Seminar Sophus Lie
1 (1991) 73–80

On Locally compact groups whose set of compact subgroups is inductive.

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The main concern of this paper is the set $\mathfrak{K}$ of compact subgroups of a locally compact group. We are interested in those groups which have inductive $\mathfrak{K}$, where $\mathfrak{K}$ is partially ordered by set inclusion. The main theorems of this paper will give a characterization of such groups.

Definition 1. Let $\mathfrak{M}$ be a set, and let $\{F_i\}_{i\in I}$ be a family of subsets of $\mathfrak{M}$. We call $\{F_i\}_{i\in I}$ a *tower* if at least one of the two inclusions

$$X_i \subseteq X_j \qquad \text{or} \qquad X_j \subseteq X_i$$

holds for all $i, j \in I$.

Definition 2. Let G be a locally compact group. If the set

$$\bigcup_{i\in I} K_i$$

has compact closure for every tower $\{K_i\}_{i\in I}$ of compact subgroups of G then G is called an *ICS group*, where ICS is the short form of *inductive set of compact subgroups.*

It is clear that a locally compact group has an inductive set of compact subgroups if and only if it is an ICS group. Hence, the aim of this paper is to give a classification of ICS groups.

Obviously, every ICS group G has maximal compact subgroups, and every compact subgroup of G is contained in a maximal compact subgroup of G. Since the topological structure of G is largely determined by its maximal compact subgroups, it is important to guarantee the existence of maximal compact subgroups for the biggest possible class of groups. There are some results about the existence of maximal compact subgroups in locally compact groups. Probably the most famous of all shows that a Lie group G has maximal compact subgroups if G/G_0 is finite (see [Ho], Theorem 3.1 on page 180, for example).

Of course, there are many groups which have no maximal compact subgroups at all, and it is clear that all such groups are examples of non-ICS groups. One may wonder then, if there do exist any standard examples of non-ICS groups. If we want to construct examples of non-ICS groups, it is only natural that we start with the case of discrete groups. For this class of groups, we may equivalently define that a group G is an ICS group iff the union of every countable tower of finite subgroups of G is finite.

There are three types of discrete non-ICS groups which are particularly important for the classification of Lie ICS groups.

Example 1. Let p be any prime. Then

$$\mathbb{Z}(p)^{(\mathbb{N})}$$

endowed with the discrete topology is not an ICS group since it is the union of a tower $\{\mathbb{Z}(p)^n\}_{n\in\mathbb{N}}$ of finite subgroups.

Example 2. Let $\{p_i\}_{i\in\mathbb{N}}$ be a sequence of pairwise distinct primes p_i. Then

$$\bigoplus_{i\in\mathbb{N}} \mathbb{Z}(p_i)$$

endowed with the discrete topology is not an ICS group. It is an interesting observation that this group is even the union of a tower of cyclic subgroups:

$$\bigoplus_{i\in\mathbb{N}} \mathbb{Z}(p_i) \;=\; \bigcup_{n\in\mathbb{N}} \left(\oplus_{i=1}^{n} \mathbb{Z}(p_i)\right) \;=\; \bigcup_{n\in\mathbb{N}} \mathbb{Z}(p_1 p_2 \cdot \ldots p_n) \;.$$

Example 3. Let p be a prime. Then the set

$$\{t \in \mathbb{T} : \; t^k = 1 \quad \text{with} \quad k = p^n \quad \text{for some} \quad n \in \mathbb{N}_0\}$$

of p-power roots of unity, when endowed with the discrete topology, is not an ICS group. We call this group *Prüfer group*, and we use the abbreviation $\mathbb{Z}(p^\infty)$ for it. Just as for the second example, we observe that $\mathbb{Z}(p^\infty)$ is the union of a tower of cyclic subgroups:

$$\mathbb{Z}(p^\infty) = \bigcup_{n\in\mathbb{N}} \mathbb{Z}(p^n) \;.$$

It is not at all clear that this set of examples of non-ICS groups suffices to give a satisfying picture of all non-ICS groups. Yet, these three types of groups lie at the heart of the following classification of Lie ICS groups.

Theorem 1. *A Lie group G is not an ICS group if and only if G contains a discrete subgroup which is isomorphic to*

$$\mathbb{Z}(p^\infty), \quad \mathbb{Z}(p)^{(\mathbb{N})} \quad \text{or} \quad \bigoplus_{i\in\mathbb{N}} \mathbb{Z}(p_i)$$

for a suitable prime p or for a suitable sequence $\{p_i\}_{i\in\mathbb{N}}$ of pairwise distinct primes p_i.

