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A method for the computation of Clebsch-Gordan coefficients

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Clebsch-Gordan series, Clebsch-Gordan coefficients

Let G be a semisimple connected Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. A finite dimensional irreducible representation of G (of $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{C}}$) with highest weight m will be denoted by $[m]$. Let be $[m']$ and $[m'']$ be two such representations with representation spaces V' and V''. In general, the tensor product $[m'] \otimes [m'']$ is a direct sum of irreducible representations $[m_i]$ with the multiplicities n_i:

$$[m'] \otimes [m''] = \sum_{i \in I} n_i [m_i] \qquad \text{(Clebsch-Gordan series).}$$

Furthermore, we will assume that for every irreducible representation $[m]$ in the representation space V is given a basis $\{g(m, p_1, p_2, \ldots) : p_1, p_2, \ldots\}$ described by parameters $p_1, p_2, \ldots$ which range over some intervals. This basis we shall call a *canonical basis* for $[m]$ in the following. (We do not give an exact definition of the canonical basis here). In $V' \otimes V''$ we have two distinguished bases: The product basis

$$\{g(m', p'_1, p'_2, \ldots) \otimes g(m'', p''_1, p''_2, \ldots) : p'_1, p'_2, \ldots, p''_1, p''_2, \ldots\}$$

and the basis consisting of the canonical bases of the irreducible constituents $[m_i]$:

$$\{g^{j_i}(m_i, p_1, p_2, \ldots) : i \in I, j_i = 1, \ldots, n_i, p_1, p_2, \ldots\}$$

The connection between these bases is described by means of the so-called *Clebsch-Gordan coefficients* CGC(...): $g^{j_i}(m_i, p_1, p_2, ...) =$

$$\sum \mathrm{CGC}(m', m'', m_i, j_i; p'_1, \ldots, p''_1, \ldots, p_1, \ldots) g(m', p'_1, \ldots) \otimes g(m'', p''_1, \ldots).$$

So, in general we have to solve two problems:
I. *The computation of the Clebsch-Gordan series* (CG series).
II. *The computation of the Clebsch-Gordan coefficients* (CGC).

Remarks to the computation of the CG series

In closed form CG series are known only in a few cases. For instance, if $G = \mathrm{SU}(2)$ the irreducible representations are described by the numbers $l = 0, 1/2, 1, \ldots$ and the CG series is given by the formula

$$[l_1] \otimes [l_2] = \sum_{k=0}^{\min\{2l_1, 2l_2\}} [l_1 + l_2 - k]. \tag{1}$$

The formula of Kostant-Steinberg gives a solution for arbitrary semisimple Lie algebras $\mathfrak{g}_{\mathbb{C}}$: It is possible to compute the multiplicity n_i of any irreducible representation in a given tensor product by means of the Weyl group of g_C. But it is very hard to work with this formula (see [1]).

A well-known method is that of Littlewood-Richardson: The decomposition of the tensor product is obtained by a graphical procedure working with Young frames. For $\mathrm{SU}(n)$ and $\mathrm{U}(n)$, this procedure can be used to prove a closed formula similar to (1) (see [3]); in these cases it is easy to calculate the CG series with the aid of a computer.

For example, the irreducible representations $D(p,q)$ of $\mathrm{SU}(3)$ are described by two natural numbers p and q (the highest weight of $D(p,q)$ is $(p+q, q, 0)$) and by the CG series which we get by

$$D(p_1, q_1) \otimes D(p_2, q_2) = \sum_{i=0}^{i_1} \sum_{k=0}^{k_1} \sum_{l=l_0}^{l_1} D(p_1 + p_2 - i - 2k + l, q_1 + q_2 - i + k - 2l)$$

$$i_1 = min\{p_2, q_1\}, k_1 = min\{p_1, p_2 + q_2 - i\}, l_1 = min\{q_1 + k - i, q_2\},$$

$$l_0 = max\{0, k + i - p_2\}.$$

A method for the computation of CGC

The classical CGC are the coefficients connected with the representations of $\mathrm{SU}(2)$. Here a canonical basis for the irreducible representation $[l]$ ($l = 0, 1/2, 1, \ldots$) can be characterized in the following way: Let

$$H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

be a basis of the Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$; the normed eigenvectors of $[l](H)$ build a canonical basis for $[l]$ (these vectors are determined by this condition up to a factor of absolute value 1; we choose this factor equal to 1). We denote the basis by $\{g(l, \mu) : \mu = 0, \ldots, 2l\}$. Here $g(l, 0)$ is the highest vector. In many papers the parameter μ ranges over the eigenvalues $-l, -l+1, \ldots, l$ of $[l](H)$. The CGC are given by the closed formula

$$\mathrm{CGC}(l_1, l_2, k; \mu_1, \mu_2, \mu) = \sqrt{\frac{\binom{2l_1}{\mu_1}\binom{2l_2}{\mu_2}\binom{2l_1}{k}\binom{2l_2}{k}}{\binom{2(l_1+l_2-k)}{\mu}\binom{2(l_1+l_2)-k+1}{k}}} \sum_{h=0}^{k} \frac{(-1)^h \binom{k}{h}\binom{\mu_1}{h}\binom{\mu_2}{k-h}}{\binom{2l_1}{h}\binom{2l_2}{k-h}}$$

