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Symplectic convexity theorems

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0. Introduction

In this note we explain some of the key results of the article [9]. They are generalizations of well known facts dating back up to seventy years. In 1923 Schur proved that the diagonal entries $a=(a_1,\ldots,a_n)$ of a Hermitean $n\times n$-matrix with eigenvalues $r=(r_1,\ldots,r_n)$ are contained in the convex hull of $\mathcal{S}_n.r$, where $\mathcal{S}_n$ is the symmetric group acting on $\mathbb{C}^n$ by permutation of coordinates. 31 years later Horn proved that each point of the convex hull can be obtained this way. In 1973 Kostant published a seminal paper in which he interpreted the Schur-Horn result as a property of adjoint orbits of the unitary group and generalized it to arbitrary compact Lie groups. More precisely, he proved that for an element X in a maximal abelian subspace $\mathfrak{t}$ in the Lie algebra $\mathfrak{k}$ of a compact Lie group K one has

$$\mathrm{pr}_{\mathfrak{t}}(\mathrm{Ad}\,K.X)=\mathrm{conv}\,\mathcal{W}.X,$$

where $\mathrm{pr}_{\mathfrak{t}}\colon\mathfrak{k}\to\mathfrak{t}$ is the orthogonal projection (w.r.t. the Killing form) and $\mathcal{W}$ is the Weyl group associated to the pair $(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$. In turn Atiyah and, independently, Guillemin and Sternberg, in 1982 gave an interpretation of Kostant's theorem as a special case of a theorem on the image of the momentum map of a Hamiltonian torus action. In that context one has a symplectic manifold (M,ω) and a smooth action $\sigma\colon G\times M\to M$ of a Lie group G on M which preserves the form ω. The space $C^\infty(M)$ carries a Lie algebra structure given by the Poisson bracket

$$\{f,h\}=\omega(\mathcal{X}_f,\mathcal{X}_h),$$

where the vector fields $\mathcal{X}_f$ and $\mathcal{X}_h$ correspond to df and dh under the isomorphism $TM\cong T^*M$ coming from ω. Moreover σ induces a natural homomorphism $\dot\sigma$ from the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathcal{V}(M)$ of vector fields on M. The action σ is called Hamiltonian if there exists a Lie algebra homomorphism $\lambda\colon\mathfrak{g}\to C^\infty(M)$ such that

$$\dot\sigma(X)=\mathcal{X}_{\lambda(X)}\qquad\forall X\in\mathfrak{t}.$$

The functions $\lambda(X)$ are called Hamiltonian. Given λ one defines the moment map $\Phi\colon M\to\mathfrak{g}^*$ via

$$\langle\Phi(m),X\rangle=\lambda(X)(m)\qquad\forall X\in\mathfrak{g}.$$

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The Atiyah-Guillemin-Sternberg (AGS) theorem then reads: If M is compact and T is a torus, then

$$\Phi(M) = \operatorname{conv} \Phi\big(\operatorname{Fix}(M)\big),$$

where $\operatorname{Fix}(M)$ is the set of T-fixed points in M.

In Kostant's situation the symplectic manifold is the adjoint orbit which can be identified with a coadjoint orbit carrying a natural symplectic form. The group is the maximal torus T in K with Lie algebra $\mathfrak{t}$ and the action is the coadjoint action. Then it is not hard to see that the corresponding moment map is the natural map $\mathrm{pr}_{\mathfrak{t}^*}\colon \mathfrak{k}^* \to \mathfrak{t}^*$ restricted to the coadjoint orbit. Some standard Lie theoretic arguments show that for $M = \mathrm{Ad}^*(K).\alpha$ the set $\operatorname{Fix}(M)$ coincides with the Weyl group orbit of α.

Since complex flag manifolds can be viewed as certain compact coadjoint orbits, the AGS-theorem proves convexity properties of complex flag manifolds. On the other hand Kostant had proved analogous results for real flag manifolds. In order to give a symplectic interpretation for those, Duistermaat in 1983 proved a convexity theorem for fixed point sets of involutions τ on symplectic manifolds which satisfy $\tau^*\omega = -\omega$.

All the symplectic convexity theorems mentioned so far were proved applying some Morse theory to a generic component function of the moment map. At that point it was essential to assume that the symplectic manifold was compact. Nevertheless, using Kostant's theorem Paneitz in 1984 and 'Olafsson in 1990 were able to prove convexity theorems which can be interpreted as symplectic convexity theorems for certain non-compact coadjoint orbits (with involution). It turns out that it actually is enough to assume that the moment map Φ is proper, i.e., the inverse images of compact sets are compact. To prove that, one first proves a local convexity theorem using a suitable normal form for Hamiltonian torus actions.

The next step is to establish a very general principle (we call it the Lokal-global-Prinzip) which shows that the local convexity theorem together with the properness always gives rise to a global convexity theorem. The basic idea for the proof of the Lokal-global-Prinzip is borrowed from the paper [3] by Condevaux, Dazord and Molino.

