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Compact elements in solvable real Lie algebras

Karl H. Hofmann

1. Observations on solvable algebras

We consider a real solvable Lie algebra $\mathfrak{g}$.

The intersection of the descending central series will be called $\mathcal{C}^\infty(\mathfrak{g})$ or $\mathcal{C}^\infty$ for short. If $\mathfrak{h}$ is a Cartan subalgebra then $\mathfrak{g} = \mathfrak{h} + \mathcal{C}^\infty$. Since the characteristic ideal $\mathcal{C}^\infty$ is contained in the commutator algebra $\mathfrak{g}'$, it is nilpotent.

All Cartan algebras in the solvable algebra $\mathfrak{g}$ are conjugate.

1.1. Definition. Fixing a Cartan algebra $\mathfrak{h}$ we shall write

$$\mathfrak{e} \stackrel{\text{def}}{=} \mathfrak{h} \cap \mathcal{C}^\infty. \tag{1}$$

1.2. Lemma. *All roots vanish on $\mathfrak{e}$.*

Proof. We assume the contrary and find an element $X \in \mathfrak{e} = \mathfrak{h} \cap \mathcal{C}^\infty$ such that there is a root λ on $\mathfrak{h}_\mathbb{C}$ with $\lambda \stackrel{\text{def}}{=} \lambda(X) \neq 0$. By the definition of the root λ, there is a smallest natural number n such that $(\operatorname{ad} X - \lambda\cdot\mathbf{1})^n(\mathfrak{g}_\mathbb{C}) = \{0\}$. Hence there is a $Y_0 \in \mathfrak{g}_\mathbb{C}^\lambda$ such that $Y \stackrel{\text{def}}{=} (\operatorname{ad} X - \lambda\cdot\mathbf{1})^{n-1} Y_0 \neq 0$. Hence $(\operatorname{ad} X)Y = \lambda\cdot Y$ and $(\operatorname{ad} X)^n Y = \lambda^n\cdot Y \neq 0$ for all $m \in \mathbb{N}$. But $X \in \mathcal{C}^\infty$ and thus $\operatorname{ad} X$ is nilpotent, i.e., there is a natural number m with $(\operatorname{ad} X)^m = 0$. This contradiction proves the claim. ∎

1.3. Lemma. *Under the circumstances of* Lemma 1.2, *$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{e}$ with a suitable vector space complement $\mathfrak{h}_1$ for $\mathfrak{e} \in \mathfrak{h}$. Every element $X \in \mathfrak{h}$ then decomposes uniquely as $X = X_1 + Y$ with $X_1 \in \mathfrak{h}_1$ and $Y \in \mathfrak{e}$. Also, X is regular if and only if X_1 is regular. In particular, the regular elements of $\mathfrak{h}_1$ are dense in $\mathfrak{h}_1$.*

Proof. By Lemma 1.2 we have $\lambda(X) = \lambda(X_1)$ for every root λ. Since $X \in \mathfrak{h}$ is regular iff $\lambda(X) \neq 0$ for all roots λ, the assertion follows. ∎

1.4. Lemma. (i) *If $X \in \mathfrak{g}$ is such that* $\operatorname{ad} X$ *is nilpotent as well as semisimple, then X is central.*

(ii) *If $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ and $S \subseteq \mathfrak{g}$ is the set of all elements X with* $\operatorname{ad} X$ *semisimple, then $S \cap \mathfrak{n} = \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$.*

Proof. (i) If $\operatorname{ad} X$ is semisimple and nilpotent, then $\operatorname{ad} X = 0$ and thus X is central.

(ii) Since $X \in \mathfrak{n}$ implies that $\operatorname{ad} X$ is nilpotent, by (i) we conclude that $S \cap \mathfrak{n} \subseteq \mathfrak{z}$. Conversely, $X \in \mathfrak{z}$ implies $\operatorname{ad} X = 0$. In particular, $X \in S$. But $\mathfrak{z} \subseteq \mathfrak{n}$ and thus $\mathfrak{z} \subseteq S \cap \mathfrak{n}$. ∎

Note $\mathcal{C}^\infty \subseteq \mathfrak{n}$.

2. Compact elements in solvable algebras

2.1. Definition. (i) Set

$$\text{(2)} \qquad \operatorname{comp}(\mathfrak{g}) = \{X \in \mathfrak{g} : \operatorname{Spec}(\operatorname{ad} X) \subseteq i\mathbb{R} \text{ and } \operatorname{ad} X \text{ is semisimple}\}.$$

If no confusion is possible we shall briefly write comp for this set.

(ii) We fix a Cartan subalgebra and set

$$\text{(3)} \qquad \mathfrak{t} = \mathfrak{h} \cap \operatorname{comp}.$$

∎

The elements of comp are precisely the elements $X \in \mathfrak{g}$ for which $e^{\mathbb{R}\cdot \operatorname{ad} X}$ is relatively compact in $\operatorname{Aut}(\mathfrak{g}) \subseteq \operatorname{Gl}(|\mathfrak{g}|)$. If $\mathfrak{g}$ is solvable and G is the simply connected Lie group associated with $\mathfrak{g}$ then $\exp \mathbb{R}{\cdot}X$ is isomorphic to $\mathbb{R}$ for all $X \in \mathfrak{g}$, in particular for those in comp. The adjoint group $\operatorname{Ad}(G)$ agrees with $\operatorname{Inn}(\mathfrak{g}) \subseteq \operatorname{Aut}(\mathfrak{g})$. Our arguments will remain entirely within $\mathfrak{g}$ and its group $\operatorname{Inn}(\mathfrak{g})$ of inner automorphisms.

Let $\Gamma = \langle e^{\operatorname{ad} \mathcal{C}^\infty} \rangle = e^{\operatorname{ad} \mathcal{C}^\infty} \subseteq \operatorname{Inn}(\mathfrak{g})$.

2.2. Lemma. (i) $\mathfrak{t}$ *is a central subalgebra of* $\mathfrak{h}$.

(ii) $\mathfrak{t}$ *contains the center* $\mathfrak{z}$ *of* $\mathfrak{g}$.

(iii) $\operatorname{comp} = \Gamma{\cdot}\mathfrak{t}$.

Proof. (i) Every compact element X in $\mathfrak{h}$ generates a semisimple $\operatorname{ad} X$ with purely imaginary spectrum. In particular, $\operatorname{ad} X|\mathfrak{h}$ is nilpotent and semisimple and thus X is central in $\mathfrak{h}$ by 1.4(i). It follows that the set $\mathfrak{t}$ is central in $\mathfrak{h}$. Clearly $\mathfrak{t}$ is closed under scalar multiplication. In order to show that $\mathfrak{t}$ is closed under addition, take $X_j \in \mathfrak{t}$, $j = 1, 2$. Then $[X_1, X_2] = 0$ and thus $[\operatorname{ad} X_1, \operatorname{ad} X_2] = \operatorname{ad}[X_1, X_2] = 0$. Hence

$$\overline{e^{\mathbb{R}\cdot \operatorname{ad}(X_1+X_2)}} \subseteq \overline{e^{\mathbb{R}\cdot \operatorname{ad} X_1}}\cdot\overline{e^{\mathbb{R}\cdot \operatorname{ad} X_2}}$$

is compact. Thus $X_1 + X_2 \in \operatorname{comp} \cap \mathfrak{h} = \mathfrak{t}$.

(ii) If $X \in \mathfrak{z}$, then $\operatorname{ad} X = 0$ whence $X \in \operatorname{comp}$. Also, $\mathfrak{z} \subseteq \mathfrak{h}$ since the center is contained in every Cartan algebra. Thus $\mathfrak{z} \subseteq \mathfrak{t}$.

(iii) The Cartan algebras of $\mathfrak{g}$ are conjugate under Γ. Every compact element element X of $\mathfrak{g}$ is semisimple, and thus is contained in a Cartan algebra of $\mathfrak{g}$ (namely, in each Cartan algebra of its centralizer $\mathfrak{z}(X, \mathfrak{g})$). Thus there is a $\gamma \in \Gamma$ such that $\gamma^{-1}X \in \mathfrak{h}$. Then $X \in \gamma{\cdot}\mathfrak{h} \cap \operatorname{comp} = \gamma(\mathfrak{h} \cap \operatorname{comp}) = \gamma{\cdot}\mathfrak{t}$. Thus $\operatorname{comp} \subseteq \Gamma{\cdot}\mathfrak{t}$. The converse is clear. ∎

The linear span of $\operatorname{comp}\mathfrak{g}$ is invariant under automorphisms and is therefore a fully characteristic ideal of $\mathfrak{g}$ (i.e, it is invariant under all derivations). We want to investigate the structure of this ideal and shall therefore assume until further notice that

$$\mathfrak{g} = \operatorname{span}(\operatorname{comp}\mathfrak{g}). \tag{4}$$

Let A denote the set of nonzero weights of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}^{\alpha}$, $\alpha \in A$ the respective weight spaces of $\mathfrak{g}_{\mathbb{C}}$. Now set $\mathfrak{t}^{+} = \mathfrak{g} \cap \bigoplus_{\alpha\in A} \mathfrak{g}_{\mathbb{C}}^{\alpha}$. This definition yields immediately the following remark:

2.3. Remark. The vector space $\mathfrak{t}^{+}$ is the same as the Fitting one component of $\operatorname{ad} X$ for any regular $X \in \mathfrak{t}$. For the action of the torus $\Omega = \overline{e^{\operatorname{ad}\mathfrak{t}}}$ on $\mathfrak{g}$, the vector space $\mathfrak{t}^{+}$ is precisely the span of all simple nonzero nontrivial modules. ■

Write $\mathfrak{z}(\mathfrak{t}, \mathfrak{g})$ for the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$.

2.4. Lemma. (i) $\mathfrak{g} = \mathfrak{t} + \mathcal{C}^{\infty}(\mathfrak{g})$ *and* $\mathcal{C}^{\infty} = \mathfrak{g}'$.

(ii) $\mathfrak{z}(\mathfrak{t}, \mathfrak{g}) = \mathfrak{h} = \mathfrak{t} + \mathfrak{e}$.

(iii) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}^{+}$.

