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Laplace transform and unitary highest weight modules

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Abstract. The unitarizable modules in the analytic continuation of the holomorphic discrete series for tube type domains are realized as Hilbert spaces obtained through the Laplace transform.

0. Introduction

Let G be a connected real semi-simple Lie group with finite center, U a maximal compact subgroup, and assume G/U is a Hermitian symmetric space. Harish Chandra constructed a family of irreducible unitary representations of G, called the holomorphic discrete series, realized on holomorphic sections of some vector bundles over G/U, square-integrable with respect to the invariant measure on G/U.

The vector bundles are associated to irreducible (finite-dimensional) representations of U, with some restriction on the dominant weight of the representation in order to have non-trivial L^2-sections. However the formulae for the action of G make sense for all values of the dominant weight, and Harish Chandra indicated the possibility that some of these modules (or some submodules) might be unitarizable ([10]).

This problem was completely solved ([5],[11]), but the proofs are of algebraic nature and use case-by-case arguments. Moreover there is no concrete realization of the corresponding Hilbert spaces. For more recent work in this direction, see [2], [6], [12]. A few years earlier the special case of line bundles (associated to characters of U) had been studied, both from the algebraic point of view [19] and the analytic counterpart [17]. We follow here the second approach. To avoid complications, we restrict our attention to tube type domains. We obtain a characterization and a realization of the unitarizable modules in terms of some (operator-valued) measure on a cone $\overline{\Omega}$. In some cases, we are able to completely determine the corresponding measures.

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1. Geometric preliminaries

Let V be a Euclidean Jordan algebra (for this notion and further properties, see [7]). For sake of simplicity, V is assumed to be simple. Let $< \, , \, >$ be the inner product, e the neutral element and let $n = \dim V$.

Let Ω be the associated cone. Denote by L the connected component of the automorphism group of the cone Ω, i.e.

$$L = \Big\{ l \in GL(V) \mid l\Omega = \Omega \Big\}^0$$

The groupe L is closed under conjugation, hence is reductive. The subgroup $K = \big\{ l \in L \mid le = e \big\}$ is a maximal compact subgroup of L, and coincides with the connected component of the automorphism group of V.

If $x \in V$, we denote by $L(x)$ the mapping $y \mapsto xy$, and by $P(x) = 2L(x)^2 - L(x^2)$ the so called quadratic representation of V.

If $x \in \Omega$, then $P(x)$ is symmetric with respect to the inner product and belongs to L. Every element of L can be written as $g = kP(x)$, for some $k \in K$ and some $x \in \Omega$. This is in fact the Cartan decomposition of L. For further use, notice the formula $\exp 2L(x) = P(\exp x)$, for $x \in V$, where on the right handside exp stands for the exponential map in the Jordan algebra V.

Let Δ be the Koecher norm function (also called determinant). Δ is a polynomial of degree r, where r is the rank of the Jordan algebra V (r is also the rank of the symmetric space $\Omega \simeq L/K$). Up to a positive constant, Ω has a unique L-invariant measure given by

$$d^*x = \Delta(x)^{-m} dx \quad ,$$

where $m = \frac{n}{r}$ (m turns out to be an integer or half an integer). For further use, notice the formula $\mathrm{Det}(P(x)) = \Delta(x)^{2n/r}$, for $x \in V$.

The Iwasawa decomposition also has a specific realization. Fix a *Jordan frame* , i.e. $e = c_1 + c_2 + \ldots + c_r$, where the $(c_i)_{1 \le i \le r}$ form a complete orthogonal system of primitive idempotents. Let $R = \big\{ a = \sum_{i=1}^r a_i c_i \ , a_i \in \mathbb{R} \big\}$. The space $\mathfrak{a} = \big\{ L(a) | a \in R \big\}$ is a Cartan subspace in $\mathfrak{p} = \big\{ L(x) | x \in V \big\}$. Let $A = \{ \exp L(a), a \in R \}$ be the corresponding Lie subgroup, which can also be viewed as $A = \{ P(a), a = \sum_{i=1}^r a_i c_i \ , a_i > 0 \}$. Now for $1 \le i < j \le r$, let

$$V_{ij} = \big\{ x \in V \mid c_i x = c_j x = \frac{1}{2} x \big\} \qquad .$$

Then $V = \oplus_{j=1}^r \mathbb{R} c_j \bigoplus_{1 \le i < j \le r} V_{ij}$ (*Peirce decomposition*). For $x, y \in V$, write $x \,\square\, y = L(xy) + [L(x), L(y)]$, and for $1 \le i < j \le r$ let $\mathfrak{n}_{ij} = V_{ij} \,\square\, c_i$. Then $\mathfrak{n} = \bigoplus_{1 \le i < j \le r} \mathfrak{n}_{ij}$ is the Iwasawa nilpotent subalgebra associated to the Weyl chamber $\mathfrak{a}^+ = \{ L(a), \ a = \sum_{i=1}^r a_i c_i \ , a_1 < a_2 < \ldots < a_r \}$. Let N be the analytic subgroup of L with $Lie(N) = \mathfrak{n}$.

