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A globality theorem for Lie-wedges that are bounded by a hyperplane-ideal

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Abstract

We consider a Lie group G containing a normal subgroup $N \trianglelefteq G$ such that $G/N \cong \mathbb{R}$, i.e the Lie algebra $\mathfrak{n}$ is a hyperplane ideal in $\mathfrak{g}$. A natural question that arises in this context is the following: Suppose we are given a Lie-wedge W which is contained in a halfspace bounded by $\mathfrak{n}$. Under which conditions is W global in G? We will prove globality for all pointed wedges W such that there exists another wedge $W' \subseteq \mathfrak{n}$ which is global in N and satisfies $(W \cap \mathfrak{n}) \supseteq int_{\mathfrak{n}}(W') \cup \{0\}$. Especially our result applies to the groups and Lorentzian wedges considered by Levichev and Levicheva in this volume. As another application, we solve the globality problem of the Heisenberg-algebra, i.e. we give a complete characterization of all Lie-wedges that are global in the Heisenberg-group.

1. The globality theorem

Theorem 1.1. *Let G be a Lie group, $N \trianglelefteq G$ a normal subgroup such that $G/N \cong \mathbb{R}$. Let $W \subseteq \mathfrak{n}$ be a Lie wedge which is global in N and $X \in \mathfrak{g} \setminus \mathfrak{n}$ such that the edge $H(W) = W \cap (-W)$ is $ad\,X$-invariant. We define the following set:*

$$\mathcal{W} = \left\{ W' \subseteq \mathfrak{n} \mid W' \text{ global in } N \text{ and } (\exists \varepsilon > 0)(\forall t \in [0, \varepsilon]) e^{t\,ad\,X} W \subseteq W' \right\}.$$

If $\bigcap \mathcal{W} = W$ then the Lie wedge $W + \mathbb{R}^+ X$ is global in G.

Proof. The $ad\,X$-invariance of $H(W)$ implies that $e^{t\,ad\,Y} X \in X + H(W)$ for all $Y \in H(W)$. Since W is a Lie wedge and $H(W) = H(W + \mathbb{R}^+ X)$ it follows that $W + \mathbb{R}^+ X$ is invariant under the adjoint action of its edge, hence a Lie wedge. Let us denote with $\alpha(t)$ the inner automorphism of G induced by $\exp(tX)$, i.e. $\alpha(t)(g) = \exp(tX) g \exp(-tX)$ and $\alpha(t)(\exp Y) = \exp(Ad(\exp(tX)Y) = \exp(e^{t\,ad\,X} Y)$. We use the notation $\prod_{i=1}^n g_i = g_1 g_2 \cdots g_n$ and set for $t \geq 0$

$$S_t = \left\{ \left(\prod_{i=1}^n \alpha(t_i)(g_i) \right) \exp(tX) \mid t_i \in [0, t],\ g_i \in \exp(W) \right\} \tag{1}$$

and $S = \bigcup_{t \geq 0} S_t$. We will prove the following:

(i) S is a semigroup because for $s,t \geq 0$ we have $S_t S_s \subseteq S_{t+s}$. Indeed, if $g = (\prod_{i=1}^n \alpha(t_i)(g_i)) \exp(tX) \in S_t$ and $h = (\prod_{j=1}^m \alpha(s_j)(h_j)) \exp(sX) \in S_s$ then

$$\begin{aligned} gh &= \left(\prod_{i=1}^n \alpha(t_i)(g_i)\right) \exp(tX) \left(\prod_{j=1}^m \alpha(s_j)(h_j)\right) \exp(sX) \\ &= \left(\prod_{i=1}^n \alpha(t_i)(g_i)\right) \alpha(t) \left(\prod_{j=1}^m \alpha(s_j)(h_j)\right) \exp((s+t)X) \\ &= \left(\prod_{i=1}^n \alpha(t_i)(g_i)\right) \left(\prod_{j=1}^m \alpha(t+s_j)(h_j)\right) \exp((s+t)X) \in S_{t+s}. \end{aligned}$$

(ii) $L(S) \supseteq W + \mathbb{R}^+X$ for obviously $\exp(W) \cup \exp(\mathbb{R}^+X) \subseteq S$.

