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Triangular Lie bialgebras and matched pairs for Lie algebras of real vector fields on $\mathbf{S}^1$

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Abstract: Real Lie bialgebras and matched pairs for a Lie algebra of formal vector fields and for Lie algebras of smooth vector fields with fixed zeros on the circle are constructed.

1. Introduction

The construction of a quantum group generalization of the group $\mathrm{Diff}_+(S^1)$ (the group of orientation preserving C^∞-diffeomorphisms of S^1) is an open problem. On the classical level we have the corresponding problem of the construction of Poisson Lie groups or of matched pairs for the diffeomorphism group (cf. [D], [Ta]).

There is some progress in this direction on the Lie algebra level. In [B-M], [M] and [T] various matched pair structures for certain Lie algebras of complex vector fields on S^1 are obtained. The choice of different function spaces for the Lie algebra of vector fields sheds a different light on the problem. In [M] the author considers polynomial vector fields and recursive sequences for the dual Lie algebra. In [B-M] certain spaces of analytic functions are considered. However, these structures are not compatible with the subalgebra of real vector fields. Therefore, they do not admit an integration to the matched pair group level because there is no complexification of $\mathrm{Diff}_+(S^1)$ (cf. [P-S]).

In this article we classify additional Lie bialgebras and matched pair structures and obtain the structures of [B-M], [M] as special cases. Every structure is parametrized by a subalgebra h and by a moment b of h (or $[h,h]$) of the Lie algebra g for which the bilinear form $h \times h \ni (x,y) \mapsto b([x,y])$ is invertible. The inverse plays the role of the classical r-matrix. In the case $h = g$ the matched pair structures are self-dual. For the Lie algebra of vector fields we first choose a space of semi-infinite formal power series. Such completions of the Witt algebra occur in conformal field theory (cf. [W1]). We give an explicit description of the structures and single out the real structures. Next we choose Lie algebras g_c of

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smooth functions with fixed zeros. These function spaces allow the construction of real matched pairs. These Lie algebras correspond to Lie groups G_c with fixed points of a certain order (see the Remark after Definition 4.2). The structures correspond to subalgebras $h_c^{(n)}$ and certain moments. We hope that some matched pairs allow an integration to matched pairs for the Lie groups G_c.

The paper is organized as follows: In Section 2 we generalize the foundations of the finite-dimensional situation to the infinite-dimensional case. In Section 3 we classify Lie bialgebra and matched pair structures for the case of semi-infinite formal vector fields on the circle. In Section 4 we obtain matched pair structures for smooth vector fields with fixed zeros on the circle.

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2. Preliminaries

In this section we generalize some well-known results about Lie bialgebras and matched pairs (cf. [B-M], [D]) to the infinite-dimensional case.

2.1. Construction of matched pairs of Lie algebras. The goal of this subsection is to introduce matched pair structures for dual pairs of infinite-dimensional vector spaces without any topology. The constructions of Propositions 2.3 and 2.4 provide the foundation of Section 4.

Definition 2.1. Let g be a (possibly infinite-dimensional) Lie algebra. We say that g' is a *dual* of g if and only if

(i) there is a bilinear form $\langle -,- \rangle : g \times g' \to C$ such that $g, g', \langle -,- \rangle$ forms a reflexive pair of vector spaces (i.e., g and g' separate the points of g' and g respectively).

(ii) the coadjoint action K of g on g' is well-defined (i.e., $\forall x \in g, y \in g', \quad \exists\, z =: ad_x^T y = -K_x y \in g'$ given by $\langle ad_x u, y \rangle = \langle u, z \rangle \ \ \forall u \in g'$).

Definition 2.2. A *matched pair* $(g_1, g_2, \alpha, \beta)$ is a pair of Lie algebras g_1, g_2 with the Lie algebra representations $\alpha : g_1 \times g_2 \to g_2$ and $\beta : g_2 \times g_1 \to g_1$ such that, for all $x, y \in g_1$ and $u, v \in g_2$,

$$\alpha_x([u,v]) = [\alpha_x(u), v] + [u, \alpha_x(v)] + \alpha_{\beta_v(x)}(u) - \alpha_{\beta_u(x)}(v),$$

and

$$\beta_u([x,y]) = [\beta_u(x), y] + [x, \beta_u(y)] + \beta_{\alpha_y(u)}(x) - \beta_{\alpha_x(u)}(y).$$

We say that $(g_1, g_2, \alpha, \beta)$ is a *coadjoint* matched pair if there is a bilinear form $\langle -,- \rangle$, such that $(g_1; g_2; \langle -,- \rangle)$ is a dual pair of vector spaces and α, β are the corresponding coadjoint actions.

Further we say that the linear map $\omega : g \to g'$ is a 2-*cocycle*, if and only if the bilinear form $\omega : g \times g \to C$ specified by $(x,y) \mapsto \langle \omega(x), y \rangle$ is antisymmetric and a 2-cocycle, i.e., $\omega([p,q],r) + \omega([q,r],p) + \omega([r,p],q) = 0 \ \ \forall p,q,r \in g$.

Matched pairs can be constructed by invertible 2-cocycles (cf., [W2, page 49]).

Proposition 2.3. *Let g be a Lie algebra, g' a dual to g, and ω a 2-cocycle for g. Further suppose that ω has an inverse map $r : g' \to g$. Set*

$$\begin{array}{lcll} [x,y]_{g'} & := & \omega([r(x),r(y)]) & x,y \in g', \\ \alpha_e & := & -ad_e^T & e \in g, \\ \beta_x & := & r\ \alpha_{r(x)}\omega & x \in g'. \end{array}$$

Then (g, g', α, β) forms a coadjoint matched pair of Lie algebras.

We remark that $r : g' \to g$ is a Lie algebra isomorphism and g is self dual with respect to the bilinear form $\langle \omega(-), - \rangle$. Therefore we can identify g and g' and reformulate Proposition 2.3 and obtain a self-dual representation.

Proposition 2.3. a. *Under the the assumptions of Proposition 2.3, if $\alpha' := r \circ \alpha \circ (id \times \omega)$ then (g, g, α', α') with $\alpha' := r\alpha\omega$ and (g, g', α, β) are isomorphic matched pairs of Lie algebras.*

The next proposition generalizes Proposition 2.3. We show that every invertible r-matrix of a subalgebra allows the construction of a matched pair.

Let g be a Lie algebra with dual g', and let h be a subalgebra with dual $h' \subset g'$. We introduce the following notation:
$h^\perp := \{x \in g' | \langle h, x \rangle = 0\}$ while $h'^\perp := \{e \in g | \langle e, h' \rangle = 0\}$.

