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The geometry of nilpotent coadjoint orbits of convex type in hermitian Lie algebras

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Introduction

The unitarizable highest weight representations of simple Lie groups have been classified in an algebraic manner more than ten years ago (cf. [34], [5], [16], [17], [6]). Nevertheless, no uniform analytic way of constructing these representations is known. Pioneering work in this direction has been done by Rossi and Vergne [33]. Other approaches are given for example in [7], [4], [18] and [27].

It is our goal to construct the unitary highest weight representations in an analytic fashion which is based on the orbit method. The coadjoint orbits we want to use are the orbits of convex type. These are the coadjoint orbits which are contained in a proper generating cone invariant under the coadjoint action. The basic idea is that unitary highest weight modules can be extended to contraction representations of certain complex semigroups having the original group as a kind of Shilov boundary. This leads to a one sided boundedness for the spectrum of the operators given by the derived representation. Following the idea that the spectrum of the representation has a close connection to the spectrum of its classical counterpart, i.e., the range of the Hamiltonian functions on the coadjoint orbit, one is led to the consideration of coadjoint orbits of convex type. This connection has been studied in detail in [24]. So far we are only able to carry out the program for *nilpotent* orbits. In that case it results in analytic realizations of all highest weight modules whose lowest K-types are one dimensional.

For the construction of the representations one proceeds in several steps. The first step is to use the detailed knowledge of the geometry of nilpotent orbits of convex type to construct a polarization of a suitable open dense subset of the orbit. This subset is in its own right a coadjoint orbit for a subgroup Q_{red} for

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which the polarization is invariant. The next step is to construct a Hilbert space and a representation of $Q_{\rm red}$ following the scheme of geometric quantization. This Hilbert space turns out to be a space of holomorphic functions on a Siegel domain of the second kind. It is a key observation that this Hilbert space can be reinterpreted as a space of holomorphic functions on a Siegel domain of the third kind on which the big group acts. The last step is to identify the resulting representation of $Q_{\rm red}$ as the restriction of a highest weight representation by comparing the reproducing kernels coming from the construction with the ones predicted by the general realization theory given in [25]. The first step is described in the present paper. The two others are dealt with in [14] and [15]. The paper is organized as follows:

I. Cayley homomorphisms of reductive Lie algebras
II. (H_1)-homomorphisms and invariant cones
III. The classification of the nilpotent orbits of convex type
IV. The fine structure of the Jacobson-Morosow parabolics
V. Moment maps
VI. The symplectic geometry of $\mathcal{O}_k$ and $\mathcal{M}_k$
VII. Examples

We explain the contents in a little more detail. In Section I we establish some refinements of well known results on $\mathfrak{sl}_2$-triples which will be used to describe the geometry of conjugacy classes of *real* nilpotent elements. The main point is to keep track of the Cartan decompositions when studying homomorphisms between real reductive Lie algebras. Homomorphisms preserving the Cartan decomposition are called Cayley homomorphisms. In the case of quasihermitean Lie algebras, which are characterized by the fact that there exists a complex structure on $\mathfrak{p}$ induced by a central element of $\mathfrak{k}$, one can also consider (H_1)-homomorphisms. These are Cayley homomorphisms preserving also the complex structure on the $\mathfrak{p}$-parts. Quasihermitean Lie algebras are singled out as the class of algebras which admit generating Ad-invariant convex cones with compactly embedded edge. The key results of Section II describe how invariant cones behave under (H_1)-homomorphisms. We show that this description can be used to see that nilpotent elements of convex type correspond to $\mathfrak{sl}_2$-triples which are at the same time (H_1)-homomorphisms. The Sekiguchi correspondence between nilpotent orbits and $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$ ([32]) then yields a bijection between nilpotent orbits of convex type and $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^+ \cup \mathfrak{p}^-$.

In the third section we use the results of Section II to give a complete classification of the nilpotent orbits of convex type in a hermitean simple Lie algebra $\mathfrak{g}$. It turns out that there are exactly $2r+1$ such orbits, where r is the real rank of $\mathfrak{g}$ and each invariant cone in $\mathfrak{g}$ contains $r+1$ such orbits. The ordering of the orbits lying in an invariant cone via inclusion of closures is linear. Using the Sekiguchi correspondence, we see that the nilpotent orbits lying in a fixed cone correspond to $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^+$ which have been classified by Muller, Rubenthaler and Schiffmann in [22].

Section IV gives a detailed account of the structure of the parabolic subalgebras associated to $\mathfrak{sl}_2$-triples which arise from nilpotent orbits of convex type via the Jacobson-Morosow Theorem. These structural results are mostly based on the work of Koranyi and Wolf ([19], [36]). Our presentation follows Satake's book [29] from which we extract and reorganize all the material we will need in this and the papers [14,15].

The structure of the Jacobson-Morosow parabolics then leads to a series of natural moment maps between various subalgebras. A closer inspection of these maps restricted to coadjoint orbits shows the existence of an open dense submanifold $\mathcal{M}$ of the nilpotent orbit which is a symplectic homogeneous space in its own right and is later shown to admit an invariant polarization. We show in particular that an invariant polarization for the whole group exists only in very special cases (when the orbit of convex type is maximal and G/K is a tube domain).

In Section VI we provide explicit information on the symplectic geometry of the nilpotent orbit and the manifold $\mathcal{M}$. In particular we show that $\mathcal{M}$ has a symplectic potential and determine an invariant complex polarization for $\mathcal{M}$. The final section contains various examples illustrating the theory.

As a standard reference we use [29]; several of the results there we reinterpret and strengthen to suit our point of view, in particular with our application to nilpotent orbits of convex type in mind.

I. Cayley homomorphisms of reductive Lie algebras

Definition I.1. Let $(\mathfrak{g}, \theta)$ and $(\widetilde{\mathfrak{g}}, \widetilde{\theta})$ be two reductive Lie algebras with a fixed Cartan involutions. A homomorphism $\kappa\colon \mathfrak{g} \to \widetilde{\mathfrak{g}}$ is called a *Cayley homomorphism* if $\kappa \circ \theta = \widetilde{\theta} \circ \kappa$. ■

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}$ be the Cartan decompositions associated to θ and $\widetilde{\theta}$. Then $\kappa\colon \mathfrak{g} \to \widetilde{\mathfrak{g}}$ is a Cayley homomorphism if and only if one has $\kappa(\mathfrak{k}) \subseteq \widetilde{\mathfrak{k}}$ and $\kappa(\mathfrak{p}) \subseteq \widetilde{\mathfrak{p}}$.

The following lemma which is a generalization of a theorem of Mostow ([21, Th. 6]) shows that the inclusion of a semisimple Lie algebra into a reductive one can always be made a Cayley inclusion if one chooses the Cartan involution on the bigger algebra in an appropriate way.

Recall that a subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called *compactly embedded* if the group of inner automorphisms of $\mathfrak{g}$ generated by $e^{\mathrm{ad}\,\mathfrak{h}}$ is relatively compact in $\mathrm{Aut}(\mathfrak{g})$.

Lemma I.2. *Let $\widetilde{\mathfrak{g}}$ be a semisimple Lie algebra and $\mathfrak{g} \subseteq \widetilde{\mathfrak{g}}$ a reductive subalgebra with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that $\mathfrak{z}(\mathfrak{g})$ is compactly embedded. Then there exists a Cartan decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}$ with $\mathfrak{k} \subseteq \widetilde{\mathfrak{k}}$ and $\mathfrak{p} \subseteq \widetilde{\mathfrak{p}}$.*

Proof. First we use Mostow's Theorem ([21, Th. 6]) to see that there exists a Cartan decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}$ extending the Cartan decomposition of the commutator algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. Let $\widetilde{\theta}$ denote the corresponding involution.

Set $\mathfrak{a} := \mathfrak{z}_{\widetilde{\mathfrak{g}}}([\mathfrak{g}, \mathfrak{g}])$. Then $\mathfrak{a} \subseteq \widetilde{\mathfrak{g}}$ is a $\widetilde{\theta}$-invariant, and hence reductive, subalgebra which intersects $\mathfrak{g}$ in $\mathfrak{z}(\mathfrak{g})$. The orthogonal (w.r.t. the Killing form) projection $p\colon \widetilde{\mathfrak g} \to [\mathfrak a, \mathfrak a]$ commutes with $\widetilde\theta$ and its restriction to $\mathfrak a$ is a surjective homomorphism with kernel $\mathfrak z(\mathfrak a)$. Therefore $p\big(\mathfrak z(\mathfrak g)\big)$ is a compactly embedded abelian subalgebra of $[\mathfrak a, \mathfrak a]$. Using [10, Ex. VI.A.8(ii)], we now find a Cartan decomposition $\mathfrak a = \mathfrak k_{\mathfrak a} + \mathfrak p_{\mathfrak a}$ such that $\mathfrak z(\mathfrak g) \subseteq p\big(\mathfrak z(\mathfrak g)\big) + \mathfrak z(\mathfrak a) \subseteq \mathfrak k_{\mathfrak a}$ and $\mathfrak p_{\mathfrak a} \subseteq [\mathfrak a, \mathfrak a]$. On the other hand we have the Cartan decomposition $\mathfrak a = (\widetilde{\mathfrak k} \cap \mathfrak a) + (\widetilde{\mathfrak p} \cap \mathfrak a)$. Since Cartan decompositions of $\mathfrak a$ are conjugate under inner automorphisms, there exists $\gamma \in \langle e^{\operatorname{ad} \mathfrak a}\rangle$ with $\gamma(\mathfrak k_a) = \widetilde{\mathfrak k} \cap \mathfrak a$ and $\gamma(\mathfrak p_{\mathfrak a}) = \widetilde{\mathfrak p} \cap \mathfrak a$. In particular this implies $\gamma\big(\mathfrak z(\mathfrak g)\big) \subseteq \widetilde{\mathfrak k}$.

Now $\gamma(\mathfrak k) = \gamma\big(\mathfrak z(\mathfrak g)\big) + (\mathfrak k \cap \mathfrak g') \subseteq \widetilde{\mathfrak k}$ and $\gamma(\mathfrak p) = \mathfrak p \subseteq \widetilde{\mathfrak p}$ since $\mathfrak p \subseteq [\mathfrak g, \mathfrak g]$ shows that the Cartan decomposition of $\gamma(\mathfrak g)$ induced by that of $\mathfrak g$ is compatible with the Cartan decomposition of $\widetilde{\mathfrak g}$. Hence $\widetilde{\mathfrak g} = \gamma^{-1}(\widetilde{\mathfrak k}) + \gamma^{-1}(\widetilde{\mathfrak p})$ is a Cartan decomposition of $\widetilde{\mathfrak g}$ satisfying the requirements of the lemma. ∎

In this paper we will primarily be concerned with Cayley inclusions of $\mathfrak{sl}(2, \mathbb R)$ into some reductive Lie algebra $\mathfrak g$. So let $\mathfrak g_0 := \mathfrak{sl}(2, \mathbb R)$ and

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then

$$[U, T] = 2H, [U, H] = -2T, [H, T] = 2U, [H, X] = 2X, [H, Y] = -2Y, [X, Y] = H.$$

Every triple (H, X, Y) of elements in a Lie algebra $\mathfrak g$ satisfying the above commutation relations is called an $\mathfrak{sl}_2$*-triple*. We fix the Cartan decomposition $\mathfrak g_0 = \mathfrak k_0 + \mathfrak p_0$ with $\mathfrak k_0 = \mathbb R U$ and $\mathfrak p_0 = \mathbb R T + \mathbb R H$ on $\mathfrak g_0$. An $\mathfrak{sl}_2$-triple on which the Cartan involution of $\mathfrak g$ agrees with this involution will be called a *Cayley triple.*

Given a reductive Lie algebra $\mathfrak g$ and an element $X \in \mathfrak g$ we call X *nilpotent* if it belongs to the commutator algebra $[\mathfrak g, \mathfrak g]$ and is nilpotent there (in the usual sense). Then a trivial extension of the Jacobson-Morosow Theorem (cf. [35, Prop. 1.3.5.3]), says that a non-zero element X in a reductive Lie algebra $\mathfrak g$ is nilpotent if and only if it is part of an $\mathfrak{sl}_2$-triple (H, X, Y). Using Lemma I.2 and the fact that the Cartan involutions on $\mathfrak g$ are conjugate, we see that the adjoint orbit of a nilpotent element in $\mathfrak g$ always contains an element X which belongs to a Cayley triple (H, X, Y).

Fix a Cayley triple (H, X, Y) in $\mathfrak g = \mathfrak k + \mathfrak p$ and choose a maximal abelian subspace $\mathfrak a$ of $\mathfrak p$ containing H. Then we have a system $\Sigma = \Sigma(\mathfrak g, \mathfrak a)$ of restricted roots and there is a positive system Σ^+ such that $\mu(H) \geq 0$ for all $\mu \in \Sigma^+$. This means that we declare a Weyl chamber containing H in its closure to be the positive one. Let $\mathfrak g^\mu$ be the root spaces and set $\mathfrak n = \sum_{\mu \in \Sigma^+} \mathfrak g^\mu$. Then $[H, X] = 2X$ implies that

$$X \in \sum_{\mu(H) = 2} \mathfrak g^\mu \subseteq \mathfrak n.$$

Recall that the restriction of the adjoint representation to $\mathfrak{s} = \mathbb{R}H + \mathbb{R}X + \mathbb{R}Y$ is completely reducible and each irreducible summand is determined by a highest weight vector which is annihilated by X and an eigenvector for H. The corresponding eigenvalue is a non-negative integer and gives the isomorphy class of the simple $\mathfrak{s}$-module. Accordingly we decompose $\mathfrak{g} = \mathfrak{g}^{[0]} + \mathfrak{g}^{[1]} + \ldots$ into isotypic $\mathfrak{s}$-submodules. In particular we see that $\operatorname{ad} H$ is diagonalizable with integer eigenvalues. Thus the action of H on $\mathfrak{g}$ gives a grading of $\mathfrak{g}$ via

$$\mathfrak{g}(\operatorname{ad} H; k) := \{Z \in \mathfrak{g} \colon [H, Z] = kZ\}, \quad k \in \mathbb{Z}.$$

We set

$$\mathfrak{q} := \sum_{j \geq 0} \mathfrak{g}(\operatorname{ad} H; j), \quad \mathfrak{l} := \mathfrak{g}(\operatorname{ad} H; 0), \quad \text{and} \quad \mathfrak{u} := \sum_{j > 0} \mathfrak{g}(\operatorname{ad} H; j).$$

Then we have

$$\mathfrak{q} = \sum_{\mu(H) \geq 0} \mathfrak{g}^{\mu}, \quad \mathfrak{l} = \sum_{\mu(H) = 0} \mathfrak{g}^{\mu}, \quad \text{and} \quad \mathfrak{u} = \sum_{\mu(H) > 0} \mathfrak{g}^{\mu},$$

and therefore $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. We call it the *Jacobson-Morosow parabolic* associated to the Cayley triple (H, X, Y). Note here that a different choice of Σ^+ with $\mu(H) \geq 0$ for $\mu \in \Sigma^+$ leads to the same parabolic.

Proposition I.3. *Let (H, X, Y) be a Cayley triple, $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ its Jacobson-Morosow parabolic and $\mathfrak{s} = \mathbb{R}H + \mathbb{R}X + \mathbb{R}Y$. Then*

(i) $\mathfrak{z}_{\mathfrak{g}}(H) \cap \mathfrak{z}_{\mathfrak{g}}(X) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$.

(ii) *The centralizer $\mathfrak{z}_{\mathfrak{g}}(X)$ of X in $\mathfrak{g}$ is contained in $\mathfrak{q}$ and given by*

$$\mathfrak{z}_{\mathfrak{g}}(X) = \mathfrak{z}_{\mathfrak{u}}(X) + \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \sum_{j \geq 0} \mathfrak{z}_{\mathfrak{g}(\operatorname{ad} H; j)}(X) = \mathfrak{z}_{\mathfrak{l}}(X) + \sum_{j > 0} \mathfrak{z}_{\mathfrak{g}(\operatorname{ad} H; j)}(X).$$

(iii) $\mathfrak{z}_{\mathfrak{u}}(X) = \sum_{j > 0} \mathfrak{z}_{\mathfrak{g}(\operatorname{ad} H; j)}(X) = \mathfrak{z}_{\mathfrak{g}}(X) \cap [\mathfrak{g}, X]$.

Proof. (i) Assume that $Z \in \mathfrak{z}_{\mathfrak{g}}(X) \cap \mathfrak{z}_{\mathfrak{g}}(H)$. The general representation theory of $\mathfrak{sl}(2, \mathbb{R})$ applied to the representation on $\mathfrak{g}$ then shows that also $[Z, Y] = 0$ so that $Z \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$.

(ii) We note first that obviously

$$\mathfrak{z}_{\mathfrak{u}}(X) + \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \subseteq \mathfrak{z}_{\mathfrak{g}}(X) \cap \mathfrak{q}.$$

For the converse we remark that the subalgebra $\mathfrak{z}_{\mathfrak{g}}(X)$ is invariant under $\operatorname{ad} H$. Therefore we assume that $Z \in \mathfrak{z}_{\mathfrak{g}}(X) \cap \mathfrak{g}(\operatorname{ad} H; j)$. If $j = 0$, part (i) shows $Z \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$. If $j > 0$ we have $Z \in \mathfrak{u}$ by definition. So it only remains to exclude the case $j < 0$. To that effect we only need to remark that $\mathfrak{z}_{\mathfrak{g}}(X)$ consists of highest weight vectors of the representation of $\mathfrak{sl}(2, \mathbb{R})$ on $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathbb{R}H$ and the corresponding weights are all non-negative.

(iii) The first equality follows immediately from the proof of (ii) and the second from (ii) and the theory of $\mathfrak{sl}(2, \mathbb{R})$-representations applied to the representation on $\mathfrak{g}$. ∎

Remark I.4. Suppose that $\mathfrak{g}$ is semisimple. For $Z \in \mathfrak{z}_{\mathfrak{g}}(X)$ the linear map $\operatorname{ad} X \operatorname{ad} Z$ is nilpotent since $\operatorname{ad} X$ is is nilpotent and commutes with $\operatorname{ad} Z$. Let B be the Killing form of $\mathfrak{g}$. Then $B(X,Z) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Z) = 0$ so that X and $\mathfrak{z}_{\mathfrak{g}}(X)$ are orthogonal under the Killing form. The non-degeneracy and the invariance of the Killing form show that $\mathfrak{z}_{\mathfrak{g}}(X) = [X, \mathfrak{g}]^{\perp}$, where $^{\perp}$ denotes the orthogonal complement with respect to the Killing form. Thus we have

$$X \in \mathfrak{z}_{\mathfrak{u}}(X) = \mathfrak{z}_{\mathfrak{g}}(X) \cap \mathfrak{z}_{\mathfrak{g}}(X)^{\perp}. \qquad \blacksquare$$

Lemma I.5. *Let (H,X,Y) be a Cayley triple in $\mathfrak{g}$ and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ its Jacobson-Morosow parabolic. Suppose that (H', X, Y') is another $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Then there exists a $Z \in \mathfrak{z}_{\mathfrak{u}}(X)$ such that*

$$e^{\operatorname{ad} Z}.H = H' \quad \text{and} \quad e^{\operatorname{ad} Z}.Y = Y'.$$

Proof. [3, Ch. VIII, §11, no. 1, Lemme 4] ■

Remark I.6. It follows from Lemma I.5 that the Jacobson-Morosow parabolic of a Cayley triple (H,X,Y) only depends on the X-part of the triple. ■

Theorem I.7. *Let G be a connected real reductive Lie group and (H,X,Y) be a Cayley triple in the Lie algebra $\mathfrak{g}$ of G. We set $\mathfrak{s} = \mathbb{R}H + \mathbb{R}X + \mathbb{R}Y$ and denote the Levi decomposition of the corresponding Jacobson-Morosow parabolic subgroup by $Q = UL$. Then*

(i) $Z_U(X) = \exp\big(\mathfrak{z}_{\mathfrak{u}}(X)\big)$.

(ii) $Z_G(X) = Z_U(X) \rtimes Z_G(\mathfrak{s})$.

(iii) $Z_G(X) \subseteq Q$.

Proof. (i) The connectedness of $Z_U(X)$ follows from the fact that U is a unipotent algebraic group and that all algebraic subgroups of unipotent groups are connected. Since $\mathfrak{u}$ is nilpotent and hence U is exponential, (i) follows.
(ii) It is clear that $Z_U(X)$ and $Z_G(\mathfrak{s})$ are contained in $Z_G(X)$. The action of $e^{\operatorname{ad} \mathbb{R}H}$ on $Z_U(X)$ and $Z_G(\mathfrak{s})$ shows that these two groups have trivial intersection. Moreover it is clear that $Z_U(X)$ is normalized by $Z_G(\mathfrak{s})$. Thus it only remains to show that $Z_G(X) \subseteq Z_U(X) Z_G(\mathfrak{s})$. To this end we let $g \in Z_G(X)$. Then (H,X,Y) and $(\operatorname{Ad}(g)H, X, \operatorname{Ad}(g)Y)$ satisfy the hypotheses of Lemma I.5 so that one can find a $Z \in \mathfrak{z}_{\mathfrak{u}}(X)$ with $e^{\operatorname{ad} Z} H = \operatorname{Ad}(g)H$ and $e^{\operatorname{ad} Z} Y = \operatorname{Ad}(g)Y$ which implies $\exp(-Z)g \in Z_G(\mathfrak{s})$. This proves the claim.
(iii) Let $g \in Z_G(\mathfrak{s})$. Then g normalizes $\mathfrak{l} = \ker(\operatorname{ad} H)$ and also

$$\mathfrak{u} = \sum_{j>0} \mathfrak{g}(\operatorname{ad} H; j).$$

Thus g normalizes $\mathfrak{q}$ and hence is in Q (cf. [35, Th. I.2.1.1]). Since $Z_U(X) \subseteq U \subseteq Q$ holds by definition, the assertion follows from (ii). ■

Remark I.8. Theorem I.9 shows in particular that the adjoint orbit $\mathcal{O}_X = \mathrm{Ad}(G).X$ of X has a bundle structure with compact base space G/Q and fibre isomorphic to $Q/Z_G(X)$. More precisely we have

$$\mathcal{O}_X \cong G/Z_G(X) \cong G \times_Q \big(Q/Z_G(X)\big),$$

where $G \times_Q \big(Q/Z_G(X)\big)$ is the fiber product with respect to the right action of Q on G via translation and the left action of Q on $Q/Z_G(X)$ also via translation.■

II. (H_1)-homomorphisms and invariant cones

We call a reductive Lie algebra $\mathfrak{g}$ *quasihermitean* if it contains only compact or hermitean simple ideals. In this section $\mathfrak{g}$ always denotes a quasihermitean reductive Lie algebra and G a simply connected group with $\mathbf{L}(G) = \mathfrak{g}$.

Definition II.1. (cf. [29]) Let $\mathfrak{g}$ be a reductive quasihermitean Lie algebra. An element $H_0 \in \mathfrak{g}$ is called an H-element if $\mathfrak{z}_{\mathfrak{g}}(H_0) = \ker \mathrm{ad}\, H_0$ is a maximal compactly embedded subalgebra of $\mathfrak{g}$ and $\mathrm{Spec}(\mathrm{ad}\, H_0) = \{0, i, -i\}$. The pair $(\mathfrak{g}, H_0)$ is called a reductive Lie algebra of *hermitean type*.

Let $(\mathfrak{g}, H_0)$ be a reductive Lie algebra of hermitean type and set $\mathfrak{k} := \ker \mathrm{ad}\, H_0$. Then $\mathfrak{p} := [H_0, \mathfrak{g}]$ is a uniquely determined complement for $\mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition. Note that $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{k}$. In the complexification $\mathfrak{g}_{\mathbb{C}}$, the endomorphism $\mathrm{ad}\, H_0$ is diagonalizable and we obtain

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^-,$$

where $\mathfrak{p}^{\pm}$ is the $\pm i$-eigenspace of $\mathrm{ad}\, H_0$. ■

Note that $\frac{1}{2}U$ is an H-element in $\mathfrak{sl}(2, \mathbb{R})$. In view of this example we should rather speak of U-elements, but we stick to Satake's notation since we will have to refer to [29] on various occasions.

Remark II.2. Let $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$, where $\mathfrak{z}(\mathfrak{g})$ is the center, $\mathfrak{g}_0$ is the maximal compact semisimple ideal, and $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ are the non-compact simple ideals. Then an element $H_z + H_0 + \sum_{j=1}^k H_j$ is an H-element if and only if $H_0 = 0$ and H_j is an H-element in $\mathfrak{g}_j$. It follows in particular that the number of H-elements associated to a fixed Cartan decomposition in the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is 2^k since every simple hermitean Lie algebra contains exactly two H-elements associated to a fixed Cartan decomposition. ■

We recall some basic facts about invariant cones (cf. [11, Ch. III] and [23]): As was mentioned in the introduction, quasihermitean reductive Lie algebras are precisely the class of reductive Lie algebras that admit invariant cones having the maximal compact ideal as largest vector subspace. More precisely, closed convex cones W with non-empty interior, invariant under inner automorphisms and such that $W \cap (-W)$ is the largest compact ideal in the algebra. Quasihermitean Lie algebras admit compactly embedded Cartan subalgebras.

Fix a quasihermitean Lie algebra $\mathfrak{g}$ and a compactly embedded Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{g}$. Then there exists a unique Cartan involution θ on $\mathfrak{g}$ such that $\mathfrak{t}$ is contained in the corresponding maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the root system associated to the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. A root α is called *compact* if the corresponding rootspace $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ is contained in $\mathfrak{k}_{\mathbb{C}}$ and *non-compact* otherwise. We write Δ_k for the set of compact roots and Δ_p for the set of non-compact roots. A positive system Δ^+ of the root system Δ is called $\mathfrak{k}$-adapted, if the set Δ_p^+ of positive non-compact roots is invariant under the Weyl group $\mathcal{W}_{\mathfrak{k}}$ of the pair $(\mathfrak{k}, \mathfrak{t})$. Such positive systems always exist. Given a $\mathfrak{k}$-adapted positive system we define two convex cones in $\mathfrak{t}$ via

$$C_{\min} = \operatorname{cone}\{i[\overline{X}, X]\colon X \in \mathfrak{g}_{\mathbb{C}}^{\alpha}, \alpha \in \Delta_p^+\},$$

where $\operatorname{cone}(E)$ denotes the smallest closed convex cone containing E, and

$$C_{\max} = \{X \in \mathfrak{t}\colon (\forall \alpha \in \Delta_p^+)\, i\alpha(X) \geq 0\}.$$

Then $C_{\min}$ is actually contained in $C_{\max}$. Invariant cones in $\mathfrak{g}$ are uniquely determined by their intersections with $\mathfrak{t}$ and each $\mathcal{W}_{\mathfrak{k}}$-invariant cone between $C_{\min}$ and $C_{\max}$ occurs as such an intersection. In particular, there is a minimal invariant cone $W_{\min}$ with $C_{\min} = W_{\min} \cap \mathfrak{t}$ and a maximal invariant cone $W_{\max}$ with $C_{\max} = W_{\max} \cap \mathfrak{t}$.

