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Covariant Bimodules and Differential Calculi on Quantum Groups of Type B, C, D

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1. Introduction

Non-Commutative differential calculi on quantum groups were first studied by S. L. Woronowicz [W1], [W2]. His seminal paper [W2] provides a general framework for such calculi. Following the general ideas of A. Connes the basic objects in this approach are differential forms rather than vector fields. It seems that there is no functorial way to construct differential calculi for general quantum groups. In the meantime concrete examples of covariant differential calculi on quantum spaces have been constructed by several authors such as [CS], [J], [PW], [W1], [W2] and others, but only very few papers (cf. e.g. [PW]) are concerned with the classification of such calculi.
In this work we present some results which classify all bicovariant bimodules and bicovariant differential calculi on quantum groups of type B, C and D. The crucial assumption is that the differentials of the coordinate functions form a basis for the left module of differential forms. The result on classification are Theorem 5 and 6. The latter may be rephrased by saying that there is up to isomorphism a unique N^2-dimensional bicovariant differential calculus on quantum groups of type B_n, C_n, D_n provided that the deformation parameter q is not a root of unity and that $N \geq 3$, where $N = 2n+1$ for B_n and $N = 2n$ for C_n and D_n.
In case C_1 ($N = 2$, $Sp_q(1) = SL_{q^2}(2)$) there are precisely two non-isomorphic 4-dimensional bicovariant differential calculi which correspond to the $4D_\pm$ calculi on $SU_q(2)$ constructed by Woronowicz [W2]. Details of proofs and additional related results will appear elsewhere.

2. Basic definitions and preliminaries

Let us collect some definitions, notations and facts which are used in the sequel. They are more or less standard, see e.g. [W2]. Throughout we use the convention to sum over repeated indices. We assume that all algebras are complex and have a unit denoted by 1, and that algebra homomorphisms preserve the units. q is always a fixed non-zero complex number. Let A be a Hopf algebra. The comultiplication and the counit are denoted by Δ and ε, respectively.
A linear space Γ is said to be a *left comodule* over A if there exists a linear mapping $\Phi_L : \Gamma \to A \otimes \Gamma$ such that:

$$(id \otimes \Phi_L)\Phi_L \quad = \quad (\Delta \otimes id)\Phi_L$$

$$\text{and} \qquad (\varepsilon \otimes id)\Phi_L = id.$$

In a similar way a *right comodule* over A is defined. A linear space Γ is called a *bicomodule* if there are linear mappings Φ_L and Φ_R such that Γ is a left and a right comodule over A and both actions commute:

$$(id \otimes \Phi_R)\Phi_L = (\Phi_L \otimes id)\Phi_R.$$

A bimodule Γ is called *bicovariant* if Γ possesses the structure of a bicomodule over A, say Φ_L and Φ_R, which is compatible with the module structure of Γ:

$$\begin{aligned} \Phi_L(a\omega b) &= \Delta(a)\Phi_L(\omega)\Delta(b), \\ \Phi_R(a\omega b) &= \Delta(a)\Phi_R(\omega)\Delta(b) \qquad \forall a, b \in A, \ \ \omega \in \Gamma. \end{aligned}$$

A first order differential calculus (or briefly, a *differential calculus*) over A is a pair (Γ, d), where Γ is a bimodule over A and $d{:}\, A \to \Gamma$ is a linear mapping such that

$$d(xy) = dx \cdot y + x \cdot dy \quad for \quad x, y \in A$$

and Γ is the linear span of $x \cdot dy$, where $x, y \in A$. Two differential calculi (Γ_1, d_1) and (Γ_2, d_2) over A are said to be *isomorphic* if there is a bimodule isomorphism $\psi{:}\, \Gamma_1 \to \Gamma_2$ such that $\psi d_1 = d_2$.
We say that a differential calculus (Γ, d) over A is *bicovariant* if (Γ, Φ_R, Φ_L) is a bicovariant bimodule over A such that

$$\begin{aligned} \Phi_L(da) &= (id \otimes d)\Delta a \\ \text{and} \qquad \Phi_R(da) &= (d \otimes id)\Delta a \qquad \text{for} \quad a \in A. \end{aligned}$$