(iii) $L(S) \subseteq W + \mathbb{R}^+X$. This is the non-trivial part of the proof. Let $Y \in L(S)$ then there exist sequences $m_j \in \mathbb{R}^+$ and $g_j \in S$ such that $\lim_j g_j = 1$ and $Y = \lim_j m_j \log g_j$. Now $g_j = a_j \exp(t_j X)$ with $t_j \geq 0$, $\lim_j a_j = 1$ and $\lim_j t_j = 0$ because $G/N \cong \mathbb{R}$. Let an arbitrary $W' \in \mathcal{W}$ be given, then there is an $\varepsilon > 0$ such that $e^{s\,ad\,X} W \subseteq W'$ for all $s \in [0, \varepsilon]$. Since $\lim_j t_j = 0$ we have $t_j < \varepsilon$ for all sufficiently large j. For these j we have

$$a_j = \prod_i \alpha(s_i) \exp(w_i) = \prod_i \exp(e^{s_i\,ad\,X} w_i) \in \langle \exp(W') \rangle\,.$$

Applying the formula for the Campbell-Hausdorff-multiplication $*$ we obtain

$$\begin{aligned} m_j \log g_j &= m_j \log(a_j \exp(t_j X)) = m_j (\log a_j * t_j X) \\ &= \underbrace{m_j \log(a_j)}_{\in \mathfrak{n}} + m_j t_j X + m_j r_j \end{aligned} \tag{2}$$

with $r_j \in \mathfrak{n}$ and $\|r_j\| \leq |t_j|\, \|\log a_j\|$ for a suitable norm on $\mathfrak{g}$. Now the existence of $Y = \lim m_j \log g_j$ implies that $m_j t_j \geq 0$ converges, hence is bounded. Thus $\|m_j r_j\| \leq |m_j t_j|\, \|\log a_j\| \to 0$ since $a_j \to 1$, and we conclude that $m_j \log a_j$ also converges. Now the globality of W' in N yields $\lim_j m_j \log a_j \in W'$ which implies $Y \in W' + \mathbb{R}^+X$. Since $W' \in \mathcal{W}$ was arbitrary, it follows that

$$L(S) \subseteq \bigcap_{W' \in \mathcal{W}} (W' + \mathbb{R}^+X) = W + \mathbb{R}^+X, \tag{3}$$

which proves (iii).

From (i)–(iii) we deduce the globality of $W + \mathbb{R}^+X$ in G, because we have proved the existence of a semigroup S with the prescribed tangent wedge. ■

Remark 1.2. *The sets S_t defined in the previous proof are bigger than they ought to be. Indeed, it would be sufficient to take*

$$\widetilde{S}_t = \{\alpha(t_1)(g_1) \cdots \alpha(t_n)(g_n) \exp(tX) \mid 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq t,\ g_i \in \exp(W)\}$$

in order to obtain a semigroup via $\widetilde{S} = \bigcup_{t\geq 0} \widetilde{S}_t$. But this does not change the proof of (iii). The main difference is that with our choice, the sets $T_t = S_t \exp(-tX) \subseteq N$ are subsemigroups of N whereas the corresponding sets $\widetilde{T}_t = \widetilde{S}_t \subseteq N$ need not be subsemigroups of N. If N is abelian then obviously $S_t = \widetilde{S}_t$ because we may rearrange the factors $\alpha(t_i)\exp(w_i)$ such that $t_i \leq t_{i+1}$, but in the non-abelian case $S_t \setminus \widetilde{S}_t$ may be non-empty.

The assumption that $\bigcap \mathcal{W} = W$ may not be weakened which will be shown by a counter example.