This line of argument gives strengthened versions of the AGS-theorem as well as the Duistermaat theorem. Moreover it has the advantage that in order to derive Duistermaats theorem one no longer needs to essentially redo the proof of the AGS theorem but only to establish the right local convexity theorem and then apply the Lokal-global-Prinzip. When applied to coadjoint orbits the strengthened convexity theorems, just as in Kostant's case, yield an "abstract" convexity statement. In order to give a "concrete" description of that convex image one again has to use Lie theoretic arguments. The situation becomes more complicated for non-compact coadjoint orbits because it is no longer clear that the convex image is spanned by extreme points. In fact, it turns out that the image of the moment map which is closed, convex and locally polyhedral is always a sum of the convex hull of its extreme points and a convex cone which may be interpreted as the cone of limit directions of the set. In terms of Lie theory the extreme points come up as a Weyl group orbit whereas the limit cone is given by certain roots. As special cases one finds Paneitz's and 'Olafsson's theorems (cf. [9]). The purpose of this note is to describe the Lokal–global–Prinzip and how it can be used to prove general convexity theorems.

1. Closed convex sets

In this section we collect some facts on the geometry of convex sets which are needed to prove the convexity theorems for non–compact manifolds.

Let V be a finite dimensional real vector space and $C \subseteq V$ a closed convex set. For $x \in C$ we define the *subtangent wedge*

$$L_x(C) := \overline{\mathbb{R}^+(C-x)}$$

(cf. [8, Ch. I]). Note that this set deserves to be called a wedge since it is a closed convex cone in V. It follows immediately from the definition that

$$C \subseteq x + L_x(C) \qquad \forall x \in C.$$

We say that a closed convex set C is *locally polyhedral* if for every $x \in C$ there exists a neighborhood U such that $U \cap C = U \cap \big(x + L_x(C)\big)$ and $L_x(C)$ is polyhedral. We remark that one can show that the condition that $L_x(C)$ is polyhedral is superfluous. It follows from the condition that C is locally polyhedral. For a closed convex set C we write $C^\star := \{\omega \in V^* : \omega(C) \subseteq \mathbb{R}^+\}$ for the dual cone.

Proposition 1.1.

(i) $\lim(C) := \{v \in V : C + v \subseteq C\}$ *is a closed convex cone in* V.

(ii) $H(C) := \{v \in V : C + v = C\}$ *is a vector subspace of* V. ∎

The cone $\lim(C)$ is the cone of all limited directions of C and the sets $c + H(C)$ are maximal affine subspaces contained in C. A subset $F \subseteq C$ is said to be *extremal* if $tx + (1-t)y \in F$, $t \in]0,1[$ and $x, y \in C$ implies that $x, y \in F$.

A *face* is a convex extremal set, an *extreme point* is a point $e \in C$ such that $\{e\}$ is a face. We note that, if C is locally polyhedral, a point $x \in C$ is an *extreme point* if and only if the wedge $L_x(C)$ is pointed. We write $\mathrm{Ext}(C)$ for the set of extreme points of C. Note that $\mathrm{Ext}(F) = \mathrm{Ext}(C) \cap F$ holds for every face F of C.

We define the set $\widetilde{\mathrm{Ext}}(C)$ as the union of all faces of minimal dimension. This is the set of the "most extremal points" of C. Note that $\widetilde{\mathrm{Ext}}(C) = \mathrm{Ext}(C)$ if and only if $H(C) = \{0\}$.

The following proposition shows that $\lim(C)$ is determined by the subtangent cones in the most extremal points.

Proposition 1.2. *Let C be a closed convex set. Then*

(i) $\lim(C) = \bigcap_{x \in \widetilde{\mathrm{Ext}}(C)} L_x(C)$.

(ii)
$$C = \operatorname{conv} \widetilde{\operatorname{Ext}}(C) + \lim C = \bigcap_{x \in \widetilde{\operatorname{Ext}}(C)} \big(x + L_x(C)\big). \qquad \blacksquare$$

Proposition 1.2 says that in order to calculate C it is enough to know $\lim(C)$ and $\widetilde{\operatorname{Ext}}(C)$. Moreover it shows that in order to calculate $\lim(C)$, we only have to know the subtangent cones $L_x(C)$ in the most extremal points. It follows in particular that the locally polyhedral set that is associated to any local convexity data via the "Lokal-global-Prinzip" (cf. Section **3**), we only have to know the local convexity data for the most extremal points. If C is also compact, then $\lim C = \{0\}$ and C is the convex hull of the most extremal points.

The following fact is an ingredient of the Local Convexity Theorem in Section **2**.

Lemma 1.3. *Let $\alpha\colon V \to V'$ a linear mapping of finite dimensional vector spaces and $C \subseteq V$ a polyhedral cone. Then $\alpha\colon C \to \alpha(C)$ is an open mapping.* $\blacksquare$

2. Local convexity theorems

In this section we want to obtain local convexity results for the moment mapping. For that purpose we describe the moment mapping locally in coordinates. In Lemma 2.1 we find a local normal form for a Hamiltonian torus action, and in Lemma 2.2 we use it to get the moment mapping in coordinates. With the Duistermaat convexity theorem in mind we do this also for symplectic manifolds with an antisymplectic involution.