(iv) $\mathcal{C}^{\infty} = \mathfrak{e} \oplus \mathfrak{t}^{+}$.

(v) $\langle \mathfrak{t}^{+} \rangle$ *is an ideal of* $\mathfrak{g}$.

Proof. (i) By Lemma 2.2(iii) we have $\operatorname{comp}(\mathfrak{g}) = \Gamma\cdot\mathfrak{t}$, and $\Gamma\cdot\mathfrak{t} = e^{\operatorname{ad}\mathcal{C}^{\infty}}\mathfrak{t} \subseteq \mathfrak{t} + \mathcal{C}^{\infty}$ since every summand except the first of the exponential series contains a factor from $\mathcal{C}^{\infty}$, and since $\mathcal{C}^{\infty}$ is an ideal. Thus $\mathfrak{g} = \operatorname{span}\operatorname{comp} \subseteq \mathfrak{t} + \mathcal{C}^{\infty} \subseteq \mathfrak{g}$ and the first assertion follows. In order to see the second we recall that $\mathcal{C}^{\infty} \subseteq \mathfrak{g}'$ always and that $\mathfrak{g}/\mathcal{C}^{\infty}$ is isomorphic to a subalgebra of $\mathfrak{t}$ by what we just saw. Thus $\mathfrak{g}/\mathcal{C}^{\infty}$ is abelian and hence $\mathcal{C}^{\infty} \supseteq \mathfrak{g}'$. ■

(ii) By Lemma 1.3 there is a regular element X of $\mathfrak{g}$ contained in $\mathfrak{t}$. Thus $\mathfrak{h}$ is the nilspace of $\operatorname{ad} X$. But $\operatorname{ad} X$ is semisimple, thus $\mathfrak{h} = \ker \operatorname{ad} X = \mathfrak{z}(X, \mathfrak{g}) \supseteq \mathfrak{z}(\mathfrak{t}, \mathfrak{g})$. Now $\mathfrak{h} \subseteq \mathfrak{z}(\mathfrak{t}, \mathfrak{g})$ by Lemma 2.2(i). This proves the first equality. By (i) We have $\mathfrak{g} = \mathfrak{t} + \mathcal{C}^{\infty}$ and thus $\mathfrak{h} = \mathfrak{h} \cap (\mathfrak{t} + \mathcal{C}^{\infty}) = \mathfrak{t} + (\mathfrak{h} \cap \mathcal{C}^{\infty}) = \mathfrak{t} + \mathfrak{e}$ by the modular law and the definition of $\mathfrak{e}$ in (1).

(iii) is the Fitting decomposition of $\operatorname{ad} X$ for a regular X inside $\mathfrak{t}$.

(iv) By (iii) we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{t}^{+}$. If X is regular and contained in $\mathfrak{t}$, then $\operatorname{ad} X|\mathfrak{t}^{+}$ is an automorphism. Thus $\mathfrak{t}^{+}$ is contained in every term of the descending central series and thus in $\mathcal{C}^{\infty}$. Hence $\mathcal{C}^{\infty} = (\mathfrak{h} + \mathfrak{t}^{+}) \cap \mathcal{C}^{\infty} = (\mathfrak{h} \cap \mathcal{C}^{\infty}) + \mathfrak{t}^{+} = \mathfrak{e} + \mathfrak{t}^{+}$ by the modular law and the definition of $\mathfrak{e}$ again.

(v) We note that $[\mathfrak{h}, \mathfrak{t}^{+}] \subseteq \mathfrak{t}^{+}$. Therefore $[\mathfrak{h}, \langle \mathfrak{t}^{+} \rangle] \subseteq \langle \mathfrak{t}^{+} \rangle$. Hence $\langle \mathfrak{t}^{+} \rangle$ is an $\mathfrak{h} + \mathfrak{t}^{+}$-module and thus a $\mathfrak{g}$-module in view of (iii). Hence $\langle \mathfrak{t}^{+} \rangle$ is an ideal. ■

We shall set $L[\Gamma] \overset{\text{def}}{=} \{X \in \mathfrak{g} : e^{\mathbb{R}\cdot\operatorname{ad} X} \subseteq \Gamma\}$.

2.5. Lemma. (i) *If A is a subgroup of a Lie group B, then $A = \langle \exp \mathfrak{a} \rangle$ holds for a Lie algebra $\mathfrak{a} \subseteq \mathfrak{b} = L(B)$ if and only if A is analytic and $\mathfrak{a} = L(A)$.*

(ii) *If $\mathfrak{v} \subseteq \mathfrak{b}$ is a vector space then $A = \langle \exp \mathfrak{v} \rangle$ if an only if $\langle \mathfrak{v} \rangle = \mathfrak{a}$.*

(iii) $\Gamma = \langle e^{\operatorname{ad} \mathfrak{t}^+} \rangle$.

(iv) $\mathcal{C}^\infty = \langle \mathfrak{t}^+ \rangle$.

(v) $L[\Gamma] = \mathfrak{z} + \langle \mathfrak{t}^+ \rangle$.

Proof. (i) The "if" portion is clear. Suppose now that $A = \langle \exp \mathfrak{a} \rangle$. Then A is pathwise connected and hence is analytic by the Theorem of Yamabe and Gôto. Hence there is a unique intrinsic Lie group topology on A making A into a Lie group A_ℓ with exponential function $\exp | L(A) \colon L(A) \to A_\ell$. Here $X \in L(A)$ iff $\exp \mathbb{R}{\cdot}X \subseteq A$. This implies $\mathfrak{a} \subseteq L(A)$. Assume for the moment that $\mathfrak{a} \neq L(A)$. Then the analytic subgroup generated by $\mathfrak{a}$ in the Lie group A_ℓ is proper in A_ℓ, since proper subalgebras generate proper analytic subgroups. On the other hand this subgroup is $\langle \exp \mathfrak{a} \rangle = A$. This is a contradiction since $A = A_\ell$ as sets. Hence $\mathfrak{a} = L(A)$ follows and the claim is proved.

(ii) Firstly, suppose $A = \langle \exp \mathfrak{v} \rangle$. Then $X \in \mathfrak{v}$ implies $\exp \mathbb{R}{\cdot}X \subseteq \langle \exp \mathfrak{v} \rangle = A$ and this implies $X \in L(A) = \mathfrak{a}$. It follows that $\langle \mathfrak{v} \rangle \subseteq \mathfrak{a}$. Then $A = \langle \exp \mathfrak{v} \rangle \subseteq \langle \exp \langle \mathfrak{v} \rangle \rangle \subseteq \langle \exp \mathfrak{a} \rangle = A$. Thus $A = \langle \exp \langle \mathfrak{v} \rangle \rangle$ and hence, by (i) we conclude $\langle \mathfrak{v} \rangle = \mathfrak{a}$ as asserted.

Secondly suppose that $\langle \mathfrak{v} \rangle = \mathfrak{a}$. Then $\langle \exp \mathfrak{v} \rangle \subseteq \langle \exp \langle \mathfrak{v} \rangle \rangle = \langle \exp \mathfrak{a} \rangle = A$. On the other hand, $\mathfrak{v} \subseteq L(\langle \exp \mathfrak{v} \rangle)$ by the definition of the right hand side, and thus $\mathfrak{a} = \langle \mathfrak{v} \rangle \subseteq L(\langle \exp \mathfrak{v} \rangle)$ and therefore $A = \langle \exp \mathfrak{a} \rangle \subseteq \langle \exp L \langle \exp \mathfrak{v} \rangle \rangle \rangle = \langle \exp \mathfrak{v} \rangle$ by (i).

(iii) Clearly $\Gamma_1 \overset{\text{def}}{=} \langle e^{\operatorname{ad} \mathfrak{t}^+} \rangle$ is contained in Γ and is normal by 2.4(v). If $X \in \mathfrak{t}$, then the centralizer of X contains $\mathfrak{e}$ as $[\mathfrak{t}, \mathfrak{e}] = \{0\}$ by 2.2(i). Thus for the isotropy group Γ_X at X we have $\mathfrak{e} \subseteq L[\Gamma_X]$. Hence $\Gamma = \Gamma_1 \Gamma_X$. If $\gamma \in \Gamma$, then $\Gamma = \gamma \Gamma \gamma^{-1} = \Gamma_1 \Gamma_{\gamma X}$. Thus $\Gamma = \Gamma_1 \Gamma_X$ for all $X \in \operatorname{comp}$ by 2.2(iii). Hence $\Gamma{\cdot}X = \Gamma_1.X$ for all $X \in \operatorname{comp}$. But $e^{\operatorname{ad} \langle \mathfrak{t}^+ \rangle} X \subseteq X + \langle \mathfrak{t}^+ \rangle$ by 2.4(v). Thus $\operatorname{comp} \subseteq \mathfrak{t} + \langle \mathfrak{t}^+ \rangle$ and hence $\mathfrak{g} = \operatorname{span} \operatorname{comp} \subseteq \mathfrak{t} + \langle \mathfrak{t}^+ \rangle$. Thus $\mathfrak{g}/\langle \mathfrak{t}^+ \rangle$ is an abelian algebra isomorphic to a subalgebra of $\mathfrak{t}$ and thus $\mathcal{C}^\infty = \mathfrak{g}' \subseteq \langle \mathfrak{t}^+ \rangle$ in view of 2.4(i). We conclude $\mathcal{C}^\infty = \langle \mathfrak{t}^+ \rangle$ and thus $\Gamma = e^{\operatorname{ad} \langle \mathfrak{t}^+ \rangle}$. The Lie algebra generated by $\operatorname{ad} \mathfrak{t}^+$ in $\operatorname{Der} \mathfrak{g}$ is $\operatorname{ad} \langle \mathfrak{t}^+ \rangle$. By (ii) we obtain $\Gamma_1 = \langle e^{\operatorname{ad} \mathfrak{t}^+} \rangle = \exp^{\langle \operatorname{ad} \mathfrak{t}^+ \rangle} = \Gamma$.

(iv) was proved in the process of proving (iii).

