Seminar Sophus Lie
3 (1993) 35–46

New sufficient criteria for the globality of Lie wedges

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Abstract

We consider a simply connected Lie group G with Lie algebra $\mathfrak{g}$ and Lie semigroups $S, T \subseteq G$ with tangent wedges $L(S)$, resp. $L(T)$. We give some sufficient conditions for $W := L(S) + L(T)$ to be a global Lie wedge. The first theorem applies to the situation where $G = NH$ is a semidirect decomposition and W is adapted to this decomposition in the sense of $L(S) \subseteq \mathfrak{n}$ and $L(T) \subseteq \mathfrak{h}$ while the second theorem applies to the case where the semigroup generated by S and T admits a product decomposition $\langle S \cup T \rangle = ST$. As an application, we prove that in a three-dimensional Lie algebra every Lie wedge that lies in the intersection of two distinct halfspace-semialgebras is global in the corresponding simply connected Lie group.

1. Introduction

First we recall some definitions and notations concerning Lie wedges, Lie semigroups and orders. In what follows, G denotes an arbitrary Lie group and $\mathfrak{g}$ its Lie algebra. We consider only real Lie algebras.

Definition 1.1. *Suppose $S \subseteq G$ is a subsemigroup, then we define the tangent wedge $L(S)$ by $X \in L(S)$ if and only if there exist sequences $(s_j)_{j \in \mathbb{N}} \subseteq S$ and $m_j \in \mathbb{R}^+$ such that*

$$\lim_j s_j = 1, \qquad \textit{and} \qquad X = \lim_j m_j \log s_j,$$

where log *denotes the logarithm which is well-defined on a sufficiently small neighborhood of* 1.
A subsemigroup $S \subseteq G$ is a Lie semigroup *iff it is reconstructible from its tangent wedge, i.e.*

$$S = \overline{\langle \exp L(S) \rangle}.$$

Remark 1.2. *Four our purposes the former definition turns out to be more convenient than the equivalent (see [1])*

$$L(S) \overset{def}{=} \left\{ X \in \mathfrak{g} \mid \exp \mathbb{R}^+ X \subseteq \overline{S} \right\}.$$

*supported by a DFG-grant

In any case it's worth to note that $L(S) = L(\overline{S})$. *Therefore we will be able to restrict most of our considerations to dense subsemigroups of Lie semigroups.*

Definition 1.3. *Let* $\mathfrak{g}$ *be a Lie algebra,* $W \subseteq \mathfrak{g}$ *is a* Lie wedge*, if it satisfies*

$$\overline{W} = W, \quad W + W = W, \quad \mathbb{R}^+ W = W \text{ and } e^{ad\, H(W)} W = W,$$

where $H(W) \overset{def}{=} W \cap -W$ *is called the* edge of W *and* ad *is the adjoint representation.*
Let G *be a Lie group with Lie algebra* $\mathfrak{g}$. *A Lie wedge* $W \subseteq \mathfrak{g}$ *is* global in G, *if there exists a subsemigroup* $S \subseteq G$ *with* $W = L(S)$. *A Lie wedge is* global, *if it is global in the simply connected Lie group corresponding to* $\mathfrak{g}$.

The starting point of the Lie theory of semigroups was the observation that for any subsemigroup $S \subseteq G$ the tangent wedge $L(S) \subseteq \mathfrak{g}$ is a Lie wedge. Conversely, if $W \subseteq \mathfrak{g}$ is a Lie wedge, then $S(W) \overset{def}{=} \overline{\langle \exp_G W \rangle}$ is the smallest Lie semigroup with $L(S(W)) \supseteq W$, but the inclusion may be proper. Equality holds if and only if W is global in G. The so-called *halfspace semialgebras* are of special importance, because they are maximal global Lie wedges:

Definition 1.4. *A* halfspace-semialgebra *in a Lie algebra* $\mathfrak{g}$ *is a halfspace such that its bounding hyperplane is a subalgebra. An intersection of halfspace-semialgebras is called an* intersection-semialgebra.

Finally every semigroup gives rise to a left-invariant partial order on the whole group:

Definition 1.5. *Let* $S \subseteq G$ *be a subsemigroup, then we denote with* $\leq_S$ *the left invariant order on* G *induced by* S:

$$x \leq_S y \quad \textit{iff } xS \ni y.$$

For $x, y \in G$ *the* order interval $[x,y]_S$ *is defined as* $[x,y]_S = xS \cap yS^{-1}$.

2. A globality theorem for wedges that are adapted to a semidirect decomposition

Lemma 2.1. *Suppose* G *is a Lie Group,* N *a closed normal subgroup,* H *a closed subgroup satisfying* $N \cap H = \{1\}$ *and* $NH = \overline{NH}$. *Assume* $g_j = n_j h_j$ *with* $n_j \in N$, $h_j \in H$ *and* $\lim_j g_j = 1$. *Then* h_j *converges to* 1 *and so does* n_j.