Closely associated to this Iwasawa decomposition is a parametrization of Ω. Let

$$V^+ = \big\{ u = \sum_{i=1}^r u_j c_j + \sum_{j<k} u_{jk} \quad | \quad u_j > 0, u_{jk} \in V_{jk} \big\} \ .$$

For $u^{(j)} \in \bigoplus_{k=j+1}^{r} V_{jk}$ let $\tau(u^{(j)}) = \exp(2u \square c_j)$, and for $u = \sum_{i=1}^{r} u_i c_i + \sum_{j<k} u_{jk}$ in V^+, let $b_j = c_1 + \ldots + c_{j-1} + u_j c_j + c_{j+1} + \ldots + c_r$, $1 \leq j \leq r$, $u^{(j)} = \sum_{k=j+1}^{r} u_{jk}$, $1 \leq j \leq r-1$, and define

$$t(u) = P(b_1)\tau(u^{(1)})P(b_2)\tau(u^{(2)}) \cdots \tau(u^{(r-1)})P(b_r).$$

Proposition 1.1. *The map $u \mapsto t(u)e$ is a bijection from V_+ onto Ω. If*

$$x = \sum_{j=1}^{r} x_j c_j + \sum_{j<k} x_{jk}$$

is the Peirce decomposition of $x = t(u)e$, then

$$x_j = u_j^2 + \frac{1}{2} \sum_{k=1}^{j-1} \| u_{kj} \|^2 ,$$

$$x_{jk} = u_j u_{jk} + 2 \sum_{l=1}^{j-1} u_{lj} u_{lk} .$$

The invariant measure on Ω is given by :

$$\int_{\Omega} f(x) d^*x = 2^r \int_{V_+} f(t(u)e) \prod_{j=1}^{r} u_j^{-d(j-1)-1} du_j \prod_{1 \leq j < k \leq r} du_{jk} .$$

For later use, we need a careful analysis of the L orbits in $\overline{\Omega}$. For p, $0 \leq p \leq r$, let $e_p = \sum_{i=1}^{p} c_i$, and let $\mathcal{O}_p$ be the orbit under L of e_p. Observe that $\mathcal{O}_0 = \{0\}$, and $\mathcal{O}_r = \Omega$.

Proposition 1.2. $\overline{\Omega} = \bigsqcup_{0 \leq p \leq r} \mathcal{O}_p$.

Each orbit can be parametrized in a way similar to the parametrization of Ω. We need some more notations. For $0 \leq p \leq r-1$, let L_p be the stabilizer of e_p, and $\mathfrak{l}_p$ its Lie algebra. Now let

$$\mathfrak{a}_p = \{L(\sum_{1 \leq i \leq p} a_i c_i), a_i \in \mathbb{R}\}$$

$$\mathfrak{n}_p = \bigoplus_{1 \leq i \leq p} \left(\bigoplus_{i+1 \leq j \leq r} \mathfrak{n}_{ij} \right)$$

It is easily checked that $\mathfrak{l} = \mathfrak{l}_p \oplus \mathfrak{a}_p \oplus \mathfrak{n}_p$. On the group level, we get the density of $L_p A_p N_p$ in L, where A_p and N_p are the analytic subgroups of L corresponding to respectively $\mathfrak{a}_p$ and $\mathfrak{n}_p$ (see e.g. [17]). To state this result in a way similar to Proposition 1.1, let

$$V_p^+ = \sum_{i=1}^{p} \mathbb{R}^+ c_i \ \oplus \ \mathfrak{n}_p .$$

For $u = \sum_{i=1}^{p} u_i c_i + \sum u_{ij} \in V_p^+$, let as above

$$t(u) = P(b_1)\tau(u^{(1)})P(b_2)\tau(u^{(2)}) \cdots \tau(u^{(p-1)})P(b_p),$$

where $u^{(i)} = \sum_{j=i+1}^{r} u_{ij}, 1 \leq i \leq p$.

Proposition 1.3. *Let $1 \le p \le r-1$. The mapping $u \mapsto t(u)e_p$ from V_p^+ into V is a one-to-one map into a dense subset $\mathcal{O}_p'$ of $\mathcal{O}_p$. There exists a unique (up to a positive scalar) relatively invariant measure ν_p on $\mathcal{O}_p$. The complementary set $\mathcal{O}_p \setminus \mathcal{O}_p'$ has ν_p measure 0. In the corresponding coordinates ν_p is given by*

$$\int_{\mathcal{O}_p} f(x)d\nu_p(x) = \int_{V_p^+} f(t(u)e_p) \prod_{j=1}^{p} u_j^{(p-j+1)d-1} du_j \prod_{\substack{1\le i\le p \\ i<j\le r}} du_{ij} \ .$$

The relative invariance is expressed by the following formula :

$$\int_{\mathcal{O}_p} f(lx)d\nu_p(x) = (\det l)^{-s_p} \int_{\mathcal{O}_p} f(x)d\nu_p(x) \ , \forall l \in L,$$

where $s_p = \frac{r}{2n}dp$ (d is the common dimension of all $\mathfrak{n}_{ij}$, for $1 \le i < j \le r$). For a proof, see [17] or [7], p. 134.