(v) The analytic subgroup Γ of $\operatorname{Aut} \mathfrak{g}$ has the Lie algebra $\operatorname{ad} \langle \mathfrak{t}^+ \rangle$ by (iii) and (i) above. Hence the Lie algebra $L[\Gamma]$ is exactly $\langle \mathfrak{t}^+ \rangle + \mathfrak{z}$. ∎

2.6. Lemma. *Let $\mathfrak{j}$ denote the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{e}$. Then*

$$\mathfrak{j} = \mathfrak{z} \cap \mathfrak{e} = \mathfrak{z} \cap \mathfrak{g}' = \mathfrak{t} \cap \mathfrak{e} = \mathfrak{t} \cap \mathfrak{g}'. \tag{5}$$

Proof. Firstly, as $\mathfrak{e} = \mathfrak{h} \cap \mathfrak{g}'$ we have $\mathfrak{z} \cap \mathfrak{e} = \mathfrak{z} \cap \mathfrak{h} \cap \mathfrak{g}' = \mathfrak{z} \cap \mathfrak{g}'$ and $t \cap \mathfrak{e} = \mathfrak{t} \cap \mathfrak{h} \cap \mathfrak{g}' = \mathfrak{t} \cap \mathfrak{g}'$.

Secondly, since $\mathfrak{j} \subseteq \mathfrak{e} \subseteq \mathfrak{h}$ and $\mathfrak{t}^+$ is an $\mathfrak{h}$-module, we have $[\mathfrak{j}, \mathfrak{t}^+] \subseteq \mathfrak{j} \cap \mathfrak{t}^+ \subseteq \mathfrak{e} \cap \mathfrak{t}^+ = \{0\}$. It follows that $[\mathfrak{j}, \langle \mathfrak{t}^+ \rangle] = [\mathfrak{t}, \mathfrak{g}'] = \{0\}$. Since $\mathfrak{t}$ is central in $\mathfrak{h}$ we also know $[\mathfrak{j}, \mathfrak{t}] = \{0\}$. Then $[\mathfrak{j}, \mathfrak{g}] = [\mathfrak{j}, \mathfrak{t} + \mathfrak{g}'] = \{0\}$, i.e., $\mathfrak{j} \subseteq \mathfrak{z}$. Thus $\mathfrak{j} \subseteq \mathfrak{z} \cap \mathfrak{g}'$, and since $\mathfrak{z} \cap \mathfrak{g}'$ is an ideal in $\mathfrak{g}$, the equality $\mathfrak{j} = \mathfrak{z} \cap \mathfrak{g}'$ follows. Thirdly,

$\mathfrak{z} \subseteq \mathfrak{t}$. Hence $\mathfrak{z} \cap \mathfrak{g}' \subseteq \mathfrak{t} \cap \mathfrak{g}'$. On the other hand, $X \in \mathfrak{t} \cap \mathfrak{g}'$ implies that $\operatorname{ad} X$ is both nilpotent and semisimple. Hence $X \in \mathfrak{z}$ by Lemma 1.4. Thus $\mathfrak{t} \cap \mathfrak{g}' \subseteq \mathfrak{z}$. Hence $\mathfrak{z} \cap \mathfrak{g}' = \mathfrak{t} \cap \mathfrak{e}$. ∎

We also need the following lemmas for the purpose of induction with respect to dimension.

2.7. Lemma. *Let* $\mathfrak{a}$ *denote an ideal of* $\mathfrak{g}$ *and write* $\mathfrak{g}_1 = \mathfrak{g}/\mathfrak{a}$. *Then* $\mathfrak{g}_1$ *is spanned by* $\operatorname{comp}(\mathfrak{g}_1)$.

Proof. We know that $\mathfrak{h}_1$ is a Cartan algebra of $\mathfrak{g}_1$. Furthermore, all elements of $(\mathfrak{t} + \mathfrak{a})/\mathfrak{a}$ are semisimple and have purely imaginary spectrum, hence are contained in $\mathfrak{t}_1 = \operatorname{comp} \mathfrak{h}_1$. Since $\mathfrak{g}$ is spanned by $\operatorname{comp} = \Gamma{\cdot}\mathfrak{t}$ and then $\mathfrak{g}_1$ is spanned by $\operatorname{comp}(\mathfrak{g}_1) = \Gamma_1{\cdot}\mathfrak{t}_1$. ∎

2.8. Lemma. *Let* $\mathfrak{a}$ *be a nonzero minimal ideal (which exists as* $\mathfrak{g}$ *is solvable).*
(i) *Then there are two mutually exclusive cases:*

(a) $[\mathfrak{t}, \mathfrak{a}] = \{0\}$. *Then* $\mathfrak{a}$ *is central and* $\dim \mathfrak{a} = 1$.

(b) $[\mathfrak{t}, \mathfrak{a}] \neq \{0\}$. *Then* $\mathfrak{a} \subseteq \mathfrak{t}^+$ *and* $\dim \mathfrak{a} = 2$.

in both cases, $[\mathfrak{g}', \mathfrak{a}] = \{0\}$.

(ii) $\mathfrak{z} \cap \mathfrak{g}' = \{0\}$ *if and only if all nonzero minimal ideals are of type* (b). *In this case,* $\mathfrak{h}$ *is abelian.*

Proof. We prove (i): Case (a): We know $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{t}, \mathfrak{g}) = \mathfrak{h}$. Also $[\mathfrak{a}, \mathfrak{t}^+] \subseteq \mathfrak{a} \cap \mathfrak{t}^+ \subseteq \mathfrak{h} \cap \mathfrak{t}^+ = \{0\}$. Then also $[\mathfrak{a}, \mathfrak{g}'] = [\mathfrak{a}, \langle \mathfrak{t}^+ \rangle] = \{0\}$ whence $\{0\} = [\mathfrak{a}, \mathfrak{t} + \mathfrak{g}'] = [\mathfrak{a}, \mathfrak{g}]$, i.e., $\mathfrak{a} \subseteq \mathfrak{z}$.

Case (b): We briefly consider the compact connected abelian group $\Omega \overset{\text{def}}{=} \overline{e^{\operatorname{ad} \mathfrak{t}}}$. Then $\mathfrak{g}$ is an Ω-module. Its unique fixed point module is $\mathfrak{h} = \mathfrak{z}(\mathfrak{t}, \mathfrak{g})$. Its unique effective submodule (spanned by all nontrivial simple submodules) is $\mathfrak{t}^+$. Now $\mathfrak{a}$ is an Ω-module. Hence $\mathfrak{a}$ is of the form $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{h}) \oplus (\mathfrak{a} \oplus \mathfrak{t}^+)$. The assumption $[\mathfrak{t}, \mathfrak{a}] \neq \{0\}$ implies $\mathfrak{a} \oplus \mathfrak{t}^+ \neq \{0\}$. Since Ω is connected, any nonzero submodule of the effective module $\mathfrak{t}^+$ is at least 2-dimensional. But $\mathfrak{a}$, as a minimal ideal of $\mathfrak{g}$, is at most two dimensional. Hence $\mathfrak{a} \cap \mathfrak{t}^+ = \mathfrak{a}$ and thus $\mathfrak{a} \subseteq \mathfrak{t}^+ \subseteq \mathfrak{g}'$. In this case, $\dim \mathfrak{a} = 2$.

In order to show $[\mathfrak{g}', \mathfrak{a}] = \{0\}$ consider the representation $\pi\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{a}) \cong \mathfrak{gl}(2, \mathbb{R})$, $\pi(X)(Y) = [X, Y]$. It image $\pi(\mathfrak{g})$ contains $\pi(\mathfrak{t}) \cong \mathfrak{so}(2) \subseteq \mathfrak{gl}(2, \mathbb{R})$ and is solvable. But $\mathbb{R}{\cdot}E_2 \oplus \mathfrak{so}(2) = \{\begin{pmatrix} r & s \\ -s & r \end{pmatrix} : r, s \in \mathbb{R}\}$, the unique Cartan algebra containing $\mathfrak{so}(2)$ is also a unique Borel subalgebra (i.e., maximal solvable subalgebra) containing $\mathfrak{so}(2)$. Thus $\pi(\mathfrak{g})$ is contained in an abelian algebra and thus $\pi(\mathfrak{g}') = 0$. This means $[\mathfrak{g}', \mathfrak{a}] = \{0\}$.

(ii) If $\mathfrak{j} = \mathfrak{z} \cap \mathfrak{g}' \neq \{0\}$ then every one-dimensional vector subspace $\mathfrak{a}$ of $\mathfrak{j}$ is a nonzero minimal ideal of type (a). Conversely, if those minimal ideals exist, then $\mathfrak{j} \neq \{0\}$ by (i).

Now we assume $\mathfrak{j} = \{0\}$ and show $[\mathfrak{h}, \mathfrak{h}] = \{0\}$. We prove the claim by induction. The situation is known through simple inspection up to dimension 3. We assume that the claim is proven for all dimensions below $n > 3$ and consider $\mathfrak{g}$ with $\dim\mathfrak{g} = n$. Let $\mathfrak{a}$ denote a nonzero minimal ideal. Since $\mathfrak{j} = \{0\}$ then $\mathfrak{a}$ is of type (b). In particular, $\mathfrak{a} \subseteq \mathfrak{t}^+ \subseteq \mathfrak{g}'$. From Lemma 2.7 we know that $\mathfrak{g}_1 \overset{\rm def}{=} \mathfrak{g}/\mathfrak{a}$ satisfies our general hypothesis to be spanned by its compact elements. Suppose that $\mathfrak{j}_1$ is the maximal ideal of $\mathfrak{g}_1$ contained in $\mathfrak{h}_1 \cap \mathfrak{g}_1' = \frac{\mathfrak{h}+\mathfrak{a}}{\mathfrak{a}} \cap \frac{\mathfrak{g}'}{\mathfrak{a}} = \frac{(\mathfrak{h}+\mathfrak{a})\cap\mathfrak{g}'}{\mathfrak{a}} = \frac{(\mathfrak{h}\cap\mathfrak{g}')+\mathfrak{a}}{\mathfrak{a}} = \frac{\mathfrak{j}\oplus\mathfrak{a}}{\mathfrak{a}}$ (by the modular law). Now $\mathfrak{j}$ was assumed to be $\{0\}$. Hence $\mathfrak{j}_1 = \{0\}$. Thus the induction hypothesis applies to $\mathfrak{g}/\mathfrak{a}$ and shows that the Cartan algebra $\frac{\mathfrak{h}+\mathfrak{a}}{\mathfrak{a}}$ is abelian. Thus $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{a}$. But then $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}\cap\mathfrak{a} = \{0\}$. The induction is complete. ∎