Proposition 2.4. *Let g be a Lie algebra with dual g', let h be a subalgebra with dual $h' \subset g'$, and let $g' = h' \dot{+} h^\perp$ and $g = h \dot{+} h'^\perp$ be direct sum decompositions of g' and g respectively, as vector spaces. Further let $\omega : h \to h'$ be a 2-cocycle with inverse $r : h' \to h$. Set*

$$\begin{array}{lcl} [x,y]_{g'} & := & \left\{ \begin{array}{cl} \omega([r(x),r(y)]) & x,y \in h' \\ -ad_{r(x)}^T y & x \in h', y \in h^\perp \\ 0 & x,y \in h^\perp\ , \end{array} \right. \\ \alpha_e & := & -ad_e^T \qquad\qquad e \in g\ , \\ \beta_x(e) & := & \left\{ \begin{array}{cl} rP_{h'}\alpha_{r(x)}\omega(e) & x \in h', e \in h \\ P_{h'^\perp}[r(x),e] & x \in h', e \in h'^\perp \\ r(\alpha_e x) & x \in h^\perp, e \in g. \end{array} \right. \end{array}$$

Then (g, g', α, β) is a matched pair of Lie algebras. Here $P_{h'^\perp}$ denotes the projection to $h'^\perp$, and $P_{h'}$ denotes the projection to h'.

Example 2.5. Let $g = W = lin\{e_i | i \in \mathbb{Z}\}$ with $[e_i, e_j] = (j - i)e_{i+j}$ be the 2-sided Witt algebra with dual $g' = W' = lin\{f_i | i \in \mathbb{Z}\}$ where $\langle e_i, f_j \rangle = \delta_{i,j}$. Further let $h_n = lin\{e_0, e_n\}$ $(n \in \mathbb{Z}, n \neq 0)$ be Lie subalgebras with $h_n{}' = lin\{f_0, f_n\}$. Up to a nonzero scalar factor there is a unique 2-cocycle $\omega_n = f_n \wedge f_0$ with inverse $r_n = e_0 \wedge e_n$. We obtain

$$\begin{array}{lcll} [f_i, f_0] & = & (2n-i)f_{i-n} & if\ \ i \neq 0 \\ [f_i, f_n] & = & if_i & if\ \ i \neq 0, n \\ [f_i, f_j] & = & 0 & if\ \ i, j \neq 0, n\ , \end{array}$$

$$\alpha_{e_i} f_j \;=\; (2i-j) f_{j-i} \,,$$

and

$$\beta_{f_i} e_j = \begin{cases} (n-j)e_{j+n} & if \quad i=0; j\neq -n \\ je_j & \quad\;\; i=n; j\neq 0,n \\ -ne_0 & \quad\;\; i=n; j=0 \\ (i-2n)e_0 & \quad\;\; j=i-n; i\neq 0,n \\ -ie_n & \quad\;\; i=j; i\neq 0,n \\ 0 & \quad\;\; i=0; j=-n \\ & or \quad i=j=n \\ & or \quad i\neq 0,n;\; j\neq i, i-n. \end{cases}$$

These structures were first obtained by Michaelis [M], then by Taft [T].

2.2. Lie bialgebras, Classical r-matrices and CYBE. In this subsection we consider a class of infinite-dimensional graded Lie bialgebras (cf., Definition 2.9, below) for which we have at our disposal the main results of the finite-dimensional situation. The results of this subsection are applied in Section 3.

By a *locally convex Lie algebra* we mean a locally convex vector space with a continuous Lie bracket. By $\overline{\otimes}_\pi$ we denote the completed projective tensor product of locally convex vector spaces (cf. [K2, Chapter 41]).

Let $\tau, ad_x^{(2)}$ (for $x\in g$), and ξ, respectively denote the unique continuous extensions to $g\overline{\otimes}_\pi g$ and, respectively, to $g\overline{\otimes}_\pi g\overline{\otimes}_\pi g$ (cf. [K2, Chapter 41]) of the continuous linear maps $\tau, ad_x^{(2)} : g\otimes g\to g\otimes g$ and $\xi : g\otimes g\otimes g\to g\otimes g\otimes g$ given, respectively, by $\tau(x\otimes y) = y\otimes x$, $ad_x^{(2)}(y\otimes z) = [x,y]\otimes z + y\otimes [x,z]$, $\xi(x\otimes y\otimes z) = y\otimes z\otimes x$.

Definition 2.6. A *locally convex Lie bialgebra* $(g;[,];\triangle)$ is a locally convex Lie algebra $(g;[,])$ with a *cobracket* $\triangle$, i.e. a continuous linear map $\triangle : g\to g\overline{\otimes}_\pi g$, which satisfies the *anti-commutativity* condition

$$\tau\circ\triangle x = -\triangle f,$$

the *Jacobi identity*

$$(1+\xi+\xi^2)\circ(\triangle\overline{\otimes}_\pi id)\triangle = 0,$$

and the *compatibility* condition

$$\triangle([x,y]) = ad_x^{(2)}\triangle(y) + ad_y^{(2)}\triangle(x).$$

Remark 2.7. The operator $\triangle\overline{\otimes}_\pi id$ is the well-defined continuous extension of $\triangle\otimes id$ from $g\otimes(g\overline{\otimes}_\pi g)$ to $g\overline{\otimes}_\pi(g\overline{\otimes}_\pi g) = g\overline{\otimes}_\pi g\overline{\otimes}_\pi g$.

Let $\{V_i \mid i\in\mathbb{Z}\}$ be a set of finite-dimensional vector spaces. Denote by $\sum^+ V_i$ the vector space of semi-infinite direct formal sums $\{\prod_{i\le a} e_i | e_i\in V_i, a\in\mathbb{Z}\}$. We endow $\sum^+ V_i$ with the topology which arises from the isomorphism $\sum^+ V_n \cong \prod_{n<0} V_n \oplus \sum_{n\ge 0} V_n$. We define the locally convex space $\sum^- V_i$ analogously. If $V = \sum^+ V_i$, and if, $\forall i\in\mathbb{Z}$, V_i^* is the dual space of the finite-dimensional vector space V_i, then we obtain $V^* = \sum^- V_i^*$ as the continuous dual of V; and (V, V^*) forms a reflexive pair of locally convex vector spaces (cf. [K1, Chapter 30]).

Lemma 2.8. (i) $V\overline{\otimes}_\pi V \cong L(V^*, V)$,
(ii) $\sum^+ V_i \overline{\otimes}_\pi \sum^+ V_i = \sum_{i<i_0(j), j<j_0(i)} V_i \otimes V_j$,
(iii) $\sum^+ V_i \overline{\otimes}_\pi \sum^+ V_i \overline{\otimes}_\pi \sum^+ V_i = \sum_{i<i_0(j,k); j<j_0(i,k); k<k_0(i,j)} V_i \otimes V_j \otimes V_k$.

Proof. Cf. [K2, Chapter 41]. ■

Definition 2.9. A locally convex Lie algebra g is an *upper semibounded graded Lie algebra* (G^+*-Lie algebra*) if $g = \sum^+ g_i$ where the g_i are finite-dimensional subspaces of g and $[g_i, g_j] = g_{i+j}$.

In analogy to the finite-dimensional case we make the following definition.

