\mathcal{L}_{\overline{0}}, \mathcal{L}_{\overline{0}}] \subseteq \mathcal{L}_{\overline{0}}, \quad [\mathcal{L}_{\overline{1}}, \mathcal{L}_{\overline{1}}] \subseteq \mathcal{L}_{\overline{0}}, \\ [\mathcal{L}_{\overline{0}}, \mathcal{L}_{\overline{1}}] \subseteq \mathcal{L}_{\overline{1}}, \quad [\mathcal{L}_{\overline{1}}, \mathcal{L}_{\overline{0}}] \subseteq \mathcal{L}_{\overline{1}} \end{aligned}$$

and satisfying (4) and (5).

2. LIE Superalgebras: The series A, B, C, D, Q

Let $\mathcal{L} = \mathcal{L}_{\overline{0}} \oplus \mathcal{L}_{\overline{1}}$ denote a $\mathbb{Z}_2$-graded LIE algebra. Its even part $\mathcal{L}_{\overline{0}}$ is a LIE algebra. In view of $[\mathcal{L}_{\overline{0}}, \mathcal{L}_{\overline{1}}] \subseteq \mathcal{L}_{\overline{1}}$, multiplication of odd elements by even ones defines a representation $\mathrm{ad}_{\overline{0}}$ of the LIE algebra $\mathcal{L}_{\overline{0}}$ on the linear space $\mathcal{L}_{\overline{1}}$

$$(\mathrm{ad}_{\overline{0}}\, x_{\overline{0}}) x_{\overline{1}} = [x_{\overline{0}}, x_{\overline{1}}] \quad x_{\overline{0}} \in \mathcal{L}_{\overline{0}}, x_{\overline{1}} \in \mathcal{L}_{\overline{1}}.$$

$\mathrm{ad}_{\overline{0}}$ is called the *adjoint representation of the even part* $\mathcal{L}_{\overline{0}}$ *on the odd part* $\mathcal{L}_1$.

A $\mathbb{Z}_2$-graded LIE algebra $\mathcal{L} = \mathcal{L}_{\overline{0}} \oplus \mathcal{L}_{\overline{1}}$ is called *simple* , if there are no nontrivial $\mathbb{Z}_2$-graded ideals: If $I = I_{\overline{0}} \oplus I_{\overline{1}}$ denotes a $\mathbb{Z}_2$-graded ideal of $\mathcal{L}$ then we have $I = \{0\}$ or $I = \mathcal{L}$. A simple $\mathbb{Z}_2$-graded LIE algebra is called *classical*, if the representation $\mathrm{ad}_{\overline{0}}$ is completely reducible. A simple $\mathbb{Z}_2$-graded LIE algebra is *classical* iff its even part $\mathcal{L}_{\overline{0}}$ is a reductive LIE algebra.

Denote by $\mathrm{Mat}(m,n)$ the $\mathbb{Z}_2$-graded associative algebra with unit consisting of $(m+n) \times (m+n)$ block matrices with entries from K

$$\begin{pmatrix} A_0 & B_1 \\ B_0 & A_1 \end{pmatrix}.$$

We observe $\mathrm{Mat}(m,n) = \mathrm{Mat}_{-1}(m,n) \oplus \mathrm{Mat}_0(m,n) \oplus \mathrm{Mat}_1(m,n)$ (compare (1) and (2)).

The $\mathbb{Z}_2$-graded LIE algebra defined by $\mathrm{Mat}(m,n)$ is denoted by $\mathfrak{gl}(m,n)$ or $\mathfrak{pl}(m,n)$. It is called the *general linear* LIE *superalgebra.* $\mathfrak{gl}(m,n)$ admits a $\mathbb{Z}$-gradation

$$\mathfrak{gl}(m,n) = \mathfrak{gl}_{-1}(m,n) \oplus \mathfrak{gl}_0(m,n) \oplus \mathfrak{gl}_1(m,n)$$

implying the $\mathbb{Z}_2$-gradation in the natural way. We have

$$\begin{aligned} \mathfrak{gl}_{\bar{0}}(m,n) &= \mathfrak{gl}_0(m,n) \cong \mathfrak{gl}(m) \times \mathfrak{gl}(n), \\ \mathfrak{gl}_{\bar{1}}(m,n) &= \mathfrak{gl}_{-1}(m,n) \oplus \mathfrak{gl}_1(m,n). \end{aligned}$$

We shall discuss several subalgebras of $\mathfrak{gl}(m,n)$.

Example 1•. $\mathfrak{sl}(m,n)$ or $\mathfrak{spl}(m,n)$ denotes the subalgebra consisting of those block matrices for which the diagonal blocks have equal trace, i.e., satisfying the equation $\mathrm{Tr}(A_0) = \mathrm{Tr}(A_1)$. It is called the *special linear* LIE *superalgebra.* $\mathfrak{sl}(m,n)$ admits the $\mathbb{Z}$-gradation induced by $\mathfrak{gl}(m,n)$ and the corresponding $\mathbb{Z}_2$-grading. We have $\mathfrak{sl}_{\bar{0}}(m,n) \cong \mathfrak{sl}(m) \times \mathfrak{sl}(n) \times K$.