2.9. Lemma. *The Cartan algebra $\mathfrak{h}$ satisfies $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{z}\cap\mathfrak{g}'$. In particular, $\mathfrak{h}$ is nilpotent of class ≤ 2.*

Proof. Recall $\mathfrak{j} = \mathfrak{z}\cap\mathfrak{g}'$. Consider $\mathfrak{g}_1 = \mathfrak{g}/\mathfrak{j}$. Let $\mathfrak{j}^*$ contain $\mathfrak{j}$ and be such that $\mathfrak{j}^*/\mathfrak{j} = \mathfrak{z}(\mathfrak{g}_1)\cap\mathfrak{g}_1' \subseteq \mathfrak{h}_1\cap\mathfrak{g}_1' = \frac{\mathfrak{h}}{\mathfrak{j}} \cap \frac{\mathfrak{g}'}{\mathfrak{j}} = \frac{\mathfrak{h}\cap\mathfrak{g}'}{\mathfrak{j}} = \mathfrak{e}/\mathfrak{j}$. Thus $\mathfrak{j}^*$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{e}$. Then $\mathfrak{j}^* \subseteq \mathfrak{j}$ by 2.6 and thus $\mathfrak{j}^* = \mathfrak{j}$. Thus $\mathfrak{g}_1$ satisfies the hypotheses of 2.8(ii) and thus $[\mathfrak{h}_1,\mathfrak{h}_1] = \{0\}$. Hence $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{j}$. ∎

We collect some of the information which was accumulated so far:

2.10. Proposition. *Let $\mathfrak{g}$ be a real solvable Lie algebra which is spanned by its compact elements. Fix a Cartan algebra $\mathfrak{h}$ and set $\mathfrak{t} = \mathfrak{h}\cap\mathrm{comp}(\mathfrak{g})$. Then we have*

(i) $[\mathfrak{t},\mathfrak{h}] = \{0\}$,

(ii) $\mathfrak{g}' = \mathcal{C}^\infty$,

(iii) $\mathfrak{z}\cap\mathfrak{g}'$ *is the largest ideal contained in* $\mathfrak{h}\cap\mathfrak{g}'$, *and*

(iii) $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{z}\cap\mathfrak{g}'$. *In particular,* $\mathfrak{h}$ *is nilpotent of class* ≤ 2. ∎

M. WÜSTNER has constructed a 13-dimensional solvable Lie algebra spanned by its compact elements in which $\mathfrak{h} \cong \mathbb{R}^2 \oplus H_3$ with the three-dimensional Heisenberg algebra H_3 so that Part (iii) of Lemma 2.9 cannot be improved in that $\mathfrak{h}$ need not be abelian.

The structure of a solvable algebra $\mathfrak{g}$ spanned by comp therefore may be roughly summarized in the following diagram:

<table>
<tr><td colspan="5">$\mathfrak{g}$</td></tr>
<tr><td colspan="2"></td><td colspan="3">$\mathfrak{g}' = \mathcal{C}^\infty = \langle\mathfrak{t}^+\rangle$</td></tr>
<tr><td colspan="4">$\mathfrak{h}$</td><td>$\mathfrak{t}^+$</td></tr>
<tr><td colspan="2"></td><td colspan="2">$\mathfrak{e} = \mathfrak{h}\cap\mathfrak{g}'$</td><td></td></tr>
<tr><td colspan="3">$\mathfrak{t}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="2">$\mathfrak{z}$</td><td colspan="2"></td></tr>
<tr><td colspan="2"></td><td>$\mathfrak{j} = \mathfrak{z}\cap\mathfrak{g}'$</td><td colspan="2"></td></tr>
<tr><td>$\mathfrak{t}_1$</td><td>$\mathfrak{t}_2$</td><td>$\mathfrak{j}$</td><td>$\mathfrak{e}_1$</td><td>$\mathfrak{t}^+$</td></tr>
</table>

The following remarks will be used for purposes of induction:

2.11. Lemma. *Let $\mathfrak{a}$ denote an ideal of $\mathfrak{g}$. Set*

$$\Gamma_1 = \langle e^{\mathrm{ad}\,\mathfrak{g}_1'}\rangle \quad \textit{with } \mathfrak{g}_1' = (\mathfrak{g}' + \mathfrak{a})/\mathfrak{a}.$$

(i) *There is a surjective homomorphism $\varphi\colon \Gamma \to \Gamma_1$ with $\varphi(\gamma)(X+\mathfrak{a}) = \gamma(X) + \mathfrak{a}$. and $\ker\varphi = \{\gamma \in \Gamma\colon (\forall X \in \mathfrak{g})\gamma(X) - X \in \mathfrak{a}\}$. In particular, $\ker\varphi \supseteq \overline{\langle e^{\mathrm{ad}\,\mathfrak{a}}\rangle}$ with the closure taken in Γ.*

(ii) *The orbit $\Gamma_1{\cdot}(X+\mathfrak{a})$ equals $(\Gamma{\cdot}X + \mathfrak{a})/\mathfrak{a}$.*

Proof. (i) The assignment $\gamma \mapsto (X+\mathfrak{a} \mapsto \gamma(X)+\mathfrak{a})\colon \mathrm{Inn}(\mathfrak{g}) \to \mathrm{Inn}(\mathfrak{g}_1)$ is a well defined homomorphism whose kernel consists of all those γ with $\gamma(X) - X \in \mathfrak{a}$ for all $X \in \mathfrak{g}$.

Now let $\gamma = e^{\mathrm{ad}\,X}$ with $X \in \mathfrak{g}'$. Let $\gamma_1 = e^{\mathrm{ad}(X+\mathfrak{a})} \in \Gamma_1$. Take $Y \in \mathfrak{g}$ and notice $\varphi\big(\gamma(Y+\mathfrak{a})\big) = \gamma(Y)+\mathfrak{a} = e^{\mathrm{ad}\,X}Y+\mathfrak{a} = Y+[X,Y]+\frac{1}{2}[X,[X,Y]]+\cdots+\mathfrak{a}$ on the one hand and $\gamma_1(Y+\mathfrak{a}) = e^{\mathrm{ad}(X+\mathfrak{a})}(Y+\mathfrak{a}) = Y+\mathfrak{a}+[X+\mathfrak{a},Y+\mathfrak{a}] + \frac{1}{2}[X+\mathfrak{a},[X+\mathfrak{a},Y+\mathfrak{a}]]+\cdots = Y+[X,Y]+\frac{1}{2}[X,[X,Y]]+\cdots+\mathfrak{a}$ on the other. Hence $\gamma_1 = \varphi(\gamma)$. Thus $\varphi(\Gamma) \subseteq \Gamma_1$. Since Γ_1 is generated by $e^{\mathrm{ad}\,\mathfrak{g}_1} = \varphi(e^{\mathrm{ad}\,\mathfrak{g}'})$, we have $\varphi(\Gamma) = \Gamma_1$.

If $X \in \mathfrak{a}$, then $e^{\mathrm{ad}\,X}Y - Y = [X,Y] + \cdots \in \mathfrak{a}$.

(ii) Finally, for $\gamma_1 \in \Gamma_1$ we find $\gamma \in \Gamma$ with $\gamma_1 = \varphi(\gamma)$. Then $\gamma_1{\cdot}(X+\mathfrak{a}) = \varphi(\gamma){\cdot}(X+\mathfrak{a}) = \gamma{\cdot}X + \mathfrak{a}$. Hence $\Gamma_1{\cdot}(X+\mathfrak{a}) = (\Gamma{\cdot}X+\mathfrak{a})/\mathfrak{a}$. ∎

We consider the semidirect product $\mathbb{R}^2 \rtimes D$ of a plane with a compact connected group D acting irreducibly on $\mathbb{R}^2$. Let $\mathfrak{g} = \mathbb{R}^2 \rtimes \mathfrak{d}$ denote the Lie algebra and $X = (0,x)$ a vector acting nontrivially on $\mathbb{R}^2$. Let $Y = (v,0)$. Then in the subalgebra $\mathbb{R}^2 \rtimes \mathbb{R}{\cdot}x$ the subspaces $\mathbb{R}{\cdot}X = \{0\} \times \mathbb{R}{\cdot}x$ and $\mathbb{R}{\cdot}(X-Y) = \mathbb{R}.(-v,x)$ are Cartan subalgebras and hence are conjugate under $e^{\mathrm{ad}\,\mathbb{R}^2\times\{0\}}$. In particular, there is a $g \in \mathbb{R} \times \{\mathbf{1}\} \subseteq G$ with $\mathrm{Ad}(g)\mathbb{R}.(X-Y) = \mathbb{R}{\cdot}X$.

