Seminar Sophus Lie
1 (1991) 193–204

On a Semigroup in the Work of Charles Loewner

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We want to have a look at the work of one of the great and versatile mathematicians in this century, Charles Loewner. His biographical data: He was born in 1893 in Prague under the name of Karel Löwner into a large Jewish family of businessmen. He attended a German gymnasium in Prague and studied there at the "Alte Universität". After having finished his studies he spent some years at the University of Berlin (1922-28) and in Cologne (1928-30) until he returned to Prague as a professor. 1939 he emigrated to America; until this time he had published in German under the name Karl Löwner. After a break in his creativity of about 10 years his next article came out in 1948, and from now on all of his work is written in English and under the name Charles Loewner.

Until 1944 he was working and living in Louisville. Then he got a job at the department of mechanics at Brown University. For a period of 5 years he worked at the mathematical institute of Syracuse University. Finally he found a new home for himself and his family at Stanford; his death in the year of 1968 took him from the fullest of his intellectual prowess.

What kind of mathematics was Loewner interested in? He began his research by working on the Bieberbach-Conjecture and succeeded in proving it for $n = 3$. The proof includes some important properties of schlicht functions of certain regions in $\mathbb{C}$. During 1955 he spent a time as a Visiting Professor in Berkeley. He gave a course on continous groups, his lecture notes appeared as a book in 1970 ([10]). In most parts of his published work during his American period he tried to apply Lie-theory to semigroups. He was particularly interested in *one* semigroup which he took up again and again: the semigroup of n-monotonic functions. We find the starting point of this research in an article of the year 1934 [4].

1. On n-monotonic functions

Let $f\colon (a,b) \to \mathbb{R}$, $a < b$, be a function, X a symmetric real (n,n)-matrix with eigenvectors v_j and eigenvalues λ_j. If all eigenvalues of X fall into the interval (a,b), then $f(X)$ is the matrix with eigenvalues $f(\lambda_j)$ and the same eigenvectors v_j. Henceforth we endow the space of the real symmetric (n,n)-matrices with the natural order $X \leq Y$ defined by the non-negativity of the quadratic form $x^T(Y - X)x$.

Definition 1.1. A function f is called *monotonically increasing of order* n (say n*-monotonic*), if all symmetric real positive (n,n)-matrices X, Y with spectrum in (a,b) satisfy the following: $X \leq Y \Rightarrow f(X) \leq f(Y)$.

As a direct consequence of this definition we get the following: If f is n-monotonic of order $n \geq 2$, then the same is true for every lower order. The following properties are taken from an early work of Loewner's [4] on monotonic matrix functions; they are proved by use of CAUCHY's interpolation polynomials.

For different real numbers η, ξ set $\triangle(\eta,\xi) = \triangle_f(\eta,\xi) \stackrel{\text{def}}{=} \dfrac{f(\eta)-f(\xi)}{\eta-\xi}$. Let us call a sequence $a < \xi_1 < \eta_1 < \ldots < \xi_m < \eta_m < b$ an *even partition* of (a,b) *of degree* m.

Lemma 1.2. *For a function $f\colon (a,b) \to \mathbb{R}$ the following conditions are equivalent:*

1. *f is n-monotonic.*
2. *For each even partition of (a,b) of degree $m = 1,\ldots,n$ one has*

$$(*) \qquad \det\left(\triangle_f(\eta_j,\xi_k)\right)_{j,k=1,\ldots,m} \geq 0. \qquad \blacksquare$$

The cases of low order illustrate the situation:

Remark 1.3. (i) *The Case* $n=1$: $(*)$ consists of only one inequality, namely $\triangle(\eta_1,\xi_1) \geq 0$, so we get that monotonicity of order 1 is no more than usual monotonicity for real functions.
(ii) *The Case* $n=2$: The condition $(*)$ creates a pair of inequalities $(m=1,2)$:

(1) $\triangle(\eta_1,\xi_1) \geq 0, \qquad \xi_1 \neq \eta_1$

(2) $\det\begin{pmatrix} \triangle(\eta_1,\xi_1) & \triangle(\eta_1,\xi_2) \\ \triangle(\eta_2,\xi_1) & \triangle(\eta_2,\xi_2)\end{pmatrix} \geq 0, \qquad \begin{array}{l}\xi_j \neq \eta_k \text{ if } j \neq k \\ (\xi_2-\xi_1)(\eta_2-\eta_1) > 0\end{array}$

The determinant in (2) is $\triangle_{11}\triangle_{22} - \triangle_{12}\triangle_{21} \geq 0$, that is

$$\frac{[f(\eta_1)-f(\xi_1)]\,[f(\eta_2)-f(\xi_2)]}{[f(\eta_1)-f(\xi_2)]\,[f(\eta_2)-f(\xi_1)]} \geq \frac{(\eta_1-\xi_1)\,(\eta_2-\xi_2)}{(\eta_1-\xi_2)\,(\eta_2-\xi_1)}.$$