This result cannot be carried over to the bigger class of locally compact groups as can be seen in the case of the p-adic group $\mathbb{Q}_p$.

Example 4. Let p be any prime number, and let $\mathbb{Q}_p$ be the set of all sequences

$$\{z_i\}_{i\in\mathbb{Z}} \in \{0,1,2,\ldots,p-1\}^{\mathbb{Z}}$$

such that there exists an index $k \in \mathbb{Z}$ such that $z_i = 0$ obtains for all $i \leq k$. We define an addition on $\mathbb{Q}_p$ in the following way.

Let $\{a_i\}_{i\in\mathbb{Z}}$ and $\{b_i\}_{i\in\mathbb{Z}}$ be two elements of $\mathbb{Q}_p$. Then there exists an integer $k \in \mathbb{Z}$ such that $a_i = 0$ and $b_i = 0$ obtain for all $i \leq k$. We define recursively two sequences $\{s_i\}_{i\in\mathbb{Z}}$, $\{c_i\}_{i\in\mathbb{Z}} \in \mathbb{Q}_p$ by

(i) $s_i = 0$ and $c_i = 0$ for all $i \leq k$;

(ii) $s_{i+1} + p \cdot c_{i+1} = a_{i+1} + b_{i+1} + c_i$ for all $i \geq k$.

These two sets of equations uniquely determine $\{c_i\}_{i\in\mathbb{Z}}$ and $\{s_i\}_{i\in\mathbb{Z}}$. We set

$$\{a_i\}_{i\in\mathbb{Z}} + \{b_i\}_{i\in\mathbb{Z}} := \{s_i\}_{i\in\mathbb{Z}} ,$$

then, and this operation provides $\mathbb{Q}_p$ with the structure of an abelian group. If we restrict this addition to the set of sequences of $\mathbb{Q}_p$ which have finite support, we see that this addition is simply addition with carry over. Thus, for any sequence $\{z_i\}_{i\in\mathbb{Z}} \in \mathbb{Q}_p$ and for all $n \in \mathbb{N}$ we have

$$p^n \cdot \{z_i\}_{i\in\mathbb{Z}} = \{z_{i-n}\}_{i\in\mathbb{Z}} .$$

This shows that $\mathbb{Q}_p$ does not contain any elements of finite order.

We now want to endow $\mathbb{Q}_p$ with a topology which will make it a locally compact group. It is enough to give a suitable system of neighbourhoods of the identity. Set

$$K_n := \left\{ \{z_i\}_{i\in\mathbb{Z}} \in \mathbb{Q}_p \middle| \; z_i = 0 \text{ for all } i < n \right\} ,$$

then the family $\{K_n\}_{n\in\mathbb{Z}}$ is a system of neighbourhoods of the identity which makes $\mathbb{Q}_p$ a locally compact group that is not compact. We call this group the *p-adic group*. It is easy to prove that K_n is a compact subgroup of $\mathbb{Q}_p$ for all $n \in \mathbb{Z}$ (see [HR],]10, for example). In fact, they are the only compact subgroups of $\mathbb{Q}_p$.

It is obvious that $\mathbb{Q}_p$ is the union of the tower $\{K_i\}_{i\in\mathbb{Z}}$ of compact subgroups of $\mathbb{Q}_p$. Since the p-adic group is not compact, this gives an example of a locally compact non-discrete group that is not an ICS group. Moreover, since $\mathbb{Q}_p$ doesn't have any torsion elements, the p-adic group doesn't contain any discrete copy of Example 1, 2 or 3.

Yet, there is another characterization of ICS groups in the class of locally compact groups. This characterization makes use of the well-known structure of the connected component of the identity G_0 of G, which is the projective limit of connected Lie groups. We can prove the following theorem.

Theorem 2. *Let G be a locally compact group. Then G is an ICS group if and only if G/G_0 is an ICS group.*

We now want to give some sketches of the methods which we have used to prove the above theorems. For details and further information we refer to [Te].

To prove Theorem 1, we start with the classification of discrete ICS groups. Firstly, we take a closer look at abelian groups. Since discrete ICS groups should not contain any tower of finite subgroups, the torsion subgroup of an discrete abelian ICS group should be finite. The reverse of this observation gives that a discrete abelian non-ICS group has an infinite torsion subgroup. This directs our attention towards infinite abelian torsion groups. It is due to the structure theory of abelian torsion groups that the study of abelian non-ICS groups can be reduced to the study of p-groups. We have the following result about infinite abelian p-groups.