Let $V_{\mathbb{C}} = V + iV$ be the complexification of V and let T_Ω be the associated tube domain. When equipped with the Bergman metric, T_Ω is a hermitian symmetric space. Let $G = G(T_\Omega)$ be the connected component of the identity in the group of bi-holomorphic transformations of T_Ω, and let U be the stabilizer in G of the base point $ie \in T_\Omega$. An important fact is that L and U are two real forms of the same complex Lie group, namely the connected component of the identity in $Str(V_{\mathbb{C}})$, where

$$Str(V_{\mathbb{C}}) = \{g \in GL(V_{\mathbb{C}}) \mid \forall x \in V_{\mathbb{C}} \ , \ P(gx) = gP(x)g^t \},$$

with the obvious extension of P to $V_{\mathbb{C}}$.

If $g \in L$, still denote by g its complex linear extension to $V_{\mathbb{C}}$. It clearly preserves T_Ω, giving a natural map from L into G. The Cartan subspace $\mathfrak{a}$ (for the pair $(\mathfrak{l}, \mathfrak{k})$) turns out to be also a Cartan subspace for the pair $(\mathfrak{g}, \mathfrak{u})$, where $\mathfrak{g} = Lie(G)$ and $\mathfrak{u} = Lie(U)$. Another important subgroup of G is the subgroup of translations N^+. In fact to each $v \in V$, is associated the translation t_v given by $z \mapsto z + v$. N^+ is clearly isomorphic to V (as Abelian group). Moreover, the semi-direct product LN^+ (where L acts on N^+ by its natural action on V) is the group of all affine transformations of T_Ω. N^+ is a subgroup of the Iwasawa subgroup associated to the positive Weyl chamber $\mathfrak{a}^{++} = \{\sum_{i=1}^r a_i c_i \mid 0 < a_1 < a_2 < \ldots < a_r\}$. In fact the full Iwasawa subgroup is the semi-direct product NN^+. Finally, it is worth mentioning that the group G is generated by L, N^+, and the inversion $z \mapsto -z^{-1}$.

The action of G on T_Ω can be (locally) extended to an action of $G^{\mathbb{C}}$ on T_Ω. For $g \in G^{\mathbb{C}}, z \in T_\Omega$ and $g.z \in T_\Omega$, define the automorphy factor $J(g,z) = \frac{\partial(g.z)}{\partial z}$. When defined, it turns out that $J(g,z)$ is always in $L^{\mathbb{C}}$, and satisfies the *cocycle identity* $J(g_1g_2, z) = J(g_1, g_2.z)J(g_2, z)$. Obviously, if l is in L, then $J(l,z) = l$.

2. Invariant cones and Ol'shanskiĭ semigroups

An important property of the hermitian pairs is the existence of $Ad(G)$ invariant cones in $\mathfrak{g}$. Cones are assumed to be convex, closed with a nonvoid interior and proper. One of the main facts is the existence of a minimal invariant cone $C_{\min}$ and a maximal invariant cone $C_{\max}$, in the sense that any invariant cone C contains either $C_{\min}$ or $-C_{\min}$ and similarly is contained in $C_{\max}$ or $-C_{\max}$.

Theorem 2.1. *The cone $C_{\min}$ is generated (up to ± 1) by t_{e_r} viewed as an element in $\mathfrak{n}^+ = Lie(N^+)$, i.e. $C_{\min}$ is the smallest closed convex cone containing the $Ad(G)$-orbit of t_{e_r} in $\mathfrak{g}$.*

Proof. By Vinberg's theorem ([18]), $C_{\min}$ contains a (unique) ray which is invariant by a minimal parabolic subgroup. Thanks to the structure of the nilpotent factor NN^+, it is clear that this ray can only be $\pm\mathbb{R}^+ t_{e_r}$. ∎

From now on, denote by $C_{\min}$ (resp., $C_{\max}$) the minimal (resp., maximal) cone that contains t_{e_r}.

As discovered by Ol'shanskiĭ (see [16]), to any invariant cone C, it is possible to associate a semi-group $\Gamma_C = G \exp iC$ in $G^{\mathbb{C}}$. The semi-group $\Gamma_{\max} = G \exp iC_{\max}$ is exactly the semi-group of compressions of T_Ω, namely

$$\Gamma_{\max} = \{ g \in G^{\mathbb{C}} \mid g(T_\Omega) \subset T_\Omega \} .$$

Theorem 2.2. $\Gamma_{\min} \supset \{t_{iv}\}_{v \in \overline{\Omega}}$.

Proof. As $t_{e_r} \in C_{\min}$, it is easily seen, using the action of L and the convexity of $\overline{\Omega}$, that t_v is contained in $C_{\min}$ for any $v \in \overline{\Omega}$. Hence the result follows. ∎

The importance of these cones and semi-groups for highest weight representations has been noticed by Ol'shanskiĭ and in fact if $(\pi, \mathcal{H})$ is any unitary representation, let $\mathcal{H}^\infty$ be the space of $\mathcal{C}^\infty$ vectors, and let

$$C_\pi = \{ X \in \mathfrak{g} \mid \; < id\pi(X)\xi, \xi > \; \leq 0, \forall\, \xi \in \mathcal{H}^\infty \} .$$

C_π is a cone, which is non trivial if and only if π has a highest weight, and then the representation π can be extended as a (holomorphic) representation of $\Gamma_\pi = G \exp iC_\pi$ by contractions.