Proposition II.3. *Let $\mathfrak{h}$ be a quasihermitean reductive Lie algebra.*

(i) *For each H-element $H_0 \in \mathfrak{g}$ there exists one and only one maximal invariant cone $W_{\max}(H_0)$ containing H_0.*

(ii) *Each maximal invariant cone contains an H-element.*

Proof. (i) Let $\mathfrak{t}$ be a compactly embedded Cartan algebra containing H_0. Then we have $\mathfrak{k} = \mathfrak{z}_{\mathfrak{g}}(H_0)$ and $\alpha(H_0) \in \{i, -i\}$ for all $\alpha \in \Delta_p$. Therefore we find a $\mathfrak{k}$-adapted positive system with $i\alpha(H_0) > 0$ for all $\alpha \in \Delta_p^+$. This means that $H_0 \in \operatorname{int} C_{\max} \subseteq \operatorname{int} W_{\max}$. The uniqueness follows from the fact that the interiors of two different maximal invariant cones are disjoint.
(ii) It suffices to show that $C_{\max}$ contains an H-element. But that is an immediate consequence of the structure of quasihermitean reductive Lie algebras (cf. [11, III]). ■

Remark II.4.

(i) The choice of a maximal cone and the choice of an H-element are equivalent once one has fixed a compactly embedded Cartan algebra $\mathfrak{t}$. More precisely, we find that there is a unique H-element in $[\mathfrak{g}, \mathfrak{g}]$ in each maximal invariant cone contained in $\mathfrak{t}$. In particular a reductive quasihermitean Lie algebra containing k simple hermitean ideals contains exactly 2^k maximal and minimal cones.

(ii) In the proof of Proposition II.3 we used that for non-compact quasihermitean Lie algebras the choice of a $\mathfrak{k}$-adapted positive system also implies the choice of an H-element $H_0 \in \mathfrak{t}$ (modulo the center) by asking that $i\alpha(H_0) > 0$ for all $\alpha \in \Delta_p^+$. Conversely Δ_p^+ for a $\mathfrak{k}$-adapted positive system is fixed by the choice of an H-element in $\mathfrak{t}$.

(iii) The definitions show that $C_{\min}$ contains the corresponding H-element whenever it is contained in the commutator algebra of $\mathfrak{g}$.

(iv) The choice of a maximal cone determines the projection of H_0 onto the commutator algebra. ■

We will write $W_{\max}(H_0)$ for the maximal cone containing H_0 if we want to stress the relation with H_0. Similarly we write $W_{\min}(H_0)$ for the corresponding minimal cone.

Note here that we have compiled various examples in Section VII.

Definition II.5. (cf. [29]) Let $(\mathfrak{g}, H_0)$ and $(\widetilde{\mathfrak{g}}, \widetilde{H}_0)$ be two reductive Lie algebras of hermitean type.

(a) A homomorphism $\kappa: \mathfrak{g} \to \widetilde{\mathfrak{g}}$ is called an (H_1)-homomorphism if

$$\kappa \circ \operatorname{ad} H_0 = \operatorname{ad} \widetilde{H}_0 \circ \kappa.$$

(b) A homomorphism $\kappa: \mathfrak{g} \to \widetilde{\mathfrak{g}}$ is called an (H_2)-homomorphism if $\kappa(H_0) = \widetilde{H}_0$. It is clear that this implies in particular that κ is an (H_1)-homomorphism. ■

Remark II.6. (a) Note that the (H_1)-condition is equivalent to the condition that the complex linear extension $\kappa: \mathfrak{g}_{\mathbb{C}} \to \widetilde{\mathfrak{g}}_{\mathbb{C}}$ satisfies $\kappa(\mathfrak{k}_{\mathbb{C}}) \subseteq \widetilde{\mathfrak{k}}_{\mathbb{C}}$ and $\kappa(\mathfrak{p}^{\pm}) \subseteq \widetilde{\mathfrak{p}}^{\pm}$. It is also equivalent to the condition that κ is a Cayley homomorphism with the additional property that $\kappa\,|_{\mathfrak{p}}: \mathfrak{p} \to \widetilde{\mathfrak{p}}$ is complex linear with respect to the complex structures $J = \operatorname{ad} H_0|_{\mathfrak{p}}$ and $\widetilde{J} = \operatorname{ad} \widetilde{H}_0|_{\widetilde{\mathfrak{p}}}$.

(b) Compositions of (H_1)-homomorphisms yields (H_1)-homomorphisms. ■

Proposition II.7. *Let $(\mathfrak{g}, H_0)$ and $(\widetilde{\mathfrak{g}}, \widetilde{H}_0)$ be two reductive Lie algebras of hermitean type and $\kappa: \mathfrak{g} \to \widetilde{\mathfrak{g}}$ an (H_1)-homomorphism. Then $\kappa\big(W_{\min}(H_0)\big) \subseteq W_{\min}(\widetilde{H}_0)$.*

Proof. Since $\widetilde{H}_0 \in \mathfrak{z}(\widetilde{\mathfrak{k}})$, we may choose compactly embedded Cartan algebras $\mathfrak{t}$ and $\widetilde{\mathfrak{t}}$ of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ with $H_0 \in \mathfrak{t}$ and $\widetilde{H}_0 \in \widetilde{\mathfrak{t}}$ such that $\kappa(\mathfrak{t}) \subseteq \widetilde{\mathfrak{t}}$. Let $p_{\mathfrak{t}}: \mathfrak{g} \to \mathfrak{t}$ resp. $p_{\widetilde{\mathfrak{t}}}: \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{t}}$ denote the orthogonal projection onto $\mathfrak{t}$ resp. $\widetilde{\mathfrak{t}}$. Then

$$C_{\min}(H_0) = \operatorname{cone}\big(\{p_{\mathfrak{t}}(i[\overline{X}, X]) : X \in \mathfrak{p}^+\}\big)$$

and

$$C_{\min}(\widetilde{H}_0) = \operatorname{cone}\big(\{p_{\mathfrak{t}}(i[\overline{X}, X]) : X \in \widetilde{\mathfrak{p}}^+\}\big).$$

Therefore $\kappa(\mathfrak{t}) \subseteq \widetilde{\mathfrak{t}}$ and $\kappa(\mathfrak{p}^+) \subseteq \widetilde{\mathfrak{p}}^+$ imply $\kappa\big(C_{\min}(H_0)\big) \subseteq C_{\min}(\widetilde{H}_0)$ and hence the claim. ■

Lemma II.8. *Let $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ be two reductive quasihermitean Lie algebras of hermitean type with H-elements H_0 and $\widetilde{H}_0$ and $\kappa: \mathfrak{g} \to \widetilde{\mathfrak{g}}$ a Lie algebra homomorphism. Write $p_{\mathfrak{z}}$, $p_{\widetilde{\mathfrak{g}}_n}$, and $p_{\widetilde{\mathfrak{g}}'}$ for the projection onto the center, the sum $\widetilde{\mathfrak{g}}_n$ of all non-compact simple ideals, and the commutator algebra of $\widetilde{\mathfrak{g}}$. Then the following are equivalent:*

(1) *κ is an (H_1)-homomorphism.*

(2) $p_{\widetilde{\mathfrak{g}}'} \circ \kappa$ *is an* (H_1)*-homomorphism with respect to the* H*-element* $p_{\widetilde{\mathfrak{g}}'}(\widetilde{H}_0) \in \widetilde{\mathfrak{g}}'$.

(3) $p_{\widetilde{\mathfrak{g}}_n} \circ \kappa$ *is an* (H_1)*-homomorphism with respect to the* H*-element* $p_{\widetilde{\mathfrak{g}}_n}(\widetilde{H}_0)$ *of* $\widetilde{\mathfrak{g}}_n$.

For (H_2)*-homomorphisms the conclusions* $(1) \Rightarrow (2) \Rightarrow (3)$ *hold.*

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$: Suppose that κ is an (H_1)-homomorphism. Then (2) follows from the fact that the projection $p_{\widetilde{\mathfrak{g}}'}$ is an (H_2)-homomorphism and (3) from the fact that the projection $p_{\widetilde{\mathfrak{g}}_n}|_{\widetilde{\mathfrak{g}}'}\colon \widetilde{\mathfrak{g}}' \to \widetilde{\mathfrak{g}}_n$ is an (H_2)-homomorphism. The same argument applies if κ is an (H_2)-homomorphism.
$(3) \Rightarrow (1)$: Let $\widetilde{\mathfrak{g}}_c$ be the sum of all compact simple ideals in $\widetilde{\mathfrak{g}}$. For $X \in \mathfrak{g}$ we write $\kappa(X) = Y_1 + Y_2$ with $Y_1 \in \widetilde{\mathfrak{g}}_c$ and $Y_2 \in \widetilde{\mathfrak{g}}_n$. Then (3) implies

$$[\widetilde{H}_0, \kappa(X)] = [\widetilde{H}_0, Y_2] = p_{\widetilde{\mathfrak{g}}_n}\big(\kappa([H_0, X])\big).$$

On the other hand $[H_0, X] \subseteq [\mathfrak{g}, \mathfrak{g}]$ is a split element, which shows that $p_{\mathfrak{z}}\big(\kappa([H_0, X])\big) = 0$ and $p_{\widetilde{\mathfrak{g}}_c}\big(\kappa([H_0, X])\big) = 0$. Therefore $[\widetilde{H}_0, \kappa(X)] = \kappa([H_0, X])$, i.e., κ is an (H_1)-homomorphism. ∎

If W is a closed convex cone in a finite dimensional vector space, we write $H(W) := W \cap (-W)$ for the *edge of the cone* W.

Proposition II.9. *Let* $\mathfrak{g}$ *and* $\widetilde{\mathfrak{g}}$ *be two reductive quasihermitean Lie algebras and* $\kappa\colon \mathfrak{g} \to \widetilde{\mathfrak{g}}$ *a Lie algebra homomorphism. Consider the following conditions:*

(1) *There exists a maximal invariant cone* $\widetilde{W}_{\max} \subseteq \widetilde{\mathfrak{g}}$ *such that* $\kappa^{-1}(\widetilde{W}_{\max})$ *is generating in* $\mathfrak{g}$ *and* $\kappa\big(\mathfrak{z}(\mathfrak{g})\big)$ *is compactly embedded.*

(2) *There exist* H*-elements* $H_0 \in \mathfrak{g}$ *and* $\widetilde{H}_0 \in \widetilde{\mathfrak{g}}$ *such that* κ *is an* (H_1)*-homomorphism and* $\widetilde{W}_{\max} = W_{\max}(\widetilde{H}_0)$.

Then the implication $(1) \Rightarrow (2)$ *holds and* $(2) \Rightarrow (1)$ *holds if* $\mathfrak{g}$ *is semisimple without compact factors or if* κ *is an* (H_2)*-homomorphism.*

Proof. Before we prove both implications, we make some reductions. Let $\widetilde{W}_{\max} \subseteq \widetilde{\mathfrak{g}}$ be the maximal cone associated to an H-element $\widetilde{H}_0 \in \widetilde{\mathfrak{g}}_n$ and set $W := \kappa^{-1}(\widetilde{W}_{\max})$. Let $\widetilde{\kappa} := p_{\widetilde{\mathfrak{g}}_n} \circ \kappa$, where $p_{\widetilde{\mathfrak{g}}_n}$ is the projection onto the semisimple ideal $\widetilde{\mathfrak{g}}_n$. Then $\widetilde{\kappa}^{-1}(\widetilde{W}_{\max}) = \kappa^{-1}(\widetilde{W}_{\max})$ and $\kappa\big(\mathfrak{z}(\mathfrak{g})\big)$ is compactly embedded in $\mathfrak{g}$ if and only if $\widetilde{\kappa}\big(\mathfrak{z}(\mathfrak{g})\big)$ is compactly embedded in $\mathfrak{g}_n$. According to Lemma II.8, we may therefore w.l.o.g. assume that $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_n$ is semisimple and a sum of hermitean simple ideals. Note that this implies in particular that the cones $\widetilde{W}_{\max}$ are pointed.

It is clear that $\ker \kappa \subseteq H(W)$. Further the fact that $\widetilde{W}_{\max}$ is pointed yields that $\ker \kappa = H(W)$. Write $\mathfrak{g} = \ker \kappa \oplus \mathfrak{b}$, where $\mathfrak{b}$ is a complementary ideal. Then $W = \ker \kappa + (W \cap \mathfrak{b})$ is generating if and only if $W \cap \mathfrak{b}$ is generating. Moreover $\kappa\big(\mathfrak{z}(\mathfrak{b})\big) = \kappa\big(\mathfrak{z}(\mathfrak{g})\big)$ and κ is an (H_1)-homomorphism if and only this holds for $\kappa|_{\mathfrak{b}}$. Hence it suffices to assume that κ is injective, i.e., that $\mathfrak{g}$ is a subalgebra of $\widetilde{\mathfrak{g}}$.
$(1) \Rightarrow (2)$: Let $\widetilde{W}_{\max}$ be as in (1). We assume that $\mathfrak{g} \subseteq \widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_n$. Then (1) means that the cone $W = \mathfrak{g} \cap \widetilde{W}_{\max}$ is a pointed generating invariant cone in

the reductive Lie algebra $\mathfrak{g}$. According to the theory of invariant cones (cf. [23, III]), there exists an H-element H_0 in $\mathfrak{g}$ such that

$$W_{\min}(H_0) \subseteq W \subseteq W_{\max}(H_0). \tag{II.1}$$

Let $\mathfrak{k} := \ker \operatorname{ad} H_0$ and θ the associated Cartan involution of $\mathfrak{g}$. Since κ is injective and, according to our assumption, the image of $\mathfrak{z}(\mathfrak{g})$ is compactly embedded, it follows from Lemma I.2 that there exists a Cartan involution $\widetilde{\theta}$ of $\widetilde{\mathfrak{g}}$ extending θ.

We conclude in particular that $\mathfrak{k} \subseteq \widetilde{\mathfrak{k}}$. Therefore we find a compactly embedded Cartan algebra $\mathfrak{t}$ of $\mathfrak{k}$ which is contained in a compactly embedded Cartan algebra $\widetilde{\mathfrak{t}}$ of $\widetilde{\mathfrak{k}}$. We choose $\mathfrak{k}$-adapted positive systems Δ_p^+ and $\widetilde{\Delta}_p^+$ such that $\mathfrak{p}^+$ and $\widetilde{\mathfrak{p}}^+$ are the sum of the non-compact positive root spaces. Then (II.1) implies that

$$\widetilde{C}_{\max} = \widetilde{W}_{\max} \cap \mathfrak{t} = W \cap \mathfrak{t} \subseteq W_{\max} \cap \mathfrak{t} = C_{\max}.$$

If $\alpha \in \widetilde{\Delta}_p^+$ is a positive non-compact root, then $\alpha|_{\mathfrak{t}}$ is zero or a non-compact root and $i\alpha$ is non-negative on the generating cone $\widetilde{C}_{\max} \cap \mathfrak{t}$. Since no non-compact root changes its sign on $iC_{\max}$, it follows that $i\alpha$ is non-negative on $C_{\max}$, i.e., that $i\alpha|_{\mathfrak{t}} \in \Delta_p^+ \cup \{0\}$.

On the other hand $\mathfrak{p}_{\mathbb{C}} \subseteq \widetilde{\mathfrak{p}}_{\mathbb{C}}$, so that every root α in Δ_p^+ is the restriction of a root in $\widetilde{\Delta}_p^+$. According to the above argument, every root $\beta \in \widetilde{\Delta}_p$ with $\alpha = \beta|_{\mathfrak{t}}$ must be contained in $\widetilde{\Delta}_p^+$. Hence $\mathfrak{p}^+ \subseteq \widetilde{\mathfrak{p}}^+$ and similarly $\mathfrak{p}^- \subseteq \widetilde{\mathfrak{p}}^-$, whence κ is an (H_1)-homomorphism.

(2) $\Rightarrow$ (1): Suppose that $(\mathfrak{g}, H_0) \to (\widetilde{\mathfrak{g}}, \widetilde{H}_0)$ is an (H_1)-homomorphism. We may assume that $\mathfrak{g} \subseteq \widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_n$.

(a): Let us first assume that $\mathfrak{g}$ is semisimple. Then $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and we only have to show that $W = \mathfrak{g} \cap \widetilde{W}_{\max}$ is generating. Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a compactly embedded Cartan algebra of $\mathfrak{g}$. Since $\mathfrak{t} \subseteq \widetilde{\mathfrak{k}}$, there exists a compactly embedded Cartan algebra $\widetilde{\mathfrak{t}}$ of $\widetilde{\mathfrak{g}}$ contained in $\widetilde{\mathfrak{k}}$ and containing $\mathfrak{t}$. It is clear that each element of the root system Δ of $\mathfrak{g}$ may be obtained as the restriction of an element in $\widetilde{\Delta}$, the root system of $\widetilde{\mathfrak{g}}$. The (H_1)-condition implies that Δ_p^+ is obtained by restricting elements of $\widetilde{\Delta}_p^+$. Therefore $C_{\min} \subseteq \widetilde{C}_{\min} \subseteq \widetilde{C}_{\max}$ yields

$$W_{\min} = \overline{\operatorname{Ad}(G).C_{\min}} \subseteq \mathfrak{g} \cap \widetilde{W}_{\max} = W.$$

Since $\mathfrak{g}$ is semisimple without compact factors, the cone $W_{\min}$ is generating, so that W is also generating.

(b): Now we assume that κ in an (H_2)-homomorphism and that $\mathfrak{g}$ is not necessarily semisimple. Then $\kappa(H_0) = \widetilde{H}_0 \in \operatorname{int} \widetilde{W}_{\max}$ because $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_n$. Therefore $H_0 \in \operatorname{int} \kappa^{-1}(\widetilde{W}_{\max})$ and it follows in particular that the cone $\kappa^{-1}(\widetilde{W}_{\max})$ is generating. That $\kappa(\mathfrak{z}(\mathfrak{g})) \subseteq \widetilde{\mathfrak{k}}$ follows from the assumption that κ is an (H_1)-homomorphism. ∎

Theorem II.10. *Let $\mathfrak{g}$ be a quasihermitean semisimple Lie algebra and $W_{\max} \subseteq \mathfrak{g}$ a maximal invariant cone. Then the following are equivalent:*

(1) *X is a nilpotent element contained in $W_{\max}$.*

(2) *There exists an H-element H_0 in $\mathfrak{g}$ with $W_{\max} = W_{\max}(H_0)$ and a Cayley triple (H,X,Y) defining an (H_1)-homomorphism $\big(\mathfrak{sl}(2,\mathbb{R}), \frac{1}{2}U\big) \to (\mathfrak{g}, H_0)$.*

Proof. (1) $\Rightarrow$ (2): First we use the Jacobson-Morosow Theorem (cf. [35, Prop. 1.3.5.3]) to find an $\mathfrak{sl}_2$-triple (H,X,Y). Let $\kappa\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ denote the corresponding inclusion homomorphism and $\mathfrak{s}$ its image. Then $\kappa^{-1}(W_{\max}) = W_{\max} \cap \mathfrak{s}$ contains the element X. Therefore this is a non-zero closed convex pointed invariant cone in $\mathfrak{s}$, hence $W_{\max} \cap \mathfrak{s} = W_{\max}(\frac{1}{2}U)$. Now we use Proposition II.9 to see that there exists an H-element $\widetilde{H}_0 \in \mathfrak{g}$ and an H-element $\frac{1}{2}\widetilde{U} \in \mathfrak{s}$ such that the inclusion of $\mathfrak{s}$ is an (H_1)-homomorphism, $W_{\max} \cap \mathfrak{s} = W_{\max}(\frac{1}{2}\widetilde{U})$, and $W_{\max} = W_{\max}(\widetilde{H}_0)$. Since all H-elements in $W_{\max} \cap \mathfrak{s}$ are conjugate under inner automorphisms, there exists $\gamma \in \langle e^{\operatorname{ad}\mathfrak{s}}\rangle$ with $\gamma(\widetilde{U}) = U$. Then $H_0 := \gamma(\widetilde{H}_0)$ is H-element in $\mathfrak{g}$ with the same maximal cone and $(\mathfrak{s}, \frac{1}{2}U) \to (\mathfrak{g}, H_0)$ is an (H_1)-homomorphism.

(2) $\Rightarrow$ (1): It is clear that X is nilpotent and $X \in W_{\min} \subseteq W_{\max}$ follows from Proposition II.7. ■

In view of [32, Th. I.9], one has a bijection between the set of all nilpotent orbits in $\mathfrak{g}$ and the set of all nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$, where $K_{\mathbb{C}} = \langle e^{\operatorname{ad}\mathfrak{k}_{\mathbb{C}}}\rangle$. More explicitly, this correspondence is obtained by assigning to a nilpotent element X belonging to the Cayley triple (H,X,Y) the $K_{\mathbb{C}}$-orbit of the element

$$\widetilde{X} = \frac{1}{2}(X + Y - iH) \in \mathfrak{p}_{\mathbb{C}}.$$

We will refer to this bijection as the *Sekiguchi correspondence.* We identify the orbits in $\mathfrak{p}_{\mathbb{C}}$ corresponding to the nilpotent orbits in $W_{\max}$.

Theorem II.11. *For a nilpotent orbit $\mathcal{O}_X \subseteq \mathfrak{g}$ we have:*

(i) *$\mathcal{O}_X$ is of convex type if and only if $\mathcal{O}_{\widetilde{X}}$ is contained in $\mathfrak{p}^+ \cup \mathfrak{p}^-$.*

(ii) *$\mathcal{O}_X$ is contained in $W_{\max}$ if and only if $\mathcal{O}_{\widetilde{X}}$ is contained in $\mathfrak{p}^+$.*

Proof. Since the orbit $\mathcal{O}_{-X}$ corresponds to the orbit of $\frac{1}{2}(-X - Y - iH) = -\overline{\widetilde{X}}$, it follows that $\widetilde{X} \in \mathfrak{p}^+$ is equivalent to $(-X)^{\sim} \in \mathfrak{p}^-$. Therefore it suffices to prove (ii).

If X is contained in $W_{\max}$, then we use Theorem II.10 to obtain a Cayley-triple defining an (H_1)-homomorphism κ: $\big(\mathfrak{sl}(2,\mathbb{R}), \frac{1}{2}U\big) \to (\mathfrak{g}, H_0)$. Put $\mathfrak{g}_1 := \mathfrak{sl}(2,\mathbb{R})$. Then $\mathfrak{p}_1^+ = \mathbb{C}(X+Y-iH)$ and since the complex linear extension of κ maps $\mathfrak{p}_1^+$ into $\mathfrak{p}^+$ (cf. Remark II.6), it follows that $\widetilde{X} \in \mathfrak{p}^+$.

If, conversely, $\widetilde{X} \in \mathfrak{p}^+$, then $\overline{\widetilde{X}} \in \mathfrak{p}^-$, so that the complex linear extension of κ maps $\mathfrak{p}_1^{\pm}$ into $\mathfrak{p}^{\pm}$. Using Remark II.6, we see that κ is an (H_1)-homomorphism $(\mathfrak{g}_1, \frac{1}{2}U) \to (\mathfrak{g}, H_0)$. ■

For later applications we provide a proof of Exercice II.8.1 in [29].

Lemma II.12. *Let* $\rho\colon (\mathfrak{g}, H_0) \to (\widetilde{\mathfrak{g}}, \widetilde{H}_0)$ *be an* (H_1)*-homomorphism of semi-simple Lie algebras of hermitean type. Then* ρ *is an* (H_1)*-homomorphism with respect to the* H*-element* $\mathrm{Ad}(g).\widetilde{H}_0$, $g \in \widetilde{G}$ *if and only if* $g \in Z_{\widetilde{G}}\big(\rho(\mathfrak{g})\big)\widetilde{K}$.

Proof. First let $g \in Z_{\widetilde{G}}\big(\rho(\mathfrak{g})\big)\widetilde{K}$ and $g = xk$, where $k \in \widetilde{K}$ and $x \in Z_{\widetilde{G}}\big(\rho(\mathfrak{g})\big)$. Let $H_0' := \mathrm{Ad}(g).\widetilde{H}_0 = \mathrm{Ad}(x).\widetilde{H}_0$. Then the corresponding Cartan decomposition is given by

$$\mathfrak{k}' := \mathrm{Ad}(x).\widetilde{\mathfrak{k}} \quad \text{and} \quad \mathfrak{p}' := \mathrm{Ad}(x).\widetilde{\mathfrak{p}}.$$

Since $\mathrm{Ad}(x)$ centralizes $\rho(\mathfrak{g})$, we have $\rho(\mathfrak{k}) \subseteq \mathfrak{k}'$ and $\rho(\mathfrak{p}) \subseteq \mathfrak{p}'$. Moreover

$$\begin{aligned}
\mathrm{ad}\,\big(\mathrm{Ad}(g).\widetilde{H}_0\big) \circ \rho &= \mathrm{ad}\,\big(\mathrm{Ad}(x).\widetilde{H}_0\big) \circ \rho \\
&= \mathrm{Ad}(x) \circ \mathrm{ad}\,\widetilde{H}_0 \circ \mathrm{Ad}(x)^{-1} \circ \rho \\
&= \mathrm{Ad}(x) \circ \mathrm{ad}\,\widetilde{H}_0 \circ \rho \\
&= \mathrm{Ad}(x) \circ \rho \circ \mathrm{ad}\,H_0 \\
&= \rho \circ \mathrm{ad}\,H_0,
\end{aligned}$$

i.e., ρ is an (H_1)-homomorphism with respect to H_0 and $\mathrm{Ad}(g).\widetilde{H}_0$.

If, conversely,

$$\mathrm{ad}\,\big(\mathrm{Ad}(g).\widetilde{H}_0\big) \circ \rho = \rho \circ \mathrm{ad}\,H_0,$$

then we may w.l.o.g. assume that $g = \exp Y$ with $Y \in \widetilde{\mathfrak{p}}$ since $\widetilde{G} = \exp(\widetilde{\mathfrak{p}})\widetilde{K}$. Then

$$\mathrm{ad}\,\big((e^{\mathrm{ad}\,Y}).\widetilde{H}_0\big) \circ \rho = \rho \circ \mathrm{ad}\,H_0.$$

This leads to $\rho(\mathfrak{k}) \subseteq e^{\mathrm{ad}\,Y}.\widetilde{\mathfrak{k}}$, hence to $e^{-\mathrm{ad}\,Y}.\rho(\mathfrak{k}) \subseteq \widetilde{\mathfrak{k}}$, so that [35, Lemma 1.1.3.7] implies that $[Y, \rho(\mathfrak{k})] = \{0\}$. Similarly $e^{-\mathrm{ad}\,Y}.\rho(\mathfrak{p}) \subseteq \widetilde{\mathfrak{p}}$ implies $[Y, \rho(\mathfrak{p})] = \{0\}$, whence $Y \in \mathfrak{z}_{\widetilde{\mathfrak{g}}}\big(\rho(\mathfrak{g})\big)$. This completes the proof. ∎

III. The classification of nilpotent orbits of convex type

In this part we introduce the subject proper of this paper, the nilpotent orbits of convex type. We restrict our attention to simple hermitean Lie algebras so that we can identify $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ via the Killing form B. Note right away that the more general considerations from Section I come to bear even here when we have to study various subalgebras.

Definition III.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. A linear functional $\omega \in \mathfrak{g}^*$ is said to be *of convex type* if the coadjoint orbit $\mathcal{O}_\omega = \mathrm{Ad}^*(G).\omega$ is contained in a pointed closed convex cone. ∎

From now on we fix a simple hermitean Lie algebra $\mathfrak{g}$ and identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Cartan Killing form of $\mathfrak{g}$.

As in Section I we let $\mathfrak{g}_0 := \mathfrak{sl}(2,\mathbb{R})$ and fix the elements H, X, Y, T, U. Moreover, we shall freely use the notation from Section I. Recall also that $\big(\mathfrak{sl}(2,\mathbb{R}), \frac{1}{2}U\big)$ is a simple Lie algebra of hermitean type. In this case we have $\mathfrak{p}^\pm = \mathbb{C}(T \mp iH)$ and the invariant cone $W_{\max} = W_{\min}$ containing U contains X in its boundary.