Lemma 2.1. *Let (M, ω) be a symplectic manifold, $\sigma : T \times M \to M$ a Hamiltonian action of a torus T on M which is given by the Lie algebra homomorphism $\lambda\colon \mathfrak{t} \to C^\infty(M)$, and $m_0 \in M$. Then there exist a T-invariant neighborhood $\mathcal{U} \subset M$ of $T.m_0$, a subtorus T_1 of T and a symplectic vector space V with the following properties:*

(i) *$T = T_0 \times T_1$, where $T_0 := (T_{m_0})_0$ is the connected component of the stabilizer T_{m_0}.*

(ii) *There exists a symplectic covering of a T-invariant open subset $\mathcal{U}'$ of $T_1 \times \mathfrak{t}_1^* \times V \cong T^*(T_1) \times V$ onto $\mathcal{U}$ under which σ gets transformed into the action*
$$\begin{aligned}(T_0 \times T_1) \times ((T_1 \times \mathfrak{t}_1^*) \times V) &\to ((T_1 \times \mathfrak{t}_1^*) \times V)\\ ((t_0, t_1), (t_1', \beta, v)) &\mapsto (t_1 t_1', \beta, \pi(t_0)v),\end{aligned}$$
where $\pi\colon T_0 \to \operatorname{Sp}(V)$ is a linear symplectic representation.

(iii) *If, in addition, τ is an antisymplectic involution on M, i.e., $\tau^*\omega = -\omega$, $\tau(m_0) = m_0$, and all Hamiltonian functions $\lambda(X)$, $X \in \mathfrak{t}$ are invariant under τ, then the covering in* (ii) *can be chosen τ-equivariant, where τ acts on $T^*(T_1) \times V$ by*
$$\tau.(t_1', \beta, v) = ({t_1'}^{-1}, \beta, \tau_V.v),$$

and τ_V is an antisymplectic involution on V.

Proof. We give a complete proof for this lemma because only a short sketch is given in [9].
(i) Let $m_0 \in M$ and $T_0 := (T_{m_0})_0$ denote the connected component of the stabilizer T_{m_0}. This T_0 is a subtorus of dimension l and there is another subtorus T_1 of dimension $l' = \dim T - l$, such that $T \cong T_0 \times T_1$. Note that $T_{m_0} = T_0(T_1 \cap T_{m_0})$ with the finite group $\Gamma := T_1 \cap T_{m_0}$ and the orbit $T.m_0$ is isomorphic to T_1/Γ.
(ii), (iii) To treat (ii) and (iii) simultaneously, let $K := T$ if we are in the situation, where there is no τ. Otherwise let τ act on T by $\tau.t := t^{-1}$ and form the semidirect product $K := T \rtimes \{\mathbf{1}, \tau\}$. To see that this leads to an action of K on M, let $X \in \mathfrak{t}$. Then $\lambda(X) \circ \tau = \tau^*\lambda(X) = \lambda(X)$ by assumption. Hence $\tau^* d\lambda(X) = d\lambda(X)$. If $\dot\sigma(X)$ denotes the corresponding Hamiltonian vector field with $i_{\dot\sigma(X)}\omega = -d\lambda(X)$, then $\tau^*\omega = -\omega$ yields that $\tau_*\dot\sigma(X) = -\dot\sigma(X)$. We conclude that $\tau(t.m) = t^{-1}.\tau(m)$ for $t \in T$. Hence we obtain an action of K on M such that $k^*\omega \in \{\pm\omega\}$ for all $k \in K$.

Choose a K-invariant Riemannian metric on M (cf. [6]). Let

$$B := \{v \in (T_{m_0}(T.m_0))^\perp : \|v\| < \varepsilon\}.$$

Then, if ε is small enough, the exponential function

$$\mathrm{Exp}\colon B \to \mathrm{Exp}(B)$$

is a diffeomorphism (cf. [2, p.130]). We get a neighborhood $\mathcal{U} := K.\mathrm{Exp}(B)$ of m_0 in M, which is K-invariant. Note that $\mathrm{Exp}(B)$ is a slice through m_0 ([6, p.155]), i.e.,

(i) $K.\mathrm{Exp}(B) \subset \mathcal{U}$
(ii) $K.\mathrm{Exp}(B)$ is an open neighborhood of $K.m_0 = T.m_0$
(iii) $K_{m_0}.\mathrm{Exp}(B) = \mathrm{Exp}(B)$
(iv) $k.\mathrm{Exp}(B) \cap \mathrm{Exp}(B) \neq \emptyset \Rightarrow k \in K_{m_0}$.