We deduce some properties for 2-monotonic functions:
a) In discussing the details of the case $n=2$ *we may restrict ourselves to strictly monotonic functions*: Moreover, the following holds for n-monotonic functions $f\colon (a,b) \to \mathbb{R}$: If there exist $x,y \in (a,b)$ with $f(x) = f(y)$ then f is constant in (a,b).
(**Proof.** W.l.o.g. we assume $x < y$ and choose ξ,η such that $a < x < y < \xi < \eta < b$. If f is 2-monotonic and $(*)$ holds, then $\det\begin{pmatrix} \triangle(y,x) & \triangle(y,\xi) \\ \triangle(\eta,x) & \triangle(\eta,\xi)\end{pmatrix} \geq 0$. I.e., $\triangle(y,x)\cdot\triangle(\eta,\xi) - \triangle(\eta,x)\cdot\triangle(y,\xi) \geq 0$, and by $\triangle(y,x) = 0$ we get $\triangle(\eta,x)\cdot\triangle(y,\xi) \leq 0$. But since f is monotonic in the usual sense (by (i)), we know $\triangle(\eta,x) \geq 0$ and $\triangle(y,\xi) \geq 0$ such that necessarily $\triangle(\eta,x)\cdot\triangle(y,\xi) = 0$. This yields $f(y) = f(\xi)$ or $f(x) = f(\eta)$ for arbitrary real numbers $\eta,\xi > y$. The analogous procedure with $\eta,\xi < x$ shows that indeed $f =$ const. in (a,b).)
b) In addition, f is continously differentiable in (a,b): Let α,β be real numbers such that the compact interval $[\alpha,\beta] \subset (a,b)$ and the difference $|\beta-\alpha|$ is sufficiently small. Take an even partition (x_1,y,x_2,β) with $a < \alpha < x_1 <$

$y < x_2 < \beta < b$. Because f is monotonic in the usual sense (see (i)), we know already that f is continous almost everywhere and f has continous left and right derivative in $y \in (a,b)$. Take the limits $x_1 \nearrow y, x_2 \searrow y$. Then by $\triangle(y,x_1)\cdot\triangle(\beta,x_2) \geq \triangle(\beta,x_1)\cdot\triangle(y,x_2)$ we obtain $f_l{}'(y)\cdot\triangle(\beta,y) \geq \triangle(\beta,y)\cdot f_r{}'(y)$ in the limit. Since $\triangle(\beta,y) > 0$, we have proved $f_l{}'(y) \geq f_r{}'(y)$ for arbitrary $y \in (a,b)$. We want to show equality and note the following fact: By choosing suitable sequences $(x_n)_n, (x'_n)_n$ with $x_n \nearrow y, x'_n \searrow y$ we may achieve that $\triangle(x'_n, x_n)$ tends to each value L in between the right and left derivative $f_r{}'(y) \leq L \leq f_l{}'(y)$. Take ξ_n, ξ'_n such that in particular $L = f_l{}'(y)$. Then by $(*)$ we get the following inequality $0 \leq \lim_{n\to\infty} [\triangle(y,\xi_n)\cdot\triangle(\beta,\xi'_n) - \triangle(\beta,\xi_n)\cdot\triangle(y,\xi'_n)] = f_l{}'(y)f_r{}'(y) - L^2$. For $L = f_l{}'(y) > 0$ this yields $f_r{}'(y) \geq f_l{}'(y)$, and we have proved the asserted equality. So f is continously differentiable on (a,b), which implies that f is LIPSCHITZ-continous on each compact subset of (a,b). ∎

We have just seen that 2-monotonicity implies that the function is continously differentiable. For arbitrary $n \in \mathbb{N}, n \geq 2$, one can prove the following (see [4], p. 213 f)

Corollary 1.4. *If for $n \geq 2$ f is n-monotonic in (a,b) then $f \in C^{2n-3}(a,b)$, and $f^{(2n-2)}$ is bounded on every compact subinterval in (a,b). Further, f is the limit of a series of n-monotonic functions from $C^{2n-1}(a,b)$ converging uniformly on compact subsets of (a,b).*

In the proof, Loewner uses higher difference quotients. We note the following useful

Lemma 1.5. *For a rational function $g\colon (a,b) \to \mathbb{R}$, $x \mapsto \frac{1}{\alpha - x}$, $\alpha \notin (a,b)$, the n-th difference quotient looks as follows*

$$\triangle_g{}^n(\omega_1, \ldots, \omega_{n+1}) = \frac{1}{(\alpha-\omega_1)\cdots(\alpha-\omega_{n+1})},$$

where $a < \omega_1 < \omega_2 < \ldots < \omega_{n+1} < b$ is a partition of (a,b).

Proof. We show the assumption for $n = 1, 2$, the rest follows by induction:
$n = 1$:

$$\frac{g(\omega_2) - g(\omega_1)}{\omega_2 - \omega_1} = \frac{\frac{1}{\alpha-\omega_2} - \frac{1}{\alpha-\omega_1}}{\omega_2 - \omega_1} = \frac{1}{(\alpha-\omega_2)(\alpha-\omega_1)}.$$

$n = 2$:

$$\begin{aligned}\frac{\triangle_g(\omega_3,\omega_2) - \triangle_g(\omega_2,\omega_1)}{\omega_3 - \omega_1} &= \frac{\frac{1}{(\alpha-\omega_3)(\alpha-\omega_2)} - \frac{1}{(\alpha-\omega_2)(\alpha-\omega_1)}}{\omega_3 - \omega_1} \\ &= \frac{1}{\alpha-\omega_2}\cdot\frac{g(\omega_3) - g(\omega_1)}{\omega_3 - \omega_1} \\ &= \frac{1}{(\alpha-\omega_3)(\alpha-\omega_2)(\alpha-\omega_1)}.\end{aligned}$$

∎

We note the following relation between the n-th difference quotient and the n-th derivative of g in $x \in (a,b)$: If we take for the moment $\omega_1 = \ldots = \omega_{2n-1} = x$, we obtain $\triangle_g{}^n(x) = \frac{g^{(n)}(x)}{n!}$.

In the following let f be strictly increasing. We notice that 2-monotonicity can be expressed by cross-ratios. We recall the following: Let z_1, z_2, z_3, z_4 be four pointwise different complex numbers. Then the *cross-ratio* (z_1, z_2, z_3, z_4) is the image of z_1 under the Möbius-transformation $z \mapsto \left(\frac{z_2 - z_4}{z_2 - z_3}\right) \cdot \left(\frac{z - z_3}{z - z_4}\right)$ which takes z_2 to 1, z_3 to 0, and z_4 to ∞.