The structure of bicovariant bimodules over A has been completely characterized by Theorem 2.4 and 2.5 in [W2]. We briefly recall this result:

Let (Γ, Φ_L, Φ_R) be a bicovariant bimodule over A and let $\{\omega_i\}_{i \in I}$ be a basis of the vector space $\Gamma_{inv} := \{\omega \in \Gamma{:}\, \Phi_L(\omega) = 1 \otimes \omega\}$ of left-invariant elements of Γ. Then there are matrices $v = (v^i_j)$ and $f = (f^i_j)$ of elements $v^i_j \in A$ and of functionals $f^i_j \in A'$, $i, j \in I$, such that v defines a representation of the coalgebra A and f gives a representation of the algebra A. The right module structure and the right comodule structure can be written in terms of these representations

$$\begin{aligned} \omega_j\, a &= (f^i_j * a)\omega_i \\ \text{and} \qquad \Phi_R(\omega_j) &= \omega_i \otimes v^i_j \qquad \text{for} \quad j \in I \quad \text{and} \quad a \in A. \end{aligned}$$

The bimodule Γ is generated by Γ_{inv}. Further, v and f satisfy the following compatibility condition which is very crucial for our investigations:

(c) $$v^i_j(a * f^k_i) = (f^i_j * a)v^k_i \ \text{ for all } j, k \in I \text{ and } a \in A.$$

Conversely, each bicovariant bimodule can be given in that way.

3. Bicovariant differential calculi on quantum groups of type b, c, d

Throughout this section let A be one the Hopf algebras for the quantum groups B_n, C_n, or D_n as defined in [FRT]. Let $\hat{R}$ and $C = (c^i_j)$ be the corresponding matrices from [FRT]. We suppose throughout the remainder of this paper that the complex number q *is not a root of unity.*
Recall that the algebra A is the quotient $\mathbb{C}\langle u^i_j : i,j = 1,...,N\rangle/J$ of the free algebra generated by the N^2 entries of the matrix $u = (u^i_j)$ by the two-sided ideal J generated by the relations

$$\begin{aligned} \hat{R}\,(u \otimes u) &= (u \otimes u)\,\hat{R} && (1)\\ \textit{and}\quad u^t \cdot C \cdot u &= u \cdot C \cdot u^t = C\,. && (2) \end{aligned}$$

Our aim is to classify the bicovariant differential calculi (Γ, d) over A which fulfill the following assumption:

$$(*) \qquad \{du^i_j : i,j = 1,...,N\} \textit{ is a basis of the left module } \Gamma.$$

Suppose that (Γ, d) is such a differential calculus over A. Put

$$\omega_{ij} := u^r_i\, b^r_s\, du^s_j\,,$$

where $C^{-1} = (b^r_s)$ denotes the invers matrix of C. Because of $(*)$, the elements ω_{ij}, $i,j = 1,...,N$, form a basis of the vector space of left invariant forms of Γ. Moreover, we have

$$\text{(d)} \qquad du^i_j = u^i_r\, c^r_s\, d\omega_{sj}$$

Using that (Γ, d) is bicovariant, it is straightforward to compute that

$$\Phi_R\,(\omega_{ij}) = \omega_{kl} \otimes u^k_i\, u^l_j\,,$$

i.e. the representation v of the coalgebra A for the corresponding bicovariant bimodule (Γ, Φ_R, Φ_L) is given by $v = u \otimes u$. The classification of all such bimodules appears as a subproblem which might be of interest in itself.

3.1. Bicovariant Bimodules over A with $v = u \otimes u$

Let Γ be the free left module over A generated by N^2 elements ω_{ij}, $i,j = 1,...,N$. Take $v = u \otimes u$ as a representation of the coalgebra A. The following lemma gives necessary and sufficient conditions for a representation $f \in A' \otimes L(\mathbb{C}^N \otimes \mathbb{C}^N)$ of the algebra A such that f together with v defines a bicovariant bimodule over A according to WORONOWICZ' Theorem reproduced in Section 2.