Proof. We consider the factor group G/N. The quotient map $\pi: G \to G/N$ is continuous, hence $\pi|_H: H \to \pi(H) \subseteq G/N$ is a surjective, continuous homomorphism onto a locally compact group. Since $H \cap N = \{1\}$ we know that $\pi|_H$ is injective, hence bijective. Now the open mapping theorem applies and yields a continuous homomorphism $\psi: \pi(H) \to H$ which inverts $\pi|_H$.
The continuity of π and $g_j \to 1$ in G imply $\pi(g_j) = \pi(h_j) \to 1$ in G/N. Therefore $h_j = \psi(\pi(h_j)) \to 1$ by the continuity of ψ. Thus $h_j \to 1$ in G which immediately yields $n_j = g_j h_j^{-1} \to 1$ in G. ∎

Theorem 2.2. *Suppose G is a Lie group, $N = \bar{N} \trianglelefteq G$ a normal subgroup and $H = \bar{H} \leq G$ a Lie subgroup such that $N \cap H = \{1\}$ and $NH \subseteq G$ is closed. Let $\overline{S} \subseteq N$ and $\overline{T} \subseteq H$ be Lie semigroups with $int_H(T) \neq \emptyset$ and tangent wedges $L(S) \subseteq \mathfrak{n}$, $L(T) \subseteq \mathfrak{h}$. We define the following family of wedges:*

$$\mathcal{W} = \{W' \subseteq \mathfrak{n} \mid W' \textit{ global in } N,\ (\exists t_0 \in int_H T)(\forall t \in [1,t_0]_T)\, Ad(t)\, L(S) \subseteq W'\}\,.$$

Then the following holds:
If $\bigcap \mathcal{W} = L(S)$ then the Lie wedge $L(S) + L(T)$ is global in G.

Proof. Let $t \in T$, then we denote with $\alpha(t)$ the inner automorphism of G induced by $t \in T$, i.e. $\alpha(t)(g) = tgt^{-1}$ and we let

$$E_t = \{\alpha(t_1)(s_1)\ldots\alpha(t_n)(s_n)s \mid t_i \in [1,t]_T,\ s_i \in \exp L(S)\}$$

and $E = \bigcup_{t\in T} E_t$. We will prove the following:

(i) E is a semigroup because for $t, t' \in T$ we have $E_t E_{t'} \subseteq E_{tt'}$. Indeed, if $g = (\prod_{i=1}^n \alpha(t_i)(s_i))t \in E_t$ and $g' = (\prod_{j=1}^m \alpha(t'_j)(s'_j))t' \in E_{t'}$ then

$$\begin{aligned} gg' &= \left(\prod_{i=1}^n \alpha(t_i)(s_i)\right) t \left(\prod_{j=1}^m \alpha(t'_j)(s'_j)\right) t' \\ &= \left(\prod_{i=1}^n \alpha(t_i)(s_i)\right) \alpha(t) \left(\prod_{j=1}^m \alpha(t'_j)(s'_j)\right) tt' \\ &= \left(\prod_{i=1}^n \alpha(t_i)(s_i)\right) \left(\prod_{j=1}^m \alpha(tt'_j)(s'_j)\right) tt' \in E_{tt'}. \end{aligned}$$

(ii) $L(E) \supseteq L(S) + L(T)$ for obviously $\exp L(S) \cup \exp L(T) \subseteq E$.

(iii) $L(E) \subseteq L(S) + L(T)$. This is the non-trivial part of the proof. Let $Y \in L(E)$ then there exist sequences $m_j \in \mathbb{R}^+$ and $q_j \in E$ such that $\lim_j q_j = 1$ and $Y = \lim_j m_j \log(q_j)$. Now $q_j = n_j t_j$ with $n_j \in N$, $t_j \in T$ and $\lim_j n_j = 1 = \lim_j t_j$. Applying the Campbell-Hausdorff-formula we obtain

$$m_j \log q_j = m_j \log n_j t_j = m_j(\log n_j * \log t_j) = \underbrace{m_j \log n_j}_{\in\mathfrak{n}} + \underbrace{m_j \log t_j}_{\in\mathfrak{h}} + \underbrace{m_j r_j}_{\in\mathfrak{n}}\,.$$

Since $L(H) \cap L(N) = \{0\}$ the sequence $m_j \log t_j$ converges and since $t_j \in T$, $t_j \to 1$ it follows that $\lim m_j \log t_j \in L(T)$. Since $\|r_j\| \leq c\,\|\log t_j\|\,\|\log n_j\|$ for a suitable norm and positive constant c, we conclude $\lim_j r_j = 0$ for $\|\log n_j\| \to 0$ and $m_j \|\log s_j\|$ is bounded. Thus $\lim_j m_j \log n_j$ also exists. Now let $W' \in \mathcal{W}$ be given, then there is a $t_0 \in int_H(T)$ such that $Ad(t)\, L(S) \subseteq W'$ for all $t \in [1,t_0]_T$. But $t_0 \in int_H(T)$ implies the existence of an open, symmetric $U = U^{-1}$ such that $Ut_0 \cap H \subseteq T$. Therefore $H \cap U = H \cap U^{-1} \subseteq t_0 T^{-1}$. Since $t_j \to 1$, we may conclude $t_j \leq_T t_0$ for all sufficiently large j. Hence $n_j \in \langle \exp W' \rangle$ and the globality of W' in N implies $\lim m_j \log n_j \in W'$. This proves $Y \in W' + L(T)$ for all $W' \in \mathcal{W}$, hence