2.12. Lemma. *Let $X \in \mathrm{comp}\,\mathfrak{g}$ be regular and $\mathfrak{a}$ a minimal abelian ideal such that $[X,\mathfrak{a}] \neq \{0\}$. Then $X - Y \in \mathrm{comp}(\mathfrak{g})$ for all $Y \in \mathfrak{a}$.*

Proof. The element $\mathrm{ad}\,X|\mathfrak{a}$ is semisimple and has nonzero purely imaginary spectrum. Thus the subalgebra $\mathbb{R}{\cdot}X \oplus \mathfrak{a}$ is invariant under the action of the group $e^{\mathbb{R}\cdot\,\mathrm{ad}\,X}$ which transitively permutes the the one-dimensional subspaces of $\mathfrak{a}$. Hence it permutes transitively the one parameter subgroups of the analytic subgroup $A \overset{\mathrm{def}}{=} e^{\mathrm{ad}\,\mathfrak{a}}$ under conjugation. This subgroup therefore is either closed or relatively compact. In the second case, its closure would be a compact normal abelian subgroup of $\overline{\mathrm{Inn}(\mathfrak{g})}$ and would therefore be central, which is not the case on account of the nontrivial action of X on $\mathfrak{a}$. Thus A is closed in $\overline{\mathrm{Inn}\,\mathfrak{g}}$ and isomorphic to $\mathbb{R}^2$. The group $D \overset{\mathrm{def}}{=} \overline{e^{\mathbb{R}\cdot\,\mathrm{ad}\,X}} \subseteq \mathrm{int}(\mathfrak{g})$ is compact and contained in the group $M \overset{\mathrm{def}}{=} DA \subseteq \overline{\langle e^{\mathbb{R}\cdot X\oplus\mathfrak{a}}\rangle} \subseteq \mathrm{int}(\mathfrak{g})$. The product DA is semidirect. Now let $Y \in \mathfrak{a}$. Consider the subgroup $D_1 \overset{\mathrm{def}}{=} \overline{e^{\mathbb{R}\cdot\,\mathrm{ad}(X-Y)}}$. Since $\mathbb{R}{\cdot}(X-Y)$ is conjugate to $\mathbb{R}{\cdot}X$ under the action of A by the remarks preceding this lemma, we know that D_1 is conjugate to D in $\mathrm{Inn}(\mathfrak{g})$. Thus D_1 is compact. Hence $X - Y \in \mathrm{comp}(\mathfrak{g})$ as asserted. ∎

3. On some group actions

We let $\mathfrak{g}$ denote a real Lie algebra and consider a mixture of actions by translation and inner automorphisms. If Γ is a closed subgroup of $\mathrm{Inn}(\mathfrak{g})$, as before, we let $L[\Gamma]$ denote the unique Lie algebra within $\mathfrak{g}$ such that $X \in L[\Gamma]$ means $e^{\mathbb{R}\cdot X} \subseteq \Gamma$. Then $L[\Gamma]/\mathfrak{z} \cong L(\Gamma)$ is the Lie algebra of Γ. We identify the Lie algebra of Γ with $L[\Gamma]/\mathfrak{z}$.

3.1. Lemma. *Let* $\Gamma \subseteq \mathrm{Inn}(\mathfrak{g})$ *be a subgroup and* $\mathfrak{a}$ *a subalgebra invariant under* Γ. *We let* $\mathfrak{a}\rtimes\Gamma$ *denote the semidirect product Lie group with multiplication* $(Z_1, \gamma_1)(Z_2, \gamma_2) = (Z_1 + \gamma_1(Z_2), Z_1 + Z_2)$ *and we set*

$$(Z, \gamma)\cdot X = Z + \gamma(X) \text{ for } \gamma \in \Gamma,\ Z \in \mathfrak{a},\ X \in \mathfrak{g}.$$

Then $\big((Z, \gamma)\, X\big) \mapsto (Z, \gamma)\cdot X : (\mathfrak{a}\rtimes\Gamma) \times \mathfrak{g} \to \mathfrak{g}$ *is an action.*

Proof. If $(0, \eta)$ is the identity of $\mathfrak{a}\rtimes\Gamma$ then $(0, \eta)\cdot X = 0 + \eta(X) = X$ and $(Z_1, \gamma_1)\cdot\big(Z_2, \gamma_2)\cdot X\big) = Z_1 + \gamma_1(Z_2 + \gamma_2(X)) = Z_1 + \gamma_1(Z_2) + \gamma_1\gamma_2(X) = \big((Z_1, \gamma_1)(Z_2, \gamma_2)\big)\cdot X$. ∎

3.2. Definition. We call the action described in the preceding lemma *the affine action of* $\mathfrak{a}\rtimes\Gamma$. ∎

3.3. Lemma. (i) *Let A and B two subgroups of a group C. Then $A \times B$ acts on C on the left via* $(a, b)\cdot c = acb^{-1}$.

The function $a \mapsto (a, c^{-1}ac) : A \cap cBc^{-1} \to A \times B$ *implements an isomorphism onto the isotropy group* $(A \times B)_c$.

The function $(a, b) \mapsto acb\colon A \times B \to G$ *induces a bijection*

$$(A \times B)/\{(a, c^{-1}ac)\colon a \in A \cap cBc^{-1}\} \to AcB.$$

(ii) *If A, B, and C are Lie groups, then AcB is a submersed submanifold of dimension* $\dim L(A) + \dim L(B) - \dim(L(A) \cap \mathrm{Ad}(c)L(B))$

(iii) *If, under the hypotheses of* (ii),

$$\dim(L(A) \cap \mathrm{Ad}(c)L(B)) = \dim L(A) + \dim L(B) - \dim L(C) \text{ for all } c \in C,$$

then all AcB, $c \in C$ are open closed submanifolds of C. If C is connected, then $AcB = G$ for all $c \in C$.

(iv) *Suppose that C is a connected Lie group acting differentiably on a manifold M. Let B denote the isotropy subgroup of C at m. Then cBc^{-1} is the isotropy subgroup at $c\cdot m$. If A is a Lie subgroup such that*

$$(\dagger) \qquad \dim(L(A) \cap \mathrm{Ad}(c)L(B)) = \mathrm{const} \text{ for all } c \in C,$$

then $A\cdot m = C\cdot m$.

Proof. (i) The action is clear. We have $(a,b)\in(A\times B)_c$ iff $acb^{-1}=c$ iff $ac=cb$ iff $b=c^{-1}ac$. If this is satisfied, then $a=cbc^{-1}\in cBc^{-1}$, and conversely, if a satisfies this condition then $(a,c^{-1}ac)\in(A\times B)_c$.

(ii) The dimension of the submersed manifold AcB equals the dimension of the homogeneous space $(A\times B)/\{(a,c^{-1}ac)\colon a\in A\cap cBc^{-1}\}$ and its dimension is that of $(L(A)\times L(B))/\{(X,\mathrm{Ad}(c)^{-1}X)\colon X\in L(A)\cap \mathrm{Ad}(c)L(B)\}$. The assertion follows.

(iii) Under the present hypotheses, by (ii) all sets AcB are open submanifolds of C. Since they form the orbits decomposition of a group action, each of the sets AcB is also closed. The remainder is then clear.

(iv) There are bijective immersions $AB/B\to A{\cdot}m$ and $C/B\to C{\cdot}m$ and $AB/B\subseteq C/B$. The claim then follows if we establish $AB/B=C/B$. By (iii) we have $AcB=C$ for all $c\in C$, in particular $AB=C$. The assertion follows. ■

For the following discussion it is useful to recall some general facts on the smooth action of Lie groups on manifolds.

Let $(g,m)\mapsto g{\cdot}m : G\times M\to M$ denote a smooth action of a Lie group on a manifold M. Let $X\mapsto \widetilde{X}\colon\mathfrak{g}\to V(M)$ denote the map which associates with an $X\in\mathfrak{g}$ the G-invariant vector field $\widetilde{X}\colon M\to T(M)$ on M which is given by $\widetilde{X}(x)=\frac{d}{dt}\big|_{t=0}\big(\exp(t{\cdot}X){\cdot}x\big)$ for $x\in M$. Then the integral manifolds of $\widetilde{X}$ are exactly the orbits of the action $(r,x)\mapsto(\exp r{\cdot}X){\cdot}x:\mathbb{R}\times M\to M$. Thus m is a fixed point for this action iff $\widetilde{X}(m)=0$. As a consequence, we obtain

3.4. Lemma. *Let $H=G_m$ denote the isotropy group of G at m. Then $X\in\mathfrak{h}=L(H)$ if and only if $\frac{d}{dt}\big|_{t=0}\big(\exp(t{\cdot}X){\cdot}m\big)=0$.* ■

3.5. Lemma. *Suppose that Γ is an analytic subgroup of the group* $\mathrm{Inn}(\mathfrak{g})$ *of inner automorphisms of $\mathfrak{g}$ and $\mathfrak{a}$ a subalgebra invariant under Γ and let $\mathfrak{a}\rtimes\Gamma$ act on $\mathfrak{g}$ by the affine action. (See* Definition 3.2.*)*

(i) *Let $(\mathfrak{a}\rtimes\Gamma)_X$ denote the isotropy group of the affine action at X. Then*

$$L\big((\mathfrak{a}\rtimes\Gamma)_X\big)=\{(Z,Y+\mathfrak{z})\in\mathfrak{a}\rtimes L(\Gamma)=\mathfrak{a}\rtimes(L[\Gamma]/\mathfrak{z}) : [X,Y]=Z\}.$$

(ii) *The orbit $(\mathfrak{a}\rtimes\Gamma){\cdot}X$ is a submersed submanifold whose tangent space at X is $\mathfrak{a}+[L[\Gamma],X]$ when the tangent space $T_X(\mathfrak{g})$ of $\mathfrak{g}$ at X is elementarily identified with $\mathfrak{g}$.*

Proof. (i) First we note that $L(\mathfrak{a})=\mathfrak{a}$ and $L(\mathfrak{a}\rtimes\Gamma)=\mathfrak{a}\rtimes L(\Gamma)=\mathfrak{a}\rtimes(L[\Gamma]/\mathfrak{z})$. For $Z\in\mathfrak{a}$, $Y\in L[\Gamma]$, and sufficiently small $t\in\mathbb{R}$ we have $(t{\cdot}Z,0)*(0,t{\cdot}Y+\mathfrak{z})=t{\cdot}(Z,Y+\mathfrak{z})+(\frac{t^2}{2}{\cdot}[Z,Y],0)+\mathrm{O}(t^3)$ and $\exp\big((t{\cdot}Z,0)*(0,t{\cdot}Z+\mathfrak{z})\big)=(t{\cdot}Z,e^{t\cdot\,\mathrm{ad}\,Y})$. Then

$$\begin{aligned}\exp(t{\cdot}(Z,Y+\mathfrak{z})){\cdot}X+\mathrm{O}(t^2)&=\exp(t{\cdot}(Z,Y+\mathfrak{z})+\mathrm{O}(t^2)){\cdot}X\\&=t{\cdot}Z+e^{t\cdot\,\mathrm{ad}\,Y}X=X+t{\cdot}(Z+[Y,X])+\mathrm{O}(t^2).\end{aligned}$$

Hence the assertion is a consequence of Lemma 3.4.