We have seen in Theorem II.10 how one can sharpen the Jacobson-Morozow theorem for nilpotent elements of convex type. This result shows that every ray $\mathbb{R}^+X$ contained in the nilpotent orbit $\mathcal{O}_X$ in the boundary of the invariant cone $W_{\max}$ can be obtained as

$$\lim_{t\to\infty} e^{t\,\mathrm{ad}\,H}(\mathbb{R}^+U) = \mathbb{R}^+X,$$

where (H,X,Y) is an $\mathfrak{sl}_2$-triple and $U = X - Y$. Moreover, this can be arranged in such a way that $U \in \mathfrak{k}$, $H, T \in \mathfrak{p}$ holds for a Cartan decomposition of $\mathfrak{g}$ and $T = -JH = -[H_0, H]$ with respect to the complex structure $J = \mathrm{ad}\, H_0\,|_{\mathfrak{p}}$ on $\mathfrak{p}$.

Definition III.2. Let $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ be Lie algebras. Two Lie algebra homomorphisms $\rho_1, \rho_2 \colon \mathfrak{g} \to \widetilde{\mathfrak{g}}$ are said to be orthogonal if the images commute with each other. In this case we can define the homomorphism

$$\rho_1 + \rho_2 \colon \mathfrak{g} \to \widetilde{\mathfrak{g}}, \quad X \mapsto \rho_1(X) + \rho_2(X)$$

which is called the *commutative sum of* ρ_1 *and* ρ_2.

Note that the commutative sum of two (H_1)-homomorphisms between reductive Lie algebras of hermitean type is again an (H_1)-homomorphism with respect to the same H-elements. ■

Lemma III.3. *Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace. Then $\mathfrak{s} := \mathfrak{a} + J\mathfrak{a} + [\mathfrak{a}, J\mathfrak{a}] \cong \mathfrak{sl}(2,\mathbb{R})^r$, where $r = \dim \mathfrak{a}$ is the real rank of $\mathfrak{g}$. Moreover $\mathfrak{s}$ can be obtained as the range of an (H_1)-homomorphism $\mathfrak{sl}(2,\mathbb{R})^r \to \mathfrak{g}$.*

Proof. Using a system of r strongly orthogonal roots (cf. [29, p.109]), one obtains r orthogonal (H_1)-homomorphisms $\kappa_1, \ldots, \kappa_r \colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ such that $\mathfrak{a} = \sum_{j=1}^r \mathbb{R}\kappa_j(H)$. Then

$$J\mathfrak{a} = [H_0, \mathfrak{a}] = \sum_{j=1}^r \mathbb{R}\kappa_j([U,H]) = \sum_{j=1}^r \mathbb{R}\kappa_j(T)$$

and

$$[\mathfrak{a}, J\mathfrak{a}] = \sum_{j=1}^r \mathbb{R}\kappa_j(U).$$

Therefore the assertion follows from the fact that $\mathfrak{s}$ is the range of the (H_1)-homomorphism $\kappa_1 \times \ldots \times \kappa_r \colon \mathfrak{sl}(2,\mathbb{R})^r \to \mathfrak{g}$. ■

The crucial conclusion we draw from the preceding lemma is the following.

Proposition III.4. *If $\kappa\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ is an (H_1)-homomorphism, then there exists an (H_1)-homomorphism*

$$\rho\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R})^r$$

and an (H_1)-homomorphism $\widetilde{\kappa}\colon \mathfrak{sl}(2,\mathbb{R})^r \to \mathfrak{g}$ such that $\kappa = \widetilde{\kappa} \circ \rho$.

Proof. First we extend $\mathbb{R}\kappa(H)$ to a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}_0$. Then $\kappa\big(\mathfrak{sl}(2,\mathbb{R})\big) = \mathbb{R}\kappa(H) + \mathbb{R}J\kappa(H) + \mathbb{R}\kappa(U)$ and $2U = [JH, H]$. Let $\mathfrak{s} = \mathfrak{a} + J\mathfrak{a} + [\mathfrak{a}, J\mathfrak{a}]$ be as in Lemma III.3. Then $\mathfrak{s} \cong \mathfrak{sl}(2,\mathbb{R})^r$ and the inclusion $\mathfrak{s} \to \mathfrak{g}$ is an (H_1)-homomorphism. It is also clear that the corestriction of κ, $\rho\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{s}$ is an (H_1)-homomorphism. This proves the assertion. ∎

Corollary III.5. *Every nilpotent orbit of convex type meets the range of the (H_1)-homomorphism $\kappa\colon \mathfrak{s} \to \mathfrak{g}$.*

Proof. In view of the fact that different subalgebras $\mathfrak{s}$ are conjugate under $\mathrm{Ad}(K)$, this follows by combining Proposition III.4 with Theorem II.10. ∎

Remark III.6. Theorem II.10 and Corollary III.5 have an interesting interpretation in terms of hermitean symmetric spaces. Since (H_1)-homomorphisms $\mathfrak{g} \to \widetilde{\mathfrak{g}}$ correspond to strongly equivariant holomorphic maps of the hermitean symmetric spaces $G/K \to \widetilde{G}/\widetilde{K}$ (cf. [29]), Theorem II.10 shows that nilpotent orbits of convex type correspond to embeddings of the complex unit disc into G/K and Corollary III.5 shows that every embedding of a unit disc can be extended to an embedding of an r-dimensional polydisc. ∎

Proposition III.7. *Let H_j, T_j, U_j, $j = 1, \ldots, r$ denote the basis elements of $\mathfrak{sl}(2,\mathbb{R})^r$ corresponding to the basis elements in the simple $\mathfrak{sl}(2,\mathbb{R})$-factors and set $U^r := \sum_{j=1}^r U_j$.*

(i) *Let $\rho\colon \big(\mathfrak{sl}(2,\mathbb{R}), \frac{1}{2}U\big) \to \big(\mathfrak{sl}(2,\mathbb{R})^r, \frac{1}{2}U^r\big)$ be an (H_1)-homomorphism. Then*

$$\rho(U) = \sum_{j=1}^r \lambda_j U_j$$

with $\lambda_j \in \{0,1\}$.

(ii) *Conversely, for each collection $\lambda_1, \ldots, \lambda_r$ with $\lambda_j \in \{0,1\}$ there is an (H_1)-homomorphism $\rho\colon \big(\mathfrak{sl}(2,\mathbb{R}), \frac{1}{2}U\big) \to \big(\mathfrak{sl}(2,\mathbb{R})^r, \frac{1}{2}U^r\big)$ with $\rho(U) = \sum_{j=1}^r \lambda_j U_j$.*

(iii) *Let $\mathfrak{t} := \sum_{j=1}^r \mathbb{R}U_j$. Then $e^{\mathrm{ad}\,\mathfrak{t}}$ acts on the set of (H_1)-homomorphisms $\big(\mathfrak{sl}(2,\mathbb{R}), \frac{1}{2}U\big) \to \big(\mathfrak{sl}(2,\mathbb{R})^r, \frac{1}{2}U^r\big)$ by conjugation and the conjugacy class of ρ is determined by the r-tuple $(\lambda_1, \ldots \lambda_r) \in \{0,1\}^r$. In particular there are 2^r $e^{\mathrm{ad}\,\mathfrak{t}}$-conjugacy classes.*

Proof. (i) According to Proposition II.7, $\rho(U) = \sum_{j=1}^r \lambda_j U_j$ is contained in the cone $W_{\max}(\frac{1}{2}U^r)$, and therefore $\lambda_j \geq 0$ for $j = 1, \ldots, r$. This element leaves the subspace $\mathbb{R}H_j + \mathbb{R}T_j$ invariant and acts on it with the spectrum $\{\pm\lambda_j 2i\}$. Hence $\rho(H)$ and $\rho(T)$ are contained in $\sum_{\lambda_j=1}(\mathbb{R}H_j + \mathbb{R}T_j)$ and therefore $[H,T] \in \mathbb{R}U$ yields that $\lambda_j = 1$ whenever $\lambda_j \neq 0$.

(ii) Given $\sum_{j=1}^r \lambda_j U_j$ we set

$$U' = \sum_{j=1}^r \lambda_j U_j, \quad H' = \sum_{j=1}^r \lambda_j H_j, \quad T' = \sum_{j=1}^r \lambda_j T_j$$

and note that these three elements define an (H_1)-homomorphism

$$\rho\colon \big(\mathfrak{sl}(2,\mathbb{R}), \tfrac{1}{2}U\big) \to \big(\mathfrak{sl}(2,\mathbb{R})^r, \tfrac{1}{2}U^r\big), \quad \text{via} \quad \rho(U) = U', \rho(H) = H', \rho(T) = T'.$$

(iii) It follows from the above arguments that there exist exactly 2^r (H_1)-homomorphisms

$$\rho\colon \big(\mathfrak{sl}(2,\mathbb{R}), \tfrac{1}{2}U\big) \to \big(\mathfrak{sl}(2,\mathbb{R})^r, \tfrac{1}{2}U^r\big)$$

with $\rho(H) \subseteq \sum_{j=1}^r \mathbb{R}^+ H_j$. Since the image of H under an (H_1)-homomorphism is necessarily is contained in $\sum_{\lambda_j=1}(\mathbb{R}H_j + \mathbb{R}T_j)$ it follows that *each* (H_1)-homomorphism is $e^{\mathrm{ad}\,\mathfrak{t}}$ conjugate to such an (H_1)-homomorphism. On the other hand it is clear that (H_1)-homomorphisms with different $(\lambda_1, \ldots, \lambda_r)$ cannot be $e^{\mathrm{ad}\,\mathfrak{t}}$-conjugate. This implies the claim. ∎

Proposition III.7 shows that we can define an invariant $m(\rho) := |\{j\colon \lambda_j = 1\}|$, called the *multiplicity* (cf. [29, p.111]) for any

$$\rho\colon \big(\mathfrak{sl}(2,\mathbb{R}), \tfrac{1}{2}U\big) \to \big(\mathfrak{sl}(2,\mathbb{R})^r, \tfrac{1}{2}U^r\big).$$

Let $\kappa\colon \big(\mathfrak{sl}(2,\mathbb{R}), \tfrac{1}{2}U\big) \to (\mathfrak{g}, H_0)$ be an (H_1)-inclusion. Then, according to Proposition III.4, it factors over an (H_1)-inclusion

$$\rho\colon \big(\mathfrak{sl}(2,\mathbb{R}), \tfrac{1}{2}U\big) \to \big(\mathfrak{sl}(2,\mathbb{R})^r, \tfrac{1}{2}U^r\big).$$

Proposition III.8(i) below shows that we may define the *multiplicity* $m(\kappa)$ to be the multiplicity of ρ.

Proposition III.8. *Let $(\mathfrak{g}, H_0)$ be a simple Lie algebra of hermitean type with real rank r.*

(i) *The multiplicity $m(\rho)$ of an (H_1)-inclusion $\rho\colon (\mathfrak{sl}(2,\mathbb{R}), \tfrac{1}{2}U) \to (\mathfrak{g}, H_0)$ does not depend on the choice of the (H_1)-inclusion of $\mathfrak{sl}(2,\mathbb{R})^r$ in $\mathfrak{g}$ over which it factors.*

(ii) *The $\mathrm{Ad}(G)$-conjugacy classes of (H_1)-homomorphisms*

$$\big(\mathfrak{sl}(2,\mathbb{R}), \tfrac{1}{2}U\big) \to (\mathfrak{g}, H_0)$$

are classified by their multiplicity $m(\rho)$.

(iii) *The $\mathrm{Ad}(G)$-conjugacy classes of (H_1)-inclusions*

$$\rho\colon \big(\mathfrak{sl}(2,\mathbb{R}), \tfrac{1}{2}U\big) \to (\mathfrak{g}, H_0)$$

coincide with the $\mathrm{Ad}(K)$-conjugacy classes of (H_1)-inclusions

$$\rho\colon \big(\mathfrak{sl}(2,\mathbb{R}), \tfrac{1}{2}U\big) \to (\mathfrak{g}, H_0).$$

Proof. (cf. [29, p.111]) (i): Let $\mathfrak{s}$ and $\mathfrak{s}_1$ be two copies of $\mathfrak{sl}(2,\mathbb{R})^r$ which are (H_1)-inclusions over which ρ factors. Choose maximal abelian subspaces $\mathfrak{a}$ and $\mathfrak{a}_1$ of $\mathfrak{s}\cap\mathfrak{p}$ and $\mathfrak{s}_1\cap\mathfrak{p}$ containing $\rho(H)$. The centralizer $\mathfrak{g}_0 := \ker(\operatorname{ad}\rho(H))$ of $\rho(H)$ in $\mathfrak{g}$ is θ-invariant and hence a reductive subalgebra. Since $\mathfrak{a},\mathfrak{a}_1\subseteq\mathfrak{p}\cap\mathfrak{g}_0$ and $\mathfrak{g}$ has real rank r, we see that $\mathfrak{a}$ and $\mathfrak{a}_1$ are maximal abelian also in $\mathfrak{p}\cap\mathfrak{g}_0$. Thus we can find $k\in\exp(\mathfrak{k}\cap\mathfrak{g}_0)$ such that $\operatorname{Ad}(k)\mathfrak{a}=\mathfrak{a}_1$. According to Lemma III.3,

$$\mathfrak{s}=\mathfrak{a}+J\mathfrak{a}+[\mathfrak{a},J\mathfrak{a}],\quad \mathfrak{s}_1=\mathfrak{a}_1+J\mathfrak{a}_1+[\mathfrak{a}_1,J\mathfrak{a}_1]$$

and $\operatorname{Ad}(k)$ commutes with J. Therefore $\operatorname{Ad}(k).\mathfrak{s}=\mathfrak{s}_1$. Note that $k\in\exp\mathfrak{k}$ now implies that $\operatorname{Ad}(k)\colon\mathfrak{s}\to\mathfrak{s}_1$ is an (H_1)-isomorphism. Let $\mathfrak{i}$ be the ideal complement of the largest ideal of $\mathfrak{s}$ contained in $\mathfrak{g}_0$ and similarly $\mathfrak{i}_1$ for $\mathfrak{s}_1$. Then $\operatorname{Ad}(k).\mathfrak{i}=\mathfrak{i}_1$. Since the dimension of $\mathfrak{i}$ is three times the multiplicity of ρ calculated using $\mathfrak{s}$, the claim follows.
(ii), (iii): We show first that $m(\rho)=m(\operatorname{Ad}(g)\circ\rho)$ holds whenever $\operatorname{Ad}(g)\circ\rho$ is an (H_1)-homomorphism. Let $\mathfrak{s}\cong\mathfrak{sl}(2,\mathbb{R})^r$ be such that ρ factors over the embedding of $\mathfrak{s}$. Assume that $\operatorname{Ad}(g)\circ\rho$ is an (H_1)-homomorphism. Then $\rho\colon\mathfrak{sl}(2,\mathbb{R})\to\mathfrak{g}$ is an (H_1)-homomorphism with respect to $\frac{1}{2}U$ and $\operatorname{Ad}(g)^{-1}.H_0$. The ideal of $\mathfrak{s}$ generated by the image of ρ is moved by $\operatorname{Ad}(g)$ into the ideal generated by the image of $\operatorname{Ad}(g)\circ\rho$ in $\operatorname{Ad}(g).\mathfrak{s}$. Hence the dimensions of these ideals are the same and we see that both homomorphisms have the same multiplicity.

In view of Lemma II.12, $g\in KZ_G\big(\rho(\mathfrak{sl}(2,\mathbb{R}))\big)$. Let $g=k\exp(Y)$ with $k\in K$ and $[Y,\operatorname{im}\rho]=\{0\}$. Then $\operatorname{Ad}(g)\circ\rho=\operatorname{Ad}(k)\circ\rho$ and $\operatorname{Ad}(k).\mathfrak{s}$ is a subalgebra such that $\operatorname{Ad}(k)\circ\rho$ factors over $\operatorname{Ad}(k).\mathfrak{s}$. This proves (iii).

It now suffices to show that if $m(\rho)=m(\rho')$, there exists $k\in K$ with $\operatorname{Ad}(k)\circ\rho=\rho'$. Since the maximal abelian subspaces $\mathfrak{a}\subseteq\mathfrak{p}$ are conjugate under K, we may w.l.o.g. assume that both ρ and ρ' factor over the inclusion of $\mathfrak{s}\cong\mathfrak{sl}(2,\mathbb{R})^r$. Let $\mathfrak{a}^+\subseteq\mathfrak{a}$ be a Weyl chamber. Then we may also assume that $\rho(H),\rho'(H)\in\mathfrak{a}^+$. Let $\gamma_1,\ldots,\gamma_r$ denote the strictly orthogonal roots and assume that $\gamma_1\geq\gamma_2\geq\ldots\geq\gamma_r$ in the ordering corresponding to the Weyl chamber. Write $\rho(H)=\sum_{j=1}^r\lambda_jH_j$. Then $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_r$ and $\lambda_j\in\{0,1\}$. Therefore $\rho(H)=\sum_{j=1}^kH_j$, where $k=m(\rho)$ is the multiplicity. Since $m(\rho)=m(\rho')$ by assumption, it follows that $\rho=\rho'$. ■

We turn to the classification of the nilpotent orbits lying in an invariant cone $W_{\max}$. Note first that there exist exactly 2^r nilpotent orbits lying in the cone $W_{\max}(\frac{1}{2}U^r)$ in the Lie algebra $\mathfrak{sl}(2,\mathbb{R})^r$. If $\mathfrak{g}$ is simple hermitean, then it contains exactly 2 conjugacy classes of H-elements. If $\rho\colon\mathfrak{sl}(2,\mathbb{R})\to\mathfrak{g}$ is an (H_1)-homomorphism with respect to some H_0 and H_0' is another H-element, then there exists $g\in G$ with $\operatorname{Ad}(g).H_0=H_0'$ or $\operatorname{Ad}(g).H_0=-H_0'$. Hence it suffices to consider those (H_1)-homomorphisms which are associated to a fixed H-element H_0.

Theorem III.9. *Let $\mathfrak{g}$ be a simple hermitean Lie algebra of real rank r and $W_{\max}$ a maximal invariant cone. Then $W_{\max}$ contains exactly $r+1$ nilpotent orbits $\mathcal{O}_0,\ldots,\mathcal{O}_r$. They satisfy*

$$\{0\}=\mathcal{O}_0\subseteq\overline{\mathcal{O}}_1\subseteq\ldots\subseteq\overline{\mathcal{O}}_r$$

and they are contained in $W_{\min}$.

Proof. We fix an H-element H_0 contained in $W_{\max}$, a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ together with the associated subalgebra $\mathfrak{s} \cong \mathfrak{sl}(2,\mathbb{R})^r$ (cf. Lemma III.3), and a Weyl chamber $\mathfrak{a}^+ \subseteq \mathfrak{a}$. Suppose that $\gamma_1 \geq \ldots \geq \gamma_r$ is the corresponding order on the strictly orthogonal roots corresponding to $\mathfrak{s}$.

Let $\mathcal{O}_X \subseteq W_{\max}$ be a nilpotent orbit and $\rho: \left(\mathfrak{sl}(2,\mathbb{R}), \frac{1}{2}U\right) \to (\mathfrak{g}, H_0)$ the corresponding (H_1)-homomorphism (cf. Theorem II.10). We may w.l.o.g. assume that $\rho(H) \in \mathfrak{a}^+$. Then $\rho(H) = \sum_{j=1}^k H_j$ and $m(\rho) = k$. In view of Proposition III.8, this proves that every nilpotent orbit $\mathcal{O}_X$ in $W_{\max}$ can be obtained by an $\mathfrak{sl}_2$-triple (H,X,Y) as above. Hence there exist at most $r+1$ different nilpotent orbits in $W_{\max}$ which are given by $X^k := \sum_{j=1}^k X_j$. Since $X^{k-1} \in \overline{\mathcal{O}_{X^k}}$ follows from the fact that the nilpotent orbits in $\mathfrak{sl}(2,\mathbb{R})$ contain 0 in their closure, we see that $\mathcal{O}_{X^{k-1}} \subseteq \overline{\mathcal{O}_{X^k}}$.

It remains to show that $\mathcal{O}_{X^k} \neq \mathcal{O}_{X^{k-1}}$. Suppose that both orbits coincide. Then it follows from Lemma I.5 that both are contained in conjugate $\mathfrak{sl}_2$-triples. Then the associated (H_1)-homomorphisms must have the same rank which is impossible since $k \neq k-1$ (Propositions III.7, III.8). ∎

Remark III.10. A second way to prove Theorem III.9 is to show that the dimension of the centralizers of $X^0 = 0, X^1, \ldots, X^r$ are mutually different. This can be done by considering the $\mathfrak{s}$-module structure of $\mathfrak{g}$ (cf. Section IV). We will come back to this context when we study parabolics associated to the nilpotent orbits. ∎

Corollary III.11. *A simple hermitean Lie algebra* $\mathfrak{g}$ *contains exactly* $2r+1$ *nilpotent orbits of convex type.* ∎

Remark III.12. We have observed in Theorem II.11 that the nilpotent orbits contained in $W_{\max}$ correspond to $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^+$ under the Sekiguchi correspondence. These orbits have been classified by Muller, Rubenthaler and Schiffmann in [22]. ∎

IV. The fine structure of the Jacobson-Morosow parabolics

In this section we study the special structure of Jacobson-Morosow parabolics associated to (H_1)-embeddings of $\mathfrak{sl}(2,\mathbb{R})$ into some simple Lie algebra of hermitean type. So fix an (H_1)-embedding $\kappa: (\mathfrak{sl}(2,\mathbb{R}), \frac{1}{2}U) \to (\mathfrak{g}, H_0)$ and write $\mathfrak{s}^\kappa$ for its image in $\mathfrak{g}$. Further we write $H_\kappa, X_\kappa, Y_\kappa, U_\kappa, T_\kappa$ for the respective images of H, X, Y, U, T under κ. For the $\mathfrak{sl}(2,\mathbb{R})$-module $\mathfrak{g}$ defined by κ we have the isotypic decomposition

$$\mathfrak{g} = \mathfrak{g}^{[0]_\kappa} + \mathfrak{g}^{[1]_\kappa} + \mathfrak{g}^{[2]_\kappa} \tag{IV.1}$$

(cf. [29, p.90], Section I). The decomposition (IV.1) shows that the Jacobson-Morosow parabolic of the Cayley-triple $(H_\kappa, X_\kappa, Y_\kappa)$ is

$$\mathcal{U}_\kappa + \mathcal{V}_\kappa + \mathfrak{z}_{\mathfrak{g}}(H_\kappa) \tag{IV.2}$$

with $\mathcal{U}_\kappa = \mathfrak{g}(\operatorname{ad} H_\kappa; 2)$ and $\mathcal{V}_\kappa = \mathfrak{g}(\operatorname{ad} H_\kappa; 1)$. In particular it follows that $\mathcal{U}_\kappa$ and $\mathcal{V}_\kappa$ commute. The space $\mathcal{V}_\kappa$ may or may not be trivial.

Proposition IV.1. (cf. [29, p.112]) *Suppose that $\mathcal{V}_\kappa \neq \{0\}$. Then*

(i) *the representation of $\mathfrak{z}_\mathfrak{g}(H_\kappa)$ on $\mathcal{V}_k$ is irreducible and faithful, and*

(ii) $[\mathcal{V}_\kappa, \mathcal{V}_\kappa] = \mathcal{U}_\kappa$. ∎

The spaces $\mathfrak{g}^{[0]_\kappa}$ and $\mathfrak{g}^{[\mathrm{even}]_\kappa} = \mathfrak{g}^{[0]_\kappa} + \mathfrak{g}^{[2]_\kappa}$ are subalgebras of hermitean type with H-elements $H_0 - \frac{1}{2}U_\kappa$ and H_0, respectively (cf. [29, p.92]). Decompose $\mathfrak{g}^{[\mathrm{even}]_\kappa}$ in ideals

$$\mathfrak{g}^{[\mathrm{even}]_\kappa} = \bigoplus_{j=0}^{s_\kappa} \mathfrak{s}_j^{(\kappa)}$$

with $\mathfrak{s}_0^{(\kappa)}$ the largest compactly embedded ideal and $\mathfrak{s}_j^{(\kappa)}$ simple non-compact for $j > 0$. Further we order the ideals in such a way that $\bigoplus_{j=1}^{s_{o,\kappa}} \mathfrak{s}_j^{(\kappa)} \subseteq \mathfrak{g}^{[0]_\kappa}$ and $\mathfrak{s}_j^{(\kappa)} \not\subseteq \mathfrak{g}^{[0]_\kappa}$ for $j > s_{o,\kappa}$. Now we set

$$\mathfrak{g}_{1,\kappa} = \sum_{j=0}^{s_{o,\kappa}} \mathfrak{s}_j^{(\kappa)}. \tag{IV.3}$$

Proposition IV.2. (cf. [29, p.94])

(i) *$\mathfrak{g}_{1,\kappa}$ is of hermitean type with H-element $H_0 - \frac{1}{2}U_\kappa$.*

(ii) *$\sum_{j=s_{o,\kappa}+1}^{s_\kappa} \mathfrak{s}_j^{(\kappa)}$ is of hermitean type with H-element $\frac{1}{2}U_\kappa$.* ∎

Consider the partial Cayley transform $c_\kappa = e^{i\frac{\pi}{4}\operatorname{ad} T_\kappa}$. It satisfies

$$c_k(U_\kappa) = -iH_\kappa, \quad c_\kappa(H_\kappa) = -iU_\kappa, \quad c_\kappa(T_\kappa) = T_\kappa. \tag{IV.4}$$

We set

$$\mathfrak{g}_{2,\kappa} = \mathfrak{g} \cap c_\kappa^{-1}\big((\mathfrak{k} \cap \sum_{j=s_{o,\kappa}+1}^{s_\kappa} \mathfrak{s}_j^{(\kappa)})_\mathbb{C}\big). \tag{IV.5}$$

Proposition IV.3. (cf. [29, p.96]) $\mathfrak{z}_\mathfrak{g}(H_\kappa) = \mathfrak{g}_{1,\kappa} \oplus \mathfrak{g}_{2,\kappa}$. ∎

Proposition IV.4. (cf. [29, Th. III.2.3, Prop. III.4.4])

(i) *The representation of $\mathfrak{g}_{2,\kappa}$ on $\mathcal{U}_\kappa$ is faithful and irreducible.*

(ii) *$\mathfrak{g}_{1,\kappa}$ acts trivially on $\mathcal{U}_\kappa$.*

(iii) *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $G_{2,\kappa}$ the analytic subgroup of G corresponding to $\mathfrak{g}_{2,\kappa}$. Then the orbit*

$$\Omega_\kappa := \operatorname{Ad}(G_{2,\kappa}).X_\kappa$$

is an open convex cone in $\mathcal{U}_\kappa$ which is selfdual with respect to the euclidean inner product $(\cdot \mid \cdot)$ on $\mathfrak{g}$ defined by

$$(X \mid Y) = -B(X, \theta Y). \tag{IV.6}$$

The group $\operatorname{Ad}_{\mathcal{U}_\kappa}(G_{2,\kappa})$ coincides with the identity component of the linear automorphism group $G(\Omega_\kappa)$ of the cone Ω_κ. ∎

Proposition IV.5. (cf. [29, pp. 92, 102, 103, 112])

(i) $\theta c_\kappa^2 = c_\kappa^{-2}\theta$.

(ii) $I_\kappa := ic_\kappa^2\theta|_{\mathcal{V}_\kappa}$ *leaves* $\mathcal{V}_\kappa$ *invariant and defines a complex structure on this space.*

(iii) $\mathrm{ad}(H_0 - \frac{1}{2}U_\kappa)|_{\mathcal{V}_\kappa} = \frac{1}{2}I_\kappa$.