2. A counterexample

The following counterexample shows, that we cannot weaken the assumption that $\bigcap \mathcal{W} = W$. We take N to be the Heisenberg group and $G = N \rtimes \mathbb{R}$ the four-dimensional oscillator group where $\mathbb{R}$ acts on N by rotations around the center $Z(N)$. So we may identify G with the set $\mathbb{C} \times \mathbb{R} \times \mathbb{R}$ where the multiplication is given by

$$(v, s, r)(w, t, r') = \left(v + e^{ir}w,\ s + t + \frac{1}{2}\Im(\overline{v}e^{ir}w), r + r'\right). \tag{4}$$

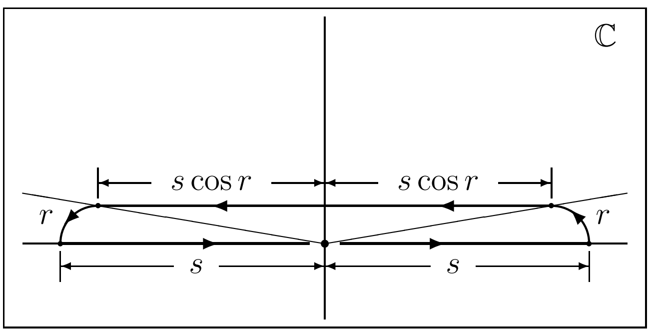
We may identify $\mathfrak{g}$ with the set $\mathbb{C} \times \mathbb{R} \times \mathbb{R}$ and let $P = (1, 0, 0)$, $Q = (i, 0, 0)$, $Z = (0, 1, 0)$, $R = (0, 0, 1)$, then $\{P, Q, Z\}$ is a standard base for the Heisenberg algebra $\mathfrak{n}$, i.e. $[P, Q] = Z$, Z is central. Now we consider the following wedge:

$$W = \mathbb{R}^+(Z - P) + \mathbb{R}^+(Z + P). \tag{5}$$

Then W is global in N because $W + (-W)$ is an abelian subalgebra of $\mathfrak{n}$, but every wedge in $\mathcal{W}$ wedge has central elements in its interior, hence generates the whole group N (see [1, V.4.10]), so $\mathcal{W} = \{\mathfrak{n}\}$ and $\bigcap \mathcal{W} = \mathfrak{n} \neq W$. The non-globality of $W + \mathbb{R}^+R$ in G follows from the following proposition:

Proposition 2.1. *Let us denote with S_r the slice $\langle \exp(W + \mathbb{R}^+R)\rangle \cap \mathbb{C} \times \mathbb{R} \times \{r\}$, then for every $r > 0$ the set $Z(N)(0, 0, 2r) = \{0\} \times \mathbb{R} \times \{2r\}$ is contained in S_{2r}, hence $Z(N) \subseteq \overline{\langle \exp W + \mathbb{R}^+R\rangle}$.*

Proof. Let $r > 0$ be given. Since $S_r(0, 0, r' - r) \subseteq S_{r'}$ for $r \leq r'$ we only have to consider small $r > 0$, so we may assume w.l.o.g. that $r < \pi/2$, i.e. $\sin(2r) > 0$. With respect to the $\mathbb{C}$-factor we will steer along the following path in order to reach all points in $\{0\} \times \mathbb{R} \times \{2r\}$:



For $s \geq 0$ and $z \geq 0$ we compute:

$$\begin{aligned}
S_{2r} &\ni (0,z,0)(s,s,0)(0,0,r)(-2s\cos r, 2s\cos r, 0)(0,0,r)(s,s,0) \\
&= (s,s+z,0)(0,0,r)(-2s\cos r, 2s\cos r, 0)(se^{ir}, s, r) \\
&= (s,s+z,0)(0,0,r)\left(-2s\cos r + se^{ir}, 2s\cos r + s + \Im(-s\cos r se^{ir}), r\right) \\
&= (s,s+z,0)(0,0,r)\left(\underbrace{-s\cos r + is\sin r}_{-se^{-ir}}, s + 2s\cos r - \frac{s^2}{2}\sin(2r), r\right) \\
&= (s,s+z,0)\left(-s, s + 2s\cos r - \frac{s^2}{2}\sin(2r), 2r\right) \\
&= \left(0, z + 2s + 2s\cos r - \frac{s^2}{2}\sin(2r) + \frac{1}{2}\Im(-s^2), 2r\right) \\
&= \left(0, z + 2s(1+\cos r) - s^2\frac{\sin(2r)}{2}, 2r\right).
\end{aligned}$$

Now $\lim_{s\to\infty} z + 2s(1+\cos r) - s^2\frac{\sin(2r)}{2} = -\infty$ thus $\{0\} \times \mathbb{R} \times \{2r\} \subseteq S_{2r}$ for any $0 < r < \pi/2$ which proves our claim. ∎

As an observation, we still prove that our example is in some sense as bad as possible.