Definition 2.10. Let g be a G^+-Lie algebra. We say that $r = \sum_{i,j,n} a_i^n \otimes b_j^n \in g\overline{\otimes}_\pi g$, $(a_i^n \otimes b_j^n \in g_i \otimes g_j)$ is a *classical r-matrix* if it satisfies the *Classical Yang Baxter Equation* (CYBE)

$$\sum_{i,j,k,l,n} [a_i^n, a_k^n] \otimes b_j^n \otimes b_k^n + a_i^n \otimes [b_j^n, a_k^n] \otimes b_l^n + a_i^n \otimes a_k^n \otimes [b_k^n, b_l^n] = 0. \tag{1}$$

Remark 2.11. Because of Lemma 2.4(iii) we have in the CYBE only finite summations in the graded components $g_i \otimes g_j \otimes g_k$.

Classical r-matrices can be constructed by certain 2-cocycles.

Proposition 2.12. *Let g be a G^+-Lie algebra, let h be a Lie subalgebra, and let $\omega : h \to h^*$ be an invertible 2-cocycle. Further let $i : h \to g$ be the imbedding operator. Then the extension $r = i\omega^{-1}i^T : g^* \to g$ of $\omega^{-1} : h^* \to h$ is a classical r-matrix for g.*

Lie bialgebras can be given by classical r-matrices.

Proposition 2.13. *Let g be a G^+-Lie algebra, and let $r \in g\overline{\otimes}_\pi g$ be a classical r-matrix. Then $\triangle : g \to g\overline{\otimes}_\pi g$ defined by $\triangle(x) := ad_x^{(2)} r$ is a well-defined continuous operator and $(g; [,]; \triangle)$ is a locally convex Lie bialgebra. (We call g a triangular G^+-Lie bialgebra (cf.* [D, page 804]*).)*

Finally we consider the connection between Lie bialgebras and coadjoint matched pairs. Both concepts are equivalent in the finite-dimensional case whereas in the infinite-dimensional situation matched pairs are more general.

Lemma 2.14. (i) *Let $(g; [,]; \triangle)$ be a G^+-Lie bialgebra. Let $[,]_{g^*} := \triangle^T$, let α be the coadjoint action of g on g^*, and let β be the coadjoint action of g^* on g. Then (g, g^*, α, β) is a coadjoint matched pair of Lie algebras.*

(ii) *Let (g, g^*, α, β) be a finite-dimensional coadjoint matched pair. Then $(g; [,]_g; [,]_{g^*}^T)$ is a Lie bialgebra.*

2.3. Real structures. Let g be a complex Lie algebra with dual g'. Let $I : g \to g$ be an antilinear involution given by $x \mapsto \overline{x}$ and let $g_{\mathbb{R}} := \{x \in g | x = \overline{x}\}$ be the real subalgebra. We say that I defines a *real structure* on g. We remark that $I^T : g' \to g'$ defines a real structure on g'. We say that a linear map $a : g \to g'$ is *real* if $a \circ I^T = I \circ a$.

Lemma 2.15. (i) *Let g be a G^+-Lie algebra (over $\mathbb{C}$) and let r be an r-matrix. Then $(g_{\mathbb{R}}; [,]|g_{\mathbb{R}}; \triangle|g_{\mathbb{R}})$ is a G^+-Lie algebra over $\mathbb{R}$ if and only if r is real.*

(ii) *Under the assumptions of Proposition 2.2 let I be an real structure of g. Then $(g_{\mathbb{R}}, {g'}_{\mathbb{R}}, \alpha|g_{\mathbb{R}}, \beta|_{{g'}_{\mathbb{R}}})$ is a real matched pair if and only if $r : h \to h'$ is real.*

3. The case of formal power series

3.1. Construction of r-matrices. In this section we consider completions of the Witt algebra W and of its finite dual W' defined in Example 2.5.

Let

$$W^+ := \sum\nolimits^+ \mathbb{C}e_i$$

$$W^- := \sum\nolimits^- \mathbb{C}f_i$$

be the spaces of semi-infinite formal linear combinations of the elements e_i and f_i, respectively. W^+ is a Lie algebra with respect to the canonical extension of the Lie bracket of W, namely,

$$\left[\sum_{i\le a} a_i e_i, \sum_{i\le b} b_i e_i\right] = \sum_{i\le a, j\le b} (j-i) a_i b_j e_{i+j}.$$

W^- is the continuous dual of W^+ (i.e. $W^{+*} = W^-$) and if we set

$$\left\langle \sum_{i=-\infty}^{a} a_i e_i, \sum_{i=b}^{\infty} b_i f_i \right\rangle := \sum_{i=-\infty}^{\infty} a_i b_i = \sum_{i=b}^{a} a_i b_i$$

then $(W^+, W^-, \langle -, - \rangle)$ forms a dual pair of vector spaces.

Now we apply the Propositions 2.12, 2.13 for the construction of Lie bialgebras. We choose $g = W^+$, $g^* = W^-$, $h = W_n^+ = \sum^+ \mathbb{C}e_{nk}$ $(n \in \mathbb{N}^+)$ and $h^* = W_n^- = \sum^- \mathbb{C}f_{nk}$. For $h^* \ni b = \sum_{i=a}^{\infty} b_i f_{in}$, $(f_{na} \neq 0)$ we consider the 2-cocycle $\omega_b^n : h \times h \to C$ defined by

$$\omega_b^n(x,y) := \langle b, [x,y] \rangle = \sum_{i,j} n(j-i) x_i y_j b_{i+j}$$

where $x = \sum_i x_i e_{in}$ and $y = \sum_i y_i e_{in}$. According to the identity $\omega_n^b(x,y) = \langle \omega_n^b(x), y \rangle$ we can view ω_b^n as a well-defined linear map $\omega_b : h \to h^*$ given by

$$\omega_b^n(e_{mn}) = \sum_k n(k-m) b_{k+m} f_{kn}.$$

Let a be an odd integer and consider the infinite system of equations in the variables $(c_i)_{i \ge \frac{-a-1}{2}}$

$$\sum_{k+l+m=s} c_k c_l b_m = \delta_{s,-1}, \quad s = -1, 0, 1, 2, ... \tag{1}$$

i.e.,

$$\begin{array}{rcl} c^2_{\frac{-a-1}{2}} b_a & = & 1 \\ 2c_{\frac{-a-1}{2}} c_{\frac{-a+1}{2}} b_a + c^2_{\frac{-a-1}{2}} b_{a+1} & = & 0 \\ \dots & . & \end{array}$$

Lemma 3.1. (i) *The equation system* (1) *has the two solutions* $\pm(c_i)_{i \geq \frac{-a-1}{2}}$.

(ii) *Conversely every semi-infinite sequence* $(c_i)_{i \geq \frac{-a-1}{2}}$ *is the solution of an equation system* (1) *for certain* (b_i).