3. Reproducing kernels and unitarity

Let (μ, V_μ) be a finite dimensional irreducible unitary representation of the maximal compact subgroup U of G. As explained before, it is convenient to consider μ as a finite dimensional (holomorphic) representation of $L^{\mathbb{C}}$. Moreover it satisfies the relation $\mu(l^*) = \mu(l)^*$, where $l^* = \overline{l}^t$, for $l \in L^{\mathbb{C}}$ (extension of the unitarity property of μ). For $g \in G$, and $z \in T_\Omega$, set $J_\mu(g,z) = \mu(J(g,z))$

Now let $\mathcal{V}_\mu$ be the space of holomorphic functions on T_Ω with values in V_μ. Define the following action of G on $\mathcal{V}_\mu$:

$$T_\mu(g) f(z) = (J_\mu(g^{-1}, z))^{-1} f(g^{-1} z) \quad ,$$

where $f \in \mathcal{V}_\mu$, $z \in T_\Omega$ and $g \in G$.

We want to discuss the existence of an invariant inner product on $\mathcal{V}_\mu$. There is in fact a natural inner product given by

$$(f,g)_\mu = \int_{T_\Omega} (\mu(P(y)^{-1})f(z)|g(z))_{V_\mu} d_*z \qquad ,$$

where $z \in T_\Omega$, $y = \Im(z)$, and d_*z is the G invariant measure on T_Ω. The invariance of the inner product by N^+ is obvious, its invariance by L is easy. It remains to check invariance by the inversion $z \mapsto -z^{-1}$. But this is a consequence of the following formula : $P(\overline{z})P(\Im(z))^{-1}P(z) = P(\Im(-z^{-1}))^{-1}$ (see [4] p. 163). Now let $\mathcal{H}_\mu = \{f \in \mathcal{V}_\mu | (f,f)_\mu < +\infty\}$. Then if $\mathcal{H}_\mu \neq \{0\}$, $(T_\mu, \mathcal{H}_\mu)$ defines a unitary representation, and in fact this is the celebrated holomorphic discrete series.

Now let $\mathcal{H}_\mu$ be an irreducible unitary representation of G, and assume there exists a continuous non trivial intertwining operator from $\mathcal{H}_\mu$ into $\mathcal{V}_\mu$, where the latter space is equipped with the compact-open topology. Then the evaluation map at any point $z \in T_\Omega$ is a continuous linear map on $\mathcal{H}_\mu$, so $\mathcal{H}_\mu$ admits a *reproducing kernel*. In fact, let $E_z : \mathcal{H}_\mu \to V_\mu$ be the evaluation map at $z \in T_\Omega$ and define

$$\mathbb{Q}_\mu(z,w) = E_z E_w^*.$$

Then $\mathbb{Q}_\mu : T_\Omega \times T_\Omega \to \mathrm{End}\,(V_\mu)$ satisfies

(3.1i) $\mathbb{Q}_\mu$ is holomorphic in z and antiholomorphic in w

(3.1ii) $$\mathbb{Q}_\mu(w,z) = \mathbb{Q}_\mu(z,w)^*$$

(3.1iii) $$\forall q \in \mathbb{N}, \forall\, (w_j)_{1\le j\le q} \in T_\Omega \ , \forall\, (\xi_j)_{1\le j\le q} \in V_\mu$$

$$\sum_i \sum_j \left(\mathbb{Q}_\mu(w_j,w_i)\xi_i|\xi_j\right)_{V_\mu} \geq 0$$

(3.1iv) $$\mathbb{Q}_\mu(g.z, g.w) = J_\mu(g,z)\mathbb{Q}_\mu(z,w)J_\mu(g,w)^*$$

A mapping $\mathbb{Q} : T_\Omega \times T_\Omega \to \mathrm{End}\,(V_\mu)$ which satisfies (3.1 i, ii, and iii) is said to be a positive definite (operator-valued) kernel (see [14]). If it moreover satisfies (3.1iv), the kernel $\mathbb{Q}$ is said to be invariant (with respect to μ).

Proposition 3.1. *Let μ be a finite dimensional holomorphic irreducible representation of $L^{\mathbb{C}}$, and let $\mathbb{Q}$ be an invariant positive definite kernel (with respect to μ). Let $\mathcal{L}_\mu$ be the span of the functions $z \mapsto \mathbb{Q}(z,w)\xi$, where w is arbitrary in T_Ω, and ξ arbitrary in V_μ. Introduce the (well defined) Hermitian form on $\mathcal{L}_\mu$ given by*

$$\Big(\sum_i \mathbb{Q}(.,w_i)\xi_i | \sum_j \mathbb{Q}(.,w'_j)\xi'_j\Big) = \sum_i \sum_j \left(\mathbb{Q}(w'_j,w_i)\xi_i|\xi'_j\right)_{V_\mu} \quad .$$

Let $\mathcal{H}_\mu$ be the usual (separate) completion of $\mathcal{L}_\mu$ with respect to this (well-defined) form. Then $\mathcal{H}_\mu$ is invariant under T_μ and the restriction of T_μ to $\mathcal{H}_\mu$ is unitary and irreducible.

For the proof, see [14]. Let us observe moreover, that $\mathcal{L}_\mu$ always contains the "highest weight vector", namely $\mathbb{Q}(z, ie)\xi_\mu$, where ξ_μ is the highest weight vector in V_μ (cf. [17]). So $\mathcal{H}_\mu$ is a highest weight representation.