Example 2•. $\mathfrak{osp}(m,n)$ consists of those block matrices satisfying the following relations

$$\begin{aligned} {A_0}^\top + A_0 &= 0, \\ {B_1}^\top - I_n B_0 &= 0, \\ {A_1}^\top I_n + I_n A_1 &= 0. \end{aligned}$$

Here $A^\top$ denotes the transpose of A, and n is assumed to be even $n = 2\ell$, and

$$I_n = \begin{pmatrix} 0 & E_\ell \\ -E_\ell & 0 \end{pmatrix},$$

where E_ℓ denotes the $\ell \times \ell$ unit matrix. The matrices of $\mathfrak{osp}(m,2\ell)$ may be written in the following form

$$\begin{pmatrix} A_0 & B_1 & B_2 \\ -{B_2}^\top & A_1 & A_{12} \\ {B_1}^\top & A_{21} & -{A_1}^\top \end{pmatrix} \quad \text{with } {A_0}^\top = -A_0,\ A_{12}^\top = A_{12},\ \text{and } A_{21}^\top = A_{21}.$$

Note that $\mathfrak{osp}(m,n)$ is a $\mathbb{Z}_2$-graded LIE algebra. It is called the *orthogonal-symplectic* LIE superalgebra. For the even part one has

$$\mathfrak{osp}_{\bar{0}}(m,n) \cong \mathfrak{o}(m) \times \mathfrak{sp}(n).$$

Example 3•. $\widetilde{\mathfrak{gl}}(m)$ consists of those block matrices satisfying $m = n$ and $A_1 = A_0$ and $B_1 = B_0$. It is called the *general linear* LIE *superalgebra of the second kind.* $\widetilde{\mathfrak{gl}}(m)$ is a $\mathbb{Z}_2$-graded LIE algebra and we have $\widetilde{\mathfrak{gl}}_{\bar{0}}(m) \cong \mathfrak{gl}(m)$.

Example 4•. $\widetilde{\mathfrak{sl}}(m)$ consists of those block matrices satisfying $m = n$ and $A_1 = A_0$, $B_1 = B_0$, and $\mathrm{Tr}(B_1) = 0$. It is called the *special linear* LIE *superalgebra of the second kind.* $\widetilde{\mathfrak{sl}}(m)$ is a $\mathbb{Z}_2$-graded LIE algebra, a subalgebra of $\widetilde{\mathfrak{gl}}(m)$, and the following relation holds: $\widetilde{\mathfrak{sl}}_{\overline{0}}(m) = \widetilde{\mathfrak{gl}}_{\overline{0}}(m) \cong \mathfrak{gl}(m)$.

The LIE *superalgebra* $\mathfrak{sl}(m,n)$ *is simple if* $m \neq n$, $m \geq 1$, $n \geq 1$.

The multiples of the unit matrix $\{\alpha E_{2m}; \alpha \in K\}$ make up a $\mathbb{Z}_2$-graded ideal of $\mathfrak{sl}(m,m)$, namely, the center.

The quotient algebra $\mathfrak{sl}(m,m)/\{\alpha E_{2m} : \alpha \in K\}$ *is simple if* $m > 1$.

The $\mathbf{A}$-series of simple LIE superalgebras is defined in analogy with the usual classification of simple LIE algebras by

$$\mathbf{A}(m,n) = \mathfrak{sl}(m+1, n+1), \quad m \neq n,\ m \geq 0,\ n \geq 0.$$
$$\mathbf{A}(m,m) = \mathfrak{sl}(m+1, m+1)/\{\alpha E_{2m+2} : \alpha \in K\}, \quad m > 0.$$

The orthogonal-symplectic LIE *superalgebra* $\mathfrak{osp}(m,n)$ *is simple if* $m \geq 1$, $n > 1$.

The series $\mathbf{B}, \mathbf{C}, \mathbf{D}$ are defined as follows

$$\mathbf{B}(m,n) = \mathfrak{osp}(2m+1, 2n), \quad m \geq 0,\ n > 0.$$
$$\mathbf{C}(n) = \mathfrak{osp}(2, 2n-2), \quad n \geq 2.$$
$$\mathbf{D}(m,n) = \mathfrak{osp}(2m, 2n), \quad m \geq 2,\ n > 0.$$

As in $\mathfrak{sl}(m,m)$, the multiples of unity $\{\alpha E_{2m}; \alpha \in K\}$ make up a homogeneous ideal in $\widetilde{\mathfrak{sl}}(m)$, namely, its center.

The quotient algebra $\widetilde{\mathfrak{sl}}(m)/\{\alpha E_{2m} : \alpha \in K\}$ *is simple if* $m \geq 3$.

The $\mathbf{Q}$-series is defined by

$$\mathbf{Q}(m) = \widetilde{\mathfrak{sl}}(m+1)/\{\alpha E_{2m+2} : \alpha \in K\}, \quad m \geq 2.$$

The LIE superalgebras $\mathbf{Q}(m)$ are often called *the* $f-d$*-algebras of* MICHAL *and* RADICATI.

The LIE *superalgebras of the series* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$, *and* $\mathbf{Q}$ *are classical* LIE *superalgebras.*

The decomposition of the even parts in a direct product of simple LIE algebras is given by the following isomorphisms

$$\mathbf{A}_{\overline{0}}(m,n) \cong \mathbf{A}(m) \times \mathbf{A}(n) \times K, \quad m \neq n$$
$$\mathbf{A}_{\overline{0}}(m,m) \cong \mathbf{A}(m) \times \mathbf{A}(m)$$
$$\mathbf{B}_{\overline{0}}(m,n) \cong \mathbf{B}(m) \times \mathbf{C}(n)$$
$$\mathbf{C}_{\overline{0}}(m) \cong \mathbf{C}(m-1) \times K$$
$$\mathbf{D}_{\overline{0}}(m,n) \cong \mathbf{D}(m) \times \mathbf{C}(n)$$
$$\mathbf{Q}_{\overline{0}}(m) \cong \mathbf{A}(m).$$