So we have a K-equivariant covering

$$T_1 \times B \to \mathcal{U}, \quad (t_1, v) \mapsto t_1.\mathrm{Exp}_{m_0} v,$$

with respect to the action

$$\begin{aligned}(T_0 \times T_1) \times (T_1 \times B) &\to (T_1 \times B)\\ ((t_0, t_1), (t_1', b)) &\mapsto (t_1 t_1', t_0 b)\end{aligned}$$

and $\tau.(t_1', b) = \big(t_1'^{-1}, d\tau(m_0)b\big)$ in case of (iii).

Let the finite group Γ act on $T_1 \times B$ by $t_0.(t_1, b) = (t_1 t_0^{-1}, t_0 b)$ and write $T_1 \times_\Gamma B$ for the set of Γ-orbits in $T_1 \times B$. Then $T_1 \times_\Gamma B$ is isomorphic to $\mathcal{U}$, and the map $p\colon T_1 \times B \to T_1 \times_\Gamma B$, $(t_1, b) \mapsto \Gamma.(t_1, b)$ is a surjective submersion. This allows us to consider the symplectic form ω relative to the direct product decomposition $T_1 \times B$. Let $d\theta_1, \ldots, d\theta_{l'}$ be a basis of the vector space of invariant

1-forms on T_1 and $b_1, \ldots, b_{l''}$ a set of linear coordinate functions on B considered as functions on $T_1 \times B$. We note that the functions b_i are invariant under the action of T_1 and that they span a vector space of functions on $T_1 \times B$ which is invariant under the action of K. For this property it is esssential to choose linear functions.

Now

$$\omega_{(t_1,b)} = \sum_{\substack{i,j=1 \\ i<j}}^{l'} a_{ij}(t_1,b) d\theta_i \wedge d\theta_j + \sum_{i=1}^{l'} \sum_{j=1}^{l''} \nu_{ij}(t_1,b) d\theta_i \wedge db_j + \sum_{\substack{i,j=1 \\ i<j}}^{l''} \mu_{ij}(t_1,b) db_i \wedge db_j,$$

where $l' = \dim T_1, l'' = \dim B$ and the $a_{ij}, \nu_{ij}, \mu_{ij}$ are smooth functions on $T_1 \times B$.

Let $\lambda : \mathfrak{t} \to C^\infty(M)$ be the homomorphism corresponding to the Hamiltonian action. Then we have for $\xi, \eta \in \mathfrak{t}_1$:

$$\omega(\mathcal{X}_{\lambda(\xi)}, \mathcal{X}_{\lambda(\eta)}) = \{\lambda(\xi), \lambda(\eta)\} = \lambda([\xi, \eta]) = \lambda(0) = 0.$$

Hence the tangent space $T(T_1 \times \{b\})$ is isotropic with respect to ω and therefore $a_{ij} = 0$.

Since the action of T_1 is Hamiltonian with respect to ω, we have that $\sigma_t^* \omega = \omega$ for all $t \in T_1$, where $\sigma_t(t_1, b) = (tt_1, b)$. Moreover, the invariance of the functions b_j shows that the 1-forms db_j are also invariant under T_1. Hence

$$\begin{aligned} \omega(t_1,b) &= (\sigma_t^* \omega)(t_1,b) \\ &= \sum_{i=1}^{l'} \sum_{j=1}^{l''} (\sigma_t^* \nu_{ij})(t_1,b) d\theta_i \wedge db_j + \sum_{\substack{i,j=1 \\ i<j}}^{l''} (\sigma_t^* \mu_{ij})(t_1,b) db_i \wedge db_j \\ &= \sum_{i=1}^{l'} \sum_{j=1}^{l''} \nu_{ij}(tt_1,b) d\theta_i \wedge db_j + \sum_{\substack{i,j=1 \\ i<j}}^{l''} \mu_{ij}(tt_1,b) db_i \wedge db_j \\ &= \omega(tt_1,b), \end{aligned}$$

shows that the functions ν_{ij} and μ_{ij} are constant on the T_1-fibers.

The 2-form ω now has the form:

$$\omega_{(t_1,b)} = \sum_{i=1}^{l'} \sum_{j=1}^{l''} \nu_{ij}(b) d\theta_i \wedge db_j + \sum_{\substack{i,j=1 \\ i<j}}^{l''} \mu_{ij}(b) db_i \wedge db_j$$

Let

$$\omega^0_{(\theta,b)} = \sum_{i=1}^{l'} \sum_{j=1}^{l''} \nu_{ij}(0) d\theta_i \wedge db_j + \sum_{\substack{i,j=1 \\ i<j}}^{l''} \mu_{ij}(0) db_i \wedge db_j,$$

and note that $N := T_1 \times \{0\}$ is a compact submanifold of $T_1 \times B$ which is invariant under the K-action, and $\omega^0 \mid_N = \omega \mid_N$. The group K is a compact group acting on $T_1 \times B$ such that there exists a continuous homomorphism $\varepsilon: K \to \{1, -1\}$ with

$$k^*\omega = \varepsilon(g)\omega \quad \text{and} \quad k^*\omega_0 = \varepsilon(g)\omega_0 \qquad \forall k \in K.$$