Proposition 1. *Let G be an infinite abelian p-group. Then G contains a copy of either $\mathbb{Z}(p^\infty)$ or*

$$\mathbb{Z}(p)^{(\mathbb{N})} .$$

This gives rise to the result that an abelian torsion group G is infinite if and only if G contains a copy of

$$\mathbb{Z}(p^\infty), \quad \mathbb{Z}(p)^{(\mathbb{N})} \quad or \quad \bigoplus_{i\in\mathbb{N}} \mathbb{Z}(p_i)$$

for a suitable prime p or for a suitable sequence $\{p_i\}_{i\in\mathbb{N}}$ of pairwise distinct primes p_i. Hence, a discrete abelian group G is not an ICS group if and only if G contains a copy of

$$\mathbb{Z}(p^\infty), \quad \mathbb{Z}(p)^{(\mathbb{N})} \quad or \quad \bigoplus_{i\in\mathbb{N}} \mathbb{Z}(p_i)$$

for a suitable prime p or for a suitable sequence $\{p_i\}_{i\in\mathbb{N}}$ of pairwise distinct primes p_i. This shows for a first time the importance of the groups that we have given in Examples 1, 2 and 3.

It is due to the Theorem of Hall-Kulatilaka-Kargapolov (see [Ro], Theorem 14.3.7, pages 416-418, for example) that we can overcome the gap between abelian and non-abelian groups. This theorem, which, until now, cannot be proved without using the celebrated Theorem of Feit-Thompson, guarantees that every infinite locally finite group has an infinite abelian subgroup.

Definition 3. Let G be a group. We call G a *locally finite group* if every finite subset of G generates a finite subgroup of G.

We easily see that the union of every countable tower of finite subgroups is a locally finite group. This shows the importance of the concept of a locally finite group for the classification of discrete ICS groups.

Now, a discrete non-ICS group has at least one countable tower of finite subgroups which has infinite union H. Since H is an infinite locally finite group, the Theorem of Hall-Kulatilaka-Kargapolov guarantees that H, hence G, contains an infinite abelian subgroup, which, by the way, is a torsion group. We may thus use our knowledge of abelian non-ICS groups to prove the central theorem about discrete groups. It shows that a discrete group G, which need not be abelian, is not an ICS group if and only if G contains a subgroup which is a copy of

$$\mathbb{Z}(p^\infty), \quad \mathbb{Z}(p)^{(\mathbb{N})} \quad \text{or} \quad \bigoplus_{i\in\mathbb{N}} \mathbb{Z}(p_i)$$

for a suitable prime p or for a suitable sequence $\{p_i\}_{i\in\mathbb{N}}$ of pairwise distinct primes p_i. This line of reasoning can be continued for Lie groups. But to do so, we first need to consider central extensions of discrete groups. An investigation of those groups G which have a central subgroup N such that G/N is isomorphic to

$$\mathbb{Z}(p^\infty), \quad \mathbb{Z}(p)^{(\mathbb{N})} \quad \text{or} \quad \bigoplus_{i\in\mathbb{N}} \mathbb{Z}(p_i)$$

for a suitable prime p or for a suitable sequence $\{p_i\}_{i\in\mathbb{N}}$ of pairwise distinct primes p_i, shows that, under certain restrictions, G itself contains a subgroup of this type. We also need to consider the question of what can be said about a factor group of G if G contains a subgroup of these three types. It is easy to see that factoring by a finite subgroup does not change the type of group under consideration.

We are now going to investigate Lie groups that are ICS groups. Of all the classes of groups that we consider in this paper, the class of Lie groups surely has the richest structure theory. And it is due to this structure theory that we can continue the line of reasoning which we have begun in the case of discrete groups. It is remarkable that it is possible to generalize the results obtained in the case of discrete groups, which finally yields Theorem 1. Since this involves both the Theorem of Feit-Thompson and the structure theory of Lie groups, this is an interesting interplay between group theoretical and topological concepts. Moreover, we see that certain properties of the set of compact subgroups of a Lie group G are displayed by the set of finite subgroups of G. Results that are concerned with the interplay between the set of finite subgroups and G have a long history. For example, it is well-known that every finite extension H of a connected Lie group G contains a finite subgroup F such that H is the product of F and G, that is, we have $H = F \cdot G$ (see [Le]). This gives then

$$F/(F \cap G) \cong H/G\ .$$

To achieve Theorem 1, we have to proceed in several steps. It is crucial for each step to consider the group of automorphisms of the connected component of the identity G_0 of the group G which is under inspection. This is due to the fact that G_0 is a normal subgroup of G such that conjugation by any element of G gives an automorphism of G_0. Hence, the group of automorphisms of G_0 contains a homomorphic image of G, which puts some severe restrictions on G. Now, we have two important special cases.