$$L(E) \subseteq \bigcap_{W'\in\mathcal{W}} W' + L(T) = L(S) + L(T),$$

by our assumption on $\bigcap \mathcal{W}$ which proves (iii).

From (i)–(iii) we deduce that $L(S)+L(T)=L(E)$ is the tangent wedge of a subsemigroup $E\subseteq G$, therefore it is a Lie wedge and global in G. ■

This is a slightly more general version of Theorem 1.1 in [2]. There was an additional assumption imposed on $L(S)$ and $L(T)$ in order to ensure that $L(S)+L(T)$ is a Lie wedge, but this assumption may be dropped, because the latter follows automatically from $L(E)=L(S)+L(T)$. In fact the assumption made in [2] is necessary (but not sufficient) for $\bigcap\mathcal{W}=L(S)$ to hold. Investigating the condition $\bigcap\mathcal{W}=L(S)$, one obtains by the same arguments as in [2] the following corollary:

Corollary 2.3. *If the ideal $\mathfrak{n}$ is almost-abelian and $L(S)$, $L(T)$ are pointed wedges, then $\bigcap\mathcal{W}=L(S)$ always holds, i.e. $L(S)+L(T)$ is global in G.*

3. A globality Theorem for a sum of pointed wedges

Proposition 3.1. *Let G be a topological group and $S,T\subseteq G$. Let $t_j\in T$ and $s_j\in S$ be arbitrary sequences, then the following are equivalent:*

(i) $s_jt_j\to 1$ *iff* $s_j\to 1$ *and* $t_j\to 1$.

(ii) $(\forall V)(\exists U)S\cap UT^{-1}\subseteq V$.

(iii) $(\forall V)(\exists U)S^{-1}U\cap T\subseteq V$.

Proof. We will prove: (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i).
Suppose (i) holds, so $s_jt_j\to 1$ implies $s_j\to 1$ and $t_j\to 1$. Now we assume that there is a V such that for any U we have $S\cap UT^{-1}\not\subseteq V$. Then we take a sequence U_j of 1-neighborhoods converging to 1. According to our assumption we find an element $s_j\in(S\cap U_jT^{-1})\setminus V$ and an $t_j\in T$ with $s_j\in U_jt_j^{-1}$. Hence $s_jt_j\in U_j$ which implies $s_jt_j\to 1$ but $s_j\not\to 1$ since $s_j\in V^C$, a contradiction to (i). Thus (ii) must hold.
Next we suppose that (ii) holds. Let V be given then we choose $V'=V'^{-1}$ with $V'V'\subseteq V$. Since (ii) holds, we find a $U=U^{-1}\subseteq V'$ such that $S\cap UT^{-1}\subseteq V'$. Hence $S^{-1}\cap TU\subseteq V'$ which proves (iii) since

$$S^{-1}U\cap T\subseteq(S^{-1}\cap TU)U\subseteq V'U\subseteq V'V'\subseteq V.$$

Finally we assume that (iii) holds and that $s_jt_j\to 1$. We will prove $t_j\to 1$, i.e. $(\forall V)(\exists j_0)(\forall j\geq j_0)s_j\in V$. Let V be given, then we can find U such that $S^{-1}U\cap T\subseteq V$. Choose j_0 such that $s_jt_j\in U$ for all $j\geq j_0$. Then we have $t_j\in s_j^{-1}U$, thus $t_j\in T^{-1}U\cap S\subseteq V$ for all $j\geq j_0$ proving (i). ■

Definition 3.2. *Let $S,T\subseteq G$ be Lie semigroups. We say that S and T are* well-separated, *if one of the equivalent conditions in Proposition 3.1 holds.*
Let $W_1,W_2\subseteq V$ be two wedges in a vector space. We call W_1 and W_2 well-separated, *if $W_1\cap -W_2=\{0\}$.*

Remark 3.3. *If the Lie semigroups S and T are well-separated, then so are their tangent wedges $L(S)$ and $L(T)$. The converse is not true. It suffices to choose W_1 pointed and controllable in order to obtain a counterexample.*

Proposition 3.4. *Let W_1, W_2 be pointed and well-separated wedges, then their sum $W_1 + W_2$ is also a pointed wedge. In addition, we can find surrounding wedges $\widetilde{W_1}$ and $\widetilde{W_2}$ which are still well-separated.*