(iv) $(v,v') \mapsto -\frac{i}{4}B(v, c_\kappa^2 v') = -\frac{1}{4}(X_\kappa \mid [v,v'])$ *defines a symplectic form on* $\mathcal{V}_\kappa$.

(v) $\big(\mathfrak{sp}(\mathcal{V}_\kappa), \frac{1}{2}I_\kappa\big)$ *is a simple Lie algebra of hermitean type.*

(vi) *The adjoint action of* $\mathfrak{g}$ *induces an* (H_2)*-homomorphism* $\mathfrak{g}_{1,\kappa} \to \mathfrak{sp}(\mathcal{V}_\kappa)$.

(vii) *The adjoint action of* $\mathfrak{g}$ *induces a homomorphism* $\mathfrak{g}_{2,\kappa} \to \mathfrak{gl}(\mathcal{V}_\kappa, I_\kappa)$, *where* $\mathfrak{gl}(\mathcal{V}_\kappa, I_\kappa)$ *denotes the complex linear endomorphisms of* $\mathcal{V}_\kappa$ *with respect to* I_κ. ■

We define a skew symmetric bilinear form A_u on $\mathcal{V}_\kappa$ for *any* $u \in \mathcal{U}_\kappa$ via

$$A_u(v,v') = -\frac{1}{4}(u \mid [v,v']). \tag{IV.7}$$

and a bilinear map $A\colon \mathcal{V}_\kappa \times \mathcal{V}_\kappa \to \mathcal{U}_\kappa$ by $A(v,v') = -\frac{1}{4}[v,v']$. Consider the group

$$\mathrm{Sp}(\mathcal{V}_\kappa, A) = \{g \in \mathrm{Gl}(\mathcal{V}_\kappa)\colon A(g.v, g.v') = A(v,v')\} \tag{IV.8}$$

and its Lie algebra $\mathfrak{sp}(\mathcal{V}_\kappa, A)$. Then it is clear that $\mathrm{Sp}(\mathcal{V}_\kappa, A) \subseteq \mathrm{Sp}(\mathcal{V}_\kappa)$, where $\mathcal{V}_\kappa$ carries the symplectic form provided by Proposition IV.5(iv). But more is true:

Proposition IV.6. (cf. [29, p.132])

(i) $\mathfrak{sp}(\mathcal{V}_\kappa, A)$ *is of hermitean type with* H*-element* $\frac{1}{2}I_\kappa$.

(ii) *The adjoint action of* $\mathfrak{g}$ *induces an* (H_2)*-homomorphism*

$$\mathfrak{g}_{1,\kappa} \to \mathfrak{sp}(\mathcal{V}_\kappa, A).$$

(iii) *The real bilinear forms* $A_u I$ *on* $\mathcal{V}_\kappa$ *given by* $(v,v') \mapsto \big(u \mid A(v, Iv')\big)$ *are symmetric for all* $u \in \mathcal{U}_\kappa$ *and positive definite for all* $u \in \Omega_\kappa$. ■

For the remainder of this section we assume that $\mathfrak{g}$ is simple hermitean of real rank r and the $\mathfrak{sl}_2$-triples are the r ones occurring in the proof of Proposition III.7. More precisely, we fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, an H-element $H_0 \in \mathfrak{z}(\mathfrak{k})$, and a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ and then view $\mathfrak{a} + J\mathfrak{a} + [\mathfrak{a}, J\mathfrak{a}]$ as the range of an (H_1)-homomorphism $\mathfrak{sl}(2,\mathbb{R})^r \to \mathfrak{g}$, where $J = \mathrm{ad}(H_0)|_{\mathfrak{p}}$ (cf. Lemma III.3).

We write Σ for the set of restricted roots in $\mathfrak{a}^*$. Then, according to a theorem of Moore, Σ is a not necessarily reduced root system of type (C_r) (G/K of tube type) or of type (BC_r) (cf. [29, p.110]).

Let $\gamma_1, \ldots, \gamma_r$ denote the strictly orthogonal roots. Then $\gamma_j = 2\varepsilon_j$ and Σ is given by

$$\Sigma = \{\pm 2\varepsilon_j, \pm(\varepsilon_i \pm \varepsilon_j)\colon 1 \le i < j \le r\} \quad \text{type } (C_r)$$

and

$$\Sigma = \{\pm\varepsilon_j, \pm 2\varepsilon_j, \pm(\varepsilon_i \pm \varepsilon_j) \colon 1 \leq i < j \leq r\} \quad \text{type } (BC_r).$$

As fundamental systems we choose

$$\Upsilon = \{\mu_1, \ldots, \mu_r\} := \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{r-1} - \varepsilon_r, 2\varepsilon_r\} \quad \text{type } (C_r)$$

and

$$\Upsilon = \{\mu_1, \ldots, \mu_r\} := \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_r\} \quad \text{type } (BC_r),$$

respectively. With respect to the ordering fixed by this choice of a fundamental system we have $\gamma_1 \geq \ldots \geq \gamma_r$.

Let $H_1, \ldots, H_r \in \mathfrak{a}$ denote the coroots for the roots γ_j, i.e., $\gamma_j(H_i) = 2\delta_{ij}$. The elements H_j span $\mathfrak{a}$. We have seen in the last section that the elements $H^k := H_1 + \ldots + H_k$ belong to Cayley triples (H^k, X^k, Y^k) associated to the nilpotent orbits $\mathcal{O}_{X^k}$ of convex type, where $X^k = X_1 + \ldots + X_k$. Now we set $\mathfrak{s}^k := \mathbb{R}H^k + \mathbb{R}X^k + \mathbb{R}Y^k$ and $U^k := X^k - Y^k, T^k := X^k + Y^k$. The corresponding (H_1)-homomorphisms will be denoted by κ_k. To simplify the notation we will $\mathfrak{g}^{[j]_k}$ instead of $\mathfrak{g}^{[j]_{\kappa_k}}$ and similarly for $\mathfrak{g}^{[\mathrm{even}]}$ and other notations involving κ. In this situation we have a lot of additional information on the Jacobson-Morosow parabolic of (H^k, X^k, Y^k). We begin with a closer look at $\mathfrak{z}_\mathfrak{g}(H^k)$.

Lemma IV.7. *Let $\mu \in \Sigma^+$. Then $\mu(H^k) \in \{0, 1, 2\}$. More precisely we have*

(i) $\mu(H^k) = 0$ *if and only if μ belongs to the following list:*

$$\begin{array}{rl} \varepsilon_i - \varepsilon_j & i < j \leq k, \\ \varepsilon_i - \varepsilon_j & k < i < j, \\ \varepsilon_i + \varepsilon_j & k < i < j, \\ 2\varepsilon_i & k < i, \\ \varepsilon_i & k < i. \end{array}$$

Here of course the last line only occurs if Σ is of type (BC_r).

(ii) $\mu(H^k) = 1$ *if and only if it belongs to the following list:*

$$\varepsilon_i \pm \varepsilon_j, i \leq k < j, \qquad \varepsilon_i, i \leq k.$$

(iii) $\mu(H^k) = 2$ *if and only if it belongs to the following list:*

$$\varepsilon_i + \varepsilon_j, i < j \leq k, \qquad 2\varepsilon_i, i \leq k.$$

Proof. Let $1 \leq i < j \leq r$. Then we calculate

$$(\varepsilon_i - \varepsilon_j)(H^k) = \sum_{s=1}^k \delta_{is} - \sum_{s=1}^k \delta_{js} = \begin{cases} 1 & \text{for } i \leq k < j \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\varepsilon_i + \varepsilon_j)(H^k) = \sum_{s=1}^k \delta_{is} + \sum_{s=1}^k \delta_{js} = \begin{cases} 0 & \text{for } k < i < j \\ 1 & \text{for } i \leq k < j \\ 2 & \text{for } i < j \leq k. \end{cases}$$

Moreover, for $1 \leq i \leq r$ we have

$$\varepsilon_i(H^k) = \sum_{s=1}^k \delta_{is} = \begin{cases} 1 & \text{for } i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Now the claim follows by inspection. ∎

Consider the subsets

$$\text{(IV.9)} \qquad \begin{aligned} \Sigma_{(k,\pm)} &:= \Sigma \cap \operatorname{span}\{\mu_j \in \Upsilon : \pm(k-j) < 0\} \\ \Sigma_{(k,j)} &:= \{\mu \in \Sigma : \mu(H^k) = j\} \end{aligned}$$

of Σ. Note that $\Sigma_{(k,+)}$ and $\Sigma_{(k,-)}$ are the two root systems that one obtains from Σ by first deleting μ_k from the Dynkin diagram of Σ and then forming two root systems from the two remaining connected components.

Proposition IV.8.

(i) $\Sigma_{(k,+)}$ *is a root system of type* $(C_{(r-k)})$ *if* Σ *is of type* (C_r).

(ii) $\Sigma_{(k,+)}$ *is a root system of type* $(BC_{(r-k)})$ *if* Σ *is of type* (BC_r).

(iii) *A system of fundamental roots for* $\Sigma_{(k,+)}$ *is* $\{\mu_{k+1}, \ldots \mu_r\}$.

(iv) $\Sigma_{(k,-)}$ *is a root system of type* (A_{k-1}).

(v) *A system of fundamental roots for* $\Sigma_{(k,-)}$ *is* $\{\mu_1, \ldots \mu_{k-1}\}$.

(vi) $\big(\Sigma_{(k,+)} + \Sigma_{(k,-)}\big) \cap \Sigma = \emptyset$.

(vii) $\big(\Sigma_{(k,+)} + \Sigma_{(k,2)}\big) \cap \Sigma = \emptyset$.

(viii) *Let* $\mu \in \Sigma$. *Then* $\mu(H^k) = 0$ *if and only if* $\mu \in \Sigma_{(k,+)} \cup \Sigma_{(k,-)}$.

Proof. An inspection of the list in Lemma IV.7 yields that

$$\Sigma_{(k,-)} = \{\pm(\varepsilon_i - \varepsilon_j) : i < j \leq k\}$$

which is a root system of type (A_{k-1}), whereas

$$\Sigma_{(k,+)} = \{\pm 2\varepsilon_j, \pm(\varepsilon_i \pm \varepsilon_j) : k < i < j \leq r\} \quad \text{is of type } (C_{(r-k)})$$

or

$$\Sigma_{(k,+)} = \{\pm\varepsilon_j, \pm 2\varepsilon_j, \pm(\varepsilon_i \pm \varepsilon_j) : k < i < j \leq r\} \quad \text{is of type } (BC_{(r-k)})$$

depending on the type of Σ. The remaining claims now follow easily by inspection. ∎

Remark IV.9. The dual base of $\{\mu_1, \ldots, \mu_r\}$ is $\{H^1, \ldots, H^r\}$ if Σ is of type (BC_r). It is $\{H^1, \ldots, H^{r-1}, \frac{1}{2}H^r\}$ if Σ is of type (C_r). ∎

Set

$$\mathfrak{a}_{(k,-)} := \operatorname{span}\{H_1, \ldots, H_k\} \cap (H^k)^\perp \quad \text{and} \quad \mathfrak{a}_{(k,+)} := \operatorname{span}\{H_{k+1}, \ldots, H_r\}.$$

Proposition IV.10.

(i) $\mathfrak{a}_{(k,-)} = \operatorname{span}\{H^k, \ldots, H^r\}^\perp$.

(ii) $\mathfrak{a}_{(k,+)} = \operatorname{span}\{H^1, \ldots, H^k\}^\perp$.

Proof. This follows from

$$\begin{aligned} \operatorname{span}\{H^j : k \leq j \leq r\}^\perp &= \operatorname{span}\{H_{k+1}, \ldots, H_r, H^k\}^\perp \\ &= \operatorname{span}\{H_j : 1 \leq j \leq k\} \cap (H^k)^\perp \end{aligned}$$

and

$$\begin{aligned} \operatorname{span}\{H^j : 1 \leq j \leq k\}^\perp &= \operatorname{span}\{H_j : 1 \leq j \leq k\}^\perp \\ &= \operatorname{span}\{H_j : k+1 \leq j \leq r\}. \end{aligned}$$

∎

Proposition IV.11. *Let* $\mu \in \Sigma_{(k,\pm)}$ *and* $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$. *Then*

(i) $\mu(\mathfrak{a}_{(k,\mp)}) = \{0\}$.

(ii) $[\mathfrak{g}^\mu, \mathfrak{g}^{-\mu}] \subseteq \mathfrak{a}_{(k,\pm)} + \mathfrak{m}$.

Proof. (i) This follows immediately from Proposition IV.8, Remark IV.9 and Proposition IV.10.
(ii) Let $\mu \in \Sigma_{(k,+)}$, $Z \in \mathfrak{g}^\mu$, $Z' \in \mathfrak{g}^{-\mu}$ and B the Killing form on $\mathfrak{g}$. Then

$$B(H^j, [Z, Z']) = B([H^j, Z], Z') = \mu(H^j)B(Z, Z')$$

which is zero for $1 \leq j \leq k$. Thus

$$[\mathfrak{g}^\mu, \mathfrak{g}^{-\mu}] \subseteq \mathfrak{m} + \operatorname{span}\{H^j : 1 \leq j \leq k\}^\perp = \mathfrak{m} + \mathfrak{a}_{(k,+)}.$$

For $\mu \in \Sigma_{(k,-)}$ we argue similarly and find

$$[\mathfrak{g}^\mu, \mathfrak{g}^{-\mu}] \subseteq \mathfrak{m} + \operatorname{span}\{H^j : k \leq j \leq r\}^\perp = \mathfrak{m} + \mathfrak{a}_{(k,-)}.$$ ∎

Lemma IV.12. *Let* $\mathfrak{g}_{(k,\pm)}$ *be the subalgebra of* $\mathfrak{g}$ *generated by the root spaces* $\mathfrak{g}^\mu$ *with* $\mu \in \Sigma_{(k,\pm)}$.

(i) $\mathfrak{g}_{(k,\pm)}$ *is simple and* θ*-invariant with restricted root system* $\Sigma_{(k,\pm)}$.

(ii) $\mathfrak{g}_{(k,+)}$ *and* $\mathfrak{g}_{(k,-)}$ *commute. In particular, being simple, they intersect in* $\{0\}$.

(iii) $\mathfrak{g}_{(k,\pm)} \cap \mathfrak{a} = \mathfrak{a}_{(k,\pm)}$ *is maximal abelian in* $\mathfrak{g}_{(k,\pm)} \cap \mathfrak{p}$.

(iv) *We set* $\mathfrak{k}_{(k,\pm)} := \mathfrak{k} \cap \mathfrak{g}_{(k,\pm)}$ *and* $\mathfrak{n}_{(k,\pm)} := \mathfrak{n} \cap \mathfrak{g}_{(k,\pm)} = \sum_{\mu \in \Sigma^+_{(k,\pm)}} \mathfrak{g}^\mu$. *Then*

$$\mathfrak{g}_{(k,\pm)} = \mathfrak{k}_{(k,\pm)} + \mathfrak{a}_{(k,\pm)} + \mathfrak{n}_{(k,\pm)}$$

is an Iwasawa decomposition.

Proof. (i) It is clear that $\mathfrak{g}_{(k,\pm)}$ is θ-invariant and hence reductive. Therefore we can write

$$\mathfrak{g}_{(k,\pm)} = \bigoplus_{j=0}^{s} \mathfrak{h}_j^\pm$$

with a compact factor $\mathfrak{h}_0^\pm$ and non-compact θ-invariant simple factors $\mathfrak{h}_j^\pm$, $j = 1, \ldots, s$. Then the restricted root system for $\mathfrak{g}_{(k,\pm)}$ is the disjoint union of the restricted root systems for the $\mathfrak{h}_j$, $j \geq 1$. On the other hand we know from Proposition IV.10 and Proposition IV.8 that

$$\mathfrak{g}_{(k,\pm)} \subseteq \mathfrak{m} + \mathfrak{a}_{(k,\pm)} + \sum_{\mu \in \Sigma_{(k,\pm)}} \mathfrak{g}^\mu.$$

Therefore the restricted root system for $\mathfrak{g}_{(k,\pm)}$ is $\Sigma_{(k,\pm)}$. Thus there can only be one non-compact simple factor which we denote by $\mathfrak{h}^\pm$. But $\mathfrak{h}^\pm$ contains all the root spaces $\mathfrak{g}^\mu$ with $\mu \in \Sigma_{(k,\pm)}$. Thus it coincides with $\mathfrak{g}_{(k,\pm)}$. This proves (i)

(ii) It follows from Proposition IV.8 that the root spaces of $\mathfrak{g}_{(k,+)}$ commute with the root spaces of $\mathfrak{g}_{(k,-)}$. Thus all of $\mathfrak{g}_{(k,+)}$ commutes with $\mathfrak{g}_{(k,-)}$.

(iii) Consider the algebra $\widetilde{\mathfrak{h}}^{\pm} := \mathfrak{a} + \mathfrak{g}_{(k,\pm)}$. It is θ-invariant, hence reductive, and $\mathfrak{a}$ is maximal abelian in $\widetilde{\mathfrak{h}}^{\pm} \cap \mathfrak{p}$. Since any maximal abelian subspace of $\widetilde{\mathfrak{h}}^{\pm} \cap \mathfrak{p}$ respects the decomposition into simple factors (plus the center), we see that $\mathfrak{a}_{\pm} := \mathfrak{a} \cap \mathfrak{g}_{(k,\pm)}$ is maximal abelian in $\mathfrak{g}_{(k,\pm)} \cap \mathfrak{p}$. From Proposition IV.11 we know that $\mathfrak{a}_{\pm} \subseteq \mathfrak{a}_{(k,\pm)}$. Conversely, the real rank of $\mathfrak{g}_{(k,\pm)}$ coincides with the dimension of $\mathfrak{a}_{(k,\pm)}$. Thus $\mathfrak{a}_{\pm} = \mathfrak{a}_{(k,\pm)}$.

(iv) This follows immediately from (i) and (iii). ∎

Theorem IV.13. *$\mathfrak{z}_{\mathfrak{g}}(H^k) = \widetilde{\mathfrak{g}}_{1,k} \oplus \widetilde{\mathfrak{g}}_{2,k}$, where $\widetilde{\mathfrak{g}}_{1,k} := \mathfrak{g}_{(k,+)} \oplus \mathfrak{m}_k$, $\widetilde{\mathfrak{g}}_{2,k} := \mathbb{R}H^k \oplus \mathfrak{g}_{(k,-)}$, and $\mathfrak{m}_k \subseteq \mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$.*

Proof. We note first that $\mathfrak{z}_{\mathfrak{g}}(H^k)$ is θ-invariant hence reductive in $\mathfrak{g}$. Thus we may write

$$\mathfrak{z}_{\mathfrak{g}}(H^k) = \bigoplus_{j=0}^{s} \mathfrak{h}_j$$

with a compact factor $\mathfrak{h}_0$ and non-compact θ-invariant simple factors $\mathfrak{h}_j$, $j = 1, \ldots, s$. Then the restricted root system for $\mathfrak{z}_{\mathfrak{g}}(H^k)$ is the disjoint union of the restricted root systems for the $\mathfrak{h}_j$, $j \geq 1$. On the other hand we know from Proposition IV.8 and Lemma IV.12 that

$$\mathfrak{z}_{\mathfrak{g}}(H^k) = \mathbb{R}H^k \oplus \big(\mathfrak{m} + (\mathfrak{g}_{(k,+)} \oplus \mathfrak{g}_{(k,-)})\big).$$

But $\mathfrak{m}$ leaves each root space $\mathfrak{g}^{\mu}$ invariant, so $\mathfrak{g}_{(k,+)}$ and $\mathfrak{g}_{(k,-)}$ are ideals in $\mathfrak{z}_{\mathfrak{g}}(H^k)$. Therefore the restricted root system for $\mathfrak{z}_{\mathfrak{g}}(H^k)$ is a disjoint union of $\Sigma_{(k,+)}$ and $\Sigma_{(k,-)}$. Thus there can only be two non-compact simple factors and these have to be $\mathfrak{g}_{(k,+)}$ and $\mathfrak{g}_{(k,-)}$. Now consider $\mathfrak{h}_0$. It is contained in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ since $\mathfrak{a} = \mathbb{R}H^k + \mathfrak{a}_{(k,+)} + \mathfrak{a}_{(k,-)}$, $\mathfrak{a}_{(k,\pm)} \subseteq \mathfrak{g}_{(k,\pm)}$ and H^k is central in $\mathfrak{z}_{\mathfrak{g}}(H^k)$. Therefore the compact part $\mathfrak{m}_k := \mathfrak{k} \cap \mathfrak{h}_0$ is contained in $\mathfrak{m}$. The non-compact part $\mathfrak{p} \cap \mathfrak{h}_0$ is contained in $\mathfrak{a}$ and hence the claim follows. ∎

We will show below (Remark IV.19) that $\widetilde{\mathfrak{g}}_{i,k} = \mathfrak{g}_{i,k}$.

Proposition IV.14.

(i) $\mathfrak{g}_{(k,+)} \subseteq \mathfrak{g}^{[0]_k} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}^k)$.

(ii) $\mathfrak{z}_{\mathfrak{g}}(H^k) \subseteq \mathfrak{g}^{[\mathrm{even}]_k}$

(iii) $H_j \in \mathfrak{g}^{[2]_k}$ *for* $1 \leq j \leq k$.

Proof. (i) Since $X_j \in \mathfrak{g}^{2\varepsilon_j}$ we have $[X_j, \mathfrak{g}^{\mu}] \subseteq \mathfrak{g}^{\mu + 2\varepsilon_j}$. If $\mu \in \Sigma_{(k,+)}$ and $1 \leq j \leq k$, then $\mu + 2\varepsilon_j \notin \Sigma$ so that $[X^k, \mathfrak{g}^{\mu}] = 0$, i.e., $\mathfrak{g}^{\mu} \subseteq \mathfrak{z}_{\mathfrak{g}}(X^k)$. Therefore the algebra $\mathfrak{g}_{(k,+)}$ which is generated by these root spaces is also contained in $\mathfrak{z}_{\mathfrak{g}}(X^k)$. Now Proposition I.3 implies the claim.

(ii) This follows immediately from the definitions.

(iii) H_j belongs to the $\mathfrak{s}^k$-submodule $\mathbb{R}H_j + \mathbb{R}X_j + \mathbb{R}Y_j \subseteq \mathfrak{g}$ which is equivalent to the adjoint representation of $\mathfrak{s}^k$ and hence in $\mathfrak{g}^{[2]_k}$. ∎

Proposition IV.15. $\mathfrak{g}_{(k,+)}$ *is an ideal in* $\mathfrak{g}^{[\mathrm{even}]_k}$.

Proof. Note first that

$$\mathfrak{g}^{[\mathrm{even}]_k} = \mathfrak{z}_{\mathfrak{g}}(H^k) + \mathfrak{g}(\mathrm{ad}\, H^k; 2) + \mathfrak{g}(\mathrm{ad}\, H^k; -2)$$

and $\mathfrak{g}(\mathrm{ad}\, H^k; -2) = \theta\mathfrak{g}(\mathrm{ad}\, H^k; 2)$. Since $\mathfrak{g}_{(k,+)}$ is a θ-invariant ideal in $\mathfrak{z}_{\mathfrak{g}}(H^k)$, it suffices to show that $[\mathfrak{g}_{(k,+)}, \mathfrak{g}(\mathrm{ad}\, H^k; 2)] = \{0\}$. But that follows from Proposition IV.8(vii) since

$$\mathfrak{g}(\mathrm{ad}\, H^k; 2) = \sum_{\mu \in \Sigma_{(k,2)}} \mathfrak{g}^{\mu}. \qquad \blacksquare$$

Recall from the beginning of this section that $\mathfrak{s}_0^{(k)}$ is the maximal compactly embedded ideal in $\mathfrak{g}^{[\mathrm{even}]_k}$ and the $\mathfrak{s}_j^{(k)}$ for $j > 0$ are non-compact simple ideals such that $\bigoplus_{j=1}^{s_{o,k}} \mathfrak{s}_j^{(k)} \subseteq \mathfrak{g}^{[0]_k}$ and $\mathfrak{s}_j^{(k)} \not\subseteq \mathfrak{g}^{[0]_k}$ for $j > s_{o,k}$. In particular this implies that the $\mathfrak{s}_j^{(k)}$ for $1 \leq j \leq s_{o,k}$ are non-compact simple ideals of $\mathfrak{z}_{\mathfrak{g}}(H^k)$ which are contained in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}^k)$. Thus Theorem IV.13, Proposition IV.14 and Proposition IV.15 show that $s_{o,k} = 1$ and $\mathfrak{s}_1^{(k)} = \mathfrak{g}_{(k,+)}$. We write $\widetilde{\mathfrak{s}}^{(k)}$ for $\bigoplus_{j>1} \mathfrak{s}_j^{(k)}$.

The following proposition extends Proposition IV.2.

Proposition IV.16. ([29, Prop. III.1.3, III.1.5]) *The following pairs are reductive Lie algebras of hermitean type:*

(i) $(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}^k), H_0 - \frac{1}{2}U^k)$.

(ii) $(\mathfrak{g}^{[\mathrm{even}]_k}, H_0)$.

(iii) $(\widetilde{\mathfrak{s}}^{(k)}, \frac{1}{2}U^k)$.

(iv) $(\mathfrak{s}_0^{(k)} \oplus \mathfrak{g}_{(k,+)}, H_0 - \frac{1}{2}U^k)$. $\blacksquare$

Remark IV.17.

(i) Let $\widetilde{\mathfrak{s}}_0^{(k)} := \mathfrak{g}^{[0]_k} \cap \widetilde{\mathfrak{s}}^{(k)}$ and note that $\widetilde{\mathfrak{s}}_0^{(k)} \subseteq \mathfrak{z}_{\widetilde{\mathfrak{s}}^{(k)}}(U^k)$. Then $\mathfrak{s}_0^{(k)}$ is compactly embedded in $\mathfrak{g}^{[\mathrm{even}]_k}$, hence by Proposition IV.16(iii) annihilated by H_0. Thus it is contained in $\mathfrak{k}$.

(ii) We have the following decompositions

(a) $\mathfrak{g}^{[0]_k} = \mathfrak{s}_0^{(k)} \oplus \mathfrak{g}_{(k,+)} \oplus \widetilde{\mathfrak{s}}_0^{(k)}$.

(b) $\mathfrak{g}^{[\mathrm{even}]_k} = \mathfrak{s}_0^{(k)} \oplus \mathfrak{g}_{(k,+)} \oplus \widetilde{\mathfrak{s}}^{(k)}$.

(c) $\widetilde{\mathfrak{s}}^{(k)} = \widetilde{\mathfrak{s}}_0^{(k)} \oplus \mathfrak{g}^{[2]_k}$. $\blacksquare$

Proposition IV.18.

(i) $\mathfrak{z}_{\mathfrak{g}}(U^k) = \mathfrak{s}_0^{(k)} \oplus \mathfrak{g}_{(k,+)} \oplus (\mathfrak{k} \cap \widetilde{\mathfrak{s}}^{(k)})$.

(ii) $\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(H^k) = \big(\mathfrak{h}_0^{(k)} \oplus \mathfrak{g}_{(k,+)}\big)_{\mathbb{C}} \oplus c_k^{-1}(\mathfrak{k} \cap \widetilde{\mathfrak{s}}^{(k)})_{\mathbb{C}}$.

(iii) $\mathfrak{z}_{\mathfrak{g}}(H^k) = \mathfrak{s}_0^{(k)} \oplus \mathfrak{g}_{(k,+)} \oplus \big(\mathfrak{g} \cap c_k^{-1}(\mathfrak{k} \cap \widetilde{\mathfrak{s}}^{(k)})_{\mathbb{C}}\big)$.