3. The GRASSMANN-hull

The GRASSMANN-hull is a construction, which enables us to make a $\mathbb{Z}_2$-graded LIE algebra into a LIE algebra. Let Λ denote a finitely generated GRASSMANN algebra and let $\mathcal{L}$ be a $\mathbb{Z}_2$-graded LIE algebra. Taking the tensorproduct $\Lambda \otimes \mathcal{L}$ of $\mathbb{Z}_2$-graded algebras we have

$$\begin{aligned}(\Lambda \otimes \mathcal{L})_{\overline{0}} &= \Lambda_{\overline{0}} \otimes \mathcal{L}_{\overline{0}} + \Lambda_{\overline{1}} \otimes \mathcal{L}_{\overline{1}},\\ (\Lambda \otimes \mathcal{L})_{\overline{1}} &= \Lambda_{\overline{0}} \otimes \mathcal{L}_{\overline{1}} + \Lambda_{\overline{1}} \otimes \mathcal{L}_{\overline{0}}.\end{aligned}$$

Writing the elements of $\Lambda \otimes \mathcal{L}$ for simplicity as λx, the brackets are defined for homogeneous elements as follows: $[\lambda_\alpha x_\beta, \lambda_\gamma x_\delta] = (-1)^{\beta\gamma} \lambda_\alpha \lambda_\gamma [x_\beta, x_\delta]$, α, β, γ, $\delta \in \mathbb{Z}_2$. If $\alpha = \beta$ and $\gamma = \delta$, then we have

$$\begin{aligned}[\lambda_\gamma x_\delta, \lambda_\alpha x_\beta] &= (-1)^{\alpha\delta} \lambda_\gamma \lambda_\alpha [x_\delta, x_\beta]\\ &= (-1)^{\alpha\delta+\alpha\gamma+\beta\delta}(-1)\lambda_\alpha \lambda_\gamma [x_\beta, x_\delta]\\ &= -(-1)^{\alpha\delta+\alpha\gamma+\beta\delta+\beta\gamma}[\lambda_\alpha x_\beta, \lambda_\gamma x_\delta]\end{aligned}$$

Since $\alpha\delta + \beta\delta = \alpha\gamma + \beta\gamma = \overline{0}$ we have $[\lambda_\gamma x_\delta, \lambda_\alpha x_\beta] = -[\lambda_\alpha x_\beta, \lambda_\gamma x_\delta]$. The even part $(\Lambda \otimes \mathcal{L})_{\overline{0}}$ of the tensor product $\Lambda \otimes \mathcal{L}$ is a LIE algebra. It is called the GRASSMANN-*hull* of the LIE superalgebra $\mathcal{L}$.

Let $\mathcal{L}$ denote one of the LIE superalgebras $\mathfrak{gl}(m,n)$, $\mathfrak{sl}(m,n)$, $\mathfrak{osp}(m,n)$, $\widetilde{\mathfrak{gl}}(m)$, or $\widetilde{\mathfrak{sl}}(m)$. The GRASSMANN-hull consists of block matrices

$$\begin{pmatrix} A_0(\Lambda_{\overline{0}}) & B_1(\Lambda_{\overline{1}}) \\ B_0(\Lambda_{\overline{1}}) & A_1(\Lambda_{\overline{0}}) \end{pmatrix}. \tag{5}$$

The entries of the diagonal blocks A_0 and A_1 belong to $\Lambda_{\overline{0}}$, while the entries of the matrices B_0 and B_1 are from $\Lambda_{\overline{1}}$. We denote the GRASSMANN-hulls by $\mathfrak{gl}(m,n;\Lambda)$, $\mathfrak{sl}(m,n;\Lambda)$, $\mathfrak{osp}(m,n;\Lambda)$, $\widetilde{\mathfrak{gl}}(m;\Lambda)$, $\widetilde{\mathfrak{sl}}(m;\Lambda)$, respectively. Note that $\mathfrak{sl}(m,n;\Lambda)$ is the LIE algebra of block matrices of type (5) such that $\mathrm{Tr}(A_0(\Lambda_{\overline{0}})) = \mathrm{Tr}(A_1(\Lambda_{\overline{0}}))$. Further, $\mathfrak{osp}(m,2l;\Lambda)$ is the LIE algebra of block matrices

$$\begin{pmatrix} A_0(\Lambda_{\overline{0}}) & B_1(\Lambda_{\overline{1}}) & B_2(\Lambda_{\overline{1}}) \\ -B_2^\top(\Lambda_{\overline{1}}) & A_1(\Lambda_{\overline{0}}) & A_{12}(\Lambda_{\overline{0}}) \\ B_1^\top(\Lambda_{\overline{0}}) & A_{21}(\Lambda_{\overline{0}}) & -A_1^\top(\Lambda_{\overline{0}}) \end{pmatrix}$$

satisfying the relations $A_0^\top(\Lambda_{\overline{0}}) = A_0(\Lambda_{\overline{0}})$, $A_{12}^\top(\Lambda_{\overline{0}}) = A_{12}(\Lambda_{\overline{0}})$, $A_{21}^\top(\Lambda_{\overline{0}}) = A_{21}(\Lambda_{\overline{0}})$. Finally, $\widetilde{\mathfrak{sl}}(m;\Lambda)$ consists of block matrices

$$\begin{pmatrix} A(\Lambda_{\overline{0}}) & B(\Lambda_{\overline{1}}) \\ B(\Lambda_{\overline{1}}) & A(\Lambda_{\overline{0}}) \end{pmatrix}$$

satisfying the relation $\mathrm{Tr}\, B(\Lambda_{\overline{1}}) = 0$.