(iii) *Every solution of the equation system* (1) *is also a solution of the equation system*

$$\sum_{k+l=s} (2l + k + 1) b_k c_l = 0 \quad s \in \mathbb{Z} \ .$$

Proof. (i) The first equation has the two solutions $c_{\frac{-a-1}{2}} = \pm\sqrt{\frac{1}{b_a}}$. Both solutions admit the unique successive solution of the remaining equations.
(ii) Fix (c_i) and consider (1) as an equation system for the (b_i). This triangular equation system has a unique solution.
(iii) (1) corresponds to the formal power series identity $b(z)c(z)^2 = 1$ where $b(z) = \sum_k b_k z^{k+1}$ and $c(z) = \sum_k c_k z^k$. If we differentiate the identity $b(z)c(z)^2 = 1$ and divide the result by $c(z)$ we obtain the identity $b'(z)c(z) + 2b(z)c'(z) = 0$ which in turn yields the identities.

$$\sum_{k+l=s} (2l + k + 1) b_k c_l = 0 \quad s \in \mathbb{Z} \ .$$

∎

Consider an element $W' \ni b := b_a z^a + b_{a-1} z^{a-1} + \dots \in W'$ with $b_a \neq 0$. We say that b is *even* (respectively, *odd*) if a is *even* (respectively, *odd*).

Proposition 3.2. (i) *If* b *is odd then* ω_b^n *is invertible and as the inverse the map* $r_b^n : W^+ \to W^-$ *defined by*

$$r_b^n(f_{mn}) = \frac{1}{n} \sum_{k,l} \frac{c_l c_{-k-l-m-1}}{2m + 2l + 1} e_{kn}.$$

(ii) *If* b *is even then* ω_b *is not invertible and we have dim Ker* $\omega_b^n = 1$.

Proof. (i) It is easy to verify that r_b^n maps W^+ into W^-. Further we have

$$\omega_b^n r_b^n(f_{mn}) = \omega_b(\frac{1}{n}\sum_{k,l}\frac{c_l c_{-k-l-m-1}}{2m+2l+1}e_{kn})$$

$$= n\sum_j (j-k)b_{j+k}\frac{1}{n}\sum_{k,l}\frac{c_l c_{-k-l-m-1}}{2m+2l+1}f_{jn}$$

$$= \sum_{j,k,l} c_l c_{-k-l-m-1} b_{k+j}\frac{j-k}{2m+2l+1}$$

Because of Lemma 3.1(iii) we have

$$\sum_{\substack{k,\,l \\ j+k+l=s}} (-2l-k-2m-1+k+j)b_{k+j}c_{-k-l-m-1} = 0 \quad \text{for} \quad s\in\mathbb{Z}\ .$$

Using this identity and (1) we obtain

$$\omega_b^n r_b^n(f_{mn}) = \sum_{j,k,l} c_l c_{-k-l-m-1} b_{k+j}\frac{j-k}{2m+2l+1} = \delta_{j-m-1,-1} = \delta_{j,m}.$$

The proof of $r_b^n\omega_b^n = id$ is analogous.
(ii) For simplicity consider the case $n=1$. ω_b has the following matrix representation with respect to the bases e_j, f_j.

$$\left(\begin{array}{cccc|c|cccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & -6b_0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & -4b_0 & -5b_1 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & -2b_0 & -3b_1 & -4b_2 & \cdots \\
\hline
\cdots & 0 & 0 & 0 & 0 & -b_1 & -2b_2 & -3b_3 & \cdots \\
\hline
\cdots & 0 & 0 & 2b_0 & b_1 & 0 & -b_3 & -2b_4 & \cdots \\
\cdots & 0 & 4b_0 & 3b_1 & 2b_2 & b_3 & 0 & -b_5 & \cdots \\
\cdots & 6b_0 & 5b_1 & 4b_2 & 3b_3 & 2b_4 & b_5 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right)$$

$$=: \begin{pmatrix} O & o & D_1 \\ o^T & 0 & -d \\ D_2 & d^T & B \end{pmatrix}$$

Let $g \ni e = \sum_{i\le k} a_i e_i \in Ker\ \omega_b$. It follows that $a_1 = ... = a_k = 0$, $a_0 = \gamma \in\mathbb{C}$ and that, for $i\le 0$ the a_i are determined by the equation system $D_2 a^T = -\gamma d^T$ where $a = (..., a_{-2}, a_{-1})$. That is, $Ker\ \omega_b = 1$. ■

By Lemma 3.1 all operators ω_b^n are parametrized by semi-infinite sequences $(c_i)_{i\le l}$ (up to multiplication with -1). By Lemma 2.8(i) we can also write

$$r_b^n = \frac{1}{2}\sum_{k,l,m}\frac{c_l c_{-k-l-m-1}}{2m+2l+1}e_{kn}\wedge e_{mn}.$$

As a consequence of Proposition 2.12 and 3.2 we can classify all r-matrices which arise from invertible 2-cocycles ω_b^n on W_n^+. That is the content of the following result.

Theorem 3.3. *Let $(c_i)_{i\geq a}$ for $a \in \mathbb{Z}$ be a semi-infinite complex sequence. Then*

$$r_b^n = \frac{1}{2}\sum_{k,l,m} \frac{c_l c_{-k-l-m-1}}{2m+2l+1} e_{kn} \wedge e_{mn}$$

is a classical r-matrix.

Now we consider two special cases.

Examples 3.4. 1. n=1: We have

$$r_b^1 = \frac{1}{2}\sum_{k,l,m} \frac{c_l c_{-k-l-m-1}}{2m+2l+1} e_k \wedge e_m.$$

These r-matrices correspond those in [B-M, page 27]. But in their paper, instead considering the spaces $W^{\pm}$ of formal power series, Beggs and Majid consider certain spaces of analytic functions.

2. $b = f_u = f_{-2a-1}$, $(c_i = \pm\delta_{i,a})$: This is the simplest nontrivial case. The summation reduces to one summation parameter. We obtain

$$r^n_{u=-2a-1} = \frac{1}{2n}\sum_m \frac{1}{2m-u} e_{(u-m)n} \wedge e_{mn}.$$

3.2. Construction of Lie bialgebras and matched pairs. Next we describe the Lie bialgebra and matched pair structures which arise from the r-matrices of Theorem 3.3.

Proposition 3.5. *Set*

$$\triangle_b^n(e_i) := ad_{e_i}^{(2)} r_b^n = \frac{1}{n}\sum_{k,l,m} \frac{c_l c_{-k-l-m-1}}{2m+2l+1}(mn-i) e_{mn+i} \wedge e_{kn}.$$

Then $(W^+, \triangle_b^n)$ is a locally convex Lie bialgebra.