For the assertion on ω_0, we note that invariance under T_1 holds trivially and that the invariance under K_{m_0} follows from the fact that ω_0 corresponds to the symplectic form $\omega(m_0)$ on the tangent space $T_{m_0}(M)$. As one easily checks, the Darboux-Weinstein Theorem ([6, Section 22, p.155]) also applies in this case and gives us a K-invariant neighborhood $\mathcal{U}'$ of N in $T_1 \times B$, and a K-equivariant diffeomorphism f of $\mathcal{U}'$ into $T_1 \times B$ with $f\,|_N = \mathrm{id}_N$ and $f^*\omega^0 = \omega$. The K-equivariance of f now implies that $\mathcal{U}' = T_1 \times \mathcal{U}'_B$ and that $f(t_1, b) = \big(t_1 f_{T_1}(b), f_B(b)\big)$, where $f_B: \mathcal{U}'_B \to B$ and $f_{T_1}: \mathcal{U}'_B \to T_1$ are smooth maps. Note that f maps $\mathcal{U}'_B$ onto another section of the bundle $(T_1 \times \mathcal{U}'_B) \to \mathcal{U}'_B$.

We set $\widetilde{b_j} := f^*b_j = b_j \circ f$ and note that, according to the T_1-equivariance, the 1-forms $\widetilde{d\theta_j} := f^*d\theta_j$ are still T_1-invariant and they are extensions of left-invariant 1-forms according to the direct product decomposition

$$\mathcal{U}'' := f(T_1 \times \mathcal{U}'_B) \cong T_1 \times B'$$

with $B' := f(\mathcal{U}'_B)$. Moreover, again by the K-equivariance, the functions $\widetilde{b_j}$ span a K-invariant vector space of functions on the T-space $\mathcal{U}''$. In these coordinates we have with $\nu_{ij} := \nu_{ij}(0)$ and $\mu_{ij} := \mu_{ij}(0)$ that

$$\begin{aligned}
\omega_{(t_1,b')} &= f^*\omega^0(t_1, b') \\
&= \sum_{i=1}^{l'} \sum_{j=1}^{l''} \nu_{ij} f^*d\theta_i \wedge f^*db_j + \sum_{\substack{i,j=1\\ i<j}}^{l''} \mu_{ij} f^*db_i \wedge f^*db_j \\
&= \sum_{i=1}^{l'} \sum_{j=1}^{l''} \nu_{ij} \widetilde{d\theta_i} \wedge \widetilde{db_j} + \sum_{\substack{i,j=1\\ i<j}}^{l''} \mu_{ij} \widetilde{db_i} \wedge \widetilde{db_j}.
\end{aligned}$$

Let $W := T_{m_0}(B')$ denote the tangent space of the slice $B' = f(B)$ in m_0. Then $T_{m_0}(\mathcal{U}') \cong \mathfrak{t}_1 \oplus W$ is a symplectic vector space and the decomposition is invariant under K_{m_0}. Hence the subspace $V := \mathfrak{t}_1^\perp \cap W$ is K_{m_0}-invariant. Moreover the restriction ω_V of ω to V is non-degenerate since $\mathfrak{t}_1$ is isotropic. It also follows that $W' := V^\perp \cap W \cong \mathfrak{t}_1^*$ because ω_V provides a non-degenerate pairing between W' and $\mathfrak{t}_1$. In case of (iii), we have $\tau(t_1.m_0) = t_1^{-1}.m_0$ for all $t_1 \in T_1$ and $\tau^*\omega = -\omega$ which forces $d\tau(m_0)w = w$ for all $w \in W_1$ because this subspace is invariant under τ and paired via ω with $\mathfrak{t}_1$.

This discussion shows that, after a linear change of coordinates on B', we may assume that $\mu_{ij} = 0$ for $i \leq l'$ or $j \leq l''$ and also that $\nu_{ij} = 0$ for $j > l'$. Now

$$\omega_{(t_1,b')} = \sum_{i,j=1}^{l'} \nu_{ij} \widetilde{d\theta_i} \wedge \widetilde{db_j} + \sum_{\substack{i,j=l'+1\\ i<j}}^{l''} \mu_{ij} \widetilde{db_i} \wedge \widetilde{db_j}.$$

Next we define

$$b_i' := \sum_{j=1}^{l'} \nu_{ij} \widetilde{b_j} \quad \text{for } i = 1, \ldots, l'$$

and this leads to

$$\omega_{(t_1,b')} = \sum_{i=1}^{l'} \widetilde{d}\theta_i \wedge db'_i + \sum_{\substack{i,j=l'+1\\ i<j}}^{l''} \mu_{ij} \widetilde{db}_i \wedge \widetilde{db}_j.$$

Shrinking $\mathcal{U}'$, we may further assume that $\mathcal{U}'_B = \mathcal{U}_{\mathfrak{t}_1^*} \times \mathcal{U}_V$. Now

$$\omega_{(t_1,\beta,v)} = \omega^{T^*(T_1)}_{(t_1,\beta)} + \omega_V$$

on $T_1 \times \mathcal{U}'$ and the action of T is given by

$$((t_0,t_1),(t'_1,\beta,v)) \mapsto (t_1 t'_1, \beta, \pi(t_0)v),$$

where $\pi : T_0 \to \mathrm{Sp}(V,\omega_V)$ describes the action of T_0 on V. In case of (iii), the assertion on the action of τ follows from what we have already noted above and the fact that the action of K_{m_0} on V is linear. ∎

It follows immediately from the normal form that the manifold Q of τ-fixed points in M is a submanifold.