In all of our considerations Λ denotes an arbitrary finitely generated GRASSMANN algebra. Later it will become clear, that it is necessary to assume, that the number of generators "is not to small" with respect to m and n.

Let $p_0\colon \Lambda \to \Lambda_0 = K$ denote the canonical projection of the $\mathbb{Z}$-graded algebra Λ onto its zero component. Then p_0 defines a canonical projection of the GRASSMANN-hull $(\Lambda \otimes \mathcal{L})_{\bar{0}}$ of a LIE superalgebra $\mathcal{L}$ onto its even part $\mathcal{L}_{\bar{0}}$. We denote it once more by p_0, so that $p_0(\lambda x) = p_0(\lambda)x$. With respect to the matrix LIE algebra $\mathfrak{gl}(m,n;\Lambda)$ and its subalgebras we have

$$p_0 \begin{pmatrix} A_0(\Lambda_{\bar{0}}) & B_1(\Lambda_{\bar{1}}) \\ B_0(\Lambda_{\bar{1}}) & A_1(\Lambda_{\bar{0}}) \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}$$

with entries from K in the diagonal blocks A_0 and A_1.

In the case of $\mathfrak{osp}(m,n;\Lambda)$ the matrices A_0 make up the LIE algebra $\mathfrak{o}(m)$, while the matrices A_1 are the matrices of $\mathfrak{sp}(n)$.

In the same way we may define the GRASSMANN-hull of a $\mathbb{Z}_2$-graded associative algebra. The GRASSMANN-hull of the matrix algebra $\mathrm{Mat}(m,n)$ is denoted by $\mathrm{Mat}(m,n;\Lambda) = (\Lambda \otimes \mathrm{Mat}(m,n))_{\bar{0}}$, the elements are the block matrices (5).

4. GRASSMANN LIE groups

The series GL, SL, OSp, $\widetilde{\mathrm{GL}}$, $\widetilde{\mathrm{SL}}$. We define matrix LIE groups corresponding to the GRASSMANN-hulls of matrix LIE superalgebras. First we answer the question of invertibility of a matrix (5) from $\mathrm{Mat}(m,n;\Lambda)$.

A block matrix of type (5) *is invertible iff the matrices* $A_0 = p_0(A_0(\Lambda_{\bar{0}}))$ *and* $A_1 = p_0(A_1(\Lambda_{\bar{0}}))$ *are invertible.*

It follows that the diagonal blocks $A_0(\Lambda_{\bar{0}})$ and $A_1(\Lambda_{\bar{0}})$ are invertible for invertible block matrices. The inverse of a block matrix is written

$$\begin{pmatrix} A_0^{(-1)}(\Lambda_{\bar{0}}) & B_1^{(-1)}(\Lambda_{\bar{1}}) \\ B_0^{(-1)}(\Lambda_{\bar{1}}) & A_1^{(-1)}(\Lambda_{\bar{0}}) \end{pmatrix}$$

satisfying the equations

$$\begin{aligned}
A_0^{(-1)}(\Lambda_{\bar{0}}) &= (A_0(\Lambda_{\bar{0}}) - B_1(\Lambda_{\bar{1}})A_1(\Lambda_{\bar{0}})^{-1}B_0(\Lambda_{\bar{1}}))^{-1}, \\
A_1^{(-1)}(\Lambda_{\bar{0}}) &= (A_1(\Lambda_{\bar{0}}) - B_0(\Lambda_{\bar{1}})A_0(\Lambda_{\bar{0}})^{-1}B_1(\Lambda_{\bar{1}}))^{-1}, \\
B_1^{(-1)}(\Lambda_{\bar{1}}) &= -A_0(\Lambda_{\bar{0}})^{-1}B_1(\Lambda_{\bar{1}})(A_1(\Lambda_{\bar{0}}) - B_0(\Lambda_{\bar{1}})A_0(\Lambda_{\bar{0}})^{-1}B_1(\Lambda_{\bar{1}}))^{-1}, \\
B_0^{(-1)}(\Lambda_{\bar{1}}) &= -A_1(\Lambda_{\bar{0}})^{-1}B_0(\Lambda_{\bar{1}})(A_0(\Lambda_{\bar{0}}) - B_1(\Lambda_{\bar{1}})A_1(\Lambda_{\bar{0}})^{-1}B_0(\Lambda_{\bar{1}}))^{-1}.
\end{aligned}$$

Let $\mathrm{GL}(m,n;\Lambda)$ denote the group of units in $\mathrm{Mat}(m,n;\Lambda)$. It consists of the invertible block matrices of type (5) which for simplicity are written

$$\mathcal{A}(\Lambda) = \begin{pmatrix} A_0(\Lambda) & B_1(\Lambda) \\ B_0(\Lambda) & A_1(\Lambda) \end{pmatrix}.$$

The *superdeterminant* or BEREZIN*ian* of $\mathcal{A}$ is defined by $\operatorname{sdet}\mathcal{A}(\Lambda) = \det\big(A_0(\Lambda) - B_1(\Lambda)A_1(\Lambda)^{-1}B_0(\Lambda)\big)\det A_1^{-1}(\Lambda)$. The superdeterminant is defined on the GRASSMANN LIE group $\mathrm{GL}(m,n;\Lambda)$. It is multiplicative, i.e., sdet is a homomorphism of $\mathrm{GL}(m,n;\Lambda)$ into the group of units $K^\times$ of Λ.