Proof. The assertion follows from Propostion 2.13, Theorem 3.3 and the calculation

$$\begin{aligned}
\triangle_b^n(e_i) &= ad_{e_i} \frac{1}{n}\sum_{k,l,m} \frac{c_l c_{-k-l-m-1}}{2m+2l+1} e_{mn} \otimes e_{kn} \\
&= \frac{1}{n}\sum_{k,l,m} \frac{c_l c_{-k-l-m-1}}{2m+2l+1} [e_i, e_{mn}] \otimes e_{kn} + e_{mn} \otimes [e_i, e_{kn}] \\
&= \frac{1}{n}\sum_{k,l,m} \frac{c_l c_{-k-l-m-1}}{2m+2l+1} (mn-i) e_{mn+i} \otimes e_{kn} + (kn-i) e_{mn} \otimes e_{kn+i} \\
&= \frac{1}{n}\sum_{k,l,m} \frac{c_l c_{-k-l-m-1}}{2m+2l+1} (mn-i) e_{mn+i} \wedge e_{kn}.
\end{aligned}$$

■

Example 3.6. Let $r = r^n_{u=-2a-1}$. We obtain

$$\triangle^n_b(e_i) = \sum_m \frac{1}{2m-u} e_{mn+i} \wedge e_{(u-m)n}.$$

Proposition 3.7. *Let $b \in h^* = W^-_n = \sum^- \mathbb{C} f_{nk}$ even. Set*

$$[f_{rn}, f_{sn}]_{W^-} = \sum_j c_l c_{-r-s+j-l-1} \frac{(r-s)(2(l+j)+1)}{(2(r-j+l)+1)(2(s-j+l)+1)} f_{jn}, \quad (1)$$

$$[f_{rn}, f_q]_{W^-} = -\frac{1}{n} \sum_j c_l c_{-r-j-l-1} \frac{(2jn-q)}{2j+2l+1} f_{q-jn} \quad \text{if} \quad n \nmid q, \quad (2)$$

$$[f_p, f_q]_{W^-} = 0 \quad \text{if} \quad n \nmid p, q, \quad (3)$$

$$\alpha_{e_i} f_j = (2i-j) f_{j-i}, \quad (4)$$

$$\beta_x(e) = \begin{cases} r_c P_{h_c'} \alpha_{r_c(x)} \omega(e) & x \in h_c', e \in h_c \\ P_{h_c'^\perp}[r_c(x), e] & x \in h_c', e \in h_c'^\perp \\ r_c(\alpha_e x) & x \in h_c^\perp, e \in g_c \ . \end{cases} \quad (5)$$

Then $M_b^{(n)} = (W^+, W^-, \alpha, \beta)$ is a matched pair of Lie algebras.

Proof. We calculate the matched pair structure using Lemma 2.14(i). It suffices to examine the following formula in 3 separate cases:

$$\langle e_i, [f_p, f_q] \rangle = \langle \triangle^n_b(e_i), f_p \otimes f_q \rangle \ .$$

$$= \frac{1}{n} \sum_{k,l,m} \frac{c_l c_{-k-l-m-1}}{2m+2l+1} (mn-i)(\delta_{mn+i,p}\delta_{kn,q} - \delta_{kn,p}\delta_{mn+i,q}) f_i \quad (6)$$

Case 1: $n|p$ and $n|q$. In this case formula (1) follows upon taking $p = rn,\ q = sn$ and $i = jn$.
Case 2: $n|p$ but $n \nmid q$. In this case the first summand in (6) vanishes. Formula (2) follows upon taking $p = rn$ and $i = q - jn \neq 0$.
Case 3: $n \nmid p$ and $n \nmid q$. In this case formula (3) follows since both summands vanish in (6).

The formula for α follows from the fact that

$$\langle \alpha_{e_i} f_j, e_k \rangle = \langle -ad^T_{e_i} f_j, e_k \rangle = \langle f_j, [e_k, e_i] \rangle = (i-k)\delta_{j,i+k}.$$

■

Now let n=1 and choose the basis $g_i = \omega_b(e_i)$ in W^-. Then the matched pairs of Proposition 3.7 take on the following self-dual form.

Proposition 3.8. *Let $b \in W^-$. Then $M_b^{(1)} = (W^+, W^-, \alpha, \beta)$ is given by*

$$[e_i, e_j] = (j-i)e_{i+j} \qquad [g_i, g_j] = (j-i)g_{i+j},$$

$$\alpha_{e_i} g_j = \sum_{k,l,m} \frac{(m+i-j)(i-m)}{2m+2l+1} b_{m+i+j} c_l c_{-k-l-m-1} e_k,$$

and

$$\beta_{g_i} e_j = \sum_{k,l,m} \frac{(m+i-j)(i-m)}{2m+2l+1} b_{m+i+j} c_l c_{-k-l-m-1} g_k.$$

Example 3.9. Consider the special case where $b = f_{un} = f_{(-2a-1)n}$ (i.e., $c_i = \delta_{i,a}$). Then

$$[f_{rn}, f_{sn}] = \frac{(r-s)(2r+2s+3u)}{(2s+u)(2r+u)} f_{(r+s+u)n},$$

$$[f_{rn}, f_q] = -\frac{2rn+q+2un}{2rn+un} f_{rn+q+un} \quad if \quad n \nmid q \ ,$$

and

$$[f_p, f_q] = 0 \quad if \quad n \nmid p, q \ .$$

Now let n=1. After the basis change $g_i := (u-2i) f_{-i+u}$ (i.e., $\langle e_i, g_j \rangle = -(2j+n)\delta_{i+j+n,0}$) we get mutually non-equivalent structures parametrized by odd integers u in a self-dual form

$$[e_i, e_j] = (j-i)e_{i+j}, \qquad [g_i, g_j] = (j-i)g_{i+j},$$

$$\alpha_{e_i} g_j = \frac{(2j-u)(2i+j-u)}{2i+2j-u} g_{i+j}, \quad \text{and} \quad \beta_{g_i} e_j = \frac{(2j-u)(2i+j-u)}{2i+2j-u} e_{i+j}.$$

α, β are not compatible with the real structure described next in section 3.3.

3.3. The real case.

Consider the real structure $Ie_k := -e_{-k}$.

Remark 3.10. This involution is motivated by the function space representation $e_k = -i \exp(ikx) \frac{d}{dx}$.