Now we can compute the image of the moment map locally.

Lemma 2.2. *Let σ be the Hamiltonian action as in the lemma above and $\pi: T_0 \to \mathrm{Sp}(V)$ the corresponding symplectic action of the torus T_0 on the symplectic vector space (V,ω_V). Then there exists a complex structure I on V such that $\langle X,Y\rangle := \omega_V(IX,Y)$ defines a positive definite scalar product on V. Then $V = \bigoplus_{\alpha\in\mathcal{P}_V} V_\alpha$, where $V_\alpha := \{v \in V : (\forall Y \in \mathfrak{t}_0) Y.v = \alpha(Y)Iv\}$ and $\mathcal{P}_V := \{\alpha \in \mathfrak{t}_0^* : V_\alpha \neq \{0\}\}$. The moment map for σ is given by*

$$\Phi : T^*(T_1) \times V \to \mathfrak{t}_1^* \times \mathfrak{t}_0^* \cong \mathfrak{t}^*$$
$$\big((t_1,\beta),v\big) \mapsto \Phi(1,0,0) + \left(\beta, \frac{1}{2}\sum_{\alpha\in\mathcal{P}_V} \|v_\alpha\|^2 \alpha\right).$$

If, in addition, τ_V is an antisymplectic involution τ on V which leaves all the Hamiltonian functions $\lambda(X)(v) := \frac{1}{2}\omega(X.v,v)$, $X \in \mathfrak{t}_0$ invariant, then τ_V is antilinear, i.e., it anticommutes with I, and it leaves all subspaces V_α invariant. ∎

Now Lemma 2.1 and Lemma 2.2 together with Lemma 1.3 yield the following local convexity theorem.

Theorem 2.3. (The local convexity theorem for Hamiltonian torus actions) *Let $\sigma: T\times M \to M$ be a Hamiltonian action of a torus T on a symplectic manifold M and $m_0 \in M$. Then there exist an arbitrarily small open neighborhood $\mathcal{U}$ of m_0 and a polyhedral cone $C_{m_0} \subset \mathfrak{t}^*$ with vertex $\Phi(m_0)$ such that the following is true:*

(i) *$\Phi(\mathcal{U})$ is an open neighborhood of $\Phi(m_0)$ in C_{m_0}.*

(ii) *$\Phi : \mathcal{U} \to C_{m_0}$ is an open map*

(iii) *$\Phi^{-1}(\Phi(u)) \cap \mathcal{U}$ is connected for all $u \in \mathcal{U}$.*

If, in addition, τ is an antisymplectic involution on M which leaves all Hamiltonian function associated to the action of T invariant, then the assertion (i) *and* (ii) *of the proposition remains true for the manifold $Q := \{m \in M : \tau(m) = m\}$ of fixed points of τ and the same cones C_m, $m \in Q$.* ∎

In general (iii) is false for the submanifold Q on M. This is due to the fact that the vector space V_α^+ is a real form of the complex vector space V_α. Hence the spheres in V_α^+ are disconnected if V_α is a one–dimensional complex vector space.

3. The "Lokal-global-Prinzip" for convexity theorems

In this section we describe a general principle that allows to pass from a local convexity theorem to a global one if the convex sets involved are locally polyhedral.

Let X be a connected topological space and V a finite dimensional vector space. A continuous map $\Psi\colon X \to V$ is called *locally fiber connected* if for each $x \in X$ there exist arbitrarily small neighborhoods U of x such that

(LC) $\Psi^{-1}\big(\Psi(u)\big) \cap U$ is connected for all $u \in U$.

If $\Psi\colon X \to V$ is locally fiber connected we define an equivalence relation $\sim$ on X by saying $x \sim y$ if $\Psi(x) = \Psi(y)$ and x and y belong to the same connected component of $\Psi^{-1}\big(\Psi(x)\big)$. The topological quotient space $\widetilde{X} := X/\sim$ is called the Ψ-quotient of X. The quotient map will be denoted by $\pi\colon X \to \widetilde{X}$ and the map induced on $\widetilde{X}$ by Ψ by $\widetilde{\Psi}\colon \widetilde{X} \to V$. For $x \in X$ we write $E_x := \pi^{-1}(x) = \{y \in X\colon y \sim x\}$ for the equivalence class of x.