A 2-monotonic function 1) is monotonic in the usual sense and 2) does not decrease the cross-ratio of any 4 naturally ordered real numbers.

But we know that the proper Möbius-transformations are the only functions which preserve the cross-ratio, so we are led to the following

Proposition 1.6. *For any invertible function $f\colon (a,b) \to (A,B)$ the following conditions are equivalent:*

1. *f and f^{-1} are 2-monotonic.*
2. *f is a proper Möbius-transformation with real coefficients, $f(x) = \frac{\alpha x + \beta}{\gamma x + \delta}$, where $\alpha\delta - \beta\gamma > 0, -\frac{\delta}{\gamma} \notin (a,b)$.*

In particular, if $f\colon \mathbb{R} \to \mathbb{R}$ is invertible, then f and f^{-1} are 2-monotonic iff f is affine. ■

$\mathcal{M}_\infty$-functions

In the following discussion we use a theorem of G. Pick [11]: If $F\colon \mathbb{E} \to -i\mathbb{H}^+$ is an analytic function mapping the open unit disc $\mathbb{E} \subseteq \mathbb{C}$ into the complex half plane of positive real parts $-i\mathbb{H}^+$, the following inequality holds for every $m \in \mathbb{N}, z_\nu, z_\kappa \in \mathbb{E}$:

$$\det \left(\frac{F(z_\nu) + \overline{F(z_\kappa)}}{1 - z_\nu \overline{z_\kappa}} \right)_{\nu,\kappa=1,\ldots,m} \geq 0.$$

Let now $\mathcal{M}_n = \{f\colon (a,b) \to (a,b) \mid f \ n\text{-monotonic}\}$. Then $\mathcal{M}_n$ is a semigroup with respect to the composition of functions. We have already noticed that $\mathcal{M}_{n+1} \subseteq \mathcal{M}_n$ holds. Taking the intersection of all these semigroups $\mathcal{M}_\infty = \bigcap_{n \in \mathbb{N}} \mathcal{M}_n$, we obtain a semigroup for which we have the following characterization:

Theorem 1.7. *The following statements are equivalent:*

1. *f is monotonic in (a,b) of arbitrarily high order.*
2. *f is real analytic in (a,b) and f can be extended analytically to a function $\tilde{f}\colon U \to \mathbb{C}$, $U \subset \mathbb{C}$ a region such that the complex upper half-plane $\mathbb{H}^+$ is contained in U and $\tilde{f}(\mathbb{H}^+) \subseteq \mathbb{H}^+$.*

Proof. 2. ⇒ 1. Let $\tilde{f}: U \to \mathbb{C}$ be complex analytic. For shorter writing, we identify $f = \tilde{f}$. We make use of the property $f(\mathbb{H}^+) \subseteq f(\mathbb{H}^+)$ and define $\mu: \mathbb{E} \to \mathbb{H}^+$, $\mu(z) = i\frac{z+1}{-z+1}$ a Möbius-transformation mapping $\mathbb{E}$ onto $\mathbb{H}^+$. (Note that $\operatorname{Im}\mu(z) = \frac{1-|z|^2}{1+|z|^2} > 0$ for $z \in \mathbb{E}$.) We define $F: \mathbb{E} \to -i\mathbb{H}^+$ as the composition $F: \mathbb{E} \xrightarrow{\mu} \mathbb{H}^+ \xrightarrow{f} \mathbb{H}^+ \overset{w \mapsto -iw}{\to} -i\mathbb{H}^+$ and rewrite the entries in the Pick-matrix above in terms of $\zeta = \mu(z)$ from the upper half plane ($z \in \mathbb{E}$):

$$\begin{aligned} 1 - z_\nu \overline{z_\kappa} &= 1 - \mu^{-1}(\zeta_\nu)\,\overline{\mu^{-1}(\zeta_\kappa)} \\ &= 1 - \frac{\zeta_\nu - i}{\zeta_\nu + i} \cdot \overline{\frac{\zeta_\kappa - i}{\zeta_\kappa + i}} = \frac{2i(\overline{\zeta_\kappa} - \zeta_\nu)}{(\zeta_\nu + i)(\overline{\zeta_\kappa + i})} \\ F(z_\nu) + \overline{F(z_\kappa)} &= -if(\zeta_\nu) + (\overline{-if(\zeta_\kappa)}) \\ &= i\left((\overline{f(\zeta_\kappa)} - f(\zeta_\nu)\right). \end{aligned}$$

Pick's lemma yields

$$\begin{aligned} 0 \ \leq \ & \det\left(\frac{F(z_\nu) + \overline{F(z_\kappa)}}{1 - z_\nu \overline{z_\kappa}}\right)_{\nu,\kappa=1,\ldots,m} \\ & \frac{1}{2^m}\prod_{\nu=1}^{m}(\zeta_\nu + i)\prod_{\kappa=1}^{m}(\overline{\zeta_\kappa + i})\cdot \det\left(\frac{f(\zeta_\nu) - \overline{f(\zeta_\kappa)}}{\zeta_\nu - \overline{\zeta_\kappa}}\right) \\ & \frac{1}{2^m}\prod_{\nu=1}^{m}|\zeta_\nu + i|^2 \cdot \det\left(\frac{f(\zeta_\nu) - \overline{f(\zeta_\kappa)}}{\zeta_\nu - \overline{\zeta_\kappa}}\right). \end{aligned}$$

We know that f is analytic on $\mathbb{H}^+$ and real analytic in (a,b), too, so if ζ_ν, ζ_κ converge to pairwise different real values η_ν, ξ_κ we obtain $(*)$ $\det\left(\triangle_f(\eta_\nu, \xi_\kappa)\right)_{\nu,\kappa=1,\ldots,n} \geq 0$, and $f \in \mathcal{M}_\infty(a,b)$ by Lemma 1.2.