Proposition 3.11. *$(g_{\mathbb{R}}, \triangle|_{\mathbb{R}})$ and $(g_{\mathbb{R}}, {g'}_{\mathbb{R}}, \alpha, \beta)$ are real if and only if $c_l = \overline{c_{-l-1}}$ $(\forall l \in \mathbb{Z}$) or $c_l = -\overline{c_{-l-1}}$ $(\forall l \in \mathbb{Z}$).*

Proof. According to Lemma 2.15 $(g_{\mathbb{R}}, \triangle_{\mathbb{R}})$ and $(g_{\mathbb{R}}, {g'}_{\mathbb{R}}, \alpha, \beta)$ are real if and only if $r_b^{(n)}$ is real.
(i) Let $c_l = \pm\overline{c_{-l-1}}$ $(\forall l \in \mathbb{Z})$. Because of $I^T f_k = -f_{-k}$ we have to show that $r_b^n(f_j) = -\overline{r_b^n(f_{-j})}$ $(\forall j \in \mathbb{Z})$. The relation is satisfied if $n \nmid j$. Now let $j = mn$. We have

$$r_b^{(n)}(f_{mn}) = \frac{1}{n}\sum_{k,l}\frac{c_l c_{-k-l-m-1}}{2m+2l+1}e_{kn} = \frac{1}{n}\sum_{k,l}\frac{\overline{c_{-l-1}c_{k+l+m}}}{2m+2l+1}e_{kn}$$

$$= \frac{1}{n}\sum_{k,l}\frac{\overline{c_l c_{-k-l-m-1}}}{-2m-2l-1}e_{-kn} = -\overline{r_b^n(f_{-mn})}.$$

(ii) Let $r(f_j) = -\overline{r(f_{-j})}$. It follows that

$$\sum_l \frac{c_l c_{-k-l-m-1} - \overline{c_{-l-1}c_{k+m+l}}}{2m+2l+1} = 0 \quad \forall m.$$

Because the last equation is true for all m we obtain

$$c_l c_{-k-l-m-1} - \overline{c_{-l-1}c_{k+m+l}} = 0 \quad \forall k+m, l. \tag{1}$$

In other words there is a constant $\gamma \in \mathbb{C}$ with

$$c_l = \gamma\overline{c_{-l-1}} \quad \forall l. \tag{2}$$

¿From (1) and (2) we find that $\gamma^2 = 1$, i.e., $c_l = \pm\overline{c_{-l-1}}$ $\forall l$. ■

Remark 3.12. (i) It follows from $c_l = \pm\overline{c_{-l-1}}$ that c is a finite power series.
(ii) Let r_b^n be real. Then the operator $\omega_b^n = r_b^{-1}$ is complex, since $b = \frac{1}{c^2}$ is an infinite power series for finite c.

Example 3.13. $n = 1$, $c_0 = c_{-1} = 1$. $W_{\mathbb{R}}^+$ has the real base $u_k = \frac{i}{2}(e_k + e_{-k}), v_k = \frac{1}{2}(e_k - e_{-k})$. We obtain
$r = \sum_m \frac{1}{2m+1}(v_m - \mathrm{i}u_m) \otimes (v_{-m-1} - \mathrm{i}u_{-m-1}) + \frac{1}{2m-1}(v_m - \mathrm{i}u_m) \otimes (v_{-m+1} - \mathrm{i}u_{-m+1})$
$+(\frac{1}{2m+1} + \frac{1}{2m-1})(v_m - \mathrm{i}u_m) \otimes (v_{-m} - iu_{-m})$.

4. The case of smooth functions

In this section we apply Proposition 2.3 and 2.4 to obtain real Lie bialgebras for Lie algebras of smooth functions with fixed zeros on the circle. In [B-M] the authors obtain a class of complex matched pairs for certain Lie algebras of analytic vector fields and indicate an obstruction for the case of real structures. These structures are parametrisized by certain moments $b \in g'$. We obtain additional structures by restriction to subalgebras g_c of g and by an enlargement of g' to ${g_c}'$ (cf. 4.1) and secondly by application of the construction of proposition 2.4 (cf. 4.2).

4.1. Matched pairs M_c. First we introduce the function class D. We identify $2\pi(respectively, 4\pi)$-periodic functions with functions which are defined on S^1 (respectively, on $S^{1(2)}$, the double cover of S^1).

Definition 4.1. Let D be the class of real C^∞-functions $c(x)$, which are defined on the double cover $S^{1(2)}$ of S^1 and which have the following properties:

(i) $c(x+2\pi) = -c(x)$.

(ii) $c(x)$ has zeros $x_1, ..., x_{2k}$ of orders $p_1, ..., p_{2k}$ ($p_i \in \mathbb{N}$) with $p_1 + p_2 + ... + p_{2k} = 2u$ (u odd). (We remark that because of (i) we have $x_{j+k} = x_j + 2\pi, p_{j+k} = p_j$.)

Example 4.2. $c(x) = \sin \frac{nx}{2}$ (n odd). We have $k = n$, $x_j = (j-1)\frac{2\pi}{n}$, $p_j = 1$ ($j = 1, ..., 2n$).

For the following we fix an element $c \in D$. We consider the points x_j also as elements of S^1 (i.e., $x_j = x_{j+k}$).

Definition 4.3. (i) For $c \in D$, g_c is the subspace of $C^\infty(S^1)$ generated by all real C^∞-vector fields f for which $f(x_j) = f'(x_j) = ... = f^{(p_j-1)}(x_j) = 0$ $\forall j \in \{1, ..., k\}$.

(ii) ${g_c}'$ is the vector space of C^∞-functions on S^1 with poles at x_j of order not greater than p_j.

Remark 4.4. g_c is a Lie subalgebra with odd codimension of the Lie algebra g of all real C^∞-vector fields (s. Lemma 4.6 (i)). g_c corresponds the Lie subgroup G_c of $\mathrm{Diff}_+(S^1)$ with

$$G_c = \left\{\phi \in \mathrm{Diff}_+(S^1) \,\middle|\, \phi(x_i) = x_i, \phi'(x_j) = ... = \phi^{(p_k-1)}(x_j) = 0\right\}.$$

Now consider the linear maps $\omega_c : g_c \to {g_c}'$ and $r_c : {g_c}' \to g_c$ defined, respectively, by

$$\omega_c(f) \;=\; \frac{-2fc'}{c^3} + \frac{2f'}{c^2},$$

and by

$$r_c(p) \;=\; -\frac{1}{4}c(x)\int_x^{x+2\pi} c(y)p(y)dy.$$

Lemma 4.5. (i) *ω_c is a well-defined linear map.*

(ii) *r_c is a well-defined linear map.*

(iii) $\omega_c r_c = id_{g'}$, $r_c \omega_c = id_g$.

Proof. (i) ω_c changes the zeros at x_j of order $\geq p_j$ into poles of order $\leq p_j$. The only nontrivial case is a p_j-zero at x_j. Without loss of generality let $x_j = 0$ and $f(x) = a_1(x)x^{p_j}, c(x) = a_2(x)x^{p_j}$ ($a_i(0) \neq 0$). We obtain

$$\omega_c(f) = \frac{-2}{c^3}(fc' - f'c)$$

$$= \frac{-2}{x^{3p_j}a_2^3}(p_j x^{2p_j-1}a_1a_2 + x^{2p_j}a_1a_2' - p_j x^{2p_j-1}a_1a_2 - x^{2p_j}a_1'a_2)$$

$$= \frac{1}{x^{p_j}}\left(-\frac{2a_1(x){a_2}'(x)}{a_2(x)^3} + \frac{2{a_1}'(x)}{a_2(x)^2}\right). \tag{1}$$