However, the kernel $\mathbb{Q}_\mu(z, w)$ satisfies another important condition which is related to the remark due to Ol'shanskiĭ we mentioned above.

Proposition 3.2. *Let μ be a finite dimensional holomorphic representation of $L^{\mathbb{C}}$, and assume that $\mathbb{Q}_\mu$ is positive definite. Then $\mathbb{Q}_\mu$ satisfies*

$$(3.1v) \qquad \forall\, q \in \mathbb{N},\ \forall\, (w_j)_{1\le j\le q} \in T_\Omega\ ,\ \forall\, (\xi_j)_{1\le j\le q} \in V_\mu\ ,\forall\, y \in \overline{\Omega}$$

$$\Big(\sum_i \sum_j \mathbb{Q}_\mu(w_j + iy, w_i + iy)\xi_i|\xi_j\Big)_{V_\mu} \le \Big(\sum_i \sum_j \mathbb{Q}_\mu(w_j, w_i)\xi_i|\xi_j\Big)_{V_\mu}\ .$$

Proof. In fact, the cone $C_{T_\mu} = C_\mu$ contains $-t_{e_r}$ (see the original argument in [10]), hence $C_\mu \supset -C_{\min}$ by Theorem 2.2. (cf [16]), and from the holomorphic extension of T_μ to the Olshanskiĭ semigroup $G\exp iC_\pi$ by contractions yields $\|T_\mu(t_{-iy})\Phi\|^2 \le \|\Phi\|^2$, where $y \in \overline{\Omega}$, and $\Phi(.) = \sum_i \mathbb{Q}_\mu(., w_i)\xi_i \in \mathcal{H}_\mu$. But this is exactly the inequality we were looking for, once observed that $T_\mu(t_{-iy})\mathbb{Q}_\mu(., w) = \mathbb{Q}_\mu(., w + iy)$. ■

The conditions (3.1i-iv) completely determine (up to a positive scalar) the possible kernels (cf [4]). In fact by using the action of the translations $\{t_y\}_{y\in V}$, it is easily seen that $\mathbb{Q}$ must be of the form $\mathbb{Q}(z, w) = Q(\frac{z-\overline{w}}{2})$, where Q is a holomorphic map from T_Ω into $\mathrm{End}\,(V_\mu)$. Moreover, if one considers the origin $ie \in T_\Omega$, then from (3.1iv) we immediately see that $Q(ie)$ must commute with the operators $\mu(J(k, ie))$ for any k in the stabilizer U of ie in G. An application of Schur's lemma forces $Q(ie)$ to be a multiple of the identity. The invariance property applied to $P(y^{1/2})$, where $y \in \Omega$ shows that $Q(iy) = Q(P(y^{1/2}).ie) = \mu(P(y))$, up to a positive constant. As Q is holomorphic, the only possibility for $\mathbb{Q}$ is (up to a positive constant)

$$\mathbb{Q}(z, w) = \mu(P(\frac{z - \overline{w}}{2i})).$$

Conversely, properties (3.1i) and (3.1ii) are immediate. The invariance property can easily be established for the translations and the elements of L. For the inversion $z \mapsto -z^{-1}$, one uses the identity

$$P(\overline{w}^{\,-1} - z^{-1}) = P(\overline{w}^{\,-1})P(z - \overline{w})P(z^{-1}), \text{for } z, w \in T_\Omega$$

(cf. [7] page 200), and takes images of both sides under μ to get the desired invariance property.

Henceforth we concentrate our effort towards property (3.1v), which is crucial for discussing unitarity.

If W is a finite-dimensional Hilbert space, denote by $Herm(W)$ the the space of Hermitian operators on W and by $Herm^+(W)$ the cone of positive semidefinite Hermitian operators on W. In what follows, by a measure on $\overline{\Omega}$ with values in $Herm^+W$, we mean, following Bourbaki (see [1]), a linear map R from the space $C_c(\overline{\Omega})$ of continuous real valued functions with compact support on $\overline{\Omega}$ into $Herm(W)$, which is continuous for the usual topology on $C_c(\overline{\Omega})$, and such that for any nonnegative function φ in $C_c(\overline{\Omega})$, $R(\varphi) \in Herm^+(W)$.

Theorem 3.3. *Let W be a finite dimensional Hilbert space and let $q : \Omega \to Herm^+(W)$ be a continuous map with the property (3.1v). Then there exists a unique measure R on $\overline{\Omega}$, with values in $Herm^+(W)$, such that :*

$$q(y) = \int_{\overline{\Omega}} e^{-(y|v)} dR(v) \ ,$$

for all $y \in \Omega$.