1. ⇒ 2. To prove the converse, we assume $f \in \mathcal{M}_\infty(a,b)$. Let $[\alpha,\beta] \subseteq (a,b)$ be a compact subinterval of (a,b) and choose an arbitrary odd partition $\alpha \leq \omega_1 < \omega_2 < \ldots < \omega_{2n-1} \leq \beta$. In a very technical proof Loewner points out that there is exactly *one* rational function $\rho = \rho_{n-1}$ where both the denominator and the numerator polynom have maximal degree $n-1$ and which interpolates f in the points ω_j of the partition: $\rho(\omega_j) = f(\omega_j)$ for all $j = 1, \ldots, 2n-1$ ([4], pp. 194–213). ρ has the following properties: It is a rational function with only simple poles on the real axis, $\alpha_j \in \mathbb{R} \setminus (a,b)$, all having negative residues. ρ can be written in the form $\rho(x) = c + a_0 x + \sum_{j=1}^{n-1} \frac{a_j}{\alpha_j - x}$ with real numbers $a_j \geq 0$ and $\prod_{j=0}^{n-1} a_j = 0$. (The affine part $c + a_0 x$ is due to the fact that if f has no poles at all on the real axis, f must be an affine function (1.6) since f is *strictly* increasing, that is injective.) We do not want to prove this statement here even though it is at this point that the monotonicity of arbitrary high order (i.e. $(*)$) comes in. The technical details will be written down separately.

From 1.5 we know that f is C^∞ in (a,b). We want to show that f is real analytic and use the following well known criterion: A C^∞-function f is analytic in $[\alpha,\beta]$ if $(\exists r > 0)\,(\forall n \in \mathbb{N})\,(\forall x \in [\alpha,\beta])\,|f^{(n)}(x)| \leq r^n n!$. So we are looking for such an upper bound for $f^{(2n-2)}(x)$. We replace f by ρ and compute (for an arbitrary partition $(\omega_j)_j$) the $(2n-2)$-th difference quotient $\triangle_\rho^{2n-2} \overset{\text{def}}{=} \triangle_\rho^{2n-2}(\omega_1, \ldots, \omega_{2n-1})$ of ρ, $n \geq 2$. Lemma 1.5 yields $\triangle_\rho^{2n-2} =$

$\sum_{j=1}^{n-1} \frac{a_j}{(\alpha_j-\omega_1)\cdots(\alpha_j-\omega_{2n-1})} \le \frac{1}{d^{2n-3}} \sum_{j=1}^{n-1} \frac{a_j}{(\alpha_j-\omega_1)(\alpha_j-\omega_2)}$ where $d = \min\{\alpha - a, b - \beta\}$. We observe $|\frac{f^{(2n-2)}(x)}{(2n-2)!}| = |\triangle_f{}^{2n-2}(x,\ldots,x)| \le |\triangle_\rho{}^{2n-2}(\omega_1,\ldots,\omega_{2n-1})|$ if the points ω_j lie in a sufficiently small neighbourhood of x. Since f is continously differentiable in (a,b), the first difference quotient of f is bounded. Because of the interpolation property, the same must yield for the first difference quotient of ρ which we compute by Lemma 1.5. So we can find an upper bound $C > 0$ such that $\sum_{j=1}^{n-1} \frac{a_j}{(\alpha_j-\omega_1)(\alpha_j-\omega_2)} \le C+1$. Taking this together we have found an upper bound for $|f^{(2n-2)}(x)|$ in the form as needed in the analyticity criterion above. So we have that the function f is analytic in (a,b).

Since ρ is a rational function with real coefficients and all poles lying on the real axis, ρ *is* the restriction of a holomorphic function $\tilde\rho\colon \mathbb{C} \setminus \{\text{poles}\} \to \mathbb{C}$.

We remark that $\tilde\rho$ is "positive" in the following sense: Take $z \in \mathbb{H}^+$ that is $\operatorname{Im} z > 0$. Then $a + a_0 z \in \mathbb{H}^+$ because $a_0 > 0$. Also $\alpha_j - z \in -\mathbb{H}^+$ implies $\frac{1}{\alpha_j - z} = \frac{\alpha_j - \overline{z}}{|\alpha_j - z|^2} \in \mathbb{H}^+$ ($\operatorname{Im}(-\overline{z}) = \operatorname{Im} z > 0$). Now $a_j \ge 0$ and $\tilde\rho$ is the sum of complex numbers in the upper half plane.