Theorem 3.5. *Let $\overline{S}$ and $\overline{T} \subseteq G$ be well-separated Lie semigroups with pointed tangent wedges $L(S)$ and $L(T)$. We further assume that $\langle S \cup T\rangle = ST$, then $L(S) + L(T)$ is global in G.*

Proof. Let $g \in \langle S \cup T\rangle$, then we may write $g = st$ with $s \in S$ and $t \in T$. If we have a sequence $g_j = s_j t_j \to 1$ and $Y = \lim_j m_j \log g_j$, then the well-separateness of S and T implies $t_j \to 1$ and $s_j \to 1$. Let W_S and W_T be well-separated wedges surrounding $L(S)$, resp. $L(T)$. Then

$$m_j \log(s_j t_j) = \underbrace{m_j \log s_j}_{\in W_S} + \underbrace{m_j \log t_j}_{\in W_T} + m_j r_j$$

Since W_S and W_T are well-separated, $W_S + W_T$ is pointed, so its dual has non-empty interior. Let $\omega \in int((W_S + W_T)^*)$, then there exists a constant $c > 0$ such that $\omega(w) \geq c\,\|w\|$ for all $w \in W_S + W_T$. Thus

$$\begin{aligned} \omega(m_j \log g_j) &= m_j \omega(\log s_j) + m_j \omega(\log t_j) + m_j \omega(r_j) \\ &\geq m_j (c\,\|\log s_j\| + c\,\|\log t_j\| - M\,\|\log s_j\|\,\|\log t_j\|) \end{aligned}$$

with some positive constant $M > 0$. But the function

$$f(x, y; c, M) = cx + cy - Mxy = M\left(\frac{c^2}{M^2} - \left(x - \frac{c}{M}\right)\left(y - \frac{c}{M}\right)\right)$$

is positive for all $(x, y) \in \left[0, \frac{2c}{M}\right] \times \left[0, \frac{2c}{M}\right]$. So we may conclude $\omega(Y) \geq 0$. Since $\omega \in int((W_S + W_T)^*)$ was arbitrary, we obtain $Y \in \left(int((W_S + W_T)^*)\right)^* = W_S + W_T$. Since this holds for arbitrary W_S and W_T (close enough to $L(S)$, resp. $L(T)$), we conclude

$$Y \in \bigcap_{W_S, W_T} W_S + W_T = L(S) + L(T),$$

proving our claim. ∎

Remark 3.6. *A problem always occurring in this kind of proof is getting rid of the term $m_j r_j$. This can easily be done if $S \subseteq Z(G)$ because then $r_j \equiv 0$. Another idea applies if S is already an invariant subsemigroup, i.e. $Ad(G)\,L(S) = L(S)$. In that case we may conclude $\log s_j \in L(S)$ for s_j sufficiently close to 1. Therefore $r_j \in Ad(G) \log s_j \subseteq L(S)$, implying $m_j \log g_j \in L(S) + W_T$. So we may drop the assumption that $L(T)$ is pointed.*

It may be a difficult task to prove the well-separateness of two Lie semigroups S and T. We will need the following characterization of globality from [3, II.11; IV.1.]:

Theorem 3.7. *A Lie generating wedge $W \subseteq \mathfrak{g}$ is global in G if and only if the analytic subgroup H corresponding to $H(W) = W \cap -W$ is closed in G and there exists a function $f \in C^\infty(G)$ such that*

$$\langle df(g), d\lambda_g(1)X\rangle > 0 \textit{ for all } X \in W \setminus H(W),\ g \in G,$$

i.e. $df(g) \circ d\lambda_g(1) \in \mathrm{algint} W^* = \mathrm{int}_{W^*-W^*} W^*$. The function f is called W-monotone, it is constant on all cosets gH.

In view of the preceding theorem and the globality result for subwedges of global wedges in [1, VI.5.2] the following is an interesting application:

Proposition 3.8. *Suppose $W \subseteq \mathfrak{g}$ is a global Lie wedge and $\mathfrak{h} = W \cap -W$ its edge. Suppose $\overline{S}$ and $\overline{T}$ are Lie semigroups such that*

(i) $L(S) \subseteq W$,
(ii) $L(S) \cap \mathfrak{h} = \{0\}$,
(iii) $L(T) \subseteq \mathfrak{h}$,
Then $\overline{S}$ and $\overline{T}$ are well-separated.

Proof. We have to prove: If $s_j t_j \to 1$, then $s_j \to 1$ and thus $t_j \to 1$. Since W is global, there exists a W-positive function $f: G \to \mathbb{R}$ satisfying $f(1) = 0$, $f(gT) = f(g)$ and $df(g) \circ d\lambda_g(1) \in \mathrm{algint} W^* \subseteq \mathrm{int}\, L(S)^*$. If $s_j t_j \to 1$, then $f(s_j) = f(s_j t_j) \to 0$ thus $s_j \to 1$, because f is strictly $L(S)$-positive, proving our claim. ■

Corollary 3.9. *Suppose $W \subseteq \mathfrak{g}$ is a global Lie wedge with edge $\mathfrak{h} = W \cap -W$. Suppose $\overline{S}$ and $\overline{T}$ are pointed Lie semigroups with the following properties:*

(i) $L(S) \subseteq W$,
(ii) $L(S) \cap \mathfrak{h} = \{0\}$,
(iii) $L(T) \subseteq \mathfrak{h}$,
(iv) $\langle S \cup T\rangle = ST$.
Then $L(S) + L(T)$ is global in G.