Proof. (i) In view of Remark IV.17, we only have to show that $\mathfrak{z}_{\mathfrak{g}}(U^k)$ is contained in $\mathfrak{g}^{[\mathrm{even}]_k}$. But this is clear since U^k acts as an injective mapping on $\mathfrak{g}^{[1]_k}$.

(ii) and (iii) are an immediate consequence of (i) and $c_k(H^k) = -iU^k$. $\blacksquare$

Remark IV.19. Now we can identify $\widetilde{\mathfrak{g}}_{1,k}$ and $\widetilde{\mathfrak{g}}_{2,k}$ in terms of our root data. We have $\mathfrak{g}_{1,k} = \mathfrak{s}_0^{(k)} \oplus \mathfrak{g}_{(k,+)}$. It is shown in [29, p.113] that

$$\mathfrak{g}_{2,k} = \mathbb{R}H^k \oplus \mathfrak{g}_{(k,-)} = \widetilde{\mathfrak{g}}_{2,k}$$

(cf. Th. IV.13). Combining this with Theorem IV.13 and Proposition IV.18, we see that $\mathfrak{s}_0^{(k)} = \mathfrak{m}_k$ and

$$\mathfrak{g}_{1,k} = \mathfrak{m}_k \oplus \mathfrak{g}_{(k,+)} = \widetilde{\mathfrak{g}}_{1,k}. \qquad \blacksquare$$

Remark IV.20. $\mathfrak{z}_{\mathfrak{g}_{(k,-)}}(X^k) = \mathfrak{k}_{(k,-)} = \mathfrak{k}_{2,k} = \widetilde{\mathfrak{s}}_0^{(k)}$, where the first equality follows from [29, Th. III.2.3] and the second is [29, Lemma III.2.2]. $\blacksquare$

In our special situation it is also easy to determine whether we can apply Proposition IV.1.

Proposition IV.21. (cf. [29, p.112]) *$\mathcal{V}_k = \{0\}$ if and only if $k = r$ and Σ is of type (C_r).* $\blacksquare$

Proposition IV.22. *Let $\mathfrak{q}_k = \mathfrak{l}_k + \mathfrak{u}_k$ be the Jacobson-Morosow parabolic for the Cayley triple (H^k, X^k, Y^k) and denote the generalized Heisenberg algebra $\mathcal{U}_k + \mathcal{V}_k$ by $\mathfrak{h}_k$. Then*

(i) *$\mathfrak{q}_k$ is a maximal parabolic in $\mathfrak{g}$.*

(ii) $[\mathcal{U}_k, \mathcal{V}_k + \mathcal{U}_k + \mathfrak{g}_{(k,+)}] \subseteq [\mathcal{U}_k, \mathfrak{h}_k + \mathfrak{g}_{1,k}] = \{0\}$.

Proof. (i) μ_k is the only root in Υ which does not vanish on H^k.
(ii) Proposition IV.8, (IV.1), and Proposition IV.15. $\blacksquare$

Proposition IV.23.

(i) $\mathfrak{z}_{\mathfrak{g}}(X^k) = \mathfrak{z}_{\mathfrak{l}_k}(X^k) + \mathfrak{z}_{\mathfrak{u}_k}(X^k)$.

(ii) $\mathfrak{z}_{\mathfrak{l}_k}(X^k) = \mathfrak{s}_0^{(k)} \oplus \mathfrak{g}_{(k,+)} \oplus \widetilde{\mathfrak{s}}_0^{(k)} = \mathfrak{g}_{1,k} \oplus \mathfrak{k}_{2,k}$.

(iii) $\mathfrak{z}_{\mathfrak{u}_k}(X^k) = \mathfrak{u}_k$.

Proof. (i) This claim follows from the fact that $\mathfrak{z}_{\mathfrak{g}}(X^k)$ is invariant under $\operatorname{ad} H^k$ and Proposition I.3.
(ii) $\mathfrak{z}_{\mathfrak{l}_k}(X^k) = \mathfrak{g}_{1,k} \oplus \mathfrak{k}_{2,k}$ by Proposition IV.4 and $\mathfrak{s}_0^{(k)} \oplus \mathfrak{g}_{(k,+)} \oplus \widetilde{\mathfrak{s}}_0^{(k)} = \mathfrak{g}_{1,k} \oplus \mathfrak{k}_{2,k}$ follows from Remark IV.19 and Remark IV.20.
(iii) This follows from Proposition IV.22. $\blacksquare$

Lemma IV.24. *The map $\operatorname{ad}(X^k)\colon \theta\mathcal{U}_k \to \mathfrak{p}_{2,k}$ is a bijection.*

Proof. The representation theory of $\mathfrak{sl}(2,\mathbb{R})$ shows that $\operatorname{ad}(X^k)$ is injective on the eigenspaces of $\operatorname{ad} H^k$ for negative eigenvalues and in particular on $\theta\mathcal{U}_k$.

Moreover, also by $\mathfrak{sl}(2,\mathbb{R})$-representation theory, we have $[\mathcal{U}_k, \theta\mathcal{U}_k] \subseteq \mathfrak{l}_k$ and therefore

$$\operatorname{ad} X^k(\theta\mathcal{U}_k) \subseteq \mathfrak{g}^{[2]} \cap \mathfrak{l}_k = \mathfrak{p}_{2,k}$$

(cf. Proposition IV.23).

Now the assertion follows from $\dim \mathfrak{p}_{2,k} = \dim \mathcal{U}_k$ which in turn follows from the fact that the map $\mathfrak{g}_2 \to \mathcal{U}_k$, $Y \mapsto [Y, X^k]$ is surjective with kernel $\mathfrak{k}_{2,k}$. $\blacksquare$

Remark IV.25. The argument in the above proof even yields

$$\mathfrak{g}^{[2]_k} = \theta\mathcal{U}_k + \mathfrak{p}_{2,k} + \mathcal{U}_k. \qquad \blacksquare$$

For later use we record some facts concerning the transformation properties of the various spaces we encountered so far under the Cayley transform.

Proposition IV.26. (cf. [29, p.95])

(i) $\mathfrak{k}_{\mathbb{C}}(c_k^2; -1) \subseteq \mathfrak{g}_{\mathbb{C}}^{[2]_k}$.

(ii) $\mathfrak{p}_{\mathbb{C}}(c_k^2; -1) \subseteq \mathfrak{g}_{\mathbb{C}}^{[2]_k}$.

(iii) c_k *induces a bijection* $\mathfrak{k}_{\mathbb{C}}(c_k^2; 1) \to \mathfrak{k}_{\mathbb{C}}(c_k^2; 1)$.

(iv) c_k *induces a bijection* $\mathfrak{p}_{\mathbb{C}}(c_k^2; 1) \to \mathfrak{p}_{\mathbb{C}}(c_k^2; 1)$.

(v) c_k *induces a bijection* $\mathfrak{k}_{\mathbb{C}}(c_k^2; -1) \to \mathfrak{p}_{\mathbb{C}}(c_k^2; -1)$. $\blacksquare$

Let I_κ denote the complex structure in $\mathcal{V}_\kappa$ (Proposition IV.5). Then we write $\mathcal{V}_\kappa^{\pm}$ for the $\pm i$-eigenspaces of the complex linear extension of I_κ to $(\mathcal{V}_\kappa)_{\mathbb{C}}$.

Proposition IV.27.

(i) $c_k\big((\mathcal{U}_k)_{\mathbb{C}}\big) = \mathfrak{p}^+ \cap \mathfrak{g}_{\mathbb{C}}^{[2]_k}$.

(ii) $c_k(\mathcal{V}_k^+) = \mathfrak{p}^+ \cap \mathfrak{g}_{\mathbb{C}}^{[1]_k}$.

(iii) $c_k\big(\mathfrak{p}_{1,k}^+ + \mathcal{V}_k^+ + (\mathcal{U}_k)_{\mathbb{C}}\big) = c_k\big((\mathfrak{p}_{\mathbb{C}}^+ \cap \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{s}^k)) + \mathcal{V}_k^+ + (\mathcal{U}_k)_{\mathbb{C}}\big) = \mathfrak{p}^+$.

(iv) $c_k(\mathcal{V}_k^- + \theta\mathcal{V}_k^+) = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{C}}^{[1]_k}$.

(v) $c_k\big((\mathfrak{g}_{2,k} \cap \mathfrak{p})_{\mathbb{C}}\big) = c_k(\mathbb{C}H^k + (\mathfrak{p} \cap \mathfrak{g}_{(k,-)})_{\mathbb{C}}) = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{C}}^{[2]_k}$.

(vi) $c_k\big((\mathfrak{k}_{1,k})_{\mathbb{C}} + (\mathcal{V}_k^- + \theta\mathcal{V}_k^+) + (\mathfrak{g}_{2,k})_{\mathbb{C}}\big) = \mathfrak{k}_{\mathbb{C}}$.

Proof. (i) [29, p.97], (ii) [29, p.101], (iii) [29, p.104], (iv) [29, p.105], (v) [29, p.96], (vi) [29, p.104]. $\blacksquare$

Remark IV.28. $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{s}^k) = \mathfrak{k} \cap (\mathfrak{m}_k \oplus \mathfrak{g}_{(k,+)} \oplus \mathfrak{k}_{(k,-)}) = \mathfrak{k}_{1,k} + \mathfrak{k}_{2,k} = \mathfrak{l} \cap \mathfrak{k}$, according to Remark IV.17 and Proposition IV.18. Thus we have

$$c_k^{-1}(\mathfrak{k}_{\mathbb{C}}) = \big(\mathfrak{k}_{(k,+)} \oplus \mathfrak{m}_k \oplus \mathfrak{g}_{(k,-)} \oplus \mathbb{R}H^k\big)_{\mathbb{C}} + \mathcal{V}_k^- + \theta\mathcal{V}_k^+$$

(cf. [29, p.104]). $\blacksquare$

Proposition IV.29. $\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X^k) \cap c_k^{-1}(\mathfrak{k}_{\mathbb{C}}) = \mathfrak{z}_{\mathfrak{k}_{\mathbb{C}}}(\mathfrak{s}^k) + \mathcal{V}_k^-$.

Proof. According to Remark IV.28 and Proposition IV.23, we have

$$\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X^k) \cap c_k^{-1}(\mathfrak{k}_{\mathbb{C}}) = (\mathfrak{k}_{(k,+)} \oplus \mathfrak{m}_k \oplus \mathfrak{k}_{(k,-)})_{\mathbb{C}} + \mathcal{V}_k^- = \mathfrak{z}_{\mathfrak{k}_{\mathbb{C}}}(\mathfrak{s}^k) + \mathcal{V}_k^-. \qquad \blacksquare$$

V. Moment maps

In this section we fix $k \in \{1, \ldots, r\}$, the corresponding nilpotent orbit $\mathcal{O}_k = \mathrm{Ad}(G).X^k$ and all the other entities depending on k (cf. Section IV). We will be interested in the following chain of subalgebras:

$$\mathcal{U}_k \longrightarrow \mathfrak{h}_k \longrightarrow \mathfrak{q}_{k,\mathrm{red}} := \mathfrak{h}_k \rtimes \mathfrak{g}_{2,k} \longrightarrow \mathfrak{q}_k = \mathfrak{h}_k \rtimes \mathfrak{l}_k \longrightarrow \mathfrak{g}, \tag{V.1}$$

where $\mathfrak{h}_k := \mathcal{U}_k + \mathcal{V}_k$ and the corresponding chain

$$\text{(V.1}'\text{)} \qquad \mathfrak{g}^* \xrightarrow{\Phi_{\mathfrak{q}_k}} \mathfrak{q}_k^* \xrightarrow{\Phi_{\mathfrak{q}_{k,\mathrm{red}}}} \mathfrak{q}_{k,\mathrm{red}}^* \xrightarrow{\Phi_{\mathfrak{h}_k}} \mathfrak{h}_k^* \xrightarrow{\Phi_{\mathcal{U}_k}} \mathcal{U}_k^*$$

of restriction maps. In order to make this system of moment maps and its encoded symplectic information available for our study of $\mathcal{O}_k$, we transfer it to $\mathfrak{g}$ using the Killing form B and the Cartan involution θ. We start with a few general observations: For the moment let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan involution θ and Killing form B. Then $\mathfrak{g}$ has an inner product $(\cdot \mid \cdot)$ defined by

$$(X \mid Y) = -B(X, \theta Y) \quad \forall X, Y \in \mathfrak{g}.$$

Moreover we have two linear isomorphisms $\psi\colon \mathfrak{g} \to \mathfrak{g}^*$ and $\varphi\colon \mathfrak{g} \to \mathfrak{g}^*$ defined by

$$\langle \psi(X), Y \rangle = B(X, Y) \quad \forall X, Y \in \mathfrak{g}$$

and

$$\langle \varphi(X), Y \rangle = (X \mid Y) \quad \forall X, Y \in \mathfrak{g}.$$

Then $\varphi = -\psi \circ \theta$ and ψ is G-equivariant, where G acts on $\mathfrak{g}$ via the adjoint and on $\mathfrak{g}^*$ via the coadjoint action. This shows that

$$\text{(V.2)} \qquad (\mathrm{Ad}(g)X \mid Y) = (X \mid \mathrm{Ad}(\theta(g)^{-1})Y) \quad \forall X, Y \in \mathfrak{g}, g \in G.$$

This equation implies

$$\text{(V.3)} \qquad \mathrm{Ad}^*(g) \circ \varphi = \varphi \circ \mathrm{Ad}\left(\theta(g)^{-1}\right).$$

Now let $\mathfrak{b} \subseteq \mathfrak{g}$ be a subspace. Then we may view $\mathfrak{b}^*$ as a subspace of $\mathfrak{g}^*$ extending $\xi \in \mathfrak{b}^*$ to $\mathfrak{g}$ by 0 on

$$\mathfrak{b}^\perp = \{Y \in \mathfrak{g} \colon (\forall X \in \mathfrak{h})\ (X \mid Y) = 0\}.$$

A simple calculation shows $\mathfrak{b}^* = \varphi(\mathfrak{b})$. Let $\mathrm{p}_{\mathfrak{b}}\colon \mathfrak{g} \to \mathfrak{b}$ be the orthogonal projection w.r.t. $(\cdot \mid \cdot)$. It is clear that $\mathrm{p}_{\mathfrak{b}}$ is selfadjoint w.r.t. the inner product. Now suppose that $N \subseteq G$ is a subgroup such that $\mathrm{Ad}(N)$ leaves $\mathfrak{b}$ invariant. Then the selfadjointness of $\mathrm{p}_{\mathfrak{b}}$ and equation (V.2) show that the map

$$\rho_{\mathfrak{b}}\colon N \times \mathfrak{b} \to \mathfrak{b}, \quad (n, X) \mapsto n.X := \mathrm{p}_{\mathfrak{b}} \circ \mathrm{Ad}(\theta n)(X)$$

is a group action. Let $\Phi_{\mathfrak{b}}\colon \mathfrak{g}^* \to \mathfrak{b}^*$ be the restriction map. We define a group action

$$\rho_{\mathfrak{b}^*}\colon N \times \mathfrak{b}^* \to \mathfrak{b}^*, \quad (n, f) \mapsto n.f := \Phi_{\mathfrak{b}} \circ \mathrm{Ad}^*(n)(f).$$

Then one easily checks

$$\text{(V.4)} \qquad \varphi(n.X) = n.\varphi(X).$$

and

$$\text{(V.5)} \qquad \Phi_{\mathfrak{b}} \circ \varphi = \varphi \circ \mathrm{p}_{\mathfrak{b}}.$$

Proposition V.1. *Let $N \subseteq G$ be a subgroup and $\mathfrak{a} \subseteq \mathfrak{b}$ subspaces of $\mathfrak{g}$. Suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathrm{Ad}(N)$-invariant. Then*

(i) *$\mathfrak{a}^*$ and $\mathfrak{b}^*$ are $\mathrm{Ad}^*\left(\theta(N)\right)$-invariant.*

(ii) *$\Phi_{\mathfrak{a}}\colon \mathfrak{b}^* \to \mathfrak{a}^*$ is equivariant w.r.t. the actions $\rho_{\mathfrak{b}^*}$ and $\rho_{\mathfrak{a}^*}$.*

(iii) *$\mathrm{p}_{\mathfrak{a}}\colon \mathfrak{b} \to \mathfrak{a}$ is equivariant w.r.t. the actions $\rho_{\mathfrak{b}}$ and $\rho_{\mathfrak{a}}$.*

Proof. This follows from the above by a simple calculation. ∎

Now we return to our special situation and note that using φ we can replace (V.1$'$) by

$$\text{(V.1}''\text{)} \qquad \mathfrak{g} \ \overset{p_{\mathfrak{q}_k}}{\longrightarrow} \ \mathfrak{q}_k \ \overset{p_k}{\longrightarrow} \ \mathfrak{q}_{k,\mathrm{red}} \ \overset{p_{\mathfrak{h}_k}}{\longrightarrow} \ \mathfrak{h}_k \ \overset{p_{\mathcal{U}_k}}{\longrightarrow} \ \mathcal{U}_k \quad .$$

The kernels of the respective projections are given by the following proposition.

Proposition V.2.

(i) $\mathfrak{q}_k^{\perp} = \theta\mathfrak{h}_k$.

(ii) $\mathfrak{q}_{k,\mathrm{red}}^{\perp} \cap \mathfrak{q}_k = \mathfrak{g}_{1,k}$.

(iii) $\mathfrak{h}_k^{\perp} \cap \mathfrak{q}_{k,\mathrm{red}} = \mathfrak{g}_{2,k}$.

(iv) $(\mathcal{U}_k)^{\perp} \cap \mathfrak{h}_k = \mathcal{V}_k$.

Proof. (i) $\mathfrak{h}_k$ is nilpotent, hence B vanishes on $\mathfrak{h}_k \times \mathfrak{h}_k$. On the other hand $\theta\mathfrak{h}_k$ and $\mathfrak{l}_k$ are sums of different root spaces $\mathfrak{a}$, respectively and therefore orthogonal w.r.t. B. This shows " $\supseteq$ " and equality follows for dimensional reasons.
(ii) $\mathfrak{g}_{1,k}$ and $\mathfrak{g}_{2,k}$ are orthogonal which shows " $\supseteq$ ". Equality follows again by counting dimensions.
(iii) This follows from parts (i) and (ii).
(iv) Again one uses that different root spaces are $(\cdot \mid \cdot)$-orthogonal. ■

For the following we recall the parabolic subgroup Q_k of G associated to the parabolic subalgebra $\mathfrak{q}_k$. We write $Q_k = H_k \rtimes L_k$ for the Levi decomposition of Q_k and $G_{1,k}$, $G_{2,k}$, H_k and Q_{red} for the analytic subgroups corresponding to $\mathfrak{g}_{1,k}$, $\mathfrak{g}_{2,k}$, $\mathfrak{h}_k$ and $\mathfrak{q}_{\mathrm{red}}$ respectively. We also recall from [35, Lemma 1.2.4.5] that $Q_k = Z_K(\mathfrak{a})(Q_k)_0$.

Proposition V.3. *$p_{\mathfrak{q}_k}, p_k, p_{\mathfrak{h}_k}$ and $p_{\mathcal{U}_k}$ are Q_k-equivariant w.r.t. the ρ-actions.*

Proof. The only thing that remains to be checked is that $\mathfrak{q}_{\mathrm{red}}$ is invariant under Q_k. Since $\mathfrak{q}_{\mathrm{red}}$ is an ideal of $\mathfrak{q}$, it is invariant under $(Q_k)_0$. Moreover $\mathfrak{h}_k$ is invariant under Q_k, so that is suffices to show that $\mathfrak{g}_{2,k}$ is invariant under $Z_K(A)$. We have $\mathfrak{g}_{2,k} = \mathbb{R}H^k \oplus \mathfrak{g}_{(k,-)}$. The element H^k is fixed by $Z_K(A)$, and since all the real root spaces are $Z_K(A)$-invariant, the same holds for the subalgebra $\mathfrak{g}_{(k,-)}$ which is generated by certain real root spaces (Lemma IV.12). ■

Next we consider the G-orbit $\mathcal{O}_k$ of X^k in $\mathfrak{g}$. Note first that

$$\mathrm{Ad}(G).X^k = \rho_{\mathfrak{g}}(G, X^k)$$

since G is θ-invariant. In order to take advantage of the above equivariance properties, we use the Bruhat decomposition of G w.r.t. Q_k:

$$\text{(V.6)} \qquad G = \bigcup_{w \in \mathcal{W}(\Sigma)} Q_k m_w \theta H_k = \bigcup_{w \in \mathcal{W}(\Sigma)} H_k L_k m_w \theta H_k,$$

where $\mathcal{W}(\Sigma) \cong N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \cong \{-1,1\}^r \rtimes \mathcal{S}_r$ is the Weyl group of the restricted root system Σ and m_w a representative of w in $N_K(\mathfrak{a})$. We do not claim that

(V.6) is a disjoint union. Now θH_k stabilizes X^k in the $\rho_{\mathfrak g}$-action and the element $w = \big((\varepsilon_1, \ldots, \varepsilon_r), \sigma\big) \in \{-1,1\}^r \times \mathcal{S}_r$ acts on $X^k = \sum_{j=1}^k X_j$ via

$$w.X^k = \sum_{\substack{j\in\{1,\ldots,k\}\\ \varepsilon_{\sigma(j)}=1}} X_{\sigma(j)} + \sum_{\substack{j\in\{1,\ldots,k\}\\ \varepsilon_{\sigma(j)}=-1}} \theta X_{\sigma(j)}.$$

Thus we have

$$\mathrm{p}_{\mathfrak{q}_k}(w.X^k) = \sum_{\substack{j\in\{1,\ldots,k\}\\ \varepsilon_{\sigma(j)}=1}} X_{\sigma(j)}.$$

The group $G_{1,k}$ acts trivially on $\mathcal{U}_k$ and since the root space $\mathfrak{g}^{2\varepsilon_k}$ is one-dimensional, it is pointwise fixed by $Z_K(A)$. So, in view of $Q = Q_{\rm red} G_{1,k} Z_K(A)$ and Proposition V.3, we obtain

$$\mathrm{p}_{\mathfrak{q}_k}(\mathcal{O}_k) = \bigcup_{j=0}^{k} \rho_{\mathfrak{q}_k}(Q_{k,\rm red}, X^j), \tag{V.7}$$

where $X^0 = 0$.

Remark V.4.

(i) Recall the open cone Ω_k from Proposition IV.4 and its automorphism group $G(\Omega_k)$. It follows from [29, Th. III.2.3] that $\mathcal{U}_k$ is a euclidean Jordan algebra with unit X^k and Jordan frame $(X_1.\ldots, X_k)$. Here by a Jordan frame we mean a maximal set of pairwise orthogonal idempotents whose sum is the unit element.

(ii) Part (i) allows us to apply [7, Prop. IV.3.1] to Ω_k. It says in particular that

$$\overline{\Omega}_k = \bigcup_{j=0}^{k} G(\Omega_k)_0.X^j$$

and characterizes $G(\Omega_k)_0.X^j$ as the set of elements $Y \in \overline{\Omega}_k$ whose rank (in the Jordan algebra sense) is j. ■

Lemma V.5.

(i) $\rho_{\mathfrak{q}_k}(L_k, X^j) = G(\Omega_k)_0.X^j$.

(ii) $\rho_{\mathfrak{q}_k}(\exp(X'), X) = X + [\theta X', X] \in X + \mathfrak{l}_k$ *for all* $X, X' \in \mathcal{U}_k$.

(iii) $\rho_{\mathfrak{q}_k}(\exp(Y), X + Z) = X + [\theta Y, X] + (Z + \frac{1}{2}[\theta Y, [\theta Y, X]]) \in X + \mathcal{V}_k + \mathfrak{l}_k$ *for all* $Y \in \mathcal{V}_k, X \in \mathcal{U}_k$ *and* $Z \in \mathfrak{l}_k$.

Proof. (i) Since L_k is θ-invariant we see that $\mathrm{Ad}(\theta L_k).X^j \subseteq \mathcal{U}_k$ so that $\rho_{\mathfrak{q}_k}(L_k, X^j) = \mathrm{Ad}(L_k).X^j$ which in turn is equal to $G(\Omega_k)_0.X^j$ since the groups $Z_K(A)$ and $G_{1,k}$ fix X^k.
The assertions (ii) and (iii) follow from the eigenspaces decomposition of $\mathrm{ad}(H^k)$ and the fact that different root spaces are $(\cdot \mid \cdot)$-orthogonal. ■

Similarly we find

Lemma V.6.

(i) $\rho_{\mathfrak{q}_{k,\mathrm{red}}}(L_k, X^j) = G(\Omega_k)_0.X^j$.

(ii) $\rho_{\mathfrak{q}_{k,\mathrm{red}}}(\exp(X'), X) = X + [\theta X', X] \in X + \mathfrak{g}_{2,k}$ *for all* $X, X' \in \mathcal{U}_k$.

(iii) $\rho_{\mathfrak{q}_{k,\mathrm{red}}}(\exp(Y), X + Z) = X + [\theta Y, X] + \big(Z + \frac{1}{2}\mathrm{p}_{\mathfrak{q}_{k,\mathrm{red}}}([\theta Y, [\theta Y, X]])\big) \in X + \mathcal{V}_k + \mathfrak{g}_{2,k}$ *for all* $Y \in \mathcal{V}_k, X \in \mathcal{U}_k$ *and* $Z \in \mathfrak{g}_{2,k}$.

Proof. The only thing we still have to note is that $[\theta X', X] \in \mathfrak{g}_{2,k}$ for all $X, X' \in \mathcal{U}_k$. But that follows from [29, (3.41)]. ∎

The action $\rho_{\mathfrak{q}_{k,\mathrm{red}}}$ will be the one we use primarily. Therefore we abbreviate it by $\rho_k \colon Q_{k,\mathrm{red}} \times \mathfrak{q}_{k,\mathrm{red}} \to \mathfrak{q}_{k,\mathrm{red}}$. Similarly, we write p_k for $\mathrm{p}_{\mathfrak{q}_{k,\mathrm{red}}}$.

Lemma V.7.

(i) $\rho_{\mathfrak{h}_k}(L_k, X^j) = G(\Omega_k)_0.X^j$.

(ii) $\rho_{\mathfrak{h}_k}(\exp(X'), X) = X$ *for all* $X, X' \in \mathcal{U}_k$.

(iii) $\rho_{\mathfrak{h}_k}(\exp(Y), X) = X + [\theta Y, X] \in X + \mathcal{V}_k$ *for all* $Y \in \mathcal{V}_k$ *and* $X \in \mathcal{U}_k$. ∎

Lemma V.8.

(i) $\rho_{\mathcal{U}_k}(L_k, X^j) = \rho_{\mathcal{U}_k}(G_{2,k}, X^j)$.

(ii) $\rho_{\mathcal{U}_k}(\exp(X'), X) = X$ *for all* $X, X' \in \mathcal{U}_k$.

(iii) $\rho_{\mathcal{U}_k}(\exp(Y), X) = X$ *for all* $Y \in \mathcal{V}_k$ *and* $X \in \mathcal{U}_k$. ∎

Proposition V.9.

(i) $\mathrm{p}_{\mathcal{U}_k}(\mathcal{O}_k) = \overline{\Omega}_k$.

(ii) *The set* $\mathcal{O}_k' := \mathrm{p}_{\mathcal{U}_k}^{-1}(\Omega_k) \cap \mathcal{O}_k = \mathrm{Ad}(\theta Q_{k,\mathrm{red}}).X^k$ *is open and dense in* $\mathcal{O}_k$.