Proposition 2.2. *For any $r > 0$ we have $S_r = N \times \{r\}$.*

Proof. First we will prove $\mathrm{pr}_{\mathbb{C}}(S_{2r}) = \mathbb{C}$: We have for $0 \leq t \leq r$ and $s \geq 0$:

$$(0,0,t)(s,s,0)(0,0,r-t) \in S_r \text{ and } (0,0,t)(-s,s,0)(0,0,r) \in S_r.$$

Hence $\mathrm{pr}_{\mathbb{C}}(S_r) \supseteq C_r^1 \cup C_r^2$ with

$$C_r^1 = \{v \mid \arg(v) \in [0,r]\} \text{ and } C_r^2 = -C_r^1 = \{v \mid \arg(v) \in [\pi, \pi + r]\}.$$

Thus $\mathrm{pr}_{\mathbb{C}}(S_{2r}) \supseteq (C_r^1 + e^{ir}C_r^2) \cup (C_r^2 + e^{ir}C_r^1)$. Now

$$C_r^2 + e^{ir}C_r^1 = \{\arg(v) \in [\pi, \pi+r]\} + \{\arg(v) \in [r, 2r]\} = \{\arg(v) \in [r, \pi+r]\},$$

i.e. a halfspace E_+, and because of $C_r^2 = -C_r^1$ we have $C_r^1 + e^{ir}C_r^2 = -E_+$ which proves $\mathrm{pr}_{\mathbb{C}}(S_{2r}) \supseteq \mathbb{C}$. From the previous proposition we know that $\{0\} \times \mathbb{R} \times \{r\} \subseteq S_r$ so that we may deduce $S_{3r} = H \times \{r\}$ by right-multiplication with appropriate elements of $\{0\} \times \mathbb{R} \times \{r\}$. ∎

This counterexample is minimal with respect to the dimension of G, because if the hyperplane ideal $\mathfrak{n}$ has dimension ≤ 2 then it is either abelian or the two-dimensional non-abelian algebra. In both cases every wedge $W \subseteq \mathfrak{n}$ is global in N, (see [1, V.4.13] for the non-abelian case). So if we take a pointed wedge W, then $\bigcap \mathcal{W} = W$ and if the edge $H(W) \neq \{0\}$ then W must be a halfspace. Since $H(W)$ is $ad\,X$-invariant and $\mathfrak{n} \setminus H(W)$ is disconnected, W must be $ad\,X$-invariant itself which implies $W \in \mathcal{W}$. Thus there is no counterexample of lower dimension. It is also possible to construct a nilpotent counterexample, just take $\mathfrak{n}\rtimes_D\mathbb{R}$ with $\mathfrak{n}$ the Heisenberg-algebra, W as in the former example and $D \in Der(\mathfrak{n})$ is

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the base $\{P, Q, Z\}$.

3. Applications of the globality Theorem

The following corollary ([1, VI.5.2]) enables us to prove the globality in many cases:

Corollary 3.1. *Suppose $W_2 \subseteq \mathfrak{g}$ is global in G and $W_1 \subseteq W_2$ satisfies*

(i) $W_1 \cap H(W_2) \subseteq H(W_1)$.

(ii) The analytical subgroup $\langle \exp H(W_1)\rangle$ is closed in G.

Then W_1 is also global in G.

Remark 3.2. *If W_2 is pointed, then these conditions are trivially satisfied.*

We will now give a sufficient condition for the equality $\bigcap \mathcal{W} = W$.