In the second line of (1) the (p_j+1)-poles for the first and the third summand cancel and so we have a p_j-pole. Consequently ω_c is well-defined on g_c.
(ii) Let $p \in {g_c}'$. $r_c(p)$ is smooth and 2π-periodic because of Def. 4.1(i). Obviously $r_c(p)$ has at least a p_j-zero at x_j. That is $r_c(p) \in g_c$.
(iii)

$$\omega_c r_c(f) = -\frac{1}{4}\omega_c\left(c(x)\int\limits_x^{x+2\pi} c(y)p(y)dy\right) = -\frac{1}{4}\frac{-2c'(x)c(x)\int_x^{x+2\pi} c(y)p(y)dy}{c(x)^3}$$

$$-\frac{1}{4}\frac{2c(x)c'(x)\int_x^{x+2\pi} c(y)p(y)dy - c(x)(c(x+2\pi)p(x+2\pi) - c(x)p(x))}{c(x)^2}$$

$$= -\frac{1}{4}\frac{-2\cdot 2c(x)p(x)}{c(x)^2} = p(x)$$

and

$$r_c\omega_c(f) = r_c(\frac{-2fc'}{c^3} + \frac{2f'}{c^2}) = -\frac{1}{4}c(x)\int_x^{x+2\pi} \frac{-2fc'}{c^2} + \frac{2f'}{c}dy$$

$$= -\frac{1}{4}c(x)\int_x^{x+2\pi} 2(\frac{f}{c})'dy = -\frac{1}{4}2c(x)\left(\frac{f(x+2\pi)}{c(x+2\pi)} - \frac{f(x)}{c(x)}\right) = f(x) \quad .$$

∎

Lemma 4.6. (i) *g_c is a Lie subalgebra of the Lie algebra g of all real C^∞-vector fields on S^1.*

(ii) *${g_c}'$ is a dual of g_c (cf. Def.* 2.1*) with respect to the bilinear form*

$$\langle f(x), p(x)\rangle = \int_{S^1} f(x)p(x)dx \ .$$

(iii) *ω_c is a* 2*-cocycle (cf.* 2.1*).*

Proof. (i) One has to show that $[f,g]$ has a zero of order p_j at x_j. This follows from the formula

$$[f,g]^{(n)} = fg^{(n+1)} + \sum_{i=0}^{n-1}\left[\binom{n}{i} - \binom{n}{i+1}\right] f^{(i+1)}g^{(n-i)} - f^{(n+1)}g.$$

(ii) Clearly g_c, (respectively, ${g_c}'$) separates the points of ${g_c}'$, (respectively, g_c). Then $K(f)p = fp' + 2f'p$ acts invariantly on g because if f has an r^{th}-order zero at x_j and if p has an s^{th}-order pol at x_j then $K(f)p$ has at most a pole of order $s-r-1 \geq -1 \geq -p_j$ at x_j.
(iii) We have

$$\langle \omega_c(f), g\rangle = \int_{S^1} \frac{-2fgc'}{c^3} + \frac{2f'g}{c^2}dy = \int_{S^1} \frac{-1}{c^2}(fg)' + \frac{2f'g}{c^2}dy$$

$$= -\int_{S^1} \frac{1}{c^2}(fg' - f'g)dy = -\left\langle \frac{1}{c^2}, [f,g]\right\rangle.$$

The assertion follows from the Jacobi identity. ■

Remark 4.7. We have $\langle \omega_b(f), g\rangle = -\langle b, [f,g]\rangle$ (cf. the proof of Lemma 4.6(iii)). Because of the behavior of c at the poles x_i, $b = \frac{1}{c^2} \notin {g_c}'$ and therefore b is not a moment of g_c but we can consider b as a moment of $[g_c, g_c]$.

Proposition 4.8. *Let $c(x) \in D$, and let*

$$[p,q]_{g_{c'}} = \frac{2c'p + cp'}{4}\int_x^{x+2\pi} cqdy - \frac{2c'q + cq'}{4}\int_x^{x+2\pi} cpdy,$$

$$\alpha_f p = fp' + 2f'p,$$

and

$$\beta_p f = c\int_x^{x+2\pi} \frac{f(cp' + 2c'p)}{4}dy + \frac{1}{2}c^2 fp + \frac{1}{4}(c'f - f'c)\int_x^{x+2\pi} cpdy.$$

Then $M_c = (g_c, {g_c}', \alpha, \beta)$ is a real matched pair of Lie algebras.

Proof. The assumptions of Proposition 2.3 are satisfied by Lemmas 4.5 and 4.6. One calculates the above formulas by Proposition 2.3. ■

Because of $[p,q]_{g_{c'}} = \omega_c([r_c(p), r_c(q)])$ we can identify g_c and ${g_c}'$. Applying Proposition 2.1.a we obtain a self-dual representation for M_c.

Theorem 4.9. *Set*

$$\alpha'_f g = \frac{1}{2}c(x)\int_x^{x+2\pi}\left(\frac{3fg'c'}{c^2} + \frac{fgc''}{c^2} - \frac{3fg(c')^2}{c^3} + \frac{2f'gc'}{c^2} - \frac{2f'g'}{c} - \frac{fg''}{c}\right)dy.$$

Then $(g_c, g_c, \alpha', \alpha')$ and M_c are isomorphic matched pairs of Lie algebras.

Proof. One calculates the above expression for α' by $\alpha' = r\alpha\omega$. ∎

Example 4.10. Let $c(x) = \sin\frac{x}{2}$. Then g_c turns out to be the space of real analytic functions with $f(0) = 0$ while $g_c{}'$ is the space of real analytic functions possibly having a simple pole at 0. We obtain $r_c(p)(x) = \sin\frac{x}{2}\int_x^{x+2\pi}\sin\frac{y}{2}f(y)dy$ and

$$\alpha'{}_f g = \frac{1}{2}c(x)\int_x^{x+2\pi}\frac{(3fg' + 2f'g)\cos y}{\sin^2 y} - \frac{fg(1+2\cos^2 y)}{\sin^3 y} - \frac{2f'g' + g''f}{\sin y}dy.$$

Remark 4.11. With respect to some weaker topology of the Lie algebra g_c and a certain completion $g_c\overline{\otimes}g_c$ of the algebraic tensor product $g_c \otimes g_c$, the matched pairs M_c correspond to r-matrices

$$g_c\overline{\otimes}g_c \ni r = r(x,y) = -\frac{1}{4}\chi(x,y)c(x)c(y),\ \chi(x,y) = \begin{cases} -1 \ if \ y \in [0,x) \\ +1 \ if \ y \in [x,2\pi) \end{cases}$$

and Lie bialgebras $(g_c, \triangle : g_c \to g_c\overline{\otimes}g_c)$ with

$$\triangle(f) = \frac{1}{4}\chi(x,y)(c(x)c(y)(f'(x) + f'(y)) - f(x)c'(x)c(y) - f(y)c(x)c'(y)).$$

4.2. Matched pairs $M_c^{(n)}$. Finally we generalize Proposition 4.8 in the spirit of Proposition 2.4. Fix an element $c(x) \in D$ and consider also $c_n(x) := c(nx)$ as an element of D. Let c_n have zeros x_j of order p_j $(j = 1,...,2nk)$ with $p_1 + ... + p_{2nk} = 2nu$ and u odd. We have $x_{j+k} = x_j + \frac{2\pi}{n}$ and $p_j = p_{j+k}$. We can consider the elements $x_j \in S^{1(2)}$ as elements of S^1 (i.e., $x_{j+nk} = x_j + 2\pi$).