Lemma 1. *Suppose that Γ is a bicovariant bimodule over A with $v = u \otimes u$. Let f, where $f = (f^{ij}_{kl})$, be the corresponding represenation of the algebra A on the* *vector space* $\mathbb{C}^N \otimes \mathbb{C}^N$. *Define a linear mapping* $T = (T^{ijk}_{lmn}) \in L(\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N)$ *by*

$$T^{ijk}_{lmn} := f^{jk}_{lm}(u^i_n)\,. \tag{3}$$

Then we have

$$\begin{aligned} T\,(u \otimes u \otimes u) &= (u \otimes u \otimes u)\,T\,, & (4)\\ \hat{R}_{12}\,T_{234}\,T_{123} &= T_{234}\,T_{123}\,\hat{R}_{34}\,, & (5)\\ T \cdot D\,(T) &= D\,(T) \cdot T = I\,, & (6) \end{aligned}$$

where

$$D\,(T)^{ijk}_{lmn} := b^l_r\,T^{rij}_{mns}\,c^s_k\,.$$

Conversely, if $T \in L(\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N)$ *satisfies the equations* (4), (5) *and* (6), *then there is a unique bicovariant bimodule over* A *with* $v = u \otimes u$ *and* f *such that* (3) *holds.*

In Lemma 1 and in the sequel we shall use the following notation: Lower indices always refer to the components of the tensor product $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ where the corresponding operators act, i.e.

$$(\hat{R}_{12})^{abcd}_{rstu} = \hat{R}^{ab}_{rs} \cdot \delta^c_t \cdot \delta^d_u, \qquad (T_{234})^{abcd}_{rstu} = \delta^a_r \cdot T^{bcd}_{stu} \text{ etc.}$$

The equations (5) and (6) in Lemma 1 express the fact that the functionals f^{ij}_{kl} preserve the relations (1) and (2). In the proof of the second part of Lemma 1 they are needed to ensure that the functionals f^{ij}_{kl} defined first on the free algebra $\mathbb{C}\langle u^i_j : i,j = 1,...,N\rangle$ via (3) annihilate the ideal J.

The crucial condition in Lemma 1 is equation (4). It is equivalent to the compatibility condition (c) formulated in Section 2. By results of N. YU. RESHETIKHIN [R], there is a homomorphism of the centralizer algebra of $u \otimes u \otimes u$ on the Birman-Wenzl-Murakami algebra [BW]. If $N \geq 3$, then this homomorphism is faithful and we are led to an algebraic problem in the BWM-algebra. Its solution is stated as

Theorem 2. *Suppose that* $N \geq 3$. *Up to the sign all solutions* T *of equations* (4), (5) *and* (6) *are a two-parameter-family and* 16 *additional solutions. Four of the latter solutions are*

$$\hat{R}_{12}\,\hat{R}_{23}\,,\ \hat{R}^{-1}_{12}\,\hat{R}^{-1}_{23}\,,\ \hat{R}_{12}\,\hat{R}_{23} - 2x^{-1}\,K_{23}\,K_{12}\,,\ \hat{R}^{-1}_{12}\,\hat{R}^{-1}_{23} - 2x^{-1}\,K_{23}\,K_{12}\,,$$

where

$$K := I + Q^{-1}(\hat{R}^{-1} - \hat{R}), \quad Q := q - q^{-1}, \quad x := 1 + Q^{-1}(r - r^{-1}), \quad r = \ q^{N-1}$$

for B_n, D_n *and* $r = -q^{N+1}$ *for* C_n.
The two-parameter family is given by

$$X_{23}\,\hat{R}^{-1}_{12}\,\hat{R}_{23}\,X^{-1}_{12}\,,$$

where X is an arbitrary regular linear transformation in the linear span of I, $\hat{R}$ and $\hat{R}^{-1}$.