Definition 3.3. *Let V be a vector space and $W_1, W \subseteq V$ two wedges. We say that W_1 is* surrounding *W (in V), if $W \setminus H(W) \subseteq int_V(W_1)$ or equivalently $W \subseteq int_V(W_1) \cup H(W)$.*

Proposition 3.4. *Assume $W \subseteq \mathfrak{n}$ is pointed and there exists a pointed, global wedge $W_1 \subseteq \mathfrak{n}$ surrounding W in $\mathfrak{n}$. Then $W = \bigcap \mathcal{W}$.*

Proof. If $v \in \mathfrak{n}\backslash W$ then there exists a wedge W_2 surrounding W with $v \notin W_2$. Therefore $W' := W_1 \cap W_2$ is global and since W' surrounds W, we can find an $\varepsilon > 0$ such that $e^{[0,\varepsilon]\, ad\, X} \subseteq W'$ no matter what the action of $ad\, X$ on $\mathfrak{n}$ is. Thus $W' \in \mathcal{W}$ and $v \notin \bigcap \mathcal{W}$ which proves the assertion. ■

So we may state

Corollary 3.5. *Let G be a Lie group containing a normal subgroup N such that $G/N \cong \mathbb{R}$. Let $W \subseteq \mathfrak{n}$ be a pointed wedge such that there exists a global wedge W_1 surrounding W in $\mathfrak{n}$. Then for every $X \in \mathfrak{g} \setminus \mathfrak{n}$ the Lie wedge $W + \mathbb{R}^+X$ is global in G.*

Proof. Since W is pointed, contained in W_1, it is global in N and its edge $H(W) = \{0\}$ is trivially $ad\, X$-invariant. Proposition 3.4 yields $W = \bigcap \mathcal{W}$, so we may apply Theorem 1.1. ■

The next question which arises is: Suppose we are given a Lie wedge bounded by the hyperplane ideal $\mathfrak{n}$, i.e. $W \subseteq \mathfrak{n} + \mathbb{R}^+X$ for an $X \in \mathfrak{g} \setminus \mathfrak{n}$. When are these wedges global? In view of corollary 3.1 one should try to find a suitable wedge $W' \subseteq \mathfrak{n}$ satisfying the assumptions of Theorem 1.1 such that $W' + \mathbb{R}^+X \supseteq W$. If we assume that W is pointed then the intersection $W \cap \mathfrak{n}$ will play an important role.

4. A Lemma on pointed wedges

If (X, d) is a metric space, we may endow the set $\mathcal{C}(X)$ of compact subsets with the so-called Vietoris topology which is defined through the metric

$$d(A,B) = \max\left(\max\{d(a,B) \mid a \in A\}, \max\{d(b,A) \mid b \in B\}\right).$$

where $d(a,B) = \min\{d(a,b) \mid b \in B\}$ is the distance of a from B.

Proposition 4.1. *Assume X is a metric space and $(K_n)_{n\in\mathbb{N}} \subseteq X$ a decreasing sequence of compact sets. Let $K = \bigcap_{n\in\mathbb{N}} K_n$, then $K_n \to K$ in the Vietoris topology.*

Proposition 4.2. *Suppose $f\colon X \to X$ is a continous function and $K_n \to K$ in the Vietoris topology. Then $f(K_n) \to f(K)$ in the Vietoris topology.*

Lemma 4.3. *Let V be a vector space, $C \subseteq V$ a pointed, (closed, convex) cone, H a supporting hyperplane of C and the cone $\widetilde{C} \subseteq H$ is surrounding $C \cap H$ in H, i.e. $int_H(\widetilde{C}) \supseteq (C \cap H) \setminus \{0\}$. Then there exists an $x \in V \setminus H$ such that $C \subseteq \widetilde{C} + \mathbb{R}^+ x$.*

Proof. Let $\alpha \in H^\perp \cap C^*$ and pick an arbitrary x_0 with $\alpha(x_0) = 1$. Pick $\omega \in int(C^*)$ and let $K = C \cap \omega^{-1}(1)$, i.e. K is compact and $C = \mathbb{R}^+ K$. We will construct an invertible linear map $\psi : V \to V$ with the following properties:

(i) $\psi|_H = id_H$,

(ii) $\psi(K) \subseteq \widetilde{C} + \mathbb{R}^+ x_0$.