Proof. According to Proposition 3.8 the Lie semigroups $\overline{S}$ and $\overline{T}$ are well-separated, hence we may apply Theorem 3.5. ■

4. Some preliminaries on almost abelian groups

Since we are interested in having nice product decompositions of the form $\langle S \cup T\rangle = ST$, we investigate at first semigroups in the two-dimensional non-abelian Lie group because we will use these semigroups as building blocks for larger semigroups. In addition the geometric interpretation of these results is of its own interest. At first we need some information about almost abelian Lie algebras:

Definition 4.1. *A Lie algebra $\mathfrak{g}$ is called* almost abelian *if there is a linear form $\alpha: \mathfrak{g} \to \mathbb{R}$ such that the bracket is given by*

$$[X, Y] = \alpha(X)Y - \alpha(Y)X.$$

If $\alpha \neq 0$ then $\mathfrak{g}$ is called truly almost abelian.

Proposition 4.2. *For every $n \geq 2$ there exists up to isomorphism a unique truly almost abelian Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g} = n$ and a unique Lie group G with Lie algebra $\mathfrak{g}$. The group is $G \cong \mathbb{R}^{n-1} \rtimes \mathbb{R}^*_+$ with group multiplication*

$$(x, s)(y, t) = (x + sy, st)\,,$$

the Lie algebra is $\mathfrak{g} \cong \mathbb{R}^{n-1} \rtimes \mathbb{R}$ with bracket $[(x, s), (y, t)] = (sy - tx, 0)$ and the exponential function is given by

$$\exp(x, s) = \left(\frac{e^s - 1}{s} x, e^s\right) = (f(s)x, e^s).$$

Theorem 4.3. *Let $\mathfrak{g}$ be a Lie algebra, then the following are equivalent:*

(i) $\mathfrak{g}$ is almost abelian.
(ii) Every hyperplane is a subalgebra.
(iii) Every wedge is a Lie wedge.
(iii) Every wedge is a Lie semialgebra.
(iv) Every Lie wedge is global.
(v) For every Lie wedge W the following equality holds:

$$\overline{\langle \exp W \rangle} = \exp W.$$

If W is a polyhedral wedge in an almost abelian Lie algebra, then the semigroup $\exp W$ possesses nice product decompositions. This can be verified by some elementary computations, but it is best elucidated by the following geometric considerations on the two-dimensional non-abelian Lie group:
The group of motions of the real line is the unique two-dimensional non-abelian Lie group. It may be represented as $G = \mathbb{R} \rtimes \mathbb{R}^*_+$ with group operation

$$(b, a)(b', a') = (b + ab', aa').$$

The identity element is $1 = (0, 1)$ and the group may be visualised as the upper half plane in $\mathbb{R}^2$. The unique normal subgroup $N = \mathrm{comm}(G)$ corresponds to the horizontal line $\{(b, 1) \mid b \in \mathbb{R}\}$ This is a convenient representation because of the geometric significance of fundamental objects such as one-parameter subgroups and left and right cosets:

- If $g = (b, a) \in G \setminus \{1\}$, then the one-parameter subgroup containing g is just the line through g and 1. The one-parameter subgroups are the lines containing 1.