Remark: All bimodules of the continuous family are isomorphic.
Recall that the Hopf algebras $Sp_q(1)$ and $SL_{q^2}(2)$ coinside, so in this case $N = 2$ it is sufficient to treat our problem for the Hopf algebra $A = SL_q(2)$. Then the matrices C and $\hat{R}$ occuring in our results are of the following form

$$C = \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ -q^{\frac{1}{2}} & 0 \end{pmatrix}, \qquad \hat{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Q & q & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 3. *Suppose that $A = C_1$. Up to the sign all solutions T of equations (4), (5) and (6) are a one-parameter-family and 4 additional solutions, namely*

$$q^{-1}\hat{R}_{12}\,\hat{R}_{23}\,,\; q\hat{R}_{12}^{-1}\,\hat{R}_{23}^{-1}\,,\; q^{-1}\hat{R}_{12}\,\hat{R}_{23} - 2x^{-1}\,K_{23}\,K_{12}\,,\; q\hat{R}_{12}^{-1}\,\hat{R}_{23}^{-1} - 2x^{-1}\,K_{23}\,K_{12}\,,$$

where $K := \hat{R} - qI$, $\quad x := -q - q^{-1}$ and Q as above.
The one-parameter-family is given by

$$X_{23}\,\hat{R}_{12}^{-1}\,\hat{R}_{23}\,X_{12}^{-1}\,,$$

where X is an arbitrary regular linear transformation in the linear span of I and $\hat{R}$.

3.2. Bicovariant Differential Calculi

Having a bicovariant bimodule with $v = u \otimes u$, we define du^i_j by equation (d) and we extend d to the free algebra $\mathbb{C}\langle u^i_j : i,j = 1,...,N\rangle$ by the Leibniz rule. In order to get a differential calculus over A, d has to annihilate the ideal J. We formulate this property in terms of the operator T.

Lemma 4. *Suppose that (Γ, d) is a bicovariant differential calculus over A which fulfills assumption $(*)$ and $N \geq 3$. Then the linear operator T as in Lemma 1 satisfies the equations (4), (5), (6) and moreover*

$$\begin{aligned} tr^1_1 T &= -I, \text{ and} && (7) \\ tr^{12}(T_{234}\hat{R}_{12}) + \hat{R}_{12}^{-1} &= T\hat{R}_{23} + K_{12}\hat{R}_{23}, \text{ with} && (8) \\ (tr^1_1 T)^{bc}_{st} &= b^z_x c^z_y T^{xbc}_{yst}, \text{ and} \\ (tr^{12} S)^{abc}_{rst} &= b^a_z c^x_y S^{xybc}_{zrst}. \end{aligned}$$

Conversely, if T fulfills the equations (4) – (8), then (d) defines a bicovariant differential calculus over A which satisfies the condition $(*)$.

Solving our system of equations (4) – (8) we get our main result on classification of bicovariant differential calculi.

Theorem 5. *For $N \geq 3$ there exists up to isomorphism precisely one bicovariant differential calculus over A which satisfies the condition $(*)$. The corresponding linear mapping T is given by $T = X_{23}\,\hat{R}_{12}^{-1}\,\hat{R}_{23}\,X_{12}^{-1}$, where $X = (r-Q)I + \hat{R} - \frac{Q}{r^2-1}K$ and Q and r are as above.*

In case $A = C_1$ Lemma 4 remains valid if equation (8) is modified, that is, if $\hat{R}_{12}^{-1}$ is replaced by $q\hat{R}_{12}^{-1}$ and if K is chosen as in Theorem 3. In this case we obtain the following classification theorem

Theorem 6. *Let $A = C_1$. There exist up to isomorphism precisely two bicovariant differential calculi over A which satisfy the condition $(*)$. The corresponding linear mappings T are given by*

$$T_+ = X_{23}\,\hat{R}_{12}^{-1}\,\hat{R}_{23}\,X_{12}^{-1} \quad \textit{and} \qquad T_- = -Y_{23}\,\hat{R}_{12}^{-1}\,\hat{R}_{23}\,Y_{12}^{-1},$$

where

$$X = (q^2 - Q)I + \hat{R} \quad \textit{and} \quad Y = (q^2 + Q)I - \hat{R}.$$

It can be shown that the above calculi are of the form $da = \tilde{\omega}a - a\tilde{\omega}$, where $\tilde{\omega}$ is the unique up to scalar multiples right and left invariant element of Γ. This approach for the construction of differential calculi was used in [CS] and also in [J].

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