Since $\mathbb{R}^+ K = C$ we may then conclude that $\psi(C) = \psi(\mathbb{R}^+ K) \subseteq \widetilde{C} + \mathbb{R}^+ x_0$, hence

$$C \subseteq \psi^{-1}(\widetilde{C}) + \mathbb{R}^+ \psi^{-1}(x_0) = \widetilde{C} + \mathbb{R}^+ x$$

with $x = \psi^{-1}(x_0)$ because of (i).
Now we construct the map ψ: We denote the projection along x_0 onto H with pr_H, i.e. $\mathrm{pr}_H(v) = v - \alpha(v)x_0$ since $\alpha(x_0) = 1$. Now K is compact and so is $\mathrm{pr}_H(K)$. We denote with K_r the slices $K \cap \alpha^{-1}([0, r])$ which are also compact for any $r > 0$. Since $K_r \to K_0 = K \cap H \subseteq int(\widetilde{C})$ for $r \to 0$ in the Vietoris topology, there is an $r_0 > 0$ such that $\mathrm{pr}_H(K_{r_0}) \subseteq \widetilde{C}$. Since $int_H(\widetilde{C}) \neq \emptyset$ and $\mathrm{pr}_H(K)$ is compact, we can find some $w \in int_H(\widetilde{C})$ such that

$$r_0 w + \mathrm{pr}_H(K) \subseteq \widetilde{C}.$$

Now let $\psi(v) = \mathrm{pr}_H(v) + \alpha(v)w + \alpha(v)x_0$, then $\psi|_H = id_H$, $\psi(x_0 - w) = -w + w + x_0 = x_0$, hence ψ is surjective and therefore invertible. If $v \in K$ then either $\alpha(v) \leq r_0$ which implies

$$\psi(v) = \underbrace{\mathrm{pr}_H(v)}_{\in\widetilde{C}} + \underbrace{\alpha(v)w}_{\in\widetilde{C}} + \alpha(v)x_0 \in \widetilde{C} + \mathbb{R}^+ x_0$$

or $\alpha(v) \geq r_0$ and therefore

$$\psi(v) = \underbrace{\mathrm{pr}_H(v) + r_0 w}_{\in\widetilde{C}} + \underbrace{(\alpha(v) - r_0)w}_{\in\widetilde{C}} + \alpha(v)x_0 \in \widetilde{C} + \mathbb{R}^+ x_0,$$

thus $\psi(K) \subseteq \widetilde{C} + \mathbb{R}^+ x_0$. ∎

5. Globality of wedges that are bounded by a hyperplane ideal

Now we have all the tools to prove

Theorem 5.1. *Let G be a Lie group containing a normal subgroup N with $G/N \cong \mathbb{R}$. Suppose $W \subseteq \mathfrak{g}$ is a pointed wedge such that*

(i) $\mathfrak{n}$ *is a supporting hyperplane of* W*,*

(ii) There exists a pointed wedge $W_1 \subseteq \mathfrak{n}$ such that W_1 is surrounding $W \cap \mathfrak{n}$ in $\mathfrak{n}$ and W_1 is global in N.

Then W is global in G.

Proof. Since W_1 is surrounding $W \cap \mathfrak{n}$ in $\mathfrak{n}$, we may choose a wedge W_2 such that W_1 is surounding W_2 and W_2 is surrounding $W \cap \mathfrak{n}$ in $\mathfrak{n}$. According to Lemma 4.3 we can find an $X_0 \in \mathfrak{g} \setminus \mathfrak{n}$ such that $W \subseteq W_2 + \mathbb{R}^+ X_0$, the latter being global in G in view of Corollary 3.5. Now Corollary 3.1 yields the globality of W. ■

Remark 5.2. *In view of the counterexample, it is clear that we cannot drop the assumption that there is a global wedge W' surrounding $W \cap \mathfrak{n}$. Nevertheless this condition is not necessary for globality, because if we consider the invariant wedge W in the oscillator-algebra, then this wedge is naturally global, but there is no pointed global wedge W' surrounding $W \cap \mathfrak{n} = \mathbb{R}^+ Z$.*

6. The globality problem of the Heisenberg-algebra

Let us consider the Heisenberg-algebra $\mathfrak{g}$ with the standard base $\{P, Q, Z\}$ and brackets $[P, Q] = Z$, $[P, Z] = [Q, Z] = 0$. We identify it with the Heisenberg group G via the Campbell-Hausdorff-multiplication. From [1] we know, that the hyperplane-subalgebras of $\mathfrak{g}$ play an important role. A half-space which is bounded by a subalgebra is called a halfspace-semialgebra. The following is an immediate consequence of [1, V.5.41]:

Corollary 6.1. *If W is Lie generating and not contained in a halfspace-semialgebra, then W is controllable.*

The hyperplane subaglebras of $\mathfrak{g}$ may be described as follows:

Proposition 6.2. *A hyperplane $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra iff $\mathfrak{z} \subseteq \mathfrak{h}$. Any proper subalgebra is contained in a hyperplane subalgebra.*

Now we are ready to solve the globality problem of the Heisenberg-algebra.