If π denotes the partition $a < \omega_1 < \ldots < \omega_{2n-1} < b$, we denote by ρ_π the interpolation function which belongs to this partition. The set Ω of all partitions of (a,b) is a directed set with respect to the inclusion. We shall show that the net $(\rho_\pi)_{\pi\in\Omega}$ has a subnet converging to an analytic function $\varphi\colon U \to \mathbb{C}$, where $U \stackrel{\text{def}}{=} \mathbb{C} \setminus (]-\infty, a] \cup [b, \infty[)$. Then φ must be positive, too, since all the ρ_π's are positive. Let $K \subseteq U$ a compact set and $z \in K$. We compute $|\rho'_\pi(z)| \le a_0 + \sum_{j=1}^{n-1} \frac{a_j}{|\alpha_j - z|^2} = a_0 + \sum_{j=1}^{2n-1} \frac{a_j}{(\alpha_j-\omega_1)(\alpha_j-\omega_2)} \frac{(\alpha_j-\omega_1)(\alpha_j-\omega_2)}{|\alpha_j - z|^2} \le \triangle_\rho(\omega_1,\omega_2)\cdot R(z)$, where $R(z) = \max_{j=1,\ldots,2n-1}\{1, \frac{(\alpha_j-\omega_1)(\alpha_j-\omega_2)}{|\alpha_j-z|^2}\}$. Since K is compact, we have $(\exists C_1 > 0)\,(\forall \pi \in \Omega)\,|\rho_\pi(z)| \le C_1$, i.e., ρ_π is uniformly bounded on compact subsets of U. We apply Montel's theorem and get that the set of all ρ_π is relatively compact with respect to the topology of uniform convergence on compact sets. Thus $(\rho_\pi)_{\pi\in\Omega}$ has a convergent subnet. That is, there is a *cofinal* mapping $\alpha\colon J \to \Omega$ such that $\lim_{j\in J} \rho_{\alpha(j)} = \varphi$. Because α is cofinal, for each $x \in (a,b)$ we find an index $j_0 \in J$ such that $j \ge j_0$ implies that x belongs to the partition $\alpha(j)$. Hence $\varphi(x) = \lim_{k\in J} \rho_{\alpha(k)}(x) = \rho_{\alpha(j)}(x)$ for $j \ge j_0$. But $\rho_{\alpha(j)}(x) = f(x)$ for these j since ρ_π interpolates f at the points $\alpha(j)$ and x belongs to $\alpha(j)$. Thus $\varphi(x) = f(x)$ for $x \in (a,b)$. ■

Examples 1.8. 1. Let $\mu > 0, f\colon (-1,1) \to \mathbb{R}$, $x \mapsto (x+1)^\mu$. It is obvious that f is the restriction of a function $\mathbb{C} \to \mathbb{C}$ which is analytic on $\mathbb{C} \setminus \{-1\}$. We have $f(\mathbb{H}^+) \subseteq \mathbb{H}^+$ only if $0 < \mu \le 1$. Thus $f \in \mathcal{M}_\infty$ iff $0 < \mu \le 1$.

2. $g\colon (-1,1) \to \mathbb{R}$, $g(x) = \log(x+1)$ is in $\mathcal{M}_\infty$.

3. In particular, the group of proper projective transformations (with real coefficients) is included in $\mathcal{M}_\infty$; these are exactly the schlicht n-monotonic functions.

We want to make sure that $\mathcal{M}_n$ is a subsemigroup of a Lie-group and recall again some results from [4]. (In the very long proofs Loewner uses again properties from Cauchy's interpolation polynomials.)

Theorem 1.9. 1. *For* $f \in C^{2n-1}(a,b)$ *and* $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ *we set the quadratic form* $Q_{f,x}(\xi) \stackrel{\text{def}}{=} \sum_{j,k=1}^{n} \frac{f^{(j+k-1)}(x)}{(j+k-1)!}\xi_j\xi_k$. *Then the following equivalence holds:*
$f \in \mathcal{M}_n \quad \Leftrightarrow \; (\forall x \in (a,b))\, Q_{f,x}(\xi) \geq 0$ *and* $f((a,b)) \subseteq (a,b)$.

2. *Let* $x_0 \in (a,b)$ *and* $a = (a_1, \ldots, a_{2n-1}) \in \mathbb{R}^{2n-1}$ *be real numbers, such that the quadratic form* $Q_a(\xi) \stackrel{\text{def}}{=} \sum_{j,k=1}^{n} a_{j+k-1}\xi_j\xi_k$ *is non-negative. Then there is an* n*-monotonic function* $f \in C^{2n-1}$ *such that the relation* $\frac{1}{j!}f^{(j)}(x_0) = a_j$ *holds for all* $j = 1, \ldots, 2n-1$.

2. The set of n-monotonic functions as a subsemigroup of a LIE-group

We shall from now on concentrate on the case $(a,b) = (-1,1)$, $f(0) = 0$. As a consequence of the preceding section we have that a function $f\colon (-1,1) \to (-1,1)$ is n-monotonic iff $f \in C^{2n-1}(-1,1)$ (w.l.o.g.) and the quadratic form $Q_{f,x}(\xi)$ is non-negative for every $x \in (-1,1)$. We compute the Taylor coefficients c_j of $g \circ f \in C^{2n-1}(-1,1)$ for $f, g \in C^{2n-1}(-1,1)$, both n-monotonic with $f(0) = g(0) = 0$, mapping the interval $(-1,1)$ into itself. We set $a_j = \frac{f^{(j)}(x)}{j!}, b_j = \frac{g^{(j)}(f(x))}{j!}, j = 1, \ldots, 2n-1$ and get

$$\begin{array}{rclcl}
c_1 & = & (g(f(x))' & = & a_1 b_1 \\
c_2 & = & (g(f(x))^{(2)} & = & a_1 b_2 + a_2 {b_1}^2 \\
c_3 & = & & = & a_1 b_3 + 2a_2 b_1 b_2 + a_3 {b_1}^3 \\
\vdots & & & & \\
c_{2n-1} & = & a_1 b_{2n-1} & + \cdots + & a_k \sum\limits_{\substack{\nu_1 + \cdots + \nu_k \\ = 2n-1}} (b_{\nu_1} b_{\nu_2} \ldots b_{\nu_k}) + \cdots + a_{2n-1}{b_1}^{2n-1}
\end{array}$$

We define the set Γ_{2n-1} as the set of vectors $(a_1, \ldots, a_{2n-1}) \in \mathbb{R}^{2n-1}$ with first entry $a_1 > 0$. We also note that the Taylor coefficients $a_1 > 0, b_1 > 0$ because every n-monotonic function which is not constant, is *strictly* monotonically increasing (1.3). The equations above define a monoid operation.