Proof. (i) According to Lemma V.8 and equation (V.6) we have in view of Remark V.4

$$\mathrm{p}_{\mathcal{U}_k}(\mathcal{O}_k) = \mathrm{p}_{\mathcal{U}_k}\big(\rho_{\mathfrak{g}}(G, X^k)\big) = \bigcup_{j=0}^{k} G_{2,k}.X^j = \overline{\Omega}_k.$$

(ii) The formula follows from (i), (V.7), and Lemma V.8. Moreover it is clear that the set is open in $\mathcal{O}_k$ since Ω_k is open in $\mathcal{U}_k$. To show that it is also dense it suffices to show that the open Bruhat cell $\mathcal{B} = Q_k \theta H_k$ satisfies $\mathrm{p}_{\mathcal{U}_k}\big(\rho_{\mathfrak{g}}(\mathcal{B}, X^k)\big) = \Omega_k$. But that is clear since θH_k fixes X^k and

$$\mathrm{p}_{\mathcal{U}_k}\big(\rho_{\mathfrak{g}}(Q_k, X^k)\big) = \rho_{\mathcal{U}_k}(Q_k, X^k) = \rho_{\mathcal{U}_k}(L_k, X^k) = \Omega_k. \qquad ∎$$

We want to gain more insight into the geometry of $\mathcal{O}_k$ via the study of the fibers and images of the various moment maps. We will essentially restrict ourselves to the dense open subset of $\mathcal{O}_k$ given in Proposition V.9.

Proposition V.10.

(i) $\mathrm{p}_{\mathcal{U}_k}^{-1}(X) \cap (\mathcal{U}_k + \mathcal{V}_k) = X + \mathcal{V}_k \subseteq \mathrm{p}_{\mathfrak{h}_k}(\mathcal{O}_k)$ *for all* $X \in \Omega_k$.

(ii) $\mathrm{p}_{\mathfrak{h}_k}(\mathcal{O}_k) \supseteq \Omega_k + \mathcal{V}_k$.

(iii) $p_{\mathfrak{h}_k}(\mathcal{O}'_k) = \Omega_k + \mathcal{V}_k$.

Proof. (i) Since $p_{\mathcal{U}_k}$ is Q_k-equivariant, the fibers of $p_{\mathcal{U}_k}$ are permuted by Q_k. In particular we have

$$p_{\mathcal{U}_k}^{-1}(X) = l.p_{\mathcal{U}_k}^{-1}(X^k)$$

for $X = l.X^k$. Thus it suffices to prove the claim for $X = X^k$.

According to Lemma V.7, we have to show that $\mathrm{ad}(X^k)\colon \theta\mathcal{V}_k \to \mathcal{V}_k$ is surjective. But that follows immediately from $\mathfrak{sl}(2)$-theory applied to the $\mathfrak{sl}_2$-triple (H^k, X^k, Y^k).

(ii) follows directly from (i).

(iii) Again the inclusion " $\supseteq$ " is clear and for the converse we calculate

$$\Omega_k + \mathcal{V}_k \subseteq p_{\mathfrak{h}_k}\big(\mathcal{O}_k \cap p_{\mathcal{U}_k}^{-1}(\Omega_k)\big) = p_{\mathfrak{h}_k}(\mathcal{O}'_k). \qquad \blacksquare$$

Lemma V.11. *Let $Y \in \mathcal{V}_k$ and $\widetilde{Y} \in \mathcal{V}_k$ be the uniquely determined element with $\mathrm{ad}(X^k)\theta\widetilde{Y} = Y$. Then*

$$p_k(\mathcal{O}_k) \cap (X^k + Y + \mathfrak{g}_{2,k}) = X^k + Y + p_k(\mathrm{ad}(\theta\widetilde{Y})^2 X^k) + \mathrm{ad}(X^k)\theta\mathcal{U}_k.$$

Proof. According to Lemma V.6 and the Bruhat decomposition (V.6) any element of $p_k(\mathcal{O}_k) \cap p_{\mathcal{U}_k}^{-1}(X^k)$ can be written as

$$X^k + [\theta X', X^k] + [\theta Y', X^k] + p_k([\theta Y', [\theta Y', X^k]])$$

with $X' \in \mathcal{U}_k$ and $Y' \in \mathcal{V}_k$ since p_k is equivariant w.r.t. the ρ-actions. But this element gets mapped to $X^k + [\theta Y', X^k]$ under $p_{\mathfrak{h}_k}$, i.e., its $\mathfrak{h}_k$-component is $X^k + [\theta Y', X^k]$. If $[\theta Y', X^k] = Y$, then $Y' = -\widetilde{Y}$. On the other hand $[\theta X', X^k]$ does not effect the $p_{\mathfrak{h}_k}$ projection and can be varied freely. $\blacksquare$

Proposition V.12. *The fibers of the map*

$$p_{\mathfrak{h}_k}\colon p_k(\mathcal{O}'_k) \to p_{\mathfrak{h}_k}(\mathcal{O}'_k) = \Omega_k + \mathcal{V}_k$$

are affine spaces of dimension $\dim\mathcal{U}_k$. *More precisely, for $Z \in p_k(\mathcal{O}'_k)$ with $p_{\mathcal{U}_k}(Z) = X$ we have*

$$p_k(\mathcal{O}'_k) \cap p_{\mathfrak{h}_k}^{-1}\big(p_{\mathfrak{h}_k}(Z)\big) = Z + \mathrm{ad}(X)\theta\mathcal{U}_k.$$

Proof. As in the proof of Proposition V.10, we may restrict ourselves to the case $X = X^k$. Then $Z \in X^k + Y + \mathfrak{g}_{2,k}$ and according to Lemma V.11, the fiber in question is given by

$$X^k + Y + p_k(\mathrm{ad}(\theta\widetilde{Y})^2 X^k) + \mathrm{ad}(X^k)\theta\mathcal{U}_k = Z + \mathrm{ad}(X^k)\theta\mathcal{U}_k.$$

The statement about the dimension now follows from Lemma IV.24. $\blacksquare$

Remark V.14. $\dim \mathcal{O}_k = 2\dim \mathcal{U}_k + \dim \mathcal{V}_k$. In fact, we calculate

$$\begin{aligned}\dim \mathcal{O}_k &= \dim \mathfrak{g} - \dim \mathfrak{z}_{\mathfrak{g}}(X^k)\\ &= (\dim \mathfrak{l}_k + 2\dim \mathfrak{u}_k) - (\dim \mathfrak{z}_{\mathfrak{l}_k}(X^k) + \dim \mathfrak{u}_k)\\ &= (\dim \mathfrak{l}_k - \dim \mathfrak{z}_{\mathfrak{l}_k}(X^k)) + \dim \mathfrak{u}_k\\ &= \dim \mathcal{U}_k + (\dim \mathcal{V}_k + \dim \mathcal{U}_k)\\ &= 2\dim \mathcal{U}_k + \dim \mathcal{V}_k.\end{aligned}$$

■

Now we consider the diagram

$$\begin{array}{ccccc} \mathcal{O}'_k & \stackrel{\mathrm{p}_{\mathcal{U}_k}}{\longrightarrow} & \mathrm{p}_{\mathcal{U}_k}(\mathcal{O}'_k) & = \Omega_k \\ \downarrow \mathrm{p}_k & & \uparrow \mathrm{p}_{\mathcal{U}_k} & \\ \mathrm{p}_k(\mathcal{O}'_k) & \stackrel{\mathrm{p}_{\mathfrak{h}_k}}{\longrightarrow} & \mathrm{p}_{\mathfrak{h}_k}(\mathcal{O}'_k) & = \Omega_k + \mathcal{V}_k \end{array}$$

It clearly commutes and the dimension of the manifolds involved are

$$\begin{aligned}\dim \mathcal{O}'_k &= 2\dim \mathcal{U}_k + \dim \mathcal{V}_k\\ \dim \Omega_k &= \dim \mathcal{U}_k\\ \dim(\Omega_k + \mathcal{V}_k) &= \dim \mathcal{U}_k + \dim \mathcal{V}_k\\ \dim \mathrm{p}_k(\mathcal{O}'_k) &= 2\dim \mathcal{U}_k + \dim \mathcal{V}_k\end{aligned}$$

Thus $\mathrm{p}_k\colon \mathcal{O}'_k \to \mathrm{p}_k(\mathcal{O}'_k)$ is a surjective submersion of equidimensional manifolds. In fact, more is true:

It follows from (V.4) that we can use the map $\varphi\colon \mathfrak{g} \to \mathfrak{g}^*$ to transport the natural G-invariant Poisson structure on $\mathfrak{g}^*$ over to a Poisson structure on $\mathfrak{g}$ which is invariant under the $\rho_{\mathfrak{g}}$ action. Therefore the symplectic leaves of $\mathfrak{g}$ are precisely the G-orbits w.r.t. $\rho_{\mathfrak{g}}$. Similarly we introduce a $Q_{k,\mathrm{red}}$-invariant Poisson structure on $\mathfrak{q}_{k,\mathrm{red}}$ for which the symplectic leaves are the $Q_{k,\mathrm{red}}$-orbits w.r.t. $\rho_{\mathfrak{q}_{k,\mathrm{red}}}$. Then φ is a Poisson isomorphism and the map p_k is a Poisson morphism since $\Phi_{\mathfrak{q}_{k,\mathrm{red}}}$ is one. Therefore the G-orbit $\mathcal{O}_k$ and the $Q_{k,\mathrm{red}}$-orbit $\mathrm{p}_k(\mathcal{O}'_k)$ are symplectic submanifolds of $\mathfrak{g}$ and $\mathfrak{q}_{k,\mathrm{red}}$, respectively. Moreover $\mathcal{O}'_k$ is a symplectic submanifold of $\mathcal{O}_k$.

Proposition V.15. *The map* $\mathrm{p}_k\colon \mathcal{O}'_k \to \mathrm{p}_k(\mathcal{O}'_k)$ *is a symplectic diffeomorphism.*

Proof. We first recall that p_k is $Q_{k,\mathrm{red}}$-equivariant w.r.t. the ρ-actions. Moreover $Q_{k,\mathrm{red}}$ acts transitively on both $\mathcal{O}'_k$ and $\mathrm{p}_k(\mathcal{O}'_k)$ and the symplectic structures are both $Q_{k,\mathrm{red}}$-invariant. Finally the map p_k can be viewed (via φ) as the moment map of the $Q_{k,\mathrm{red}}$-action on $\mathcal{O}'_k$. Thus [9, p.185] implies that $\mathrm{p}_k\colon \mathcal{O}'_k \to \mathrm{p}_k(\mathcal{O}'_k)$ is a symplectic covering. To conclude the argument we note that the stabilizers of X^k in $Q_{k,\mathrm{red}}$ w.r.t. the $\rho_{\mathfrak{g}}$ and the $\rho_{\mathfrak{q}_{k,\mathrm{red}}}$ actions agree, whence the equivariance shows that the fibers are trivial. ■

Since the symplectic manifold $\mathrm{p}_k(\mathcal{O}'_k)$ will play an important role in this paper we abbreviate it by $\mathcal{M}_k$ and denote its symplectic form by $\omega^{(k)}$.

Note that Proposition V.15 obviously implies that also

$$\mathrm{p}_{\mathfrak{q}_k}\colon \mathcal{O}'_k \to \mathrm{p}_{\mathfrak{q}_k}(\mathcal{O}'_k) \quad \text{and} \quad \mathrm{p}_k\colon \mathrm{p}_{\mathfrak{q}_k}(\mathcal{O}'_k) \to \mathrm{p}_k(\mathcal{O}'_k)$$

are symplectic diffeomorphisms.

The above results on the structure of $\mathcal{O}_k$ should be compared to

Proposition V.16. $\mathcal{O}_k \cap \mathfrak{h}_k = \Omega_k$.

Proof. Since $\mathcal{O}_k$ is of convex type, we know that it is contained in a proper generating closed convex invariant cone $W \subseteq \mathfrak{g}$. In particular it contains $X^k \in \mathcal{U}_k$ and therefore its $G(\Omega_k)_0$-orbit Ω_k. We apply [14, Lemma I.14] to $W \cap \mathfrak{h}_k$ and find that $W \cap \mathfrak{h}_k = \overline{\Omega}_k$.

Let $c_j := X_j = X^j - X^{j-1}$ for $j = 1, \ldots, r$. Then $(c_1, \ldots, c_k)$ is a Jordan frame in $\mathcal{U}_k$. Suppose that $Y \in \mathcal{O}_k \cap \Omega_k$. Then there exists an element $g \in G(\Omega_k)_0$ and $j \in \{0, \ldots, k\}$ such that $g.Y = X^j$ (Remark V.4). Since $\rho_U : G_{2,k} \to G(\Omega_k)_0$ is surjective, we conclude that $Y \in \mathcal{O}_j$, hence that $j = k$ and therefore that $Y \in G(\Omega_k)_0.X^k = \Omega$. Thus $\mathcal{O}_k \cap \mathcal{U}_k = \Omega_k$. ■

Remark V.17. The proof of Proposition V.16 even shows that the nilpotent orbits of convex type $\mathcal{O}_j$ with $j > k$ do not intersect $\mathfrak{h}_k$ and that for $j \leq k$ the intersection consists of the set of all elements of rank j in the closed cone $\overline{\Omega}_k$. ■

VI. The symplectic geometry of $\mathcal{O}_k$ and $\mathcal{M}_k$

In this section we give details concerning the symplectic structure of $\mathcal{O}_k$ and its various projections. We start with a few general facts on the symplectic structure of coadjoint orbits.

Lemma VI.1. *Let H be a Lie group and $f \in \mathfrak{h}^*$. Then the following statements are equivalent:*

(1) *f vanishes on the Lie algebra $\mathfrak{h}_f$ of the stabilizer H_f of f under the coadjoint action.*

(2) *The left invariant 1-form $\widetilde{\alpha}^{(f)}$ on H defined by f is the pullback of an H-invariant 1-form $\alpha^{(f)}$ on the coadjoint orbit $\mathcal{O}_f = \mathrm{Ad}^*(H).f$ under the orbit map $s: H \to \mathcal{O}_f, h \mapsto h.f := \mathrm{Ad}^*(h).f$.*

Proof. If such an $\alpha^{(f)}$ exists, then it has to satisfy

$$(*) \qquad \alpha^{(f)}_{h.f}\big(h.(\mathrm{ad}^*(X).f)\big) = \langle f, X\rangle \quad \forall h \in H, X \in \mathfrak{h}$$

since the orbit map s is H-equivariant. So, if (1) holds, the formula

$$\alpha^{(f)}_f(\mathrm{ad}^*(X).f) = \langle f, X\rangle$$

actually defines a linear functional α_f on $T_f(\mathcal{O}_f) = \mathrm{ad}^*(\mathfrak{h}).f$. Since

$$\text{(VI.1)} \qquad \mathrm{Ad}^*(h) \circ \mathrm{ad}^*(X) = \mathrm{ad}^*(\mathrm{Ad}(h).X) \circ \mathrm{Ad}^*(h)$$

and

$$\langle f, X\rangle = \langle \mathrm{Ad}^*(h).f, \mathrm{Ad}(h)X\rangle,$$

we can move this form around with the H-action to define a 1-form $\alpha^{(f)}$ on $\mathcal{O}_f$ via $(*)$. It follows from its invariance that this form is smooth and has $\widetilde{\alpha}^{(f)}$ as its pullback under s. This proves (2).

Conversely, suppose that (2) holds. Then $X \in \mathfrak{h}_f$ shows $\mathrm{ad}^*(X).f = 0$ so that $(*)$ implies $\langle f, X\rangle = 0$, i.e., (1). ■

Consider the canonical Poisson structure on the dual $\mathfrak{h}^*$ of a Lie algebra $\mathfrak{h}$. Given $f \in \mathfrak{h}^*$ the cotangent space $T_f^*(\mathfrak{h}^*)$ is identified with $\mathfrak{h}$ and the Poisson tensor Λ_f at f is given by

$$\Lambda_f(X,Y) = \langle f, [X,Y]\rangle \quad \forall X,Y \in \mathfrak{h}.$$

Therefore the associated bundle map

$$\Lambda^\sharp : T^*(\mathfrak{h}^*) \cong \mathfrak{h}^* \times \mathfrak{h} \to T(\mathfrak{h}^*) \cong \mathfrak{h}^* \times \mathfrak{h}^*$$

is given by

$$\Lambda^\sharp(f,X) = (f, -\operatorname{ad}^*(X).f)$$

and we also write $\Lambda_f^\sharp(X) = -\operatorname{ad}^*(X).f$ for this fact. The characteristic distribution $\{C_f \subseteq T_f(\mathfrak{h}^*) : f \in \mathfrak{h}^*\}$ is given by

$$C_f = \Lambda_f^\sharp\big(T_f(\mathfrak{h}^*)\big) = \operatorname{ad}^*(\mathfrak{h}).f.$$

We denote the annihilator of C_f in $T_f^*(\mathfrak{h}^*) \cong \mathfrak{h}$ by $\mathrm{Ann} C_f$ and note that

$$\mathrm{Ann} C_f = \ker \Lambda_f^\sharp = \{X \in \mathfrak{h} : \operatorname{ad}^*(X).f = 0\} = \mathfrak{h}_f.$$

Thus we have a canonical isomorphism $C_f^* \cong \mathfrak{h}/\mathfrak{h}_f$. The map $\Lambda_f^\sharp$ induces an isomorphism $\ell_f : C_f^* \to C_f$ and one has a symplectic form ω_f on C_f defined by

$$\omega_f(a,b) = \Lambda_f\big(\ell_f^{-1}(a), \ell_f^{-1}(b)\big), \quad \forall a,b \in C_f.$$

This means that the corresponding symplectic form on C_f^* is given by

$$\widetilde{\omega}_f\big(X + \mathfrak{h}_f, Y + \mathfrak{h}_f\big) = \langle f, [X,Y]\rangle.$$

The symplectic forms ω_f for f in a coadjoint orbit of $\mathfrak{h}$ form the H-invariant symplectic form $\omega^{(f)}$ on that orbit which turns it into a symplectic leaf of the Poisson manifold $\mathfrak{h}^*$.

Lemma VI.2. *Let H be a Lie group and $f \in \mathfrak{h}^*$. Suppose that $\mathfrak{h}_f \subseteq \ker f$. Then the form $\alpha^{(f)}$ constructed in* Lemma VI.1 *satisfies*

(i) $\alpha_{f'}^{(f)}(\operatorname{ad}^*(X).f') = \langle f', X\rangle$ *for all* $f' \in \mathcal{O}_f$.

(ii) $-2d\alpha^{(f)} = \omega^{(f)}$.

Proof. (i) Let $f' = h.f$. Then, using (VI.1), we calculate

$$\begin{aligned}\alpha_{f'}^{(f)}(\operatorname{ad}^*(X).f') &= \alpha_{h.f}^{(f)}(\operatorname{ad}^*(X).(h.f)) \\ &= \alpha_f^{(f)}(h^{-1}.(\operatorname{ad}^*(X)(h.f))) \\ &= \alpha_f^{(f)}(\operatorname{ad}^*(h^{-1}.X).f) \\ &= \langle f, h^{-1}.X\rangle \\ &= \langle f', X\rangle.\end{aligned}$$

(ii) We recall the orbit map $s: H \to \mathcal{O}_f$ from Lemma VI.1. The definitions show that

$$(s^*\omega^{(f)})_{\mathbf{1}}(X,Y) = \langle f, [X,Y]\rangle.$$

Since s is a submersion, it suffices to show that $s^*(d\alpha^{(f)}) = d(s^*\alpha^{(f)}) = s^*\omega^{(f)}$. On the other hand the H-invariance shows that we have to test $d(s^*\alpha^{(f)})$ and $s^*\omega^{(f)}$ only against left invariant vector fields, i.e., elements of $\mathfrak{h}$:

$$\begin{aligned} 2d(s^*\alpha^{(f)})(X_1,X_2) &= X_1.\alpha(X_2) - X_2.\alpha(X_1) - \alpha([X_1,X_2]) \\ &= -\alpha([X_1,X_2]) = -\langle f, [X_1,X_2]\rangle. \end{aligned}$$

This proves the lemma. ∎

Now we consider a semisimple Lie algebra $\mathfrak{g}$ with Cartan involution θ and Killing form B. We retain the notation from Section V and transport the Poisson structure from $\mathfrak{h}^*$ to $\mathfrak{h}$ via $\varphi: \mathfrak{h} \to \mathfrak{h}^*$. Here $T_X^*(\mathfrak{h})$ gets identified with $\mathfrak{h}^*$ and hence the Poisson tensor on $\mathfrak{h}$ is given by

$$\Lambda_X(a,b) = \langle \varphi(X), [\varphi^{-1}(a), \varphi^{-1}(b)]\rangle, \quad \forall X \in \mathfrak{h}, a, b \in \mathfrak{h}^*.$$

Lemma VI.3. *The bundle map*

$$\Lambda^\sharp: T^*(\mathfrak{h}) \cong \mathfrak{h} \times \mathfrak{h}^* \to T(\mathfrak{h}) \cong \mathfrak{h} \times \mathfrak{h}$$

associated to Λ *is given by*

$$\Lambda^\sharp(X,f) = \big(X, -\operatorname{ad}(X)\big(\theta(\varphi^{-1}(f))\big)\big), \quad \forall X \in \mathfrak{h}, f \in \mathfrak{h}^*.$$

Proof. Let $a, b \in \mathfrak{h}^*$ and $X \in \mathfrak{h}$. Then we calculate

$$\begin{aligned} \langle \Lambda_X^\sharp(a), b\rangle &= \Lambda_X(a,b) \\ &= \langle \varphi(X), [\varphi^{-1}(a), \varphi^{-1}(b)]\rangle \\ &= \big(X \mid \operatorname{ad}\big(\varphi^{-1}(a)\big)\varphi^{-1}(b)\big) \\ &= \big(\operatorname{ad}\big(\theta(\varphi^{-1}(a))\big)X \mid \varphi^{-1}(b)\big) \\ &= \langle \operatorname{ad}\big(\theta(\varphi^{-1}(a))\big)X, b\rangle \\ &= -\langle \operatorname{ad}(X)\big(\theta(\varphi^{-1}(a))\big), b\rangle. \end{aligned}$$

∎

From Lemma VI.3 it is clear that

$$C_X = \operatorname{ad}(\theta\mathfrak{h})X = \operatorname{ad}(X)\theta\mathfrak{h}. \tag{VI.2}$$

We will write $C_{X,\mathfrak{h}}$ for C_X when we want to emphasize the subalgebra $\mathfrak{h}$ for which we consider the Poisson structure. The annihilator $\operatorname{Ann}C_{X,\mathfrak{h}}$ of $C_{X,\mathfrak{h}}$ in $T_X^*(\mathfrak{h}) \cong \mathfrak{h}^*$ is

$$\operatorname{Ann}C_{X,\mathfrak{h}} = \ker \Lambda_X^\sharp = \{f \in \mathfrak{h}^* : \theta\mathfrak{h} \subseteq \ker\big(\operatorname{ad}^*(X).f\big)\}.$$

Proposition VI.4. *Let* $(\varphi^{-1})^*\colon \mathfrak{h}^* \to (\mathfrak{h}^*)^* \cong \mathfrak{h}$ *denote the dual map of* $\varphi^{-1}\colon \mathfrak{h}^* \to \mathfrak{h}$. *Then we have* $(\varphi^{-1})^* \circ \Lambda^\sharp_{\varphi(X)} = \Lambda^\sharp_X \circ \varphi$.

Proof. For $f \in \mathfrak{h}^*$ and $Y \in \mathfrak{h}$ we calculate

$$\begin{aligned}\langle f, \Lambda^\sharp_X \circ \varphi(Y)\rangle &= \Lambda_X(\varphi(Y), f)\\ &= \langle \varphi(X), [Y, \varphi^{-1}(f)]\rangle\\ &= \langle -\operatorname{ad}^*(Y)\big(\varphi(X)\big), \varphi^{-1}(f)\rangle\\ &= \langle \Lambda^\sharp_{\varphi(X)}(Y), \varphi^{-1}(f)\rangle\\ &= \langle f, (\varphi^{-1})^* \circ \Lambda^\sharp_{\varphi(X)}(Y)\rangle.\end{aligned}$$

■

Proposition VI.4 implies that φ^* induces an isomorphism $\varphi^*\colon C_{\varphi(X)} \to C_X$ and φ an isomorphism $\varphi\colon \mathrm{Ann} C_{\varphi(X)} \to \mathrm{Ann} C_X$. In fact, we have a commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ann} C_X & \longrightarrow & \mathfrak{h}^* & \xrightarrow{\Lambda^\sharp_X} & C_X & \longrightarrow & 0\\ & & \uparrow\varphi & & \uparrow\varphi & & \uparrow\varphi^* & & \\ 0 & \longrightarrow & \mathrm{Ann} C_{\varphi(X)} & \longrightarrow & \mathfrak{h} & \xrightarrow{\Lambda^\sharp_{\varphi(X)}} & C_{\varphi(X)} & \longrightarrow & 0\end{array}$$

which then gives a commutative diagram

$$\begin{array}{ccc} C_X^* & \xrightarrow{\ell_X} & C_X\\ \uparrow\overline{\varphi} & & \uparrow\varphi^*\\ C_{\varphi(X)}^* & \xrightarrow{\ell_{\varphi(X)}} & C_{\varphi(X)}\end{array}$$

where $\overline{\varphi}$ is the map induced by φ on $C_{\varphi(X)}^* = \mathfrak{h}/\mathrm{Ann} C_{\varphi(X)}$.

Lemma VI.5. $\Lambda^\sharp_X \circ \varphi = \operatorname{ad}(X) \circ \theta = -\theta \circ \operatorname{ad}(\theta X)$.

Proof. $\Lambda^\sharp_X \circ \varphi(Y) = -\operatorname{ad}(X)\left(\theta\big(\varphi^{-1}(\varphi(Y))\big)\right) = \operatorname{ad}(X)\big(\theta(Y)\big)$. ■

Proposition VI.6. *The symplectic form* ω_X *on* $C_{X,\mathfrak{h}}$ *is given by*

$$\omega_X(\operatorname{ad}(X)\theta Y, \operatorname{ad}(X)\theta Z) = -B(X, \theta[Y, Z]), \quad \forall X, Y, Z \in \mathfrak{h}.$$

Proof.

$$\begin{aligned}\omega_X(\operatorname{ad}(X)\theta Y, \operatorname{ad}(X)\theta Z) &= \Lambda_X\left(\ell_X^{-1}(\operatorname{ad}(X)\theta Y), \ell_X^{-1}(\operatorname{ad}(X)\theta Z)\right)\\ &= \Lambda_X\left(\ell_X^{-1}\big(\Lambda^\sharp_X \circ \varphi(Y)\big), \ell_X^{-1}\big(\Lambda^\sharp_X \circ \varphi(Z)\big)\right)\\ &= \Lambda_X\left(\varphi(Y) + \mathrm{Ann} C_X, \varphi(Z) + \mathrm{Ann} C_X\right)\\ &= \Lambda_{\varphi(X)}\left(Y + \mathrm{Ann} C_{\varphi(X)}, Z + \mathrm{Ann} C_{\varphi(X)}\right)\\ &= \langle \varphi(X), [Y, Z]\rangle\\ &= (X \mid [Y, Z])\\ &= -B(X, \theta[Y, Z]).\end{aligned}$$

■

Lemma VI.7. *Suppose that $X \in \mathfrak{g}$ is nilpotent. Then the functional $\varphi(X) \in \mathfrak{g}^*$ vanishes on $\mathfrak{g}_{\varphi(X)} = \{Y \in \mathfrak{g} \colon \mathrm{ad}^*(Y).\varphi(X) = 0\}$.*

Proof. Let $Y \in \mathfrak{g}$. Then $\langle \varphi(X), Y\rangle = -B(X, \theta Y)$. If $Y \in \mathfrak{g}_{\varphi(X)}$, then for all $Z \in \mathfrak{g}$ we have

$$\begin{aligned} 0 &= \langle \mathrm{ad}^*(Y).\varphi(X), Z\rangle = \langle \varphi(X), [Z, Y]\rangle \\ &= -B(X, [\theta Z, \theta Y]) = -B([X, \theta Z], \theta Y). \end{aligned}$$

But the Jacobson-Morosow Theorem says in particular that there exists a $\theta Z \in \mathfrak{g}$ with $X = [X, \theta Z]$. Therefore $\langle \varphi(X), Y\rangle = -B(X, \theta Y) = 0$ for all $Y \in \mathfrak{g}_{\varphi(X)}$. ■

Proposition VI.8. *Let $X \in \mathfrak{g}$ be nilpotent and $\omega^{(X)}$ the symplectic form on $\mathcal{O}_X = \mathrm{Ad}(G).X$ induced by the canonical symplectic form $\omega^{(\varphi(X))}$ on the coadjoint orbit $\mathcal{O}_{\varphi(X)} = \varphi(\mathcal{O}_X) = \mathrm{Ad}^*(G).\varphi(X)$. Then there exists a G-invariant 1-form $\alpha^{(X)}$ on $\mathcal{O}_X$ defined by*

$$\alpha^{(X)}_X(\mathrm{ad}(X)\theta Y) = B(X, \theta Y)$$

such that $-2d\alpha^{(X)} = \omega^{(X)}$.