(ii) We recall $\exp(Z,Y+\mathfrak{z}){\cdot}X-X=t{\cdot}([Y,X]+Z)+O(t^2)$ and this implies the assertion. ■

3.6. Lemma. *Let* $\mathfrak{n}$ *denote a subalgebra of the nilradical of a Lie algebra* $\mathfrak{g}$ *and suppose that* $\mathfrak{n}$ *contains the center* $\mathfrak{z}$. *Set* $\Gamma = \langle e^{\mathrm{ad}\,\mathfrak{n}}\rangle$ *and let and* $\mathfrak{a}$ *be any ideal of* $\mathfrak{g}$. *We consider the Lie group* $\mathfrak{a}\rtimes\Gamma$, *the isotropy group* $(\mathfrak{a}\rtimes\Gamma)_X$ *of the affine action of* $\mathfrak{a}\rtimes\Gamma$ *at an element* X. *Then*

(i) $L(\{0\}\times\Gamma)\cap L\big((Y,\alpha)^{-1}(\mathfrak{a}\rtimes\Gamma)_X(Y,\alpha)\big)\cong(\mathfrak{z}(X-Y,\mathfrak{g})\cap\mathfrak{n})/\mathfrak{z}$ *for all* $(Y,\alpha)\in\mathfrak{a}\rtimes\Gamma$.

(ii) *Suppose now that, in addition,* $\mathfrak{n}$ *is an ideal,* X *a regular compact element of* $\mathfrak{g}$, *and* $\mathfrak{a}$ *a minimal ideal with* $[X,\mathfrak{a}]\neq\{0\}$. *Then for any Cartan algebra* $\mathfrak{h}$ *we have*

$$L(\{0\}\times\Gamma)\cap L\big((Y,\alpha)^{-1}(\mathfrak{a}\rtimes\Gamma)_X(Y,\alpha)\big)\cong(\mathfrak{h}\cap\mathfrak{n})/\mathfrak{z}\ \textit{for all}\ (Y,\alpha)\in\mathfrak{a}\rtimes\Gamma.$$

Proof. (i) We identify the Lie algebra of Γ with $L[\mathfrak{n}]/\mathfrak{z}$.

For $(Y,\alpha)\in\mathfrak{a}\rtimes\Gamma$ we have $(Y,\alpha)=(Y,\mathbf{1})(0,\alpha)$. Hence

$$\begin{aligned}
L(\{0\}\times\Gamma)\cap L\big((Y,\alpha)^{-1}(\mathfrak{a}\rtimes\Gamma)_X(Y,\alpha)\big)&=\\
L(\{0\}\times\Gamma)\cap L\big((0,\alpha)^{-1}(Y,\mathbf{1})^{-1}(\mathfrak{a}\rtimes\Gamma)_X(Y,\mathbf{1})(0,\alpha)\big)&=\\
L(\{0\}\times\Gamma)\cap \mathrm{Ad}(0,\alpha)^{-1}L\big((Y,\mathbf{1})^{-1}(\mathfrak{a}\rtimes\Gamma)_X(Y,\mathbf{1})\big)&\cong\\
\mathrm{Ad}(0,\alpha)L(\{0\}\times\Gamma)\cap L\big((Y,\mathbf{1})^{-1}(\mathfrak{a}\rtimes\Gamma)_X(Y,\mathbf{1})\big)&=\\
L\big((0,\alpha)(\{0\}\times\Gamma)(0,\alpha^{-1})\cap L\big((-Y,\mathbf{1})(\mathfrak{a}\rtimes\Gamma)_X(Y,\mathbf{1})\big)&=\\
L(\{0\}\times\Gamma)\cap L\big((-Y,\mathbf{1})(\mathfrak{a}\rtimes\Gamma)_X(Y,\mathbf{1})\big)&.
\end{aligned}$$

In order to determine the right portion of the intersection, we note first that $(Z,\gamma)\in(\mathfrak{a}\rtimes\Gamma)_X$ iff $Z+\gamma(X)=X$ iff $Z=X-\gamma(X)\in\mathfrak{a}$. Further, $(-Y,\mathbf{1})(\mathfrak{a}\rtimes\Gamma)_X(Y,\mathbf{1})=(\mathfrak{a}\rtimes\Gamma)_{(-Y,\mathbf{1})\cdot X}$ and $(-Y,1)\cdot X=X-Y$. Thus

$$(-Y,\mathbf{1})(\mathfrak{a}\rtimes\Gamma)(Y,\mathbf{1})=\{\big((X-Y)-\gamma(X-Y),\gamma\big):\gamma\in\Gamma\text{ and }\gamma(X-Y)\in X-Y+\mathfrak{a}\}.$$

In order to compute the Lie algebra

$$\mathfrak{b}\overset{\mathrm{def}}{=}L\big((-Y,\mathbf{1})(\mathfrak{a}\rtimes\Gamma)_X(Y,\mathbf{1})\big)=L(\mathfrak{a}\rtimes\Gamma)_{X-Y}$$

we want to apply Lemma 3.5 and recall that $\mathfrak{n}$ is nilpotent and thus $\Gamma=e^{\mathrm{ad}\,\mathfrak{n}}$ and $L(\Gamma)=L[\Gamma]/\mathfrak{z}=\mathfrak{n}/\mathfrak{z}$. Then

$$\begin{aligned}
\mathfrak{b}&=\{(Z_1,Z_2+\mathfrak{z})\in\mathfrak{a}\rtimes L(\Gamma):[X-Y,Z_2]=Z_1\}\\
&=\{([X-Y,Z],Z+\mathfrak{z}):Z\in\mathfrak{n}\text{ and }[X-Y],Z]\in\mathfrak{a}\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&L(\{0\}\times\Gamma)\cap L\big((-Y,\mathbf{1})(\mathfrak{a}\rtimes\Gamma)_X(Y,\mathbf{1})\big)\\
&=\{0\}\times L(\Gamma)\cap\{(\mathrm{ad}(X-Y)(Z),Z+\mathfrak{z}):Z\in\mathfrak{n},\mathrm{ad}(X-Y)(Z)\in\mathfrak{a}\}\\
&=\{(0,Z+\mathfrak{z}):Z\in\mathfrak{n},\ \mathrm{ad}(X-Y)(Z)=0\}=\{0\}\times\mathfrak{z}(X-Y,\mathfrak{n})/\mathfrak{z}\\
&=\{0\}\times(\mathfrak{z}(X-Y,\mathfrak{g})\cap\mathfrak{n})/\mathfrak{z}\cong(\mathfrak{z}(X-Y,\mathfrak{g})\cap\mathfrak{n})/\mathfrak{z}.
\end{aligned}$$

This proves (i).

(ii) Under the additional hypotheses, by Lemma 2.12 we know that $X-Y\in\mathrm{comp}(\mathfrak{g})$ for all $Y\in\mathfrak{a}$. Hence $X-Y$ is semisimple. Now X is regular and $Y\in\mathfrak{a}$ and $\mathfrak{a}$ as a minimal ideal is abelian hence contained in the nilradical. Thus Y is in the nilradical. Therefore $X-Y$ is regular. Therefore $\mathfrak{z}(X-Y,\mathfrak{g})$ is the Cartan algebra $\mathfrak{h}(X-Y)$ generated by $X-Y$. Since $\mathfrak{g}$ is solvable there is an inner automorphism β with $\mathfrak{h}(X-Y)=\beta(\mathfrak{h})$. As $\mathfrak{n}$ is an ideal, we have $\mathfrak{z}(X-Y,\mathfrak{g})+\mathfrak{n}=\beta(\mathfrak{h})+\beta(\mathfrak{n})=\beta(\mathfrak{h}+\mathfrak{n})$. The assertion now follows from (i). ∎

3.7. Lemma. *Let $\mathfrak{g}$ denote a solvable Lie algebra which is spanned by* comp *and let $\mathfrak{t}=\mathfrak{h}\cap\mathrm{comp}$ as in (3) for a Cartan algebra $\mathfrak{h}$. Let $X\in\mathfrak{t}$ be a regular element and $\mathfrak{a}$ a minimal ideal with $[X,\mathfrak{a}]\neq\{0\}$. If $\Gamma=e^{\mathrm{ad}\,\mathfrak{g}'}$, then $\mathfrak{a}+\Gamma\cdot X=\Gamma\cdot X$.*

Proof. We apply the preceding Lemma 3.6 with $\mathfrak{n}=\mathfrak{g}'+\mathfrak{z}$. Thus

$$L(\{0\}\times\Gamma)\cap\mathrm{Ad}\big((Y,\alpha)\big)(\mathfrak{a}\times\Gamma)_X\cong\big(\mathfrak{h}\cap(\mathfrak{z}+\mathfrak{g}')\big)/\mathfrak{z}\text{ for all }(Y,\alpha)\in\mathfrak{a}\times\Gamma.$$

Now we apply Lemma 3.3 with $C=\mathfrak{a}\rtimes\Gamma$, $A=\{0\}\times\Gamma$, $M=\mathfrak{g}$ and $m=X$, the affine action of the group $\mathfrak{a}\rtimes\Gamma$, and $B=(\mathfrak{a}\rtimes\Gamma)_X$. Condition (†) of Lemma 3.3(iv) is satisfied. We conclude $\mathfrak{a}+\Gamma\cdot X=C\cdot m=A\cdot m=\Gamma\cdot X$. ∎

We are ready for the main result. From $\Gamma=e^{\mathrm{ad}\,\mathfrak{g}'}$ we know that $\Gamma\cdot X\subseteq X+\mathfrak{g}'$ for all X. In the four-dimensional oscillator algebra, for regular elements $X\in\mathfrak{t}$, the orbits ΓX are three dimensional quadratic hyperboloids. The next theorem, however, shows that the such orbits are almost affine under rather general circumstances; they need just the subalgebra $\mathfrak{e}=\mathfrak{h}\cap\mathfrak{g}'$ as filling material in order to fill $X+\mathfrak{g}'$.

3.8. Theorem. *Suppose that $\mathfrak{g}$ is a solvable Lie algebra which is the span of its compact elements. We set $\Gamma=e^{\mathrm{ad}\,\mathfrak{g}'}$. Let $\mathfrak{h}$ be a Cartan algebra and set $\mathfrak{t}=\mathfrak{h}\cap\mathrm{comp}$. Then $\mathfrak{h}\cap\mathfrak{g}'$ is a nilpotent subalgebra of class at most two with commutator algebra contained in $\mathfrak{z}\cap\mathfrak{g}'$, and for every regular $X\in\mathfrak{t}$ we have*

$$(\dagger)\qquad(\mathfrak{h}\cap\mathfrak{g}')+\Gamma\cdot X=X+\mathfrak{g}'.$$

Proof. The first assertions were summarized in Proposition 2.10. It remains to show (†). We prove the claim by induction w.r.t. $\dim\mathfrak{g}$.