Theorem 6.3. *Suppose $W \subseteq \mathfrak{g}$ is a Lie-wedge, then the following are equivalent:*

(i) W is global in G.

(ii) W is contained in a halfspace-semialgebra.

(iii) $W^* \cap \mathfrak{z}^\perp \neq \{0\}$.

Proof. The equivalence of (ii) and (iii) follows from the classification of hyperplane subalgebras in $\mathfrak{g}$. For proving (i)$\Rightarrow$(ii), we consider a global wedge W. If W is Lie generating then Corollary 6.1 proves (ii). If W is not Lie generating, then it is contained in a proper subalgebra. Hence (ii) holds in view of Proposition 6.2. The crucial part of the proof is (iii)$\Rightarrow$(i).
We have to consider the different cases of $\dim H(W)$ seperately:

1. $\dim H(W) = 2$. In this case W is either a hyperplane-subalgebra or a halfspace-semialgebra, hence global.

2. $\dim H(W) = 1$. The invariance of W under $e^{ad\,H(W)}$ implies
$$[H(W), W] \subseteq H(W) \cap \mathfrak{z},$$
thus either $H(W) = \mathfrak{z}$ or $W \subseteq H(W) + \mathfrak{z}$. In the first case, W satisfies the conditions of corollary 3.1, and in the second case W is contained in an abelian subalgebra, so W is global in any case.

3. $\dim H(W) = 0$. Let $\omega \in W^* \cap \mathfrak{z}^\perp$, then $\mathfrak{n} := \ker \omega$ is an abelian subalgebra and $G/N \simeq \mathbb{R}$. Take an arbitrary pointed W' surrounding W in $\mathfrak{g}$, then $W' \cap \mathfrak{n}$ is pointed, surrounds $W \cap \mathfrak{n}$ in $\mathfrak{n}$ and is global, since $\mathfrak{n}$ is abelian. Therefore Theorem 5.1 applies, proving the globality of W.

Thus everything is proved. ■

7. Some remarks on Lorentzian Lie groups

In [2] Levichev and Levicheva consider the Lorentzian manifold structure on a Lie group G obtained by choosing a Lorentzian wedge $W \subseteq \mathfrak{g}$. For a closed semigroup $S \subseteq G$ we may define the partial order $\leq_S$ on G by $x \leq_S y$ iff $xS \ni y$. The notion of *future-distinguishability* is then equivalent to the condition that S is pointed, i.e. $S \cap S^{-1} = \{1\}$. Hence the Lorentzian manifold G is future-distinguishing iff the Lie wedge W is global in G.

It may happen that the Lorentzian manifold G is not geodesically complete, i.e. if we denote *Exp* the exponential function of the affine connection on G induced by the choice of a bilinear form on $\mathfrak{g}$, then Exp_1 is not defined on all of $\mathfrak{g}$. Nevertheless the Lie semigroup generated by a Lorentzian cone is always well-defined, so the future-distinguishability-property is independent of geodesically completeness. Thus it may happen that the future of an event is well-behaved although some geodesics are not infinitely extendable.

As a final remark, we mention that Theorem 5.1 applies to the cases considered in [2], because there the hyperplane ideal $\mathfrak{n}$ is either abelian or almost-abelian, and therefore every wedge $W' \subseteq \mathfrak{n}$ is global in the corresponding group (cf. [1, V.4.13]).

References

[1] Hilgert, J., K. H. Hofmann and J. D. Lawson, "Lie groups, Convex Cones and Semigroups," Oxford University Press, 1989.

[2] Levichev A., and V. Levicheva, *Distinguishability Condition and the Future Subsemigroup,* Seminar Sophus Lie **2** 1992, 205–212.

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