Also, $\mathrm{SL}(m,n;\Lambda)$ is the subgroup of $\mathrm{GL}(m,n;\Lambda)$ defined by

$$\operatorname{sdet}\mathcal{A}(\Lambda) = 1,$$

or, equivalently, by

$$\det(A_0(\Lambda) - B_1(\Lambda)A_1(\Lambda)^{-1}B_0(\Lambda)) = \det A_1(\Lambda).$$

Using the projection $p_0\colon \Lambda \to K$ we get

$$\det A_0 = p_0(\det A_0(\Lambda)) = p_0(\det A_1(\Lambda)) = \det A_1.$$

We observe that $\mathrm{GL}(m,n;\Lambda)$ and $\mathrm{SL}(m,n;\Lambda)$ are the Λ-matrix groups corresponding to the Λ-matrix LIE algebras $\mathfrak{gl}(m,n;\Lambda)$ and $\mathfrak{sl}(m,n;\Lambda)$, respectively.

$\mathrm{OSp}(m,n;\Lambda)$ is the subgroup of $\mathrm{GL}(m,n;\Lambda)$ defined by the following relations

$$\begin{aligned} A_0(\Lambda)^\top A_0(\Lambda) - B_0(\Lambda)^\top I_n B_0(\Lambda) &= E_m, \\ A_0(\Lambda)^\top B_1(\Lambda) - B_0(\Lambda)^\top I_n A_1(\Lambda) &= 0, \\ B_1(\Lambda)^\top B_1(\Lambda) + A_1(\Lambda)^\top I_n A_1(\Lambda) &= I_n. \end{aligned}$$

Applying the projection p_0 to these equations, we get

$$A_0^\top A_0 = E_m \quad \text{and} \quad A_1^\top I_n A_1 = I_n,$$

hence $p_0\colon \mathrm{OSp}(m,n;\Lambda) \to \mathrm{O}(m) \times \mathrm{Sp}(n)$.

$\mathrm{OSp}(m,n;\Lambda)$ is the Λ-matrix group corresponding to the Λ-matrix LIE algebra $\mathfrak{osp}(m,n;\Lambda)$. The projection p_0 is a homomorphism mapping the GRASSMANN LIE group $\mathrm{OSp}(m,n;\Lambda)$ onto the LIE group $\mathrm{O}(m) \times \mathrm{Sp}(n)$ corresponding to the even part of $\mathfrak{osp}(m,n)$.

We observe that $\widetilde{\mathrm{GL}}(m;\Lambda)$ denotes the group of block matrices

$$\widetilde{\mathcal{A}}(\Lambda) = \begin{pmatrix} A(\Lambda) & B(\Lambda) \\ B(\Lambda) & A(\Lambda) \end{pmatrix},$$

with $A(\Lambda) = A(\Lambda_{\overline{0}})$ and $B(\Lambda) = B(\Lambda_{\overline{1}})$.

$\widetilde{\mathrm{SL}}(m;\Lambda)$ is the subgroup of $\widetilde{\mathrm{GL}}(m;\Lambda)$ consisting of those block matrices $\widetilde{\mathcal{A}}(\Lambda)$ satisfying $\widetilde{\operatorname{sdet}}\widetilde{\mathcal{A}} = 1$. Here the *superdeterminant of the second kind* $\widetilde{\operatorname{sdet}}$ is defined on the GRASSMANN LIE group $\widetilde{\mathrm{GL}}(m;\Lambda)$ by

$$\widetilde{\operatorname{sdet}}\widetilde{\mathcal{A}}(\Lambda) = 1 + \operatorname{Tr}\log(E_m + A(\Lambda)^{-1}B(\Lambda)). \tag{6}$$

The relation (6) may be rewritten by

$$\operatorname{Tr}\log(E_m + A(\Lambda)^{-1}B(\Lambda)) = \sum_\nu \frac{1}{2\nu+1}\operatorname{Tr}(A(\Lambda)^{-1}B(\Lambda))^{2\nu+1} = 0.$$

Notice that the entries of the product matrix $A(\Lambda)^{-1}B(\Lambda)$ belong to $\Lambda_{\bar{1}}$ which implies that the series of the logarithm is finite.

The superdeterminant of the second kind is multiplicative. Thus $\widetilde{\mathrm{sdet}}$ is a homomorphism of $\widetilde{\mathrm{GL}}(m;\Lambda)$ into the group $K^\times$ of units in Λ. The projection p_0 maps the GRASSMANN LIE group $\widetilde{\mathrm{SL}}(m;\Lambda)$ onto the LIE group $\mathrm{GL}(m)$ which corresponds to the even part of $\tilde{\mathfrak{sl}}(m)$.

$\widetilde{\mathrm{GL}}(m;\Lambda)$ and $\widetilde{\mathrm{SL}}(m;\Lambda)$ are the Λ-matrix groups corresponding to the Λ-matrix LIE algebras $\tilde{\mathfrak{gl}}(m;\Lambda)$ and $\tilde{\mathfrak{sl}}(m;\Lambda)$, respectively.

The center of the Λ-matrix group $\widetilde{\mathrm{SL}}(m;\Lambda)$ consists of the even multiples of the unit matrix, and the quotient

$$\widetilde{\mathrm{SL}}(m+1;\Lambda)/\{\lambda_{\bar{0}}E_{2m+2} : \lambda_{\bar{0}} \in \Lambda_{\bar{0}}\}$$

is a GRASSMANN LIE group, which corresponds to the GRASSMANN -hull of the LIE superalgebra $\mathbf{Q}(m)$.