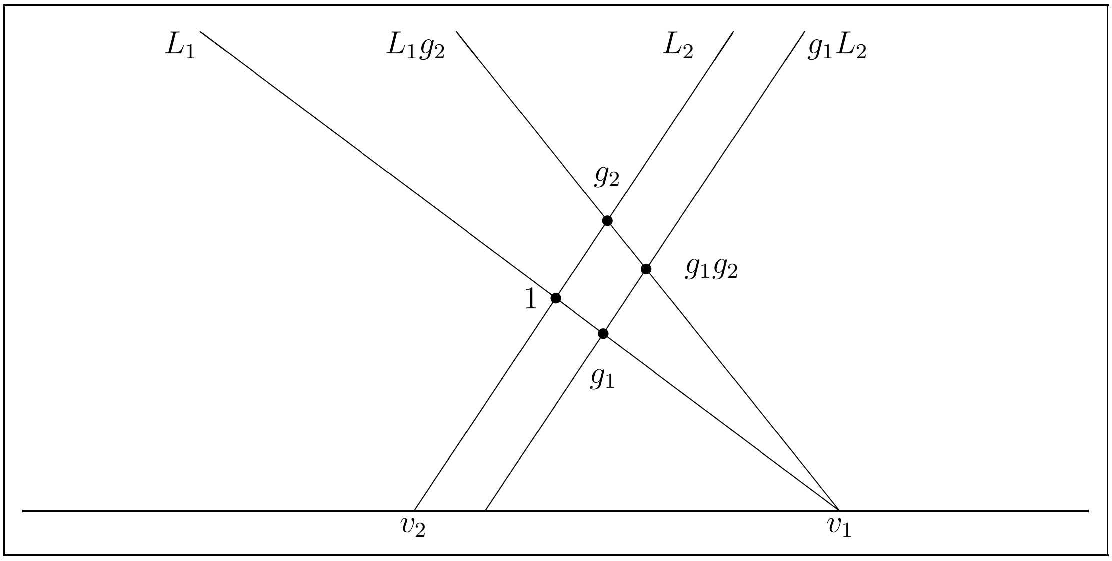

Figure 1: Geometric construction of g_1g_2

- If L is a one-parameter subgroup and $g \in G \setminus L$, then the left coset gL is the line parallel to L containing g.
- If L is a one-parameter subgroup and $g \in G \setminus L$, then the right coset Lg is the line that joins g and v_0 where v_0 is the point of intersection of L with the horizontal line $a = 0$. If L is horizontal, i.e. there is no point of intersection, then Lg is the horizontal line containing g.
- If g_1, g_2 are elements of G that do not lie on the the same one-parameter subgroup, i.e. $g_i \in L_i$ with $L_1 \neq L_2$, then one can obtain their product g_1g_2 by taking the intersection of cosets (i.e. lines):

$$g_1g_2 = g_1L_2 \cap L_1g_2,$$

Figure 1 shows how the product g_1g_2 is obtained geometrically.

A one parameter semigroup is a ray emanating from 1 and the semigroup that is generated by two such rays is the convex region inbetween. But there is a structural difference depending on whether the two rays are of the same type or not. Let us denote these ray-semigroups with S_1, S_2 and the semigroup generated by them with $S = \langle S_1 \cup S_2 \rangle$, then the following cases may occur:

(i) One of them is horizontal, w.l.o.g. S_1 is horizontal, then $S_1S_2 = S_2S_1 = S$ simply because S_1 is invariant under all inner automorphisms of G.

(ii) None of them is horizontal, either both rays meet the horizontal axis $y = 0$ or none of them does. One can assume the latter w.l.o.g. because otherwise it suffices to replace S_i by its inverse S_i^{-1}. Applying an inner automorphism one can further assume that S_1 is the vertical ray and that S_2 is located as in figure 2. Now one can easily prove $S_1S_2 = S_2S_1 = S$. Figure 2 shows how the factors $g_i, h_i \in S_i$ are determined for an arbitrary $g \in S$ such that $g_1g_2 = g = h_2h_1$.

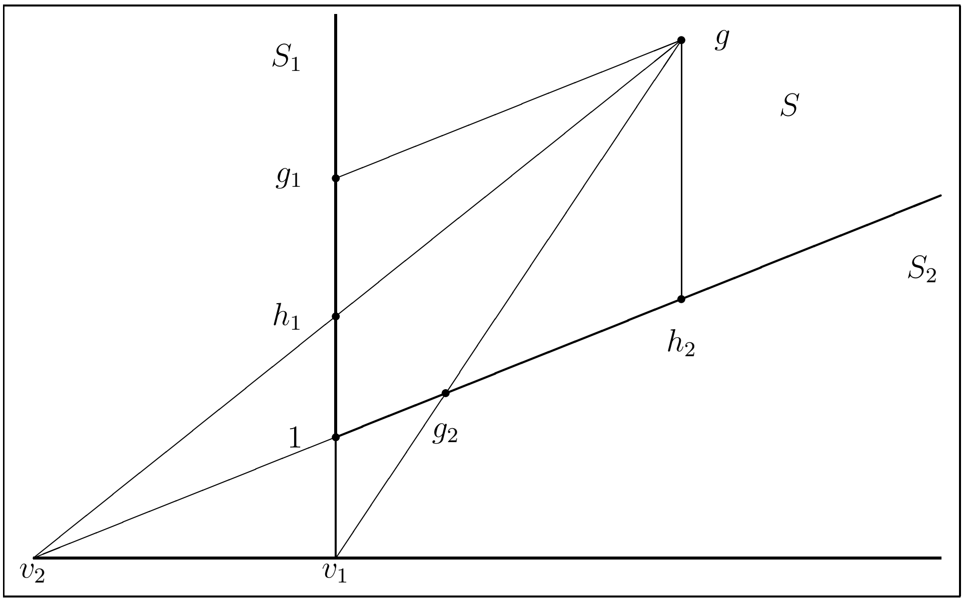

Figure 2: Geometric construction of the factors such that $g_1 g_2 = g = h_2 h_1$.