Firstly, we consider the case of a non-ICS group G such that G_0 is a compact, semisimple and connected Lie group. In this case, it is no loss of generality to assume that G/G_0 is isomorphic to $\mathbb{Z}(p^\infty)$, $\mathbb{Z}(p)^{(\mathbb{N})}$, or $\oplus_{i=1}^\infty \mathbb{Z}(p_i)$. Moreover, the study of G can be reduced to the study of a subgroup H of G with $G_0 \subseteq H$ such that conjugation by any element of H is an inner automorphism of G_0. This, then, gives rise to a discrete subgroup F of H which satisfies $F \cap G_0 = \mathrm{Z}(G_0)$ and

$$F/\mathrm{Z}(G_0) \cong H/G_0 .$$

Since $\mathrm{Z}(G_0)$ is finite and central in F, the results on central extensions, which we have achieved before, show that Theorem 1 is indeed true in this case.

Secondly, we consider the case of a non-ICS group G such that G_0 is a torus. As above, we may assume that G/G_0 is isomorphic to $\mathbb{Z}(p^\infty)$, $\mathbb{Z}(p)^{(\mathbb{N})}$, or $\oplus_{i=1}^\infty \mathbb{Z}(p_i)$. Since the group of automorphisms of a torus $\mathbb{T}^n$ is isomorphic to $\mathrm{GL}(n,\mathbb{Z})$, and since locally finite subgroups of $\mathrm{GL}(n,\mathbb{Z})$ are bounded, hence finite, there exists a subgroup H of G such that G_0 is a central subgroup of H, and such that H/G_0 is isomorphic to $\mathbb{Z}(p^\infty)$, $\mathbb{Z}(p)^{(\mathbb{N})}$, or $\oplus_{i=1}^\infty \mathbb{Z}(p)$. We may then reduce to the case that H is abelian, which immediately proves Theorem 1.

In the general case, we combine these two building blocks with a proof that proceeds by induction on the dimension of the connected component of the identity of the Lie group under inspection.

Since locally finite subgroups of Lie groups have shown their importance for the classification of Lie ICS groups, it is justified to take a closer look at them, and we are especially interested in those subgroups of G which are isomorphic to

$$\mathbb{Z}(p^\infty), \quad \mathbb{Z}(p)^{(\mathbb{N})} \quad \text{or} \quad \bigoplus_{i\in\mathbb{N}} \mathbb{Z}(p_i)$$

for a suitable prime p or for a suitable sequence $\{p_i\}_{i\in\mathbb{N}}$ of pairwise distinct primes p_i. We can show that G contains a discrete copy of $\mathbb{Z}(p^\infty)$ if and only if G/G_0 contains a copy of $\mathbb{Z}(p^\infty)$. Similiar results hold for

$$\mathbb{Z}(p)^{(\mathbb{N})} \quad \text{and} \quad \bigoplus_{i\in\mathbb{N}} \mathbb{Z}(p_i) .$$

The structure of arbitrary locally finite subgroups H of Lie ICS groups G can be clarified too. We can prove that H contains a normal abelian subgroup N which has finite index $(H:N)$. This is a generalization of a theorem of C. Jordan about locally finite subgroups of linear groups.

The classification of locally compact ICS groups demands totally different methods. We have already seen that the analogue of Theorem 1 does not hold in this class of groups. For example, take the p-adic group $\mathbb{Q}_p$, which we have introduced in Example 4.

But the structure theory of locally compact groups, which rests heavily on the theory of Lie groups, is rich enough to allow a characterization of locally compact ICS groups too. The proof of Theorem 2 is achieved in several steps.

We begin with the group $\mathrm{Aut}(C)$ of automorphisms of a closed, pointed and generating cone C in a finite dimensional real vector space.