In order to understand the algebraic background of this set-up we discuss the algebra of formal power series.

The power series algebra

Definition 2.1. Let $A = \mathbb{R}[[X]]$ denote the algebra of formal power series over the reals, let $A_n \leq A$, $n \in \mathbb{N}$, be the ideal of all such power series with vanishing coeffients $a_0, \ldots, a_{n-1}$. We have $A_n = \{\sum_{j=n}^{\infty} a_j X^j : a_n, a_{n+1} \ldots \in \mathbb{R}\}$ and obtain a filtration $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ of A.

If we set $K^n = A/A_n$, we get an n-dimensional algebra of polynomials $\sum_{j=0}^{n-1} a_j \xi_n^j$ with $\xi_n = X + A_n$. We note some properties of the algebras we have defined:

Remark 2.2. (i) With respect to the topology generated by the filter basis $\{A_n : n = 0, 1, \ldots\}$ as a basis for the 0-neighbourhoods, A is a complete topological algebra.
(ii) With this topology, A is the projective limit $\varprojlim$ of the inverse system

$$K^1 \overset{f_1}{\leftarrow} K^2 \overset{f_2}{\leftarrow} \cdots K^n \overset{\pi_n}{\leftarrow} A. \qquad \blacksquare$$

Since A is a filtered complete topological algebra, for every $q \in A_1$ there is a unique contractive morphism $\varphi(q) : K[[X]] \to A$ with $\varphi(q)(X) = q$. We shall write $p(q)$ or $p \circ q$ instead of $\varphi(p)(q)$, where $\circ$ denotes the operation of composition.

Lemma 2.3. (i) *The monoid* $E = \mathrm{Hom}(A, A)$ *of continous algebra endomorphisms of* A *has a filtration given by* $E_n = \{\alpha \in E : \alpha(X) \in A_n\}$ *and satisfying* $E_m E_n \subseteq E_{m+n}$.
(ii) *An endomorphism* $\alpha \in \mathrm{Hom}(A, A)$ *is an automorphism iff* $\alpha(X) = a_1 X + a_2 X^2 + \cdots$ *with* $a_1 \neq 0$. *Thus* $\mathrm{Aut}\, A$ *is a subset of the monoid* E_1. $\blacksquare$

If p and q are power series without absolute term, we can compose the two series. For such power series we have the following

Proposition 2.4. (i) *The set* $(A_1, \cdot)$ *of power series without absolute term* a_0 *is a monoid with respect to the operation* $(p, q) \mapsto p(q)$ *with identity* X.
(ii) *The group of units of* $(A_1, \cdot)$, *say* G, *is exactly the set of all* $a_1 X + a_2 X^2 + \ldots$ *with* $a_1 \neq 0$.

Proof. (i) We show that the function $\varphi \colon A_1 \to \mathrm{Hom}(A, A)$ is a bijective map satisfying $\varphi(q_1(q_2)) = (\varphi(q_1) \circ \varphi(q_2))$. We compute $\varphi(q_1(q_2))(X) = q_1(q_2) = \varphi(q_1)(q_2) = \varphi(q_1)(\varphi(q_2)(X)) = (\varphi(q_1) \circ \varphi(q_2))(X)$. But the elements of $\mathrm{Hom}(A, A)$ are completely determined by its image of X, so this proves the asserted property.
The inverse function to φ must associate to a continous endomorphism $\alpha \in E$, $\alpha \colon A \to A$ a power series from A_1. In view of $\varphi(q)(X) = q(X) = q$ and $\varphi(\alpha(X))(X) = \alpha(X)$ we obtain $\varphi(\alpha(X)) = \alpha$ such that the maps φ and $\alpha \mapsto \alpha(X)$ are inverses of each other. This shows (i) since $\mathrm{Hom}(A, A)$ is a monoid.
The assumption in (ii) follows directly from Lemma 2.3 (ii). $\blacksquare$

We keep in mind that we want to find out something about the n-monotonic functions as defined in the beginning of this paragraph. What we know about the algebra A_1 and its group of units may lead to results concerning $\mathcal{M}_\infty$-functions which fix the origin. Since we are interested in n-monotonic functions for arbitrary $n \in \mathbb{N}$ we may look for suitable *finitely* dimensional algebras on the formal level in this case, too, since we have seen that $f \in \mathcal{M}_n$ may be characterised by its first $2n - 1$ derivatives in 0. We note the following

Definition 2.5. Let $(K^n)_1 = A_1/A_n$, that is the set of all $\sum_{j=1}^{n-1} a_j {\xi_n}^j$.

Proposition 2.6. 1. *The map* $\pi_n : A \to K^n, \pi_n(\sum_{j=0}^{\infty} a_j X^j) = \sum_{j=0}^{n-1} a_j \xi_n^j$ *is a homomorphism.*

2. *Pick an arbitrary element* $q \in (K^n)_1$. *Then there is a unique endomorphism* $\varphi_n(q)$ *of* K^n *such that* $\varphi_n(q)(\zeta_n) = q$ *and* $\varphi_n(\pi_n(q)) = \pi_n(\varphi_n(q))$ *for* $q \in A_1$. *The composition* $(p, q) \mapsto p(q) = p \cdot q \colon K^n \times (K^n)_1 \to K^n$ *is well defined in such a fashion that* $\pi_n(p)(\pi_n(q)) = \pi_n(p(q))$, *and it provides a monoid operation on* $(K^n)_1$

3. $\pi_n \colon A \to K^n$ *induces a surjective monoid morphism of* $(A_1, \cdot) \to ((K_n)_1, \cdot)$ *with image* $(K^n)_1$. *There is a unique monoid isomorphism* $\varphi_n \colon ((K^n)_1, \cdot) \to \mathrm{Hom}(K^n, K^n)$ *given by* $\varphi_n(q)(p) = p(q)$. *Its inverse is given by* $\alpha \mapsto \alpha(\xi_n)$.