(iii) One of them, let's say S_2 meets the horizontal axis and the other one, S_1 does not. Again applying an inner automorphism of G, one may assume w.l.o.g. that the configuration is as in figures 3,4. We have $S = S_1 S_2$, but there is a large "gap" between $S_1 S_2$ and $S_2 S_1$, one can only reach the points between S_1 and the parallel ray through the intersection of S_2 with the horizontal axis, the points on this parallel ray are *not* in $S_2 S_1$. But nevertheless it's clear that $S = N_{12} S_2 S_1$ where $N_{12} = S_1 S_2 \cap N$ denotes the horizontal ray that is the intersection of S with the normal subgroup N.

Thus one can prove by repeated applications of the preceding arguments:

Proposition 4.4. *Suppose $\mathfrak{g}$ is an almost abelian Lie algebra, $S_i = \exp \mathbb{R}^+ X_i$ with $\alpha(X_i) > 0$, $T_j = \exp \mathbb{R}^+ Y_j$ with $\alpha(Y_j) < 0$ and $N_k = \exp \mathbb{R}^+ Z_k$ with $\alpha(Z_k) = 0$, $i = 1 \dots n$, $j = 1 \dots m$, $k = 1 \dots l$, then*

$$\begin{aligned}\left\langle \bigcup_{i=1}^{n} S_i \cup \bigcup_{j=1}^{m} T_j \cup \bigcup_{k=1}^{l} N_k \right\rangle &= N_1 \cdots N_k S_1 \cdots S_n T_1 \cdots T_m \\ &= N_1 \cdots N_k N_{11} N_{12} \cdots N_{nm} T_1 \cdots T_m S_1 \cdots S_n,\end{aligned}$$

with $N_{ij} = N \cap S_i T_j$, where N denotes the maximal normal subgroup of G, $N = comm(G)$.

5. Applications to three-dimensional groups

Now we are ready to prove the globality result stated at the beginning:

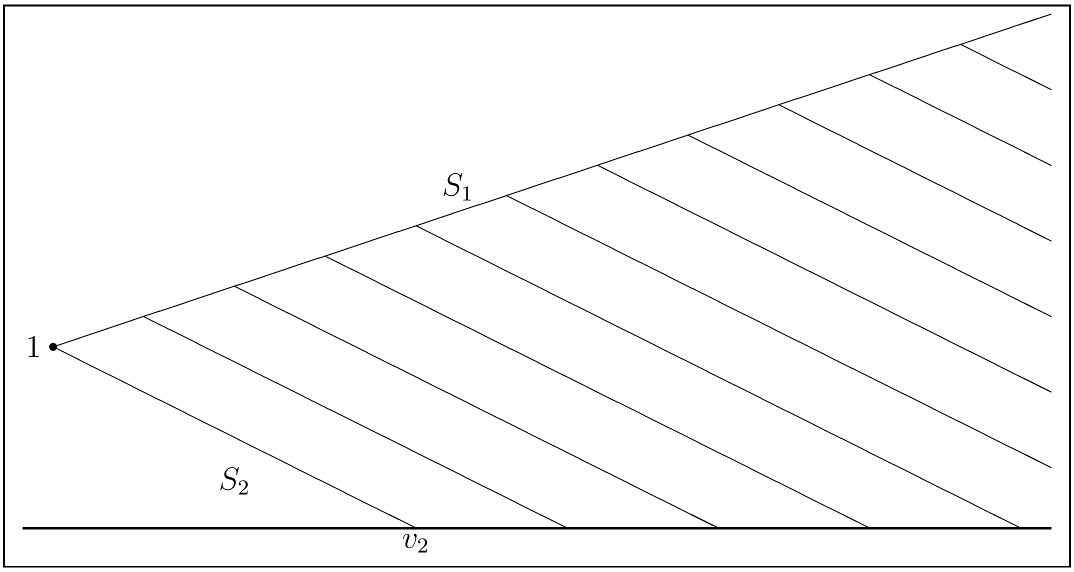

Figure 3: The set S_1S_2 regarded as union of left-translates of S_2

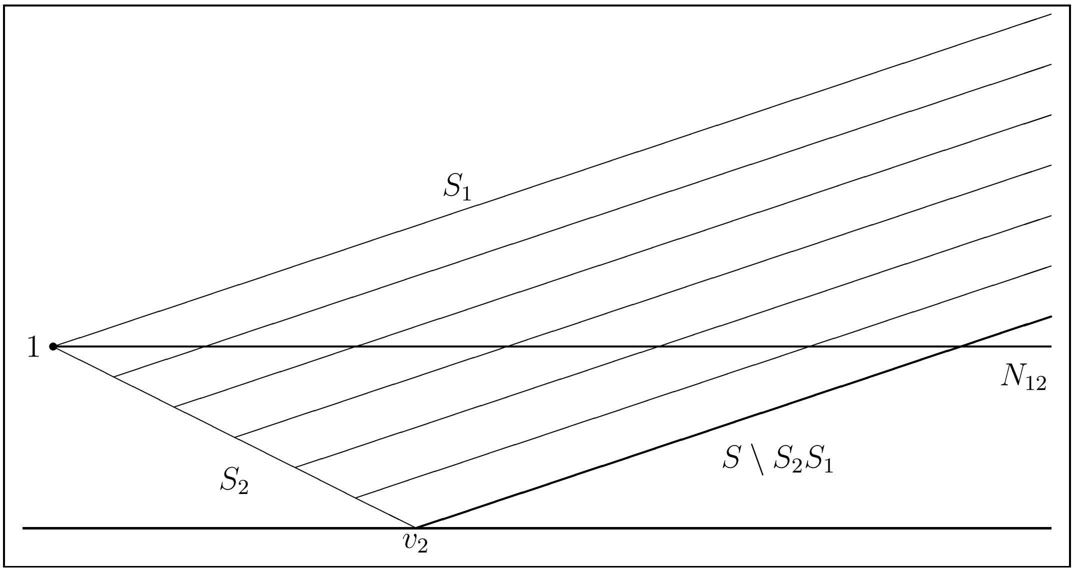

Figure 4: The set S_2S_1 visualised as union of left-translates of S_1.

Proposition 5.1. *Suppose* $\mathfrak{g}$ *is a three-dimensional Lie algebra and* $W \subseteq \mathfrak{g}$ *is a pointed Lie wedge which lies in an intersection-semialgebra, i.e.* $W \subseteq \mathfrak{h}_1^+ \cap \mathfrak{h}_2^+$, *where* $\mathfrak{h}_i^+ = \omega_i^{-1}(\mathbb{R}^+)$, $\mathfrak{h}_i = \ker \omega_i$ *is a hyperplane subalgebra and* $\mathfrak{h}_1 \neq \mathfrak{h}_2$, *i.e.* $\dim \mathfrak{h}_1 \cap \mathfrak{h}_2 = 1$. *Then* W *is global in the simply-connected Lie group corresponding to* $\mathfrak{g}$.

Proof. Let $\mathfrak{h} = \mathfrak{h}_1 \cap \mathfrak{h}_2$. If $\mathfrak{h} \cap W = \{0\}$, we may apply [1, VI.5.2] in order to prove the globality of W. Now suppose $W \cap \mathfrak{h} \neq \{0\}$. Since W is pointed, $\mathfrak{h} \backslash W \neq \emptyset$ and we can find an $\alpha \in W^* \backslash \mathfrak{h}^\perp$. Pick $0 \neq X \in \mathfrak{h} \cap W$, $0 \neq Y_i \in \ker \alpha \cap \mathfrak{h}_i$ and define $T = \exp \mathbb{R}^+ X$, $S_i = \exp \mathbb{R}^+ Y_i$. The Lie subgroups H_i corresponding to $\mathfrak{h}_i$ are almost abelian, so we may apply proposition 4.4 in order to obtain a product decomposition of $\langle S_i \cup T \rangle = N_i S_i T$. Either $N_i = \{1\}$ or $N_i = TS_i \cap \mathrm{comm}(H_i)$, in any case $L(N_i) \cap \mathfrak{h} = \{0\}$. If we let

$$S = \langle S_1 \cup N_1 \cup S_2 \cup N_2 \rangle,$$

then $L(S) = \mathbb{R}^+ Y_1 + \mathbb{R}^+ Y_2 + L(N_1) + L(N_2)$, because the right-hand side is a global Lie wedge by [1, VI.5.2]. In addition $L(T) = \mathbb{R}^+ X$ and $W \subseteq L(S) + L(T)$. By the choice of S_i and N_i, we immediately obtain

$$\langle S \cup T \rangle = ST,$$

because of $TN_iS_i = N_iS_iT$. Thus corollary 3.9 proves the globality of $L(S) + L(T)$ and hence of W. ■

As an interesting example we consider the Lie algebra $\mathfrak{g} = sl(2, \mathbb{R})$, and the Lie group $Sl(2, \mathbb{R})$. We denote G the simply connected Lie group corresponding to $\mathfrak{g}$ and $\pi: G \to Sl(2)$ the quotient homomorphism. The semigroup of all 2×2-matrices in $Sl(2)$ with nonnegative entries is denoted $Sl(2)^+$ and its tangent wedge $\mathcal{P}^+$. Then in view of [1, V.4.25], globality in $Sl(2, \mathbb{R})$ is now completely characterized:

Theorem 5.2. *A Lie wedge* $W \subseteq sl(2)$ *is global in* $Sl(2)$ *iff* W *is a subalgebra or a conjugate of* W *is contained in* $\mathcal{P}^+$.

Proof. The if-part is proved in [1, V.4.25] and the only-if-part follows immediately from the preceding proposition and the fact that $\langle \exp_G \mathcal{P}^+ \rangle \cap \ker \pi = \{1\}$, the latter being a consequence of [1, V.4.24, V.4.40]. ■

Obviously the same statement holds for $PSl(2, \mathbb{R}) = G/Z(G)$. The characterization of globality in the simply connected group G is still an open research problem.

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Received March 15, 1993
and in final form April 28, 1993