Proposition 3.1. *Let $\Psi\colon X \to V$ be a locally fiber connected map. Suppose that Ψ is a proper mapping, i.e., Ψ is closed and the sets $\Psi^{-1}(v)$, $v \in V$ are compact. Then the following assertions hold:*

(i) *The fibers $\Psi^{-1}(v)$ are locally connected.*

(ii) *$\widetilde{\Psi}^{-1}(v)$ is finite for any $v \in V$.*

(iii) *For every $x \in X$ there exists an open neighborhood U_x satisfying* (LC) *such that in addition U_x is relatively compact and $\overline{U}_x$ intersects only one component of $\Psi^{-1}\left(\Psi(x)\right)$.* ■

In the following let $\Psi\colon X \to V$ be a locally fiber connected map. A map $x \mapsto C_x$ which associates to each point in $x \in X$ a closed convex cone C_x with vertex $\Psi(x)$ in V is called *local convexity data* if for each $x \in X$ there exists an arbitrarily small open neighborhood U_x of x such that

(O) $\Psi\colon U_x \to C_x$ is an open map.

(LC) $\Psi^{-1}\big(\Psi(u)\big) \cap U_x$ is connected for all $u \in U_x$.

A convex cone is uniquely determined by its intersection with a neighborhood of its vertex. Therefore the C_x are uniquely determined once the U_x are fixed. In fact, more is true. If $U_x' \subseteq U_x$ is an open neighborhood of x, then $\Psi(U_x')$ still generates the same cone because of (O). Thus we actually have a map $x \to C_x$ which does not depend on the *choice* of the neighborhoods U_x.

Note here that the concept of local convexity data is *not* the most general one might face in this context, since it does for instance not model a spherical ball locally.

We record some of the most essential topological properties of the space $\widetilde{X}$.

Lemma 3.2. *Let $\Psi: X \to V$ be a locally fiber connected map with local convexity data $(C_x)_{x\in X}$. Suppose that Ψ is proper. Then the following assertions hold:*

(i) *C_x only depends on the equivalence class E_x of x, so we have a well defined mapping $\pi(x) \mapsto C_{\pi(x)}$.*

(ii) *$\widetilde{X}$ is a Hausdorff space.*

(iii) *If $\mathcal{U}_{\pi(x)}$ is chosen as in* (ii), *then $\widetilde{\mathcal{U}}_{\pi(x)} = \pi\left(\mathcal{U}_{\pi(x)}\right)$ is a neighborhood of $\pi(x)$ in $\widetilde{X}$ such that*

$$\widetilde{\Psi}: \widetilde{\mathcal{U}}_{\pi(x)} \to C_{\pi(x)}$$

is a homeomorphism onto its open image. ∎

We call a continuous map $\gamma: [0,1] \to \widetilde{X}$ a *regular curve* connecting $\widetilde{x}_0$ and $\widetilde{x}_1$ if $\gamma(i) = \widetilde{x}_i$ for $i = 0, 1$ and $\widetilde{\Psi} \circ \gamma$ is piecewise differentiable.

Now let $\Psi: X \to V$ and $(C_x)_{x\in X}$ be as in Lemma 3.2. Since $\widetilde{X}$ is connected and $\widetilde{\Psi}$ locally is a homeomorphism onto a connected open subset of a convex cone, any two points in $\widetilde{X}$ can be connected by a regular curve. We define $d(\widetilde{x}, \widetilde{y})$ to be the infimum of the lengths $l(\widetilde{\Psi} \circ \gamma)$ of all the curves $\widetilde{\Psi} \circ \gamma$ with γ a regular curve connecting $\widetilde{x}$ and $\widetilde{y}$. Here the length of a curve $[0,1] \to V$ is calculated with respect to an arbitrary but fixed Euclidean metric d_V on V. Obviously d is symmetric and satisfies the triangle inequality. Moreover it is clear that

$$d_V\left(\widetilde{\Psi}(\widetilde{x}), \widetilde{\Psi}(\widetilde{y})\right) \leq d(\widetilde{x}, \widetilde{y}).$$

The local convexity data now implies that d actually is a metric defining the topology of $\widetilde{X}$.

The next lemma is one place where the properness assumption enters crucially.

Lemma 3.3. *Let $\Psi: X \to V$ be a locally fiber connected map with local convexity data $(C_x)_{x\in X}$ and d_V a Euclidean metric on V. Suppose that Ψ is proper. Then for any $\widetilde{x} \in \widetilde{X}$ and any $r \in \mathbb{R}^+$ the ball*

$$B(r, \widetilde{x}) := \{\widetilde{y} \in \widetilde{X}: d(\widetilde{x}, \widetilde{y}) \leq r\}$$

is compact. ∎

The fact that a ball is compact for any radius $r \in \mathbb{R}$ is very important for the proof of the next theorem, because we can take now any two points in a compact ball.