Definition 4. A subset C of some vector space V is called a *pointed cone* iff it fulfills the following three conditions:

$$\begin{aligned} &\text{(i)} && C + C \subseteq C, \\ &\text{(ii)} && C \cap (-C) = \{0\}, \\ &\text{(iii)} && r \cdot c \in C \text{ for all } r \in \mathbb{R}^+,\ c \in C\ . \end{aligned}$$

In particular, C is a convex set. We will say that C is *generating* iff C spans V, that is, $C - C = V$. Providing V is a topological vector space, we call C a *closed, pointed cone*, if C is both a pointed cone and a closed subset of the topological space V.

We can show that $\mathrm{Aut}(C)$ is an ICS group. This is due to the fact that we can quite easily describe the set of maximal compact subgroups of $\mathrm{Aut}(C)$ in terms of fixed-points in the interior of C. We have the following proposition (see [HHL], Theorem III.2.4 for a proof of this proposition).

Proposition 2. *Let C be a closed, pointed and generating cone in a finite dimensional real vector space V. Then the following two statements are valid:*

(i) *If K is a compact subgroup of $\mathrm{Aut}(C)$ then there exists an element $x \in \mathrm{int}(C)$ such that*
$$\varphi(x) = x$$
holds for all $\varphi \in K$, that is, x is a common fixed point for K.

(ii) *The group*
$$K_x := \{\varphi \in \mathrm{Aut}(C) : \quad \varphi(x) = x\}$$
is compact for every $x \in \mathrm{int}(C)$.

It looks somewhat peculiar to start with this seemingly restricted class of groups, but it isn't. In fact, the general linear group $\mathrm{GL}(\mathbb{R}^n)$ has a homomorphic image in a group of automorphisms of some cone. The appropriate cone is the cone P of all positive semi-definite operators in the vector space A of all hermitean operators on $\mathbb{R}^n$ with respect to euclidean scalar product. We have the following action $\pi : \mathrm{GL}(\mathbb{R}^n) \to A$ of $\mathrm{GL}(\mathbb{R}^n)$ on A:

$$\pi(g)(a) := gag^*\ ,$$

where g^* denotes the adjoint map associated with g. It is easy to see that $\pi\big(\mathrm{GL}(\mathbb{R}^n)\big)$ is a subgroup of $\mathrm{Aut}(P)$. This gives rise to the proposition that $\mathrm{GL}(\mathbb{R}^n)$ is an ICS group. It is only natural that semisimple connected Lie groups G come into focus now. This is due to the fact that G modulo its center is a closed subgroup of a suitable general linear group $\mathrm{GL}(\mathbb{R}^n)$. The structure theory of Z-groups, that is of groups which have compact factor group $G/\mathrm{Z}(G)$, makes it clear then that every semisimple connected Lie group is an ICS group. This result can be generalized to connected Lie groups G by the familiar procedure which proceeds by induction on the dimension of the radical of G. Hence, every connected Lie group is an ICS group. It follows that every Lie group G with finite factor group G/G_0 is an ICS group too. Since locally compact groups G that have compact factor group G/G_0 can be approximated by Lie groups H with finite factor group H/H_0, we conclude that G is an ICS group. In a final step, we show that a locally compact group G is an ICS group if and only if G/G_0 is an ICS group. It is surprising that the last argument involves Zorn's Lemma. This is due to the fact that at one point we need to lift a tower $\{K_i\}_{i\in I}$ of compact subgroups of G/G_0 to G. Since there is no canonical choice for a compact subgroup H_i of G which satisfies $\pi(H_i) = K_i$, where $\pi : G \to G/G_0$ denotes the quotient map, we have to pick one by one a suitable subgroup H_i for each $i \in I$. Since the index set I may be somewhat complicated, we need Zorn's Lemma.

References

[HHL] Hilgert, J., Hofmann, K.H., and Lawson, J., Lie Groups, Convex Cones, and Semigroups, Oxford University Press, 1989.

[Ho] Hochschild, G., The Structure of Lie Groups, Holden-Day, Inc., San Francisco, London, Amsterdam, 1965.

[HR] Hewitt, E., and Ross, K.A., Abstract Harmonic Analysis I, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1963.

[Le] Lee, D.H., *Supplements for the Identity Component in Locally Compact Groups*, Math. Zeitschr. **104** (1968), 28–49.

[Ro] Robinson, D.J.S., A Course in the Theory of Groups, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[Te] Terp, C., Lie groups whose set of compact subgroups is inductive, doctoral thesis, Darmstadt, 1991.

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Received February 26, 1991