4. *The group of units* Γ_n *of* $(K^n)_1$ *is the set of all* $\sum_{j=0}^{n-1} a_j {\xi_n}^j$ *with* $a_0 = 0$ *and* $a_1 \neq 0$. *The group* Γ_n *is isomorphic to the group of* $\mathrm{Aut}\, K^n$. *Also* $\pi_n(\varphi(q)(Z)) = \varphi_n(\pi_n(q)(\xi_n))$.

Lie algebras associated with the power series algebra

Let $\mathrm{Der}\, A$ denote the set of all continous derivations of A. The set of all derivations of A is a Lie-algebra under $[\triangle_1, \triangle_2] = \triangle_1\triangle_2 - \triangle_2\triangle_1$.

Remark 2.7. A *continous* derivation $\triangle$ on A is uniquely determined by $\triangle(X)$.

Proof. We have $\triangle(X^n) = nX^{n-1}\triangle(X)$ since $\triangle$ is a derivation. The continuity implies $\triangle(\sum_{j=0}^{\infty} a_j X^j) = (\sum_{j=1}^{\infty} j a_j X^{j-1})\triangle(X)$, and this proves the assertion. ■

If D is a derivation on A sending X to 1, we write $p' \stackrel{\text{def}}{=} Dp$ for $p \in A$. We note the following

Lemma 2.8. *Every element* $q \in A$ *determines a unique continous derivation* $\delta(q) \colon A \to A$ *with* $\delta(q)(X) = q$, *and for* $p = \sum_{j=0}^{\infty} a_j X^j$ *we have*

$$\delta(q)(p) = \left(\sum_{j=0}^{\infty} j a_j X^{j-1}\right) q = p'q. \qquad ■$$

Proposition 2.9. (i) *The vector space* A *becomes a topological* Lie*-algebra with bracket* $[p_1, p_2] = p_1 p_2' - p_1' p_2$.

(ii) *The function* $\delta \colon A \to \mathrm{Der}\, A$ *is an isomorphism of topological* Lie*-algebras. Its inverse function is* $\triangle \mapsto \triangle(X)$.

(iii) *We have for all* $j, k \in \mathbb{N}$ *that* $[X^j, X^k] = (k - j)X^{j+k-1}$.

(iv) *If* $q \in A_m$, *then* $\delta(q)(A_n) \subseteq A_{m+n+1}$, *in particular* $\delta(A_2) \subseteq E_1$.

(v) A_1 *is a subalgebra with a flag of ideals* $A_1 \supseteq A_2 \supseteq \dots$ *such that* $A_m/A_{m+1} \cong K$. *Moreover,* $[A_m, A_n] \subseteq A_{m+n+1}$.

For a sketch of the proof see [3], p. 5.

Proposition 2.10. (i) *For* $n = 1, 2\ldots$ *the vector space* $(K^n)_1 = A_1/A_n$ *is a* LIE*-algebra with respect to the bracket* $[p,q] = pq' - p'q$*, and we have*

$$[\xi_n^j, \xi_n^k] = (k-j)\xi_n^{j+k-1} \text{ for } j,k = 1,2\ldots$$

In particular, $[\xi_n^j, \xi_n^k] = 0$ *if* $j + k > n$*.*

(ii) *The function* $\pi_n \colon A_1 \to (K^n)_1$ *maps the* LIE*-algebra* A_1 *homomorphically onto the* LIE*-subalgebra* $(K^n)_1$*.*

(iii) *For each element* $q \in K^n$ *there is a unique derivation* $\delta_n(q) \colon K^n \to K^n$ *mapping* ξ_n *to* q*, namely, the one given by* $\delta_n(q)(p) = p'q$*. If* $q \in (K^n)_1$*, then* $\delta_n(q)((K^n)_1) \subseteq (K^n)_1$*, and vice versa. If* $\mathrm{Der}_1(K^n)$ *denotes the* LIE*-algebra of all derivations mapping* $(K^n)_1$ *into itself then*

$$\delta_n \colon (K^n)_1 \to \mathrm{Der}_1(K^n),\ \delta_n(q)(\xi_n) = q$$

is an isomorphism of LIE*-algebras. Also* $\pi_n \circ \delta = \delta_n \circ \pi_n$*.*

Proof. By Proposition 2.7 we have the following homomorphism

$$\pi_n \colon \sum_{j=0}^{\infty} a_j X^j \mapsto \sum_{j=0}^{n-1} a_j {\xi_n}^j \in K^n, \text{where } \xi_n \stackrel{\text{def}}{=} X + A_n.$$

Via π_n, the n-dimensional algebra K^n is a homomorphic image of $K[[X]]$, the algebra of formal power series. For (i), (ii), we restrict π_n to A_1 and compute $\pi_n(\sum_{j=1}^{\infty} a_j X^j) = \sum_{j=1}^{n-1} a_j {\xi_n}^j \in K^n$. This yields $(K^n)_1 = \pi_n(A_1) = \mathrm{span}\{\xi_n, \ldots, \xi_n^{n-1}\}$, and the subalgebra $(K^n)_1$ is a LIE-algebra as a homomorphic image of the LIE-algebra A_1.
For (iii), we fix $q \in K^n$ and set $\delta_n(q)(p) \stackrel{\text{def}}{=} p'q$. By this definition we have $\delta_n(q)(\xi_n) = q$, and comparing $\delta_n(q) \circ \pi_n(X) = \delta_n \circ \pi_n = \delta_n(q) \circ \pi_n(X) = \delta_n(q)(\xi_n)$ with $\pi_n \circ \delta(q)(X) = \pi_n(q) = \delta_n(q)(\xi_n)$ we get $\pi_n \circ \delta = \delta_n \circ \pi_n$, since it suffices to consider the image of X. ∎

Applications to n-monotonic functions

We start with $\mathcal{M}_\infty$-functions. In view of Theorem 1.7 we know that these functions are analytic in the interval $(-1,1)$ which implies that they have a *convergent* power series near 0.