Theorem 3.4. (The “Lokal-global-Prinzip” for convexity theorems) *Let $\Psi: X \to V$ be a locally fiber connected map with local convexity data $(C_x)_{x\in X}$. Suppose that Ψ is proper. Then $\Psi(X)$ is a closed locally polyhedral convex subset* *of V, the fibers $\Psi^{-1}(v)$ are all connected, $\Psi : X \to \Psi(X)$ is an open mapping, and $C_x = \Psi(x) + L_{\Psi(x)}\big(\Psi(X)\big)$ holds for all $x \in X$.*

Proof. We want to give a very rough sketch of the proof of the convexity of $\Psi(X)$. Fix two points $\widetilde{x}_0, \widetilde{x}_1 \in \widetilde{X}$. Then for any $n \in \mathbb{N}$, there exists a regular curve γ_n connecting $\widetilde{x}_0$ and $\widetilde{x}_1$ such that

$$l(\widetilde{\Psi} \circ \gamma_n) \leq d(\widetilde{x}_0, \widetilde{x}_1) + \frac{1}{n}.$$

Then we obtain $\widetilde{x}_{\frac{1}{2}}$ as an accumulation point of the "midpoints" of the curves γ_n. This point satisfies

$$d(\widetilde{x}_0, \widetilde{x}_{\frac{1}{2}}) = d(\widetilde{x}_{\frac{1}{2}}, \widetilde{x}_1) = \frac{d(\widetilde{x}_0, \widetilde{x}_1)}{2}.$$

By bisection we get a map $\frac{n}{2^m} \mapsto \widetilde{x}_{\frac{n}{2^m}}$, which we can extend to a continuous map $\gamma : [0,1] \to \widetilde{X}$ with

$$d(\gamma(t), \gamma(t')) = d(\widetilde{x}_0, \widetilde{x}_1)|t - t'|$$

for $t, t' \in [0,1]$. This means that locally we have

$$d_V(\widetilde{\Psi} \circ \gamma(t), \widetilde{\Psi} \circ \gamma(t')) = d(\widetilde{x}_0, \widetilde{x}_1)|t - t'|,$$

which can only happen if $\widetilde{\Psi} \circ \gamma([0,1])$ is a straight line segment. ∎

Theorem 3.5 is a very useful tool in proving a lot of convexity statement. It splits the proof of a convexity result into a local part where one has to find the appropriate convexity data and a global part which consists in proving the properness of the map. Two examples are given in the next section.

The following corollary is used in the proof of the Duistermaat convexity theorem.

Corollary 3.5. *Let V be a finite dimensional real vector space and $X \subset V$ a closed connected subset such that for each $x \in X$ there exists a neighborhood U_x of x in V and a closed convex cone $C_x \subset V$ with vertex x such that $U_x \cap X = U_x \cap C_x$. Then X is convex.*

∎

4. Hamiltonian torus actions

The following theorem is a strengthened version of the convexity theorem proved by Atiyah (cf. [1]) and, independently, by Guillemin and Sternberg (cf. [5]).

Theorem 4.1. (The convexity theorem for Hamiltonian torus actions) *Let M be a connected symplectic manifold, T a torus, and $\mu: T \times M \to M$ a Hamiltonian action of T on M such that the corresponding moment mapping $\Phi: M \to \mathfrak{t}^*$ is proper. Then the following assertions hold:*

(i) *$\Phi(M)$ is a locally polyhedral convex set.*

(ii) *$\Phi : M \to \Phi(M)$ is an open mapping.*

(iii) *The inverse images of points in $\Phi(M)$ are connected.*

(iv) *Each extreme point f of $\Phi(M)$ is of the form $\Phi(x)$ for some T-fixed point $x \in M$.*

Proof. This follows immediately from Theorem 2.3 and Theorem 3.5. ■

Note that in case the symplectic manifold is compact, $\Phi(M)$ is a convex polyhedron.

Next we state a version of Duistermaat's Convexity Theorem (cf. [4]) for non-compact manifolds.

Theorem 4.2. (Duistermaat's convexity theorem for non-compact manifolds) *Let (M, ω) be a connected symplectic manifold, $\sigma : T \times M \to M$ a Hamiltonian action of a torus T on M which is given by the Lie algebra homomorphism $\lambda: \mathfrak{t} \to C^\infty(M)$. Let further τ be an antisymplectic involution on M such that all Hamiltonian functions $\lambda(X)$, $X \in \mathfrak{t}$ are invariant under τ and Q the manifold of τ-fixed points in M. Suppose that the moment mapping $\Phi: M \to \mathfrak{t}^*$ is proper. Then, for every connected component Q' of Q, we have*

$$\Phi(M) = \Phi(Q').$$

Proof. The proof of this theorem is based on Proposition 1.2 which says that a locally polyhedral closed convex set is the intersection of polyhedral cones with vertexes in the most extremal points. We note first that Corollary 3.5 implies the convexity of $\Phi(Q')$. Now the crucial point is the fact, that the polyhedral cones in the most extremal points of $\Phi(Q')$ are the same as the cones of $\Phi(M)$. The set $\Phi(Q')$ is the intersection of these cones, and $\Phi(M)$ is the intersection of these cones and perhaps other cones. This implies $\Phi(M) \subseteq \Phi(Q')$ and hence the claim. ■

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