Proof. According to Lemma VI.7 and Lemma VI.2 there exists a form $\alpha^{(\varphi(X))}$ on $\mathcal{O}_{\varphi(X)}$ with $-2d\alpha^{(\varphi(X))} = \omega^{(\varphi(X))}$. Then $\alpha^{(X)}$ is the pullback of $\alpha^{(\varphi(X))}$ under φ. A simple calculation shows that $\varphi(\mathrm{ad}(X)\theta Y) = -\mathrm{ad}^*(Y)\varphi(X)$ so the claim follows from

$$\alpha^{(\varphi(X))}_{\varphi(X)}\big(-\mathrm{ad}^*(Y)\varphi(X)\big) = -\langle \varphi(X), Y\rangle = B(X, \theta Y) \qquad ■$$

We note here that we cannot expect to be able to prove an analogue of Proposition VI.8 for the H-orbits since the centralizer $\mathfrak{h}_{\varphi(X)}$ of $\varphi(X)$ for $X \in \mathfrak{h}$ in $\mathfrak{h}$ may not be contained in $\mathfrak{g}_{\varphi(X)}$ and the conclusion of Lemma VI.7 depended on the Jacobson-Morosow Theorem which is not available for non-semisimple algebras.

Now we return to our special situation and apply the preceding results to $\mathcal{O}_k$ and $\mathcal{M}_k$.

Lemma VI.9. $C_{X^k, \mathfrak{g}} = C_{X^k, \mathfrak{q}_{k,\mathrm{red}}} = \mathfrak{h}_k + \mathfrak{p}_{2,k}$.

Proof. This can be derived from Proposition V.15 by calculating the $Q_{k,\mathrm{red}}$-orbit of X^k. We give a more direct argument: $\mathfrak{sl}_2$-theory shows that the image of $\mathrm{ad}(X^k) \colon \mathfrak{g} \to \mathfrak{g}$ is

$$C_{X^k, \mathfrak{g}} = \mathfrak{h}_k + [X^k, \theta \mathcal{U}_k] = \mathfrak{h}_k + \mathfrak{p}_{2,k} \subseteq \mathfrak{q}_{k,\mathrm{red}}$$

(Lemma IV.24). ■

It is obvious now that $\mathrm{p}_{\mathfrak{h}_k}(C_{X^k,\mathfrak{g}}) = \mathfrak{h}_k$ and $\mathrm{p}_{\mathcal{U}_k}(C_{X^k,\mathfrak{g}}) = \mathcal{U}_k$.

Proposition VI.10. *The symplectic form $\omega^{(k)}$ on $\mathcal{M}_k$ is exact. More precisely, there exists a $Q_{k,\mathrm{red}}$-invariant 1-form $\alpha^{(k)}$ on $\mathcal{M}_k$ with $-2d\alpha^{(k)} = \omega^{(k)}$ and*

$$\alpha^{(k)}_{X^k}(\mathrm{ad}(X^k)\theta Y) = B(X^k, \theta Y)$$

for all $Y \in \mathfrak{q}_{k,\mathrm{red}}$.

Proof. This follows immediately from Proposition V.15, Proposition VI.8 and Lemma VI.9 by taking pullbacks. ■

Lemma VI.11.

(i) $\mathcal{L}_{+,k}(X^k) := (\mathcal{U}_k)_{\mathbb{C}} + \mathcal{V}_k^+$ *is a Lagrangian subspace of* $(C_{X^k,\mathfrak{q}_{k,\mathrm{red}}})_{\mathbb{C}}$.

(ii) $\mathcal{L}_{-,k}(X^k) := (\mathfrak{p}_{2,k})_{\mathbb{C}} + \mathcal{V}_k^-$ *is a Lagrangian subspace of* $(C_{X^k,\mathfrak{q}_{k,\mathrm{red}}})_{\mathbb{C}}$.

Proof. For dimensional reasons it suffices to show that ω_{X^k} vanishes on the two subspaces. Let $X \in (\mathcal{U}_k)_{\mathbb{C}}$, $Y^\pm \in \mathcal{V}_k^\pm$ and $Z \in (\mathfrak{p}_{2,k})_{\mathbb{C}}$. Then we find $\widetilde{X}, \widetilde{Y}^\pm, \widetilde{Z} \in \mathfrak{q}_{k,\mathrm{red}}$ with

$$X = \mathrm{ad}(X^k)\theta\widetilde{X}, \quad Y^\pm = \mathrm{ad}(X^k)\theta\widetilde{Y}^\pm, \quad Z = \mathrm{ad}(X^k)\theta\widetilde{Z}.$$

Then $\widetilde{X} \in (\mathfrak{l}_k)_{\mathbb{C}}$, $\widetilde{Y}^\pm \in \mathcal{V}_k^\pm$ and $\widetilde{Z} \in (\mathcal{U}_k)_{\mathbb{C}}$. Now the claim follows since the below commutator spaces all belong to the orthogonal complement of X^k w.r.t. the Killing form B.

$$[\mathfrak{l}_k, \mathfrak{l}_k] \subseteq \mathfrak{l}_k, \quad [\mathfrak{l}_k, \mathcal{V}_k^+] \subseteq \mathcal{V}_k^+, \quad [\mathcal{V}_k^+, \mathcal{V}_k^+] = \{0\},$$

$$[\mathcal{U}_k, \mathcal{U}_k] = \{0\}, \quad [\mathcal{U}_k, \mathcal{V}_k^-] = \{0\}, \quad [\mathcal{V}_k^-, \mathcal{V}_k^-] = \{0\}. \qquad \blacksquare$$

Recall that $Q_{k,\mathrm{red}}$ acts transitively on $\mathcal{O}_k'$ and $\mathcal{M}_k$. The stabilizer of X^k in $Q_{k,\mathrm{red}}$ is $K_{2,k}$. Note that $K_{2,k}$ leaves both Lagrangian subspaces from Lemma VI.11 invariant (the ρ-action of $K_{2,k}$ coincides with the adjoint action). Since the Poisson structure is $Q_{k,\mathrm{red}}$-invariant we can define two $Q_{k,\mathrm{red}}$-invariant Lagrangian distributions $\mathcal{L}_{\pm,k}$ on $\mathcal{M}_k$ using the action ρ_k.

Using the semidirect product structure of $Q_k = H_k \rtimes L_k$ we obtain additional information on the form $\alpha^{(k)}$:

Consider the diffeomorphism

$$\gamma\colon H_k \times \Omega_k \to \mathcal{M}_k, \quad (h, X) \mapsto \rho_k(h, X).$$

The Q_k-action on $H_k \times \Omega_k$ induced from ρ_k via γ is given by the formula

$$(h, l).(h', X) = (hlh'l^{-1}, l.X), \quad \forall h, h' \in H_k, l \in L_k, X \in \Omega_k.$$

Pulling back $\alpha^{(k)}$ to $H_k \times \Omega_k$ via γ yields an invariant form on $H_k \times \Omega_k$ and since the action of H_k on $H_k \times \Omega_k$ simply consists of left translation in the fist argument, it is given by elements of $\mathfrak{h}_k^*$ on the H_k-orbits. We identify T^*H_k with $H_k \times \mathfrak{h}_k^*$ and $T^*(H_k \times \Omega_k)$ with $(H_k \times \Omega_k) \times (\mathfrak{h}_k^* \times \mathcal{U}_k^*)$.

Proposition VI.12.

(i) *The derivative of* γ *at* $(\mathbf{1}, X) \in H_k \times \Omega_k$ *is given by*

$$d\gamma_{(\mathbf{1},X)}(X' + Y', X'') = X'' - \mathrm{ad}(X)\theta(X' + Y') \quad \forall X', X'' \in \mathcal{U}_k, Y' \in \mathcal{V}_k.$$

(ii) *The pullback form* $\gamma^*\alpha^{(k)}$ *is given by*

$$\gamma^*\alpha^{(k)}_{(h,X)}(X' + Y', X'') = \langle \varphi(X), X' \rangle + \alpha^{(k)}_X(X''),$$

for all $X', X'' \in \mathcal{U}_k, Y' \in \mathcal{V}_k$.

Proof. (i) follows immediately from

$$\gamma(\exp Y' \exp X', X) = X - \mathrm{ad}(X).\theta(X' + Y') + \frac{1}{2} p_k(\mathrm{ad}(\theta Y')^2.X)$$

for $Y' \in \mathcal{V}_k$ and $X', X \in \mathcal{U}_k$ which in turn is an immediate consequence of Lemma V.5.
(ii) We calculate

$$\begin{aligned}
\gamma^* \alpha^{(k)}_{(h,X)}(X' + Y', X'') &= \gamma^* \alpha^{(k)}_{h.(\mathbf{1},X)}(X' + Y', X'') \\
&= \gamma^* \alpha^{(k)}_{(\mathbf{1},X)}(X' + Y', X'') \\
&= \alpha^{(k)}_X (d\gamma_{(\mathbf{1},X)}(X' + Y', X'')) \\
&= \alpha^{(k)}_X (X'' - \mathrm{ad}(X)\theta(X' + Y')) \\
&= \alpha^{(k)}_X (X'') - B(X, \theta(X' + Y')) \\
&= \alpha^{(k)}_X (X'') - B(X, \theta X') \\
&= \alpha^{(k)}_X (X'') + \langle \varphi(X), X' \rangle.
\end{aligned}$$

■

We now turn to the study of polarizations. Let $\mathfrak{h}$ be a Lie algebra and $f \in \mathfrak{h}^*$. Recall that a subalgebra $\mathfrak{b}(f)$ of $\mathfrak{h}_{\mathbb{C}}$ is called an *algebraic polarization* if (cf. [2, p.54])

(a) $\mathfrak{b}(f) + \overline{\mathfrak{b}(f)}$ is a subalgebra of $\mathfrak{h}_{\mathbb{C}}$.

(b) $\mathfrak{b}(f)$ is maximal isotropic in $\mathfrak{h}_{\mathbb{C}}$ w.r.t. the form $(X, Y) \mapsto \langle f, [X, Y] \rangle$.

Note (cf. [37, p.103]) that the maximality implies that $(\mathfrak{h}_f)_{\mathbb{C}} \subseteq \mathfrak{b}(f)$. Moreover an isotropic subalgebra containing $\mathfrak{h}_f$ is maximal iff

$$\dim_{\mathbb{C}} \big(\mathfrak{b}(f)/(\mathfrak{h}_f)_{\mathbb{C}}\big) = \dim_{\mathbb{C}} \big(\mathfrak{h}_{\mathbb{C}}/\mathfrak{b}(f)\big).$$

Let $f \in \mathfrak{q}^*_{k,\mathrm{red}}$ be the linear functional given by

$$f(u + v + X) = -(X^k \mid u), \quad \forall u \in \mathcal{U}_k, v \in \mathcal{V}_k, X \in \mathfrak{g}_{2,k} \tag{VI.3}$$

Lemma VI.13. *The subalgebra $\mathfrak{b}(f) := (\mathcal{U}_k)_{\mathbb{C}} + \mathcal{V}_k^- + (\mathfrak{k}_{2,k})_{\mathbb{C}}$ is an algebraic polarization of $f \in \mathfrak{q}^*_{k,\mathrm{red}}$.*

Proof. It is clear that $\mathfrak{b}(f) + \overline{\mathfrak{b}(f)} = (\mathcal{U}_k)_{\mathbb{C}} + (\mathcal{V}_k)_{\mathbb{C}} + (\mathfrak{k}_{2,k})_{\mathbb{C}}$ is a subalgebra. Since $[\mathfrak{k}_{2,k}, \mathcal{V}_k^-] \subseteq \mathcal{V}_k^-$, it is also clear that $\mathfrak{b}(f)$ is a subalgebra of $(\mathfrak{q}_{k,\mathrm{red}})_{\mathbb{C}}$. Moreover

$$[\mathfrak{b}(f), \mathfrak{b}(f)] \subseteq (\mathfrak{k}_{2,k})_{\mathbb{C}} + \mathcal{V}_k^- + [\mathfrak{k}_{2,k}, \mathcal{U}_k]_{\mathbb{C}} \subseteq f^{\perp}$$

because $\mathfrak{k}_{2,k} \subseteq (\mathfrak{q}_k)_f$ implies $f([\mathfrak{k}_{2,k}, \mathcal{U}_k]) = \{0\}$. This means that $\mathfrak{b}(f)$ is isotropic. Obviously we have $(\mathfrak{q}_{k,\mathrm{red}})_f = \mathfrak{k}_{2,k} \subseteq \mathfrak{b}(f)$. So to prove maximality, we simply have to check that it has the right dimension. Since $(\mathfrak{q}_{k,\mathrm{red}})_{\mathbb{C}}/\mathfrak{b}(f) \cong \big(\mathcal{V}_k^+ + (\mathfrak{g}_{2,k})_{\mathbb{C}}\big)/(\mathfrak{k}_{2,k})_{\mathbb{C}}$ and $\mathfrak{b}(f)/(\mathfrak{q}_{k,\mathrm{red}} \cap (\mathfrak{q}_k)_f)_{\mathbb{C}} \cong (\mathcal{U}_k)_{\mathbb{C}} + \mathcal{V}_k^-$, the assertion follows from $\dim \mathcal{U}_k = \dim \mathfrak{g}_{2,k} - \dim \mathfrak{k}_{2,k}$ which in turn follows from the fact that $\mathfrak{g}_{2,k}$ acts effectively on $\mathcal{U}_k$ with $\mathfrak{g}_{2,k}.X^k = \mathcal{U}_k$ and $\mathfrak{k}_{2,k} = (\mathfrak{g}_{2,k})_{X^k}$ (cf. Proposition IV.4). ■

Remark VI.14. Suppose that $\mathcal{V}_k = \{0\}$ (Proposition IV.21). Put $\mathfrak{b} := (\mathfrak{g}_{2,r} + \theta\mathcal{U}_r)_{\mathbb{C}}$. We claim that $\mathfrak{b}$ is an algebraic polarization in f, considered as an element of $\mathfrak{g}^*$.

Since $\mathfrak{b} = \overline{\mathfrak{b}}$, it is clear that $\mathfrak{b} + \overline{\mathfrak{b}}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Moreover $[\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{g}'_{2,r} + \theta\mathcal{U}_r \subseteq \ker f$ shows that $\mathfrak{b}$ is isotropic. Since

$$\dim(\mathfrak{b} \cap \mathfrak{g}) - \dim \mathfrak{g}_f = \dim \mathfrak{g}_{2,r} - \dim \mathfrak{k}_{2,r} = \dim \mathcal{U}_r = \dim \mathfrak{g} - \dim(\mathfrak{b} \cap \mathfrak{g}),$$

it follows that $\mathfrak{b}$ is maximal isotropic, hence an algebraic polarization in f. ■

Theorem VI.15. *If $\mathcal{V}_k \neq \{0\}$, then $f \in \mathfrak{g}^*$ has no algebraic polarization for $\mathfrak{g}$.*

Proof. For the sake of simplicity we omit the indices k in the following proof, f.i. we write $\mathfrak{g}_2$ instead of $\mathfrak{g}_{2,k}$ etc.

Suppose that $\mathfrak{b} \subseteq \mathfrak{g}_{\mathbb{C}}$ is a polarization in f. We recall that

$$\mathfrak{g}_f = \mathfrak{k}_2 + \mathfrak{g}_1 + \theta\mathcal{V} + \theta\mathcal{U}.$$

In view of the fact that $\mathfrak{b}$ is maximal isotropic, it follows that $(\mathfrak{g}_f)_{\mathbb{C}} \subseteq \mathfrak{b}$.

It suffices to show that $\mathfrak{b} = \mathfrak{g}_{\mathbb{C}}$. First we claim that $\mathfrak{g}_2 \subseteq \mathfrak{b}$.

As a $\mathfrak{g}_1$-module the quotient $\mathfrak{g}_{\mathbb{C}}/(\mathfrak{g}_f)_{\mathbb{C}}$ is isomorphic to $(\mathfrak{p}_2)_{\mathbb{C}} \oplus \mathcal{V}_{\mathbb{C}} \oplus \mathcal{U}_{\mathbb{C}}$. The subspace $(\mathfrak{p}_2)_{\mathbb{C}} \oplus \mathcal{U}_{\mathbb{C}}$ is annihilated by $\mathfrak{g}_1$ and the action of $\mathfrak{g}_1$ on $\mathcal{V}_{\mathbb{C}}$ is effective because $\mathcal{V}$ is an irreducible module for $\mathfrak{g}_1 + \mathfrak{g}_2$ (cf. Proposition IV.1). Let $\pi \colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}/(\mathfrak{g}_f)_{\mathbb{C}}$ denote the quotient map. Then $\pi(\mathfrak{b})$ is a $\mathfrak{g}_1$-submodule and therefore

$$\pi(\mathfrak{b}) = \big(\pi(\mathfrak{b}) \cap \mathcal{V}_{\mathbb{C}}\big) + \big(\pi(\mathfrak{b}) \cap \big(\mathcal{U}_{\mathbb{C}} + (\mathfrak{p}_2)_{\mathbb{C}}\big)\big).$$

The natural symplectic form on $\mathfrak{g}_{\mathbb{C}}/(\mathfrak{g}_f)_{\mathbb{C}}$ is given by

$$\big(Y + (\mathfrak{g}_f)_{\mathbb{C}}, Z + (\mathfrak{g}_f)_{\mathbb{C}}\big) \mapsto f([Y, Z]).$$

Hence $\mathcal{V}$ is orthogonal to $\mathfrak{p}_2 + \mathcal{U}$. Therefore the fact that $\mathfrak{b}$ is maximal isotropic shows that $\pi(\mathfrak{b}) \cap \mathcal{V}_{\mathbb{C}}$ and $\pi(\mathfrak{b}) \cap \big(\mathcal{U}_{\mathbb{C}} + (\mathfrak{p}_2)_{\mathbb{C}}\big)$ are maximal isotropic.

If $\pi(\mathfrak{b}) \cap \big(\mathcal{U}_{\mathbb{C}} + (\mathfrak{p}_2)_{\mathbb{C}}\big) \subseteq (\mathfrak{p}_2)_{\mathbb{C}}$, then counting dimensions therefore shows that $\pi(\mathfrak{b}) \cap \big(\mathcal{U}_{\mathbb{C}} + (\mathfrak{p}_2)_{\mathbb{C}}\big) = (\mathfrak{p}_2)_{\mathbb{C}}$, hence that $\mathfrak{p}_2 \subseteq \mathfrak{b}$ which in turn implies $\mathfrak{g}_2 \subseteq \mathfrak{b}$.

Suppose that $\pi(\mathfrak{b}) \cap (\mathcal{U}_{\mathbb{C}} + (\mathfrak{p}_2)_{\mathbb{C}}) \not\subseteq (\mathfrak{p}_2)_{\mathbb{C}}$. The kernel for the mapping $\widetilde{\mathrm{ad}\theta X}$ induced by $\mathrm{ad}\,\theta X$ on $\mathfrak{g}_{\mathbb{C}}/(\mathfrak{g}_f)_{\mathbb{C}}$ is given by $(\mathfrak{p}_2)_{\mathbb{C}} + \mathcal{V}_{\mathbb{C}}$ and since $\mathrm{ad}\,\theta X \colon \mathcal{U} \to \mathfrak{p}_2$ is bijective (Lemma IV.24), the same holds for $\widetilde{\mathrm{ad}\theta X}$. Thus our assumption and the invariance of $\pi(\mathfrak{b})$ under $\widetilde{\mathrm{ad}\theta X}$ yield $\pi(\mathfrak{b}) \cap (\mathfrak{p}_2)_{\mathbb{C}} \neq \{0\}$, i.e., $\mathfrak{b} \cap (\mathfrak{p}_2)_{\mathbb{C}} \neq \{0\}$.

Since $\mathfrak{g}_2 = \mathbb{R}H \oplus \mathfrak{g}'_2$, where $\mathfrak{g}'_2$ is a simple real Lie algebra (Lemma IV.12, Theorem IV.13]), the $\mathfrak{k}_2$-fixed elements in $(\mathfrak{p}_2)_{\mathbb{C}}$ are $\mathbb{C}H$ and the effective submodule is $(\mathfrak{p}'_2)_{\mathbb{C}}$, where $\mathfrak{p}'_2 := \mathfrak{g}'_2 \cap \mathfrak{p}_2$. Now we use the $\mathfrak{k}_2$-invariance of $\mathfrak{b}$ to see that

$$\mathfrak{b} \cap (\mathfrak{p}_2)_{\mathbb{C}} = (\mathfrak{b} \cap \mathbb{C}H) \oplus \big(\mathfrak{b} \cap (\mathfrak{p}'_2)_{\mathbb{C}}\big).$$

Hence one of the two summands must be non-zero. If the first one is non-zero, then $H \in \mathfrak{b}$ and we can decompose $\mathfrak{b}$ into $\operatorname{ad} H$ eigenspaces

$$\mathfrak{b} = \theta\mathcal{U}_{\mathbb{C}} + \theta\mathcal{V}_{\mathbb{C}} + (\mathfrak{g}_1)_{\mathbb{C}} + \big(\mathfrak{b} \cap (\mathfrak{g}_2)_{\mathbb{C}}\big) + (\mathfrak{b} \cap \mathcal{V}_{\mathbb{C}}) + (\mathfrak{b} \cap \mathcal{U}_{\mathbb{C}}).$$

If $\mathfrak{g}_2 \not\subseteq \mathfrak{b}$, then $\mathfrak{b} \cap (\mathfrak{p}_2)_{\mathbb{C}} = \mathbb{C}H$ and consequently $\mathfrak{b} \cap \mathcal{U}_{\mathbb{C}}$ has codimension 1 since $\mathfrak{b}$ is maximal isotropic. Then we apply the injectivity of $\operatorname{ad} \theta X \colon \mathcal{U} \to \mathfrak{p}_2$ a second time to conclude that $\dim \mathcal{U} = \dim \mathfrak{p}_2 \leq 2$. Now $\mathfrak{p}_2'$ is at most one-dimensional, so that $p_2' = \{0\}$ and therefore $\mathfrak{g}_2 = \mathbb{R}H \subseteq \mathfrak{b}$. This proves that $\mathfrak{g}_2 \subseteq \mathfrak{b}$ if $H \in \mathfrak{b}$.

Next we assume that $\mathfrak{b} \cap (\mathfrak{p}_2)_{\mathbb{C}} = \mathfrak{b} \cap (\mathfrak{p}_2')_{\mathbb{C}} \neq \{0\}$. The element X is orthogonal to $\mathfrak{b}$ with respect to the form $(Y,Z) \mapsto f([Y,Z])$ since $f([X,Z]) = B(\theta X, [X,Z]) = B([\theta X, X], Z) = -B(H,Z)$ and therefore $X \in \mathfrak{b}$ because $\mathfrak{b}$ is maximal isotropic. Thus $[X, \theta X] = H \in \mathfrak{b}$, a contradiction.

Taking all these cases together, we have proved that $\mathfrak{g}_2 \subseteq \mathfrak{b}$. Hence

$$\mathfrak{b} = \theta\mathcal{U}_{\mathbb{C}} + \theta\mathcal{V}_{\mathbb{C}} + (\mathfrak{g}_1)_{\mathbb{C}} + (\mathfrak{g}_2)_{\mathbb{C}} + (\mathfrak{b} \cap \mathcal{V}_{\mathbb{C}}).$$

Next we use the fact that $\mathfrak{b} + \overline{\mathfrak{b}}$ is a subalgebra invariant under complex conjugation, so that it can be written as $\mathfrak{h}_{\mathbb{C}}$ for $\mathfrak{h} := (\mathfrak{b} + \overline{\mathfrak{b}}) \cap \mathfrak{g}$. Then the subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ satisfies

$$\theta\mathcal{U} + \theta\mathcal{V} + \mathfrak{g}_1 + \mathfrak{g}_2 \subseteq \mathfrak{h} \subseteq \theta\mathcal{U} + \theta\mathcal{V} + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathcal{V}$$

and therefore

$$\text{(VI.4)} \qquad \mathfrak{h} = \theta\mathcal{U} + \theta\mathcal{V} + \mathfrak{g}_1 + \mathfrak{g}_2 + (\mathcal{V} \cap \mathfrak{h}).$$

In view of Proposition IV.1, the $(\mathfrak{g}_1 + \mathfrak{g}_2)$-module $\mathcal{V}$ is simple, hence $\mathfrak{h} \cap \mathcal{V} = \mathcal{V}$ holds because it is non-zero. Finally $[\mathcal{V}, \mathcal{V}] = \mathcal{U}$ (Proposition IV.1) contradicts (VI.4). ∎

So far we have seen that f has no algebraic polarization for $\mathfrak{g}$ if $\mathcal{V}_k \neq \{0\}$ and that $\mathfrak{b} = (\mathfrak{g}_2 + \theta\mathcal{U})_{\mathbb{C}}$ is a polarization if this is not the case. In [14] we will construct representations by applying the orbit method to the polarization in Lemma VI.13. The representations associated to the polarization in Remark VI.14 have been studied in [31] and also in [30].

The notion of algebraic polarization is closely related to the notion of strongly integrable complex polarizations (in the terminology of [37, p.92]) which is defined in terms of Lagrangian distributions.