5. HOPF Superalgebras

Let $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ denote a $\mathbb{Z}_2$-graded HOPF algebra. Here $\mathcal{H}$ is a $\mathbb{Z}_2$-graded associative algebra with unit—the product and the unit are considered as linear mappings $\mu: \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and $\iota: K \to \mathcal{H}$, respectively—endowed with a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a counit $\varepsilon: \mathcal{H} \to K$, and an antipode $\sigma: \mathcal{H} \to \mathcal{H}$. Here Δ and ε are homomorphisms of the corresponding $\mathbb{Z}_2$-graded algebras, σ is an antiautomorphism of $\mathcal{H}$. The following relations are satisfied $(\Delta \otimes \mathrm{id}_\mathcal{H})\circ\Delta = (\mathrm{id}_\mathcal{H} \otimes \Delta) \circ \Delta$, called the *coassociativity* of the coproduct, $(\varepsilon \otimes \mathrm{id}_\mathcal{H}) \circ \Delta = \mathrm{id}_\mathcal{H} = (\mathrm{id}_\mathcal{H} \otimes \varepsilon) \circ \Delta$, and $\mu \circ (\sigma \otimes \mathrm{id}_\mathcal{H}) \circ \Delta = \iota\varepsilon = \mu \circ (\mathrm{id}_\mathcal{H} \otimes \sigma) \circ \Delta$. Let $\nu: \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ denote the *twist* homomorphism given by $\nu(h_1 \otimes h_2) = (-1)^{|h_1||h_2|}h_2h_1$ for homogeneous elements h_1 and h_2 in $\mathcal{H}$. Then $\mathcal{H}$ is called *commutative* or *cocommutative* if the relations

$$\mu \circ \nu = \mu \quad \text{or } \nu \circ \Delta = \Delta$$

hold, respectively.

Example 1$^\square$. Put $\mathcal{H} = K[X_1, ..., X_m] \otimes \Lambda(Y_1, ..., Y_n)$. Then we have

$$\mathcal{H}_{\bar{0}} = K[X_1, ..., X_m] \otimes \Lambda_{\bar{0}}(Y_1, ..., Y_n),$$
$$\mathcal{H}_{\bar{1}} = K[X_1, ..., X_m] \otimes \Lambda_{\bar{1}}(Y_1, ..., Y_n).$$

Now $\mathcal{H}$ is a HOPF algebra with respect to the usual product, the usual unit , the coproduct $\Delta(X_\mu) = 1 \otimes X_\mu + X_\mu \otimes 1$, $\Delta(Y_\nu) = 1 \otimes Y_\nu + Y_\nu \otimes 1$, $\Delta(1) = 1 \otimes 1$, the counit $\varepsilon(X_\mu) = \varepsilon(Y_\nu) = 0$, $\varepsilon(1) = 1$, and the antipode $\sigma(X_\mu) = -X_\mu$, $\sigma(Y_\nu) = -Y_\nu$, $\sigma(1) = 1$, for $\mu = 1, \ldots, m$, $\nu = 1, \ldots, n$. The HOPF superalgebra $\mathcal{H}$ is commutative and cocommutative.

Example 2$^\square$. Let $G = \mathrm{GL}(m, n; \Lambda)$ denote the Λ-matrix group defined in Section 4. We define a block matrix of *commuting* and *anticommuting variables* by

$$\mathcal{X} = \begin{pmatrix} X & Y' \\ Y & X' \end{pmatrix}.$$

We assume

$$X = (X_{ij})_{\substack{i=1,\dots m\\ j=1,\dots m}}, \quad Y' = (Y'_{i\ell})_{\substack{i=1,\dots,m\\ \ell=1,\dots,n}},$$
$$Y = (Y_{kj})_{\substack{k=1,\dots,n\\ j=1,\dots m}}, \quad X' = (X'_{k\ell})_{\substack{k=1,\dots,n\\ \ell=1,\dots,n}}.$$

Put

$$\mathcal{H}(G) = K[X_{ij}, x, X'_{k\ell}, x'] \otimes \Lambda(Y'_{i\ell}, Y_{kj})/(x \det|X_{ij}| - 1, x' \det|X'_{k\ell}| - 1),$$

then $\mathcal{H}(G)$ is an associative and commutative $\mathbb{Z}_2$-graded algebra with unit. Using the matrix product we define the coproduct as follows:

$$\Delta(X_{ij}) = \sum_{j'=1}^{m} X_{ij'} \otimes X_{j'j} + \sum_{\ell'=1}^{n} Y'_{i\ell'} \otimes Y_{\ell'\ell},$$
$$\Delta(X'_{k\ell}) = \sum_{j'=1}^{m} Y_{kj'} \otimes Y'_{j'\ell} + \sum_{\ell'=1}^{n} X'_{k\ell'} \otimes X'_{\ell' l},$$
$$\Delta(Y'_{i\ell}) = \sum_{j'=1}^{m} X_{ij'} \otimes Y'_{j'\ell} + \sum_{\ell'=1}^{n} Y'_{i\ell'} \otimes X'_{\ell'\ell},$$
$$\Delta(Y_{kj}) = \sum_{j'=1}^{m} Y_{kj'} \otimes X_{j'j} + \sum_{\ell'=1}^{n} X'_{k\ell'} \otimes Y_{\ell' j},$$
$$\Delta(x) = x \otimes x,$$
$$\Delta(x') = x' \otimes x'.$$

For short we may write $\Delta(\mathcal{X}) = \mathcal{X} \otimes \mathcal{X}$. The counit is given by

$$\varepsilon(X_{ij}) = \delta_{ij},$$
$$\varepsilon(X'_{k\ell}) = \delta_{kl},$$
$$\varepsilon(Y'_{i\ell}) = 0.$$
$$\varepsilon(Y_{kj}) = 0,$$
$$\varepsilon(x) = 1,$$
$$\varepsilon(x') = 1.$$