Proof. First fix $\xi \in W$. Define $q_\xi(y) = (q(y)\xi|\xi)$. Clearly q_ξ is a continuous function on Ω, which satisfies

$$0 \ \leq \ \sum_i \sum_j \lambda_i \overline{\lambda}_j q_\xi(y_i + y_j + y) \ \leq \ \sum_i \sum_j \lambda_i \overline{\lambda}_j q_\xi(y_i + y_j),$$

for all $(y_i)_{1 \leq i \leq n}, y \in \Omega$, $(\lambda_i)_{1 \leq i \leq n} \in \mathbb{C}$. By Nussbaum's theorem (see [15],[17]), there exists a unique positive measure R_ξ on $\overline{\Omega}$, such that

$$q_\xi(y) = \int_{\overline{\Omega}} e^{-(y|w)} dR_\xi(w) \ .$$

Now define for $\xi, \eta \in W$

$$R_{\xi,\eta} = \frac{1}{4}[R_{\xi+\eta} - R_{\xi-\eta} + iR_{\xi+i\eta} - iR_{\xi-i\eta}] \ .$$

The way it depends on ξ, η is clearly of Hermitian nature. So there exists a measure R on $\overline{\Omega}$, mith values in $Herm(W)$, such that $R_{\xi,\eta}(.) = (R(.)\xi|\eta)$. As $R_\xi = R_{\xi,\xi}$, R has values in $Herm^+(W)$, and the result follows. The uniqueness is clear from properties of the Laplace transform. ■

It is now possible to apply this result to the reproducing kernels $\mathbb{Q}_\mu$.

Theorem 3.4. *Let μ be a finite dimensional representation of L on a vector space V_μ. Then the associated kernel $\mathbb{Q}_\mu$ is positive definite if and only if there exists a measure R_μ on $\overline{\Omega}$, with values in $Herm^+(V_\mu)$, such that*

(3.4i) $$dR_\mu(l.) = \mu(l)^{*^{-1}} dR_\mu(.)\mu(l)^{-1} \ , \forall l \in L$$

(3.4ii) $$\int_{\overline{\Omega}} e^{-\mathrm{tr} v} \ dR_\mu(v) = Id$$

Proof. The existence of such a measure, when $\mathbb{Q}_\mu$ is positive definite is clear from the preceding results. Conversely, use a change of variable to get from properties (3.4 i) and (3.4 ii) the equality $\mu(P(x)) = \int_{\overline{\Omega}} e^{-(x|w)} dR_\mu(w)$ which gives immediately $\mathbb{Q}_\mu(z, w) = \int_\Omega e^{-(\frac{z-\overline{w}}{2i}|v)} dR_\mu(v)$ proving the positive-definiteness of $\mathbb{Q}_\mu$. ■

It is possible to give a more concrete realization of the Hilbert space $\mathcal{H}_\mu$ corresponding to the kernel $\mathbb{Q}_\mu$ (according to Proposition (3.1)). In fact define $\mathcal{G}_\mu$ as the space of all measurable functions $\Phi : \overline{\Omega} \to V_\mu$, which satisfy

$$||\Phi||_\mu^2 = \int_{\overline{\Omega}} \big(dR_\mu(2v)\Phi(v)|\Phi(v)\big) < +\infty.$$

Then, after identifying two functions which are equal R_μ-almost everywhere, $\mathcal{G}_\mu$ has a Hilbert space structure for the inner product

$$(\Phi, \Psi)_{\mathcal{G}_\mu} = \int_{\overline{\Omega}} \big(dR_\mu(2v)\Phi(v)|\Psi(v)\big).$$

If $\Phi \in \mathcal{G}_\mu$ define, for $z \in T_\Omega$, $\mathcal{F}\Phi : T_\Omega \to V_\mu$

$$\mathcal{F}\Phi(z) = \int_{\overline{\Omega}} e^{i<z|v>} dR_\mu(2v)\Phi(v).$$

Let $\xi \in V_\mu$; then

$$\big|\big(dR_\mu(2v)\Phi(v)|\xi\big)_{V_\mu}\big| \le \big(dR_\mu(2v)\Phi(v)|\Phi(v)\big)^{1/2}\big(dR_\mu(2v)\xi|\xi\big)^{1/2},$$

and by applying Schwarz inequality, we get

$$| < \mathcal{F}\Phi(z)|\xi > |^2 \le \Big(\int_{\overline{\Omega}} (dR_\mu(2v)\Phi(v)\big|\Phi(v)\big)\Big(\int_{\overline{\Omega}} e^{-2<y|v>} (dR_\mu(2v)\xi|\xi)\Big) ,$$

where $z = x + iy$. This shows that the integral in the definition of $\mathcal{F}\Phi$ is (absolutely) convergent and it is then easy to verify that $\mathcal{F}\Phi$ is holomorphic. Now let $\mathcal{F}_\mu$ be the space of all (holomorphic) V_μ-valued functions of the form $\mathcal{F}\Phi$ with $\Phi \in \mathcal{G}_\mu$, and define $||\mathcal{F}\Phi||_{\mathcal{F}_\mu} = ||\Phi||_{\mathcal{G}_\mu}$. Thanks to the injectivity of the Laplace transform, $||\mathcal{F}\Phi||_{\mathcal{F}_\mu} = 0$ if and only if $\Phi = 0 \ dR_\mu \ -a.e.$, so if and only if $\mathcal{F}\Phi(z) = 0$ everywhere. Hence $\mathcal{F}_\mu$ is a Hilbert space. Moreover, the evaluation map at any point $z \in T_\Omega$ is a continuous linear (V_μ-valued) map. So $\mathcal{F}_\mu$ has a reproducing kernel $\mathbb{K}(z, w)$. By definition, there exists a measurable function $k : \overline{\Omega} \times T_\Omega \to \mathrm{End}\,(V_\mu)$, such that, for every $\xi \in V_\mu$

$$\mathbb{K}(z, w)\xi = \big(\mathcal{F}k(., w)\xi\big)(z), \quad z, w \in T_\Omega.$$