We recall what we found out about *formal* power series: The map $\varphi \colon A_1 \to \mathrm{Hom}(A,A)$ which associates with $q \in A_1$ the unique algebra endomorphism $\varphi(q)$ sending X to q is an isomorphism of monoids. Its inverse is $\alpha \mapsto \alpha(Z)$. G, the group of units of A_1 is isomorphic to $\mathrm{Aut}\, A$ under φ.

Let $f \colon (-1,1) \to (-1,1)$ belong to $\mathcal{M}_\infty$, $f(0) = 0$. Let $\mathcal{O}_0$ denote the space of functions which are real analytic in $(-1,1)$, map the latter into itself and fix the origin. Theorem 1.7 yields $f \in \mathcal{O}_0$, that is, there is an $r > 0$ such that $f(x) = \sum_{n=1}^{\infty} a_n x^n$ for every $x \in (-r,r)$, and $\sum |a_n| r^n < \infty$.

There is a canonical way to embed $\mathcal{O}_0$ into the space of formal power series without absolute term. Let $\iota \colon \mathcal{O}_0 \to \mathbb{R}[[X]]$, $\iota(f) = \sum_{n=1}^{\infty} a_n X^n$ be the map which associates to each analytic function $f \in \mathcal{O}_0$ its power series in the origin. We note that if $f, g \in \mathcal{O}_0$ then there is an $s > 0$ such that $f \circ g$ has a convergent power series in $(-s, s)$, and the power series of the composed function is the composition of the two power series of f and g. We conclude $\iota(f \circ g) = \iota(f) \circ \iota(g)$, and ι is a homomorphism.

We conclude that $\mathcal{M}_\infty$ is a subsemigroup of a group which is isomorphic to the group of units of A_1, that is $\{a_1 X + a_2 X^2 + \ldots : a_1 \neq 0\}$. The latter is isomorphic to the group of automorphisms of A. It is a LIE-group with respect to the ultrametric.

In the case of $\mathcal{M}_n$ for arbitrary $n \in \mathbb{N}$ we shall only give a sketch of our considerations here, they will be followed by a less preliminary version later.

We summarize what we have shown: We found an isomorphism of groups $\varphi_n\colon \Gamma_n \to \mathrm{Aut}(K^n)$ and an isomorphism of LIE-algebras $\delta_n\colon K^n \to \mathrm{Der}(K^n)$. In particular, Γ_n is a LIE-group by transport of structure which is isomorphic to $\mathrm{Aut}\,\mathbb{R}^n$ and which has the LIE-algebra $\mathbb{R}^n$. It is isomorphic to the following LIE-algebra $B_n = \mathrm{span}\{\xi_n, \ldots {\xi_n}^{n-1}\}$ with bracket $[\xi_j, \xi_k] = (k-j)\xi_{j+k}$ for $j, k \geq 1$. Γ_n consists of polynomials $\sum_{j=1}^{n-1} a_j \xi_n^j$ with $a_1 \neq 0$. Now any $f \in \mathcal{M}_n$ determines the vector $(f'(x), \ldots, f^{(2n-1)}(x))$ with first entry $f'(x) > 0$. The map $f \mapsto \big(f'(x), \ldots, f^{(2n-1)}(x)\big) \colon \mathcal{M}_n \to \Gamma_{2n}$ is a monoid morphism. Thus $\mathcal{M}_n$ has a homomorphic image in the LIE-group Γ_{2n}.

References

[1] Behnke, H. and F. Sommer, "Theorie der analytischen Funktionen", Springer Verlag, Berlin, Göttingen, Heidelberg, 1955.

[2] Bers, Lipman, "Charles Loewner - Collected Papers", Birkhäuser, Boston, Basel, 1988.

[3] Hofmann, K.H., D. Mittenhuber, *A memo on an algebraic group occuring in the interpretation of Loewner's work*, Seminar Notes TH Darmstadt 1991.

[4] Löwner, Karl, *Über monotone Matrixfunktionen* , Math. Zeitschrift **38** (1934), 177–216.

[5] Loewner, Charles, *Some Classes of Functions Defined by Difference or Differential Inequalities*, Nat. Adv. Comm. for Aero. Tech. Note **2065** (1950), 308–319.

[6] Loewner, Charles, *On Totally Positive Matrices*, Math. Zeitschrift **63** (1955), 338–340.

[7] Loewner, Charles, *Semigroups of Conformal Mappings*, J. Math. and Mech. **8**(1957), 278–288.

[8] Loewner, Charles, *On Generation of Monotonic Transformations of Higher Order by Infinitesimal Transformations*, J. d'Anal. Math. **70**(1963), 189–206.

[9] Loewner, Charles, *On Semigroups in Analysis and Geometry*, Bull. A.M.S. **70**(1964), 1–15.

[10] Loewner, Charles, "Theory of Continous Groups", The MIT Press, Cambridge, Massachusetts, and London, England, 1970.

[11] Pick, Georg, *Beschränkungen analytischer Funktionen durch vorgegebene Funktionswerte*, Math. Annalen **77**(1916), 7–23.

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Received August 28, 1991
and in final form October 24,1991