For short we may write $\varepsilon(\mathcal{X}) = E_{m+n}$. The coproduct in $\mathcal{H}(G)$ mirrors the matrix product, i.e., the product in the group G, the counit represents the evaluation at the unit matrix, i.e., the evaluation at the identity of the group G. Now we define the antipode, which mirrors the inverse of matrices or group elements

$$\sigma(X) = (X - Y'X'^{-1}Y)^{-1},$$
$$\sigma(X') = (X' - YX^{-1}Y')^{-1},$$
$$\sigma(Y') = -X^{-1}Y'\sigma(X, \sigma(Y) = -X'^{-1}Y\sigma(X'),$$
$$\sigma(x) = \det X,$$
$$\sigma(x') = \det X'.$$

For short we may write $\sigma(\mathcal{X}) = \mathcal{X}^{-1}$. Now $\mathcal{H}(G)$ is a commutative and not cocommutative HOPF superalgebra, which we shall denote by $\mathcal{P}(m,n)$.

Example 3$^\square$. Assume $G = \mathrm{SL}(m,n;\Lambda)$. Put

$$\begin{aligned}\mathcal{H}(G) &= \mathcal{P}(m,n)/(\operatorname{sdet}\mathcal{X} - 1)\\ &= \mathcal{P}(m,n)/(\det(X - Y'X'^{-1}Y) - \det X').\end{aligned}$$

It follows from $\operatorname{sdet}(\mathcal{X}_1\mathcal{X}_2) = \operatorname{sdet}\mathcal{X}_1 \operatorname{sdet}\mathcal{X}_2$ that the homogeneous ideal generated by $\operatorname{sdet}\mathcal{X} - 1$ is a coideal, too. We have

$$\begin{aligned}\Delta(\operatorname{sdet}\mathcal{X} - 1) &= \Delta \operatorname{sdet}\mathcal{X} - 1\otimes 1\\ &= \operatorname{sdet}(\mathcal{X}\otimes\mathcal{X}) - 1\otimes 1\\ &= \operatorname{sdet}\mathcal{X}\otimes\operatorname{sdet}\mathcal{X} - 1\otimes 1\\ &= \operatorname{sdet}\mathcal{X}\otimes\operatorname{sdet}\mathcal{X} - \operatorname{sdet}\mathcal{X}\otimes 1 + \operatorname{sdet}\mathcal{X}\otimes 1 - 1\otimes 1\\ &= \operatorname{sdet}\mathcal{X}\otimes(\operatorname{sdet}\mathcal{X} - 1) + (\operatorname{sdet}\mathcal{X} - 1)\otimes 1.\end{aligned}$$

Moreover, $\varepsilon(\operatorname{sdet}\mathcal{X}) = 1$ and $\sigma(\operatorname{sdet}\mathcal{X}) = \operatorname{sdet}\mathcal{X}^{-1}$, i.e., the ideal generated by $\operatorname{sdet}\mathcal{X} - 1$ is contained in the kernel of ε and invariant under σ. Hence it is possible to factorize Δ, ε, σ, and $\mathcal{H}(G)$ becomes a commutative and not cocommutative HOPF superalgebra, which is denoted by $\mathcal{SP}(m,n)$.

A Λ-matrix group, i.e, a subgroup of $\mathrm{GL}(m,n;\Lambda)$ is called *algebraic*, if it is the annihilator set of a $\mathbb{Z}_2-$ *graded* ideal of $\mathcal{P}(m,n)$.

Let G denote an algebraic Λ-matrix group, and let $I(G)$ denote its annihilator ideal in $\mathcal{P}(m,n)$, then $\mathcal{H}(G) = \mathcal{P}(m,n)/I(G)$ is a HOPF *superalgebra. The coproduct, the counit, and the antipode of $\mathcal{H}(G)$ are induced by factorization of the coproduct, the counit, and the antipode of $\mathcal{P}(m,n)$. The annihilator ideal $I(G)$ is a coideal of $\mathcal{P}(m,n)$, it is contained in the kernel of ε, and it is invariant under σ.*

Example 4$^\square$. The Λ-matrix group $\mathrm{OSp}(m,n;\Lambda)$ is an algebraic Λ-matrix group. Its annihilator ideal is generated by the "polynomials" $X^\top X - Y^\top I_n Y - E_m$, $X^\top Y' - YI_nX'$, and $Y'^\top Y' + X'^\top I_n X' - I_n$. The corresponding HOPF superalgebra is denoted by $\mathcal{OSpP}(m,n)$.

Example 5$^\square$. The Λ-matrix group $\widetilde{\mathrm{GL}}(m;\Lambda)$ is an algebraic Λ-matrix group. Its HOPF superalgebra is denoted by $\widetilde{\mathcal{P}}(m)$:

$$\widetilde{\mathcal{P}}(m) \cong K[X_{ij}, x]\otimes\Lambda(Y_{ij})/(x\det|X_{ij}| - 1).$$

Example 6$^\square$. The Λ-matrix group $\widetilde{\mathrm{SL}}(m;\Lambda)$ is an algebraic Λ-matrix group. Its HOPF superalgebra is denoted by $\widetilde{\mathcal{SP}}(m)$.

A HOPF superalgebra $\mathcal{H} = \mathcal{H}_{\overline{0}}\oplus\mathcal{H}_{\overline{1}}$ is called *affine* if it is commutative and finitely generated.