The assertion is true for $\dim\mathfrak{g}=0,1,2,3$ as a direct inspection shows. Now suppose that $\mathfrak{g}$ is a counterexample with smallest possible dimension.

Claim (i): $\mathfrak{z}=\mathfrak{j}$.

For the center $\mathfrak{z}$ we know $\mathfrak{z}\in\mathfrak{t}$ and we write $\mathfrak{z}=\mathfrak{j}\oplus\mathfrak{z}_1$. Then $\mathfrak{t}=\mathfrak{z}_1\oplus\mathfrak{t}_1$ and $\mathfrak{g}=\mathfrak{z}_1\oplus(\mathfrak{t}_1+\mathfrak{g}')$. If we set $\mathfrak{g}_1=\mathfrak{t}_1+\mathfrak{g}'$, then $\mathfrak{g}_1'=\mathfrak{g}'$ and $\mathfrak{e}_1\overset{\mathrm{def}}{=}\mathfrak{h}_1\cap\mathfrak{g}_1'=(\mathfrak{t}_1+\mathfrak{e})\cap\mathfrak{g}'=\mathfrak{e}$. $\mathfrak{z}(\mathfrak{g}_1)=\mathfrak{z}\cap\mathfrak{g}_1\subseteq\mathfrak{g}'$. If $\mathfrak{z}_1\neq0$, then $\dim\mathfrak{g}_1<\dim\mathfrak{g}$ and so $\mathfrak{g}_1$ is not a counterexample. As X is a regular element of $\mathfrak{t}$, we have $X=Z+X_1$ with $Z\in\mathfrak{z}_1$ and $X_1\in\mathfrak{t}_1$ with X_1 regular. Then $\mathfrak{e}+\Gamma\cdot X=\mathfrak{e}+\Gamma(Z+X_1)=\mathfrak{e}+Z+\Gamma\cdot X_1=Z+\mathfrak{e}_1+\Gamma\cdot X_1=Z+X_1+\mathfrak{g}_1'=X+\mathfrak{g}'$.

So $\mathfrak{g}$ cannot be a counterexample. Hence $\mathfrak{z}_1 = \{0\}$ and thus $\mathfrak{z} \subseteq \mathfrak{j}$. On the other hand $\mathfrak{j} \subseteq \mathfrak{z}$ by Lemma 2.6.

Step (ii): $\mathfrak{z} = \{0\}$ and $\mathfrak{h}$ is abelian. In particular, $\mathfrak{e}$ is abelian. Suppose not. By (i) above we know $\mathfrak{z} \subseteq \mathfrak{g}'$. Then we consider $\mathfrak{g}_1 = \mathfrak{g}/\mathfrak{z}$. By Lemma 2.7, $\mathfrak{g}_1$ is spanned by its compact elements. We write $\mathfrak{z}_2$ for the full inverse image of $\mathfrak{z}_1 = \mathfrak{z}(\mathfrak{g}_1)$ in $\mathfrak{g}$. Since $\mathfrak{h}$ is the full inverse image of $\mathfrak{h}_1 = \mathfrak{h}/\mathfrak{z}$ we have $\mathfrak{z}_2 \subseteq \mathfrak{h}$. Then $\mathfrak{z}_2 \cap \mathfrak{g}'$ is an ideal in $\mathfrak{e} = \mathfrak{h} \cap \mathfrak{g}'$ and is therefore contained in $\mathfrak{j} = \mathfrak{z}$ by Lemma 2.6. Then $\mathfrak{j}_1 = \mathfrak{z}_1 \cap \mathfrak{g}_1' = \frac{\mathfrak{z}_2}{\mathfrak{z}} \cap \frac{\mathfrak{g}'}{\mathfrak{z}} = \frac{\mathfrak{z}_2 \cap \mathfrak{g}'}{\mathfrak{z}} \subseteq \frac{\mathfrak{z} \cap \mathfrak{g}'}{\mathfrak{z}} = \{0\}$. Further, $\mathfrak{e}_1 = \mathfrak{h}_1 \cap \mathfrak{g}_1 = \frac{\mathfrak{h}}{\mathfrak{z}} \cap \frac{\mathfrak{g}'}{\mathfrak{z}} = \frac{\mathfrak{e}}{\mathfrak{z}}$.

Let $\Gamma_1 = e^{\operatorname{ad} \mathfrak{g}_1'} \subseteq \operatorname{Aut} \mathfrak{g}_1$. We have $\dim \mathfrak{g}_1 < \dim \mathfrak{g}$ by assumption. Then $\mathfrak{g}_1$ cannot be a counterexample to the theorem. For a regular $X \in \mathfrak{t}$ also $X + \mathfrak{z}$ is regular in $\mathfrak{g}_1$ and we obtain

$$\mathfrak{e}_1 + \Gamma_1 \cdot (X + \mathfrak{z}) = \mathfrak{z}_1 + \Gamma_1 \cdot X = X + \mathfrak{g}_1.$$

In view of Lemma 2.11 we pass to inverse images and find

$$\mathfrak{e} + \Gamma \cdot X = \mathfrak{e} + \Gamma \cdot X + \mathfrak{z} = X + \mathfrak{z} + \mathfrak{g}' = X + \mathfrak{g}'$$

as $\mathfrak{z}$ is contained in $\mathfrak{g}'$ by (i). But this contradicts the fact that $\mathfrak{g}$ is a counterexample. Thus $\mathfrak{z} = \{0\}$. Then $\mathfrak{h}$ is abelian by 2.8(ii).

Step (iii): Let $\mathfrak{a}$ be a nonzero minimal ideal of $\mathfrak{g}$. By Lemma 2.6(b) we know $\mathfrak{a} \subseteq \mathfrak{t}^+ \subseteq \mathfrak{g}'$. We consider the factor algebra $\mathfrak{g}_1 = \mathfrak{g}/\mathfrak{a}$. It is spanned by its compact elements in view of Lemma 2.7 and satisfies $\dim \mathfrak{g}_1 < \dim \mathfrak{g}$. Hence $\mathfrak{g}_1$ not a counterexample. We let $\Gamma_1 = \langle e^{\operatorname{ad}(\mathfrak{g}'/\mathfrak{a})} \rangle \subseteq \operatorname{Aut} \mathfrak{g}_1$. Then for X regular in $\mathfrak{t}$, the element $X + \mathfrak{a} \in \frac{\mathfrak{t}+\mathfrak{a}}{\mathfrak{a}}$ is compact and regular. Thus we have

$$(*) \qquad \mathfrak{e}_1 + \Gamma_1 \cdot (X + \mathfrak{a}) = (X + \mathfrak{a}) + \mathfrak{g}_1'$$

with $\mathfrak{e}_1 = \mathfrak{h}_1 \cap \mathfrak{g}_1'$.

Now we let $\mathfrak{e}^*$ denote the full inverse image of $\mathfrak{e}_1$ in $\mathfrak{g}$. Since $\mathfrak{e}_1 = \mathfrak{h}_1 \cap \mathfrak{g}_1'$ where $\mathfrak{h}_1 = \frac{\mathfrak{h}+\mathfrak{a}}{\mathfrak{a}}$ we know $\mathfrak{e}^*/\mathfrak{a} = \mathfrak{e}_1 = \frac{\mathfrak{h}+\mathfrak{a}}{\mathfrak{a}} \cap \frac{\mathfrak{g}'}{\mathfrak{a}} = \frac{(\mathfrak{h}+\mathfrak{a}) \cap \mathfrak{g}'}{\mathfrak{a}}$. Because of the modular law this equals $\frac{(\mathfrak{h} \cap \mathfrak{g}') + \mathfrak{a}}{\mathfrak{a}} = \frac{\mathfrak{e}+\mathfrak{a}}{\mathfrak{a}}$. Hence we have $\mathfrak{e}^* = \mathfrak{e} + \mathfrak{a}$. (Note that $\mathfrak{e}$ is abelian by Step (ii). By 2.8(i) we know $[\mathfrak{g}', \mathfrak{a}] = \{0\}$. Thus $[\mathfrak{e}, \mathfrak{a}] \subseteq [\mathfrak{g}', \mathfrak{a}] = \{0\}$. Hence $\mathfrak{e} + \mathfrak{a}$ abelian.)

In view of Lemma 2.11, passing to the inverse images in $(*)$ we obtain

$$(**) \qquad \mathfrak{e} + \mathfrak{a} + \Gamma \cdot X = (\mathfrak{e} + \mathfrak{a}) + (\Gamma \cdot X + \mathfrak{a}) = X + \mathfrak{a} + \mathfrak{g}' = X + \mathfrak{g}'.$$

Now the hypotheses of Lemma 3.7(iii) are satisfied whence we know that $\mathfrak{a} + \Gamma \cdot X = \Gamma \cdot X$. Thus $\mathfrak{e} + \Gamma \cdot X = X + \mathfrak{g}'$ and this again shows that $\mathfrak{g}$ is not a counterexample. Thus case (b) cannot occur either—a contradiction which finishes the proof. ∎

We shall identify the tangent space $T_X(\Gamma \cdot X)$ at X of the orbit $\Gamma \cdot X \subseteq \mathfrak{g}$ with a vector subspace of $\mathfrak{g}$ via translation.

3.9. Proposition. *Under the hypotheses of* Theorem 3.8, *the tangent space of the orbit* $\Gamma{\cdot}X$ *at the point* X *is* $T_X(\Gamma{\cdot}X)=\mathfrak{t}^+$.