For every $\Phi \in \mathcal{G}_\mu$, and $w \in T_\Omega$,

$$\begin{aligned}\big((\mathcal{F}\Phi)(w)|\xi\big)_{V_\mu} &= \big(\mathcal{F}\Phi|\mathbb{K}(., w)\xi\big)_{\mathcal{F}_\mu}\\ &= \big(\Phi|k(., w)\xi\big)_{\mathcal{G}_\mu}.\end{aligned}$$

The first term is

$$\int_{\overline{\Omega}} e^{i<w|v>} \big(dR_\mu(2v)\Phi(v)|\xi\big)_{V_\mu},$$

whereas the last is

$$\int_{\overline{\Omega}} \big(dR_\mu(2v)\Phi(v)|k(v, w)\xi\big)_{V_\mu}.$$

We easily conclude that $k(v, w)\xi = e^{-i<\overline{w}|v>}\xi$, for R_μ-almost every v in $\overline{\Omega}$. Hence

$$\mathbb{K}(z, w)\xi \quad = \quad \int_{\overline{\Omega}} e^{i<z|v>} e^{-i<\overline{w}|v>} dR_\mu(2v)\xi \quad = \quad \mathbb{Q}_\mu(z, w)\xi.$$

Hence the following conclusion :

Theorem 3.5. *Let μ be a representation of $L^{\mathbb{C}}$ such that $\mathbb{Q}_\mu$ is positive definite. Let $\mathcal{F}_\mu$ be as above. Then $\mathcal{F}_\mu$ is a Hilbert space with reproducing kernel $\mathbb{Q}_\mu(z,w)$. The space $\mathcal{F}_\mu$ is stable under T_μ and the restriction of T_μ to $\mathcal{F}_\mu$ is unitary and irreducible.*

4. Some necessary conditions for the existence of the measure R_μ

Let μ be a holomorphic finite dimensional representation of $L^{\mathbb{C}}$. Still denote by μ the restricted highest weight of the representation μ with respect to the Iwasawa decomposition considered in section **1** and by ξ_μ a non-zero highest weight vector. To be more explicit, one has

$$\mu\big(\exp 2\sum_{k=1}^{r} a_i L(c_i)\big)\ \xi_\mu = \prod_{k=1}^{r} e^{a_k m_k}\ \xi_\mu\ ,$$

for all $(a_k)_{1\le k\le r} \in \mathbb{R}$, and $\mu(n)\xi_\mu = \xi_\mu$, for all $n \in N$. The restricted highest weights are characterized by the conditions

$$\forall\ 1 \le k \le r, \quad m_k \in \mathbb{Z} \quad \text{and } m_1 \le m_2 \le \ldots \le m_r\ .$$

(cf [4], p. 167). For further use, notice the formula $\mu(P(a))\xi_\mu = \prod_{k=1}^{r} a_k^{m_k}\ \xi_\mu$, where $a = \sum_{k=1}^{r} a_k c_k,\ a_k > 0,\ \forall k, 1 \le k \le r$.

The property (3.4i) clearly shows the fact that the support of R_μ is a union of L orbits. Because of the structure of these orbits, there is an integer p, with $0 \le p \le r$, such that $\mathrm{Supp}(R_\mu) \subset \overline{\mathcal{O}}_p$ and $\mathrm{Supp}(R_\mu) \not\subset \overline{\mathcal{O}}_{p-1}$.

Theorem 4.1. *Let $\mu = (m_1, m_2, \ldots, m_r)$ as above. A necessary condition for the existence of a measure R_μ satisfying the conditions (3.4i) and (3.4ii) and such that* $\mathrm{Supp}(R_\mu) = \overline{\Omega}$ *is :*

(4.1i)
$$m_r < -\frac{d(r-1)}{2}\ .$$

A necessary condition for the existence of a measure R_μ satisfying the conditions (3.4i) and (3.4ii) and such that $\mathrm{Supp}(R_\mu) = \overline{\mathcal{O}}_p$*, for some p, $0 \le p \le r-1$ is*

(4.1ii)
$$m_{p+1} = m_{p+2} = \ldots = m_r = -\frac{dp}{2}\ .$$

Proof. Assume first that $\mathrm{Supp}(R_\mu) = \overline{\mathcal{O}}_p$, for some $p, 0 \le p \le r-1$. Consider the restriction of R_μ to $\mathcal{O}_p$ as a distribution . It must coincide with a C^∞ function. In fact, let $X \in \mathfrak{l} = Lie(L) \subset \mathfrak{gl}(V)$. It induces a vector field $\tilde{X}$ on $\mathcal{O}_p$. The invariance property (3.4i) implies the differential relation :