The HOPF *superalgebra* $\mathcal{H}(G) = \mathcal{P}(m,n)/I(G)$ *of an algebraic* Λ*-matrix group* G *is affine.*

The HOPF *superalgebras* $\mathcal{P}(m,n)$, $\mathcal{SP}(m,n)$, $\mathcal{OS}p\mathcal{P}(m,n)$, $\widetilde{\mathcal{P}}(m)$, *and* $\widetilde{\mathcal{SP}}(m)$ *are affine* HOPF *superalgebras.*

We mention the following

Structure Theorem. *Let* $\mathcal{H}$ *denote an affine* HOPF *superalgebra. Then there exists an affine* HOPF *algebra* $\mathcal{H}_0$ *and odd elements* $W_1, ..., W_s$ *such that* $\mathcal{H} \cong \mathcal{H}_0 \otimes \Lambda(W_1, \dots, W_s)$. *The isomorphism is an isomorphism of commutative superalgebras. The following relations hold:*

$$(p \otimes p) \circ \Delta = \Delta_0 \circ p,$$
$$\varepsilon = \varepsilon_0 \circ p,$$
$$p \circ \sigma = \sigma_0 \circ p.$$

Here p *denotes the canonical projection annihilating all odd elements of* $\mathcal{H}$, *i.e.,* $p: \mathcal{H} \to \mathcal{H}_0$, *and* Δ_0 ε_0, σ_0 *denote the coproduct, the counit, and the antipode of the* HOPF *algebra* $\mathcal{H}_0$, *respectively.*

The affine HOPF algebra $\mathcal{H}_0$ is the algebra of polynomial functions of an (affine) algebraic group G_0.

Example 2$^\square$ (continued). Assume $G = \mathrm{GL}(m,n;\Lambda)$, $\mathcal{H} = \mathcal{H}(G) = \mathcal{P}(m,n)$. Then one has

$$\mathcal{H}_0 \cong K[X_{ij}, x, X'_{k\ell}, x']/(x \det|X_{ij}| - 1, x' \det|X'_{kl}| - 1),$$

and

$$G_0 \cong \mathrm{GL}(m) \times \mathrm{GL}(n) = p_0(\mathrm{GL}(m,n;\Lambda)).$$

Example 3$^\square$ (continued). Assume $G = \mathrm{SL}(m,n;\Lambda)$ $\mathcal{H} = \mathcal{H}(G) = \mathcal{SP}(m,n)$. Then one has

$$\mathcal{H}_0 \cong K[X_{ij}, x, X'_{k\ell}, x']/(x \det|X_{ij}| - 1, x' \det|X'_{k\ell}| - 1, \det|X_{ij}| - \det|X'_{k\ell}|),$$

and

$$G_0 \cong \mathrm{SL}(m) \times \mathrm{SL}(n) \times K^\times = p_0(\mathrm{SL}(m,n;\Lambda).$$

Example 4$^\square$ (continued). Assume $G = \mathrm{OSp}(m,n;\Lambda)$,

$$\mathcal{H} = \mathcal{H}(G) = \mathcal{OS}p\mathcal{P}(m,n).$$

Then one has

$$\mathcal{H}_0 \cong K[X_{ij}, X'_{k\ell},]/I_0,$$

and the ideal I_0 is generated by

$$\sum_{i'=1}^{m} X_{i'i}X_{i'j} - \delta_{ij},$$

$$\sum_{k'=0}^{n'-1} (X'_{n-k',k}X'_{k'+1,\ell} - X'_{n'-k',k}X'_{n'+k'+1,\ell} - \delta_{k\ell}), \quad (n = 2n').$$

This implies $G_0 \cong \mathrm{O}(m) \times \mathrm{Sp}(n) = p_0(\mathrm{OSp}(m,n;\Lambda))$.

Example 5$^\square$ (continued). Assume $G = \widetilde{\mathrm{GL}}(m;\Lambda)$, $\mathcal{H} = \mathcal{H}(G) = \widetilde{\mathcal{P}}(m)$. Then one has

$$\mathcal{H}_0 \cong K[X_{ij}, x]/(x \det |X_{ij}| - 1)$$

and

$$G_0 \cong \mathrm{GL}(m) = p_0(\widetilde{\mathrm{GL}}(m;\Lambda)).$$

Example 6$^\square$ (continued). Assume $G = \widetilde{\mathrm{SL}}(m;\Lambda)$, $\mathcal{H} = \mathcal{H}(G) = \widetilde{\mathcal{SP}}(m)$. Then one has

$$\mathcal{H}_0 \cong K[X_{ij}, x]/(x \det |X_{ij}| - 1)$$

and

$$G_0 \cong \mathrm{GL}(m) = p_0(\widetilde{\mathrm{SL}}(m;\Lambda)).$$

Notice that the isomorphism of superalgebras stated in the structure theorem is not a canonical one. In some sense it is the choice of a coordinate system.

Example 6$^\square$ **(continued once more).** We have

$$\widetilde{\mathcal{SP}}(m) \cong K[X_{ij}, x] \otimes \Lambda(Y_{ij})/I(\widetilde{\mathrm{SL}}(m;\Lambda)).$$

The annihilator ideal $I(\widetilde{\mathrm{SL}}(m;\Lambda))$ is generated by $x \det |X_{ij}| - 1$ and $\mathrm{Tr}\log(E_m + X^{-1}Y) = \sum_\nu \frac{1}{2\nu+1} \mathrm{Tr}(X^{-1}Y)^{2\nu+1}$. Choosing instead of the Y_{ij} new odd variables W_{ij} defined by the matrix equation $W = X^{-1}Y$, then using the second relation it is possible to eliminate one of the odd variables. In this case we have $s = m^2 - 1$.

The last two sections, namely, Section 6, “Affine algebraic Supergroups” and Section 7, “The HOPF dual. Representations”, as well as the list of references are postponed to the next seminar.

References

References to the literature will be given in the sequel to this article:

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