Proof. From Lemma 3.5(ii) we know that $T_X(\Gamma{\cdot}X)=[L[\Gamma],X]$. But $L[\Gamma]=L[e^{\mathrm{ad}\,\mathfrak{g}'}]=\mathfrak{z}+\mathfrak{g}$ by Lemmas 2.4(i) and 2.5(iv,v). Now $\mathfrak{g}'=\mathfrak{e}\oplus\mathfrak{t}^+$ by Lemma 2.4(i,iv). Thus $[L[\Gamma],X]=[\mathfrak{e}\oplus\mathfrak{t}^+,X]=[\mathfrak{e},X]+[\mathfrak{t}^+,X]$. But $[\mathfrak{e},X]=\{0\}$ by Lemma 2.2(i), and $[\mathfrak{t}^+,X]=\mathrm{ad}(X)(\mathfrak{t}^+)=\mathfrak{t}^+$ by Remark 2.3. This proves the proposition. ■

This proposition shows that we cannot expect to reduce the size of the filling space $\mathfrak{e}=\mathfrak{h}\cap\mathfrak{g}'$ in Theorem 3.8, because by the preceding proposition we have exactly $(\mathfrak{h}\cap\mathfrak{g}')+T_X(\Gamma{\cdot}X)=\mathfrak{e}+\mathfrak{t}^+=\mathfrak{g}'$.

We obtain the following result as an immediate consequence of Theorem 3.8:

3.10. Theorem. *Let* $\mathfrak{g}$ *be a solvable Lie algebra which is spanned by its compact elements. Let* $\mathfrak{e}_1$ *denote any vector space complement of* $\mathfrak{z}\cap\mathfrak{g}'$ *in* $\mathfrak{e}=\mathfrak{h}\cap\mathfrak{g}'$. *Then* $\mathfrak{g}=\overline{\mathfrak{e}_1+\mathrm{comp}(\mathfrak{g})}$.

Proof. Let R denote the set of regular elements in $\mathfrak{t}$. Now, by Theorem 3.8 and Lemma 2.2(iii) we conclude $R+\mathfrak{g}'=\mathfrak{e}+\Gamma{\cdot}R\subseteq\mathfrak{e}+\mathrm{comp}(\mathfrak{g})=\big(\mathfrak{e}_1+(\mathfrak{z}\cap\mathfrak{h})\big)+\mathrm{comp}(\mathfrak{g})=\mathfrak{e}_1+\mathrm{comp}(\mathfrak{g})$ since $\mathfrak{z}+\mathrm{comp}=\mathrm{comp}$. Hence $R+\mathfrak{g}'\subseteq\overline{\mathfrak{e}+\mathrm{comp}(\mathfrak{g})}\subseteq\mathfrak{g}$. But R is dense in $\mathfrak{t}$ and $R+\mathfrak{g}'$ is dense in $\mathfrak{g}$ by Lemma 2.4(i). The assertion follows. ■

In particular we note that $\mathfrak{e}_1=\{0\}$ means $\mathfrak{t}=\mathfrak{h}$, i.e., $\mathfrak{h}\subseteq\mathrm{comp}$, and, equivalently, $\mathfrak{h}\cap\mathfrak{g}'=\mathfrak{z}\cap\mathfrak{g}'$, and that, in this case, the set comp of compact elements is dense in $\mathfrak{g}$.

We now abandon the general hypothesis that $\mathfrak{g}$ is spanned by comp and summarize that which our results yield for real solvable Lie algebras in general. Thus let $\mathfrak{g}$ denote a solvable Lie algebra. Then $\mathrm{span}\big(\mathrm{comp}(\mathfrak{g})\big)$ is a characteristic ideal $\mathfrak{g}_{\mathrm{comp}}$. Let $\mathfrak{h}$ be a Cartan algebra of $\mathfrak{g}$ and set $\mathfrak{t}=\mathfrak{h}\cap\mathrm{comp}(\mathfrak{g})$. Define $\mathfrak{t}^+$ to be the Fitting one component of $\mathrm{ad}\,X$ for a regular $X\in\mathfrak{t}$. Now $\mathfrak{h}_{\mathrm{comp}}=\mathfrak{z}(\mathfrak{t},\mathfrak{g}_{\mathrm{comp}})=\mathfrak{h}\cap\mathfrak{g}_{\mathrm{comp}}$ is a Cartan algebra of $\mathfrak{g}_{\mathrm{comp}}$ containing $\mathfrak{t}$. We set $\mathfrak{e}=\mathfrak{h}_{\mathrm{comp}}\cap\mathcal{C}^\infty(\mathfrak{g}_{\mathrm{comp}})$ and let $\mathfrak{e}_1$ denote any complement of $\mathfrak{z}\cap\mathfrak{e}$ in $\mathfrak{e}$.

3.11. Corollary. *In a solvable Lie algebra* $\mathfrak{g}$ *we have*

$$\mathfrak{g}_{\mathrm{comp}}=\mathfrak{t}+\langle\mathfrak{t}^+\rangle=\mathfrak{t}+(\mathfrak{e}\oplus\mathfrak{t}^+)$$

The subalgebra $\mathfrak{e}$ *is nilpotent of class at most* 2 *and centralizes* $\mathfrak{t}$. *It does not, in general, centralize* $\mathfrak{t}^+$ *even though all roots of* $\mathfrak{g}_{\mathrm{comp}}$ *with respect to* $\mathfrak{h}_{\mathrm{comp}}$ *vanish on* $\mathfrak{e}$.

Further, $\mathfrak{e}\oplus\mathfrak{t}^+=\langle\mathfrak{t}^+\rangle=\mathfrak{g}'_{\mathrm{comp}}=\mathcal{C}^\infty(\mathfrak{g}_{comp})$ *and* $\mathfrak{z}(\mathfrak{g}_{\mathrm{comp}})=\mathfrak{t}\cap\mathfrak{e}=\mathfrak{t}\cap\mathfrak{g}'_{\mathrm{comp}}$.

Finally,

$$\mathfrak{g}_{\mathrm{comp}}=\mathfrak{e}_1+\overline{\mathrm{comp}}.$$

Proof. This is just a summary of what has been shown in the process. ■

4. Examples

In order to see that the structural information on solvable Lie algebras which are spanned by their compact elements cannot be much improved we consider a convenient class of examples.

Let A denote a finite dimensional commutative algebra over $\mathbb{R}$ with identity and an augmentation morphism $\alpha: A \to \mathbb{R}$ whose kernel A_0 is nilpotent. If $A_0^n = \{0\}$ and $A_0^{n-1} \neq \{0\}$, then n is called the *nilpotent class of* A_0. We identify $\mathbb{R}$ with the multiples of $\mathbf{1}$.

Let $\mathfrak{k} = \mathfrak{u} \oplus \mathfrak{v}$ denote a Lie algebra such that $\mathfrak{u} = \mathbb{R}\cdot U + \mathbb{R}\cdot[IX, X]$ and $\mathfrak{v} = \mathbb{R}\cdot X \oplus \mathbb{R}\cdot IX$ with $[U, X] = -IX$ and $[U, IX] = X$. Then $\dim \mathfrak{k} = 3$ or 4 according as $[IX, X] =$ or $\neq 0$. In other words, $\mathfrak{k}$ is one of the standard low dimensional test algebras $\mathfrak{so}(3)$, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{m}_3, \mathfrak{o}_4$ where $\mathfrak{m}_3$ is the motion algebra of the euclidean plane and $\mathfrak{o}_4$ the oscillator algebra with compactly embedded Cartan algebra $\mathfrak{u}$ and real root space $\mathfrak{v}$.

4.1. Example. The Lie algebra $A\otimes\mathfrak{k}$ with bracket $[a\otimes Y, a'\otimes Y'] = aa'\otimes[Y, Y']$ is of mixed type with Levi complement $\mathbf{1}\otimes\mathfrak{k}$ if $\mathfrak{k} = \mathfrak{so}(3)$, $\mathfrak{sl}(2,\mathbb{R})$ and is solvable if $\mathfrak{k}$ is solvable. The subalgebra

$$\mathfrak{g} = (\mathbb{R}\otimes\mathfrak{u}) \oplus (A_0 \otimes \mathfrak{k})$$

is a maximal solvable subalgebra. The following statements hold

(i) The nilradical $\mathfrak{n}$ equals $A_0 \otimes \mathfrak{k}$. The nilpotent class of $\mathfrak{n}$ is that of A_0, and $\dim \mathfrak{n} = (\dim A_0)(\dim \mathfrak{k})$. Further $\mathfrak{g}' = \mathcal{C}^\infty = (A_0^2 \otimes \mathfrak{u}) \oplus (A_0 \otimes \mathfrak{v})$.

(ii) $\mathfrak{h} \overset{\text{def}}{=} A \otimes \mathfrak{u}$ is a Cartan algebra of $A \otimes \mathfrak{k}$ and of $\mathfrak{g}$. We have $[\mathfrak{h}, \mathfrak{h}] = 0$, $\mathfrak{t} \overset{\text{def}}{=} \operatorname{comp}(\mathfrak{h}) = \mathbf{1} \otimes \mathfrak{u}$, $\mathfrak{e} \overset{\text{def}}{=} \mathfrak{h} \cap \mathcal{C}^\infty = A_0 \otimes \mathfrak{u}$, and $\mathfrak{t}^+ = A_0 \otimes \mathfrak{v}$.
In particular,

$$\dim \mathfrak{h} = (\dim A)(\dim \mathfrak{u}), \quad \dim \mathfrak{t} = \dim \mathfrak{u}, \quad \dim \mathfrak{t}^+ = 2(\dim A_0).$$

(iii) $\mathfrak{g}_{\text{comp}} = \mathfrak{t} + \langle \mathfrak{t}^+ \rangle = (\mathbf{1} \otimes \mathfrak{u}) \oplus (A_0^2 \otimes \mathfrak{u}) \oplus (A_0 \otimes \mathfrak{v})$. Accordingly we have $\mathfrak{e}_{\text{comp}} = A_0^2 \otimes \mathfrak{u}$, and $\mathfrak{z}_{\text{comp}} = B \otimes \mathfrak{u}$, where $B = \{a \in A : Aa = \{0\}\}$. Further $\mathfrak{g}'_{\text{comp}} = \mathfrak{g}'$ and $\mathfrak{e}_{\text{comp}} = \mathfrak{e}$. ∎

References

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Karl H. Hofmann
Fachbereich Mathematik
Technische Hochschule Darmstadt
Schloßgartenstraße 7
D-6100 Darmstadt, Germany

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