Definition 4.12. (i) $g_c^{(n)}$ is the vector space of C^∞-vector fields on S^1 with zeros x_j of order at least p_j,

(ii) $g_c^{(n)\prime}$ is the vector space of C^∞-functions on S^1 with poles x_j of order at most p_j,

(iii) $h_c^{(n)}$ is the subspace of $g_c^{(n)}$ of $\frac{2\pi}{n}$-periodic vector fields,

(iv) $h_c^{(n)\prime}$ is the subspace of $g_c^{(n)\prime}$ of $\frac{2\pi}{n}$-periodic functions.

Lemma 4.13. (i) $g_c^{(n)}, h_c^{(n)}$ *are subalgebras of* g.

(ii) *With respect to the pairing* $\langle f(x), p(x)\rangle = \int_{S^1} f(x)g(x)dx$ *the spaces* $g_c^{(n)\prime}$, *(respectively,* $h_c^{(n)\prime}$*) are duals of* $g_c^{(n)}$ *(respectively* $h_c^{(n)}$*) (cf. Def. 2.1).*

(iii) *We have the direct decompositions* $g_c^{(n)} = h_c^{(n)} \dotplus h_c^{(n)\perp\prime}$ *and* $g_c^{(n)\prime} = h_c^{(n)\prime} \dotplus h_c^{(n)\perp}$.

Proof. (i), (ii) See the proof of Lemma 4.6(i),(ii).
(iii) Let $f \in g_c^{(n)}$. The first identity follows from the decomposition

$$f(x) = \frac{1}{n}\sum_{j=1}^{n} f(x + \frac{2\pi j}{n}) + \left(f(x) - \frac{1}{n}\sum_{j=1}^{n} f(x + \frac{2\pi j}{n}) \right).$$

The proof of the second identity is analogous. ■

Define linear maps $\omega_c^{(n)} : h_c^{(n)} \to h_c^{(n)\prime}$ and $r_c^{(n)} : h_c^{(n)\prime} \to h_c^{(n)}$ by setting

$$\omega_c^{(n)}(f) = \frac{-2f{c_n}'}{c_n^3} + \frac{2f'}{c_n^2},$$

and

$$r_c^{(n)}(p) = -\frac{1}{4}c_n(x)\int_x^{x+2\pi} c_n(y)p(y)dy.$$

Lemma 4.14. *$\omega_c^{(n)}$ is a 2-cocycle and $r_c^{(n)}$,$\omega_c^{(n)}$ are well-defined linear maps with $r_c^{(n)}\omega_c^{(n)} = id_{h_c^{(n)}}$, $\omega_c^{(n)}r_c^{(n)} = id_{h_c^{(n)\prime}}$.*

Proof. See the proof of Lemma 4.5. ■

Theorem 4.15. *Let $c \in D$ and set*

$$[p,q] = \frac{2{c_n}'p + c_n p'}{4}\int_x^{x+2\pi} c_n(y)q(y)dy - \frac{2{c_n}'q + c_n q'}{4}\int_x^{x+2\pi} c_n(y)p(y)dy$$

if $p, q \in h_c^{(n)\prime}$;

$$[p,q] = c_n^2 pq - \frac{1}{4}(c_n q' + 2{c_n}'q)\int_x^{x+2\pi} c_n p dy$$

if $p \in h_c^{(n)\prime}$, $q \in h_c^{(n)\perp}$;

$$[p,q] = 0$$

if $p, q \in h_c^{(n)\perp}$;

$$\alpha_f p = fp' + 2f'p \;;$$

and

$$\beta_x(e) = \begin{cases} r_c^{(n)} P_{h'} \alpha_{r_c^{(n)}(x)} \omega_c^{(n)}(e) & x \in h_c^{(n)\prime}, e \in h_c^{(n)} \\ P_{h'^\perp}[r_c^{(n)}(x), e] & x \in h_c^{(n)\prime}, e \in {h_c^{(n)\prime}}^\perp \\ r_c^{(n)}(\alpha_e x) & x \in {h_c^{(n)}}^\perp, e \in g_c^{(n)}. \end{cases}$$

Then $M_c^{(n)} = (g_c^{(n)}, g_c^{(n)\prime}, \alpha, \beta)$ is a real matched pair of Lie algebras.

Proof. The proof is analogous to Proposition 4.8 and follows from Lemma 4.5, 4.6 and Proposition 2.4. ■

5. Some problems

Possibly additional structures can be classified by the consideration of additional classes of subalgebras. Our method does not work in the case of subalgebras of odd dimension (e.g. $sl(2,\mathbb{C})$) and for quasi-triangular r-matrices of subalgebras.

The next step in the direction of a quantization is the integration to matched pairs of Lie groups and Poisson Lie groups.

Another question is whether there are extensions of matched pairs for g_c to matched pairs of the Lie algebra of all smooth vector fields on the circle.

References

[B-M] Beggs, E. and S. Majid, *Matched pairs of topological Lie algebras corresponding to Lie bialgebra structures on* diff(S^1) *and* diff($\mathbb{R}$), Ann. Inst. Henri Poincare, **53** no. 1 (1990), 15-34.

[D] Drinfeld, V.G., *Quantum Groups*, in: "Proceedings of the International Congress of Mathematicians Berkeley 1986", 1987.

[K1] Köthe, G., "Topologische lineare Räume", Band 1, (Grundlehren Math. Wiss. Bd. 107), Springer Verlag Berlin Heidelberg New York, 1960.

[K2] Köthe, G., "Topological vector spaces", Part 2, (Grundlehren Math. Wiss. Bd. 237), Springer Verlag Berlin Heidelberg New York, 1979.

[M] Michaelis, W., *A class of infinite-dimensional Lie bialgebras containing the Virasoro algebra*, Adv. in Math. **107** (1994), 365-392.

[P-S] Pressley, A., Segal, Gr., "Loop groups", Oxford University Press, New York 1986.

[T] Taft, E.J., *Witt and Virasoro algebras as Lie bialgebras,* Journal of Pure and Applied Algebra **87**(1993), 301-312.

[Ta] Takhtajan, L.A., *Introduction to Quantum groups*, in: "Quantum groups", Proceedings of the 8th Int. Workshop on Math. Phys. in Clausthal, ed. by Doebner, H.-D., Lecture Notes in Physics 370, Springer Verlag Berlin Heidelberg New York, 1990.

[W1] Witten, E., *Quantum fields, Grassmannians and algebraic curves*, Commun. Math. Phys. **113**(1989), 529.

[W2] Witten, E., *Coadjoint Orbits of the Virasoro-Algebra*, Commun. Math. Phys. **114**(1989), 1-53.

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