Lemma VI.16. *The Lagrangian distributions $\mathcal{L}_{\pm,k}$ on $\mathcal{M}_k$ are involutive.*

Proof. Consider the derivative $\delta \colon \mathfrak{q}_{k,\mathrm{red}} \to C_{X^k, \mathfrak{q}_{k,\mathrm{red}}}$ of the orbit map

$$Q_{k,\mathrm{red}} \to \mathcal{M}_k, \quad q \mapsto \rho_{\mathfrak{q}_{k,\mathrm{red}}}(q, X^k)$$

at the identity. According to [8, p.235], it suffices to show that the preimages of $\mathcal{L}_{\pm,k}(X^k)$ in $\mathfrak{q}_{k,\mathrm{red}}$ under the complex linear extension $\delta_{\mathbb{C}} \colon (\mathfrak{q}_{k,\mathrm{red}})_{\mathbb{C}} \to (C_{X^k, \mathfrak{q}_{k,\mathrm{red}}})_{\mathbb{C}}$ is a subalgebra of $(\mathfrak{q}_{k,\mathrm{red}})_{\mathbb{C}}$. But from Lemma V.6 we see that

$$\delta(X) = -\operatorname{ad}(X^k) \circ \theta \quad \forall X \in \mathfrak{q}_{k,\mathrm{red}}.$$

Moreover Lemma IV.24 together with [29, III(3.27)] shows that

$$\delta(\mathfrak{g}_{2,k}) = \mathcal{U}_k, \quad \delta(\mathcal{U}_k) = \mathfrak{p}_{2,k}, \quad \delta_{\mathbb{C}}(\mathcal{V}_k^{\pm}) = \mathcal{V}_k^{\pm}.$$

Since $\delta_{\mathbb{C}}^{-1}(0) = (\mathfrak{k}_{2,k})_{\mathbb{C}}$ we find

$$\delta_{\mathbb{C}}^{-1}\big(\mathcal{L}_{+,k}(X^k)\big) = \mathcal{V}_k^{+} + (\mathfrak{g}_{2,k})_{\mathbb{C}}$$

and

$$\delta_{\mathbb{C}}^{-1}\big(\mathcal{L}_{-,k}(X^k)\big) = (\mathcal{U}_k)_{\mathbb{C}} + \mathcal{V}_k^{-} + (\mathfrak{k}_{2,k})_{\mathbb{C}}$$

which proves the claim in view of Proposition IV.5(vii). ■

Consider

$$\mathcal{D}_{\pm,k} = \mathcal{L}_{\pm,k} \cap \overline{\mathcal{L}_{\pm,k}} \cap T(\mathcal{M}_k)$$

and

$$\mathcal{E}_{\pm,k} = (\mathcal{L}_{\pm,k} + \overline{\mathcal{L}_{\pm,k}}) \cap T(\mathcal{M}_k).$$

Since $\mathcal{D}_{\pm,k}(q.X^k) = q.\mathcal{D}_{\pm,k}(X^k)$, we see that $\mathcal{D}_{\pm,k}$ are distributions and hence the distributions $\mathcal{L}_{\pm,k}$ are *complex polarizations* in the sense of Woodhouse (cf. [37, p.92] and note that the definition of a polarization in [8] is weaker). Similarly we see that $\mathcal{E}_{\pm,k}$ are distributions. We recall that the polarization $\mathcal{L}_{\pm,k}$ is called *strongly integrable* if $\mathcal{E}_{\pm,k}$ is involutive.

Proposition VI.17.

(i) $\mathcal{L}_{+,k}$ *is strongly integrable iff* $\mathcal{V}_k = \{0\}$.

(ii) $\mathcal{L}_{-,k}$ *is strongly integrable.*

Proof. We use the same reasoning as in the proof of Lemma VI.16 but for $\mathcal{E}_{\pm,k}$ instead of $\mathcal{L}_{\pm,k}$. In fact, we see that

$$\delta^{-1}\big(\mathcal{E}_{+,k}(X^k)\big) = \mathcal{V}_k + \mathfrak{g}_{2,k} \quad \text{and} \quad \delta^{-1}\big(\mathcal{E}_{-,k}(X^k)\big) = \mathfrak{h}_k + \mathfrak{k}_{2,k}.$$

Since $\mathcal{V}_k + \mathfrak{g}_{2,k}$ is a Lie algebra iff $\mathcal{V}_k = \{0\}$ whereas $\mathfrak{h}_k + \mathfrak{k}_{2,k}$ always is a Lie algebra, the claim follows. ■

Recall that integrable real distributions of a real manifold are the same as real foliations.

Proposition VI.18.

(i) *The leaves of the foliation* $\mathcal{E}_{-,k}$ *are precisely the fibers of* $\mathrm{p}_{\mathcal{U}_k}\colon \mathcal{M}_k \to \Omega_k$.

(ii) *The leaves of the foliation* $\mathcal{D}_{-,k}$ *are precisely the fibers of* $\mathrm{p}_{\mathfrak{h}_k}\colon \mathcal{M}_k \to \Omega_k + \mathcal{V}_k$.

Proof. (i) Let $E = \mathrm{p}_{\mathcal{U}_k}^{-1}(X^k) \cap \mathcal{M}_k$. Then H_k acts transitively on E (Lemma V.6) and for any $q \in \exp(\mathcal{U}_k)\exp(\mathcal{V}_k)$ we have

$$T_{q.X^k}(E) = q.T_{X^k}(E) = q.(\mathfrak{p}_{2,k} + \mathcal{V}_k) = q.\mathcal{E}_{-,k}(X^k) = \mathcal{E}_{-,k}(q.X^k).$$

This shows that E is an integral manifold for $\mathcal{E}_{-,k}$. Since $G_{2,k}$ permutes the fibers and $\mathcal{E}_{-,k}$ is $G_{2,k}$ invariant by definition, this proves the claim.
(ii) According to Proposition V.12, the fiber through $Z \in \mathcal{M}_k$ is $Z + \mathrm{ad}(X)\theta\mathcal{U}_k$ with $X = \mathrm{p}_{\mathcal{U}_k}(Z)$. In particular the fiber D over X^k is $X^k + \mathfrak{p}_{2,k}$ and the tangent space $T_{X^k}(D)$ is $\mathfrak{p}_{2,k} = \mathcal{D}_{-,k}(X^k)$. As in the proof of (i), now the invariance of $\mathcal{D}_{-,k}$ and the equivariance of $\mathrm{p}_{\mathfrak{h}_k}$ prove the claim. ■

We denote the space of leaves of $\mathcal{D}_{-,k}$ by $\mathcal{M}_k/\mathcal{D}_{-,k}$ and note that it is just $\Omega_k + \mathcal{V}_k$. Similarly the space of leaves of $\mathcal{E}_{-,k}$ gets identified with Ω_k.

Remark VI.19. The polarization $\mathcal{L}_{-,k}$ is non-positive in the sense of [37, p.91]. In fact, the symplectic vector space $\mathcal{E}_{-,k}(X^k)/\mathcal{D}_{-,k}(X^k)$ is isomorphic to $\mathcal{V}_k$ with the symplectic form $\omega^{(k)}_{X^k}|_{\mathcal{V}_k\times\mathcal{V}_k}$. But Proposition VI.12 shows that

$$\begin{aligned}\gamma^*\omega^{(k)}_{X^k}(Y',Y'') &= \gamma^*\alpha^{(k)}_{X^k}([Y',Y''],0) = \langle\varphi(X^k),[Y',Y'']\rangle \\ &= -B(X^k,\theta[Y',Y'']) = -4A_{X^k}(Y',Y''),\end{aligned}$$

where $A_X(Y',Y'') = -\frac{1}{4}(X \mid [Y',Y''])$ (cf. [29, p.101]). Since $A_{X^k}I_k + iA_{X^k}$ is positive definite (cf. [loc.cit.]), the claim follows. ■

VII. Examples

Let $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H}$ and E a $\mathbb{K}$-left vector space endowed with a positive definite hermitean form $(\cdot \mid \cdot)$, i.e.,

$$\overline{(v \mid w)} = (w \mid v), \quad \lambda(v \mid w) = (\lambda v \mid w) \quad \text{and} \quad (v \mid \lambda w) = (v \mid w)\overline{\lambda}$$

and $(v \mid v) > 0$ for all $v \neq 0$. The standard form on $\mathbb{K}^n$ is

$$(x \mid y) = \sum_{j=1}^{n} x_j \overline{y}_j.$$

We write $\mathrm{U}(E)$ for the group of $\mathbb{K}$-linear isometries of E.

In addition to $(\cdot \mid \cdot)$ we consider a non-degenerate skewhermitean form $\mathcal{B}(\cdot,\cdot)$, i.e.,

$$\overline{\mathcal{B}(v,w)} = -\mathcal{B}(w,v), \quad \lambda\mathcal{B}(v,w) = \mathcal{B}(\lambda v,w) \quad \text{and} \quad \mathcal{B}(v,\lambda w) = \mathcal{B}(v,w)\overline{\lambda}.$$

Here the standard forms are

(1) $\mathbb{K} = \mathbb{R}, E = \mathbb{R}^{2n}$, $\mathcal{B}(x,y) = \sum_{j=1}^n (x_j y_{j+n} - x_{j+n} y_j)$

(2) $\mathbb{K} = \mathbb{C}, E = \mathbb{C}^{p+q}$, $\mathcal{B}(x,y) = i\left(\sum_{j=1}^p x_j\overline{y}_j - \sum_{j=p+1}^{p+q} x_j\overline{y}_j\right)$

(3) $\mathbb{K} = \mathbb{H}, E = \mathbb{H}^n$, $\mathcal{B}(x,y) = \sum_{j=1}^n x_j \mathcal{J}\overline{y}_j$, where $\mathbf{1},\mathcal{I},\mathcal{J},\mathcal{K}$ are the standard basis for $\mathbb{H}$.

We write $\mathrm{U}(\mathcal{B})$ for the group of $\mathfrak{k}$-linear $\mathcal{B}$-isometries of E. The Lie algebras corresponding to the groups introduced above will be denoted by $\mathfrak{u}(E)$ and $\mathfrak{u}(\mathcal{B})$.

Consider the endomorphism $\varphi_{\mathcal{B}} \in \mathrm{End}_{\mathbb{K}}(E)$ defined by

$$\mathcal{B}(v,w) = (\varphi_{\mathcal{B}} v \mid w) \tag{VII.1}$$

as well as the transpose operations $*$ and $\sharp$ for $(\cdot \mid \cdot)$ and $\mathcal{B}(\cdot,\cdot)$. Then we have

$$\mathfrak{u}(\mathcal{B}) = \{X \in \mathfrak{gl}(E)\colon X^\sharp = -X\} \tag{VII.2}$$

and

$$\text{(VII.3)} \qquad \varphi_{\mathcal{B}}^* = -\varphi_{\mathcal{B}}, \quad X^\sharp = \varphi_{\mathcal{B}}^{-1} X^* \varphi_{\mathcal{B}}, \quad \varphi_{\mathcal{B}}^\sharp = -\varphi_{\mathcal{B}}.$$

In the standard cases, where elements of $\operatorname{End}_{\mathbb{K}}(E)$ are given by right multiplication of matrices on row vectors we have

(1) $\varphi_{\mathcal{B}} = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$ and $\mathfrak{u}(\mathcal{B}) = \mathfrak{sp}(n, \mathbb{R})$.

(2) $\varphi_{\mathcal{B}} = i \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}$ and $\mathfrak{u}(\mathcal{B}) = \mathfrak{u}(p, q)$.

(3) $\varphi_{\mathcal{B}} = \mathcal{J}\mathbf{1}_n$ and $\mathfrak{u}(\mathcal{B}) = \mathfrak{so}^*(2n)$.

For each $X \in \mathfrak{u}(\mathcal{B})$ we have a symmetric real bilinear form $\mathcal{B}_X$ on E which is given by

$$\text{(VII.4)} \qquad \mathcal{B}_X(v, w) = -\mathcal{B}(X.v, w).$$

We set

$$\text{(VII.5)} \qquad W_{\mathcal{B}} := \{X \in \mathfrak{u}(\mathcal{B}) \colon \mathcal{B}_X \text{ positive semidefinite}\}$$

and

$$\text{(VII.6)} \qquad W_{\max,\mathcal{B}} := \{X \in \mathfrak{u}(\mathcal{B}) \colon (\forall v \in E) \mathcal{B}(v, v) = 0 \ \Rightarrow\ \mathcal{B}(X.v, v) \leq 0\}.$$

Then $W_{\mathcal{B}} \subseteq W_{\max,\mathcal{B}}$ are closed convex cones which are invariant under conjugation by elements from $\mathrm{U}(\mathcal{B})$. We assume that

$$\text{(VII.7)} \qquad \varphi_{\mathcal{B}}^2 = -\operatorname{id}.$$

Then $\varphi_{\mathcal{B}} \in W_{\mathcal{B}}$, so that the cones $W_{\mathcal{B}}$ and $W_{\max,\mathcal{B}}$ are non-trivial.

Note that (VII.7) is true for the standard cases. It follows from (VII.7) that $\sharp$ and $*$ commute. In fact, up to a renormalization of $\mathcal{B}$ the two conditions are equivalent. From this it follows that $\frac{1}{2}\varphi_{\mathcal{B}}$ is an H-element of the quasihermitean reductive algebra $\mathfrak{u}(\mathcal{B})$. To see this, note that $\mathfrak{u}(\mathcal{B})$ is invariant under $*$ and $\theta(X) = -X^*$ defines a Cartan involution on $\mathfrak{u}(\mathcal{B})$. Therefore the centralizer

$$\text{(VII.8)} \qquad \mathfrak{k}(\mathcal{B}) := \mathfrak{u}(\mathcal{B}) \cap \mathfrak{u}(E)$$

of $\varphi_{\mathcal{B}}$ in $\mathfrak{u}(\mathcal{B})$ is a maximal compactly embedded subalgebra of $\mathfrak{u}(\mathcal{B})$. Then the equality

$$\text{(VII.9)} \qquad (\operatorname{ad} \varphi_{\mathcal{B}})^2 X = -2(X^* + X) \quad \forall X \in \mathfrak{u}(\mathcal{B})$$

proves the claim.

The complex structure on

$$\text{(VII.10)} \qquad \mathfrak{p}(\mathcal{B}) = \{X \in \mathfrak{u}(\mathcal{B}) \colon X^* = X\}$$

is simply given as multiplication by $\varphi_{\mathcal{B}}$.

For $v \in E$ let P_v be the associated rank one operator, which is defined by $P_v w = (w \mid v)v$. Choose an orthonormal basis $\{v_1, \dots, v_m\}$ for E and consider the map

$$\tau\colon \sum_{j=1}^{m} \mathbb{R} v_j \to \mathfrak{u}(\mathcal{B}), \quad \sum_{j=1}^{m} x_j v_j \mapsto \frac{1}{2}\sum_{j=1}^{m} x_j(\varphi_{\mathcal{B}} P_{v_j} + P_{v_j}\varphi_{\mathcal{B}}).$$

Then $\mathfrak{t} := \tau(E)$ is a Cartan algebra of $\mathfrak{k}(\mathcal{B})$ and $\mathfrak{u}(\mathcal{B})$. Note that τ is injective for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$ but not for $\mathbb{K} = \mathbb{R}$.

Using the embedding $\mathrm{Mat}(n \times n, \mathbb{H}) \to \mathrm{Mat}(2n \times 2n, \mathbb{C})$ given by

$$A + \mathcal{J}B \mapsto \begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix}$$

for $A, B \in \mathrm{Mat}(n \times n, \mathbb{C}) \subseteq \mathrm{Mat}(n \times n, \mathbb{H})$ one has the following matrix representations for our standard cases:

(1)

$$\begin{aligned}
\mathfrak{sp}(n, \mathbb{R}) &= \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} : A, B, C \in \mathrm{Mat}(n \times n, \mathbb{R}), B^t = B, C^t = C \right\} \\
\mathfrak{k} &= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathfrak{sp}(n, \mathbb{R})\colon A^t = -A, B^t = B \right\} \\
\mathfrak{t} &= \left\{ \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \in \mathfrak{sp}(n, \mathbb{R})\colon B \text{ diagonal} \right\}
\end{aligned}$$

(2) Here we have complex block matrices according to the partion (p, q) of $p + q$.

$$\begin{aligned}
\mathfrak{u}(p, q) &= \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} : A^* = -A, D^* = -D, \right\} \\
\mathfrak{k} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{u}(p, q) \right\} \\
\mathfrak{t} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{u}(p, q)\colon A, D \text{ diagonal} \right\}
\end{aligned}$$

(3)

$$\begin{aligned}
\mathfrak{so}^*(2n) &= \left\{ \begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix} : A, B \in \mathrm{Mat}(n \times n, \mathbb{C}), A^t = -A, B^* = B \right\} \\
\mathfrak{k} &= \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathfrak{so}^*(2n)\colon A = \overline{A} = -A^\top, B = \overline{B} = B^\top \right\} \\
\mathfrak{t} &= \left\{ \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix} \in \mathfrak{k}\colon B \text{ diagonal} \right\}
\end{aligned}$$

Choose a maximal $\mathcal{B}$-isotropic subspace $F \subseteq E$ and an orthonormal basis $\{v_1, \dots, v_r\}$ of F. The number r is the real rank of $\mathfrak{u}(\mathcal{B})$. We have

$$F \oplus \varphi_{\mathcal{B}} F = (\mathbb{K}v_1 \oplus \mathbb{K}\varphi_{\mathcal{B}} v_1) \oplus \ldots \oplus (\mathbb{K}v_r \oplus \mathbb{K}\varphi_{\mathcal{B}} v_r). \tag{VII.11}$$

The $(\mathbb{K}v_j \oplus \mathbb{K}\varphi_{\mathcal{B}}v_j)$ are pairwise orthogonal w.r.t. $(\cdot \mid \cdot)$ and $\mathcal{B}$. The matrix of $\mathcal{B}$ on $(\mathbb{K}v_j \oplus \mathbb{K}\varphi_{\mathcal{B}}v_j)$ w.r.t. the basis $\{v_j, -\varphi_{\mathcal{B}}v_j\}$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. All $\mathcal{B}$-isotropic vectors are contained in $F \oplus \varphi_{\mathcal{B}}F$ since

$$E_{\mathrm{red}} := (F \oplus \varphi_{\mathcal{B}}F)^{\perp} = (F \oplus \varphi_{\mathcal{B}}F)^{\perp_{\mathcal{B}}}. \tag{VII.12}$$

We have $E_{\mathrm{red}} = \{0\}$ iff $\mathfrak{u}(E)$ is of tube type.

In the standard cases we have

(1) $\mathfrak{sp}(n,\mathbb{R})$, $r = n$, $E_{\mathrm{red}} = \{0\}$.

(2) $\mathfrak{u}(p,q)$, $p \geq q$, $r = q$, $E_{\mathrm{red}} = \mathbb{C}^{p-q}$.

(3) $\mathfrak{so}^*(2n)$, $r = \left[\frac{n}{2}\right]$, $E_{\mathrm{red}} = \begin{cases} \{0\} & \text{for } n \text{ even} \\ \mathbb{H} & \text{for } n \text{ odd} \end{cases}$.

For the skewhermitean planes $\mathbb{K}^2 := (\mathbb{K}v_j \oplus \mathbb{K}\varphi_{\mathcal{B}}v_j)$, the real rank of $\mathfrak{u}(\mathcal{B}|_{\mathbb{K}^2})$ is 1 and the corresponding (H_1)-homomorphism $\kappa_j\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{u}(\mathcal{B}|_{\mathbb{K}^2})$ is the natural inclusion. We have in particular

$$H_{\kappa_j} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad X_{\kappa_j} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Similarly (VII.11) gives an inclusion $\mathfrak{u}(\mathcal{B}|_{\mathbb{K}^2})^r \to \mathfrak{u}(\mathcal{B})$ which provides in particular an inclusion of $\mathfrak{sl}(2,\mathbb{R})^r$. This makes it rather transparent how the different $\mathfrak{sl}(2,\mathbb{R})$-subalgebras $\mathfrak{s}^k$ look like, namely exactly like they are embedded into $\mathfrak{sl}(2,\mathbb{R})^r$. Using Proposition II.7, we see that all the X_{κ_j} are contained in $-W_{\min}(\frac{1}{2}\varphi_{\mathcal{B}})$ and in fact generate it as an invariant closed convex cone. We claim that

$$W_{\max,\mathcal{B}} = W_{\max}\left(\tfrac{1}{2}\varphi_{\mathcal{B}}\right). \tag{VII.13}$$

The inclusion $\subseteq$ follows from $\varphi_{\mathcal{B}} \in W_{\max,\mathcal{B}}$. For the converse let $Y \in W_{\max}(\frac{1}{2}\varphi_{\mathcal{B}})$ and $v \in E$ be non-zero $\mathcal{B}$-isotropic. Then there exists a $g \in \mathrm{U}(\mathcal{B})$ with $g.v = v_1$ and we can extend $\{g.v, -\varphi_{\mathcal{B}}g.v\}$ to a orthogonal basis $\{w_1, w_2, \ldots, w_m\}$ for E. Then $X_{\kappa_1} \in \mathrm{End}_{\mathbb{K}}(E)$ satisfies $X_{\kappa_1}.w_2 = w_1$, $X_{\kappa_1}.w_l = 0$ for $l \neq 2$. Thus $\mathrm{Ad}(g^{-1}).X_{\kappa_1} = g^{-1}X_{\kappa_1}g \in -W_{\min}(\frac{1}{2}\varphi_{\mathcal{B}})$. But now we have

$$\begin{aligned}
\mathcal{B}(Y.v, v) &= \mathcal{B}\big(gYg^{-1}.(g.v), g.v\big) \\
&= \big(\varphi_{\mathcal{B}}gYg^{-1}.(g.v) \mid g.v\big) \\
&= -\big(gYg^{-1}.(g.v) \mid \varphi_{\mathcal{B}}g.v\big) \\
&= \big(gYg^{-1}.w_1 \mid w_2\big) \\
&= \big(gYg^{-1}X_{\kappa_1}.w_2 \mid w_2\big) \\
&= \mathrm{tr}(gYg^{-1}X_{\kappa_1}) \leq 0
\end{aligned}$$

which implies the claim.

Now we turn the structure of the Jacobson-Morosow parabolics. The (H_1)-embedding described above shows that

$$\mathfrak{z}_{\mathfrak{u}(\mathcal{B})}(H^k) = \mathfrak{z}_{\mathfrak{u}(\mathcal{B}|_{\mathbb{K}^{2k}})}(H^k) \oplus \mathfrak{u}(\mathcal{B}|_{\mathbb{K}^{2(r-k)} \oplus E_{\mathrm{red}}}). \tag{VII.14}$$

Note that H^k belongs to the $\mathfrak{sl}_2$-triple of maximal rank in $\mathfrak{u}(\mathcal{B}|_{\mathbb{K}^{2k}})$.

In view of the simple structure of H^k, as an endomorphism of $\mathbb{K}^{2k}$, it is clear that its commutant consists of those linear mappings leaving the eigenspaces of H^k invariant. Writing accordingly $\mathbb{K}^{2k} = \mathbb{K}^k \oplus \mathbb{K}^k$, we see that

$$\mathfrak{z}_{\mathfrak{u}(\mathcal{B}|_{\mathbb{K}^{2k}})}(H^k) = \left\{ \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & -A^* \end{pmatrix} \right\}$$

so that we find

$$\mathfrak{z}_{\mathfrak{u}(\mathcal{B}|_{\mathbb{K}^{2k}})}(H^k) \cong \mathfrak{gl}(k, \mathbb{K}).$$

Putting all this together we get

$$\mathfrak{z}_{\mathfrak{u}(\mathcal{B})}(H^k) \cong \mathfrak{u}(\mathcal{B}|_{E_k}) \oplus \mathfrak{gl}(k, \mathbb{K}) \cong \mathfrak{g}_{1,k} \oplus \mathfrak{g}_{2,k}, \tag{VII.15}$$

where $E_k = \mathbb{K}^{2(r-k)} \oplus E_{\mathrm{red}}$.

The corresponding parabolic $\mathfrak{q}_k$ is given by

$$\mathfrak{q}_k = \{X \in \mathfrak{u}(\mathcal{B}) \colon X.E(H^k; 1) \subseteq E(H^k; 1)\}, \tag{VII.16}$$

where one has to read $X.v$ as the matrix product $v \cdot X$ since we are dealing with a skew-field. To prove (VII.16) one checks that $\mathfrak{q}_k$ as above contains the minimal parabolic, hence is parabolic, and then that it has the correct Levi algebra.

The corresponding flag manifold $M_k := \mathrm{SU}(E)/Q_k$ is the space of all k-dimensional isotropic subspaces of E. For $k = 1$ we find in particular the isotropic part of the projective space and for $k = r$ the space of all Lagrangian subspaces. The minimal non-zero nilpotent orbit of convex type $\mathcal{O}_1$ is the image of the map

$$\{v \in V \setminus \{0\} \colon \mathcal{B}(v, v) = 0\} \to \mathfrak{u}(\mathcal{B}), \quad v \mapsto X_v,$$

where X_v is defined by

$$\mathcal{B}(Yv, v) = \mathrm{tr}(YX_v), \quad \forall Y \in \mathfrak{u}(\mathcal{B}) \tag{VII.17}$$

as can be seen from the considerations leading to (VII.13).

Note that the above class of examples covers (ignoring the trivial central factor in $\mathfrak{u}(p, q)$) all simple hermitean Lie algebras up to $\mathfrak{so}(n, 2)$ and the two exceptional ones. We conclude with some explicit information on those cases (cf. [29, pp.115–119]).

Example VII.1. For $\mathfrak{g} = \mathfrak{so}(2, n)$ we have $r = 2$.

For $k = 1$ we have $\mathfrak{z}_{\mathfrak{g}}(H^k) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}H^k \oplus \mathfrak{so}(n-2)$, $\mathcal{V}_1$ is a real module of dimension $2(n-2)$. More precisely, it is the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^{n-2}$, where the factors are endowed with the natural representations.

For $k = 2$ we have $\mathfrak{z}_{\mathfrak{g}}(H^k) \cong \mathbb{R}H^k \oplus \mathfrak{so}(1, n-1)$, $\mathcal{V}_2 = \{0\}$, and $\mathcal{U}_2 \cong \mathbb{R}^n$ is n-dimensional Minkowski space. Here the parabolic corresponds to the identification of $\mathfrak{g}$ with the Lie algebra $\mathfrak{g}(\mathcal{U}_2)$, where we consider $\mathcal{U}_2$ as a Jordan algebra. ∎

Example VII.2. For the exceptional hermitean algebra $\mathfrak{g} = \mathfrak{e}_{(6,-14)}$ one has $r = 2$.

For $k = 1$ we have $\mathfrak{z}_{\mathfrak{g}}(H^k) \cong \mathbb{R}H^k \oplus \mathfrak{su}(5,1)$, and $\mathcal{V}_1$ is a real module of dimension 20, hence a real form of $\bigwedge^3(\mathbb{C}^6)$.

For $k = 2$ we have $\mathfrak{z}_{\mathfrak{g}}(H^k) \cong \mathbb{R} \oplus \mathbb{R}H^k \oplus \mathfrak{so}(7,1)$, $\dim \mathcal{U}_2 = 8$ and $\dim \mathcal{V}_2 = 16$. Here $\mathcal{V}_2 = V_1 \otimes_{\mathbb{C}} V_2$, where $\mathbb{R}$ acts on $V_1 = \mathbb{C}$ by multiplication by i and $\mathfrak{so}(7,1)$ acts on $V_2 \cong \mathbb{C}^8$ according to the spin representation. ∎

Example VII.3. For the exceptional hermitean algebra $\mathfrak{g} = \mathfrak{e}_{(7,-25)}$ one has $r = 3$.

For $k = 1$ we have $\mathfrak{z}_{\mathfrak{g}}(H^k) \cong \mathbb{R}H^k \oplus \mathfrak{so}(10,2)$, and that $\mathcal{V}_1$ is a real module of dimension 32, hence the spin representation.

For $k = 2$ we have $\mathfrak{z}_{\mathfrak{g}}(H^k) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}H^k \oplus \mathfrak{so}(9,1)$, $\dim \mathcal{U}_2 = 10$ and $\dim \mathcal{V}_2 = 32$. Here $\mathcal{V}_2 = \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^{16}$, where $\mathfrak{so}(9,1)$ acts on $\mathbb{R}^{16}$ according to the spin representation.

For $k = 3$ we have $\mathfrak{z}_{\mathfrak{g}}(H^k) \cong \mathbb{R}H^k \oplus \mathfrak{e}_{(6,-26)}$, $\dim \mathcal{U}_3 = 27$ and $\mathcal{V}_3 = \{0\}$. This is the Jordan algebra situation, where $\mathcal{U}_3 = \mathrm{Herm}(3,\mathbb{O})$. ∎

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