(2) $$(k = 0, ..., \min\{2l_1, 2l_2\}).$$

This or an analogous formula can be proved by various methods. A possible way is the following: We look at the matrices

$$A12 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$A21 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}(2, C)$ and identify $A12$ with $[l](A12)$ ($A21$ with $[l](A21)$); by

(3) $$g(l, \mu+1) = \sqrt{\frac{\binom{2l}{\mu}}{\binom{2l}{\mu+1}}} \cdot \frac{1}{\mu+1} A21 g(l, \mu)$$

we get a recursive relation; the initial vector is the highest vector $g(l, 0)$ of $[l]$ which can be described by

(4) $$A21\big(g(l, 0)\big) = 0.$$

If a highest vector $g(l_1+l_2-k, 0)$ is computed by (4) as an element of the tensor space we can use (3) to compute a canonical basis in terms of the product vectors. Now (2) is an explicit solution of the recursive relation (3)(see [4]). The method can be generalized to a numerical computation of CGC of other groups. For this purpose we assume that a canonical basis $g_0, g_1, \ldots, g_s$ with highest vector g_0 is introduced by parameters for every irreducible representation of a given group G and the action of the root vectors A_i, B_i $(i = 1, \ldots, n)$ is known (for instance by the Gelfand-Zetlin formulas). Let $A_i(B_i)$ be root vectors corresponding to negative (positive) roots. We get all elements of the canonical basis by a suitable application of the A_i on g_0. g_0 is characterized by $B_i(g_0) = 0$, $i = 1, \ldots, n$ (see [1]). If the vector g_0 is given we have to look for an algorithm such that g_{j+1} can be calculated in the form

(5) $$g_{j+1} = c(i, j) A_i g_j + \sum_{h<j} c_h g_h.$$

More specifically, suppose that we are given a tensor product $[m'] \otimes [m'']$ of representations, and let $[m_i]$ be an irreducible constituent with multiplicity n_i. The possible highest vectors g_0 of $[m_i]$ are linear combinations of some product vectors; these product vectors are determined by the highest weight of $[m_i]$, which is a weight of $[m'] \otimes [m'']$. We apply B_i to the linear combinations of these product vectors with unknown coefficients and obtain from $B_i(g_0) = 0$ a set of homogeneous linear equations for these coefficients. The space of solutions has dimension n_i and so for $n_i > 1$ the vector g_0 must be choosen so as to satisfy further conditions. One possibility for such a choice is described in the next section for SU(3).

Now we assume that a highest vector g_0 of $[m_i]$ is given as a linear combination of product vectors:

$$g_0 = \sum_h c_h g_{1h} \otimes g_{2h}$$

(the c_h are CGC by definition). Let A_i be a suitable root vector such that

$$g_1 = cA_i(g_0) = c \sum c_h A_i(g_{1h} \otimes g_{2h}) = c \sum c_h \big((A_i g_{1h}) \otimes g_{2h} + g_{1h} \otimes (A_i g_{2h})\big).$$

Because $A_i g_{1h}$ and $A_i g_{2h}$ are known by assumption we get g_1 as a linear combination of product vectors. The coefficients are the CGC of g_1. Similarly we pass from g_j to g_{j+1} by (5) and an analogous calculation. A few years ago the procedure was programmed for the group SU(3) with the aid of the computer algebra system REDUCE (see [2]).

Computation of highest vectors in the case SU(3)

We have three operators A_i: $A21, A31, A23$ and three operators B_i: $A12, A13,\ A32$. The operators $A21, A12$ act analogously to the case of SU(2). So a subgroup SU(2) of SU(3) is selected.

In general, in the CG series $[m'] \otimes [m''] = \sum n_i [m_i]$ of SU(3) we have $n_i > 1$. It is necessary to formulate additional properties for the selection of highest vectors g_0 as solutions of $B_i(g_0) = 0$. We observe:

(i) *The irreducible representation $[m' + m'']$ is selected from the irreducible constituents $[m_i]$: It has the multiplicity 1; the set of weights of $[m'+m'']$ is identical with the set of weights of $[m'] \otimes [m'']$.*

(ii) *The canonical basis of an arbitrary representation space of $SU(3)$ provides a layer structure; the layer structure of $[m'+m'']$ can be transferred in a natural way to the product basis.*

This gives solutions of $A12(g_0) = 0$ which are similar to the SU(2)-solutions (2); these solutions are joined with the layer structure of the product basis and they are compatible with $A13(g_0) = 0, A32(g_0) = 0$. So we have a special way to solve the system $B_i(g_0) = 0$, and the solutions are related naturally to the subgroup SU(2). The exact description is given in [5].

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