$$\tilde{X} R_\mu = -\mu(X^t) \circ R_\mu - R_\mu \circ \mu(X)\ .$$

Choose vectors $X_1, X_2, \dots, X_k \in \mathfrak{l}$, such that $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_k$ form a basis of the tangent plane in a neigbourhood of some point of the the orbit $\mathcal{O}_p$ (say, e_p for example). Compute $\sum_{j=1}^k \tilde{X}_j^2 R_\mu$ near e_p using the last relation. It shows that R_μ is (near e_p) solution of a partial differential system, which is clearly elliptic. Hence, by the classical regularity results, R_μ has locally near e_p a C^∞ density w.r.t. the relatively invariant measure ν_p. From the invariance property (3.4i), this property is true everywhere on $\mathcal{O}_p$. In other words, there exists an analytic function $\rho_\mu : \mathcal{O}_p \to Herm^+(V_\mu)$, such that R_μ coincides with $\rho_\mu d\nu_p$ on $\mathcal{O}_p$. The invariance condition now reads :

$$\mu(l)^{*^{-1}} \rho_\mu(w) \mu(l)^{-1} = (\det\ l)^{s_p} \rho_\mu(lw) ,$$

for $l \in L$ and $w \in \mathcal{O}_p$.

Let $E_\mu = \rho_\mu(e_p)$. As $\mathrm{Supp}(R_\mu) = \overline{\mathcal{O}}_\mu$, $E_\mu \neq 0$. For $l \in L_p$, the invariance condition (3.4i) implies

$$E_\mu \circ \mu(l) = (\det l)^{-s_p} \mu(l)^{*^{-1}} \circ E_\mu .$$

Now let ξ_μ be a non-zero vector in V_μ of highest restricted weight μ, and consider the function $\Phi : L \to \mathbb{C}$ defined by $\Phi(l) = (E_\mu \mu(l)\xi_\mu | \mu(l)\xi_\mu)$. Recall that $L_p A_p N_p$ is dense in L, and take $l = l_p a_p n_p$, where $l_p \in L_p, a_p \in A_p$ and $n \in N_p$. Then

$$\Phi(l) = a_p^{2\mu}(E_\mu \circ \mu(l_p)\xi_\mu | \mu(l_p)\xi_\mu) = a_p^{2\mu}(\det l_p)^{-s_p}(\mu(l_p)^{*^{-1}} \circ E_\mu \xi_\mu | \mu(l_p)\xi_\mu)$$

$$= a_p^{2\mu}(\det l_p)^{-s_p}(E_\mu \xi_\mu | \xi_\mu) .$$

As $E_\mu \neq 0$, $\Phi(l)$ cannot be 0 for all $l \in L$, hence $(E_\mu \xi_\mu | \xi_\mu) \neq 0$. Now, for $(a_k)_{p+1 \le k \le r} \in \mathbb{R}^+$, consider the element

$$a = P(c_1 + c_2 + \cdots + c_p + a_{p+1}c_{p+1} + a_{p+2}c_{p+2} + \cdots + a_r c_r).$$

Now $\mu(a)\xi_\mu = a_{p+1}^{m_{p+1}} a_{p+2}^{m_{p+2}} \dots a_r^{m_r} \quad \xi_\mu$, whereas $\det(a) = (a_{p+1}a_{p+2}\dots a_r)^{\frac{2n}{r}}$. So $\Phi(a) = a^{2\mu}(E_\mu \xi_\mu | \xi_\mu) = (\det a)^{-s_p}(E_\mu \xi_\mu | \xi_\mu)$. Hence the relation

$$m_{p+1} = m_{p+2} = \ldots = m_r = -\frac{dp}{2} .$$

Let now consider the case where $\mathrm{Supp}(R_\mu) = \overline{\Omega}$. The first part of the preceding argument is still valid. In particular, the restriction of R_μ to Ω has an analytic density, say $\rho_\mu(x)$ with respect to the invariant measure d^*x. Let $E_\mu = \rho_\mu(e)$. It is still true that $(E_\mu \xi_\mu | \xi_\mu) \neq 0$, and the invariance condition now implies :

$$\rho(t(u)e) = \mu(t(u))^{*^{-1}} \rho_\mu(e)\ \mu(t(u))^{-1} \quad ,$$

for $u \in V^+$. Now the first condition (3.4ii) implies in particular

$$\int_\Omega e^{-trv} (\rho_\mu(v)\xi_\mu | \xi_\mu) d^*v < \infty .$$

Use the parametrization described in section **2** (cf [7] p. 123). As $\mu(t(u))^{-1}\xi_\mu = \prod_{j=1}^r u_j^{-m_j}\xi_\mu$, the integral converges if (and only if)

$$\int_o^{+\infty}\cdots\int_o^{+\infty}\prod_{j=1}^r u_j^{-2m_j}u_j^{-d(j-1)-1}e^{-u_j^2}du_j<\infty \quad .$$

But this happens if and only if $m_r < -\dfrac{d(r-1)}{2}$.

To finish the proof, observe that the conditions already obtained are mutually incompatible. So that, if $\mathrm{Supp}(R_\mu) = \overline{\mathcal{O}}_p$ and if ρ_μ is its density on $\mathcal{O}_p$, then the difference $dR_\mu - \rho_\mu(.)d\nu_p$ has its support contained in $\overline{\mathcal{O}}_{p-1}$ and still satisfies the condition (3.4ii). If it were non zero on $\mathcal{O}_{p-1}$, the first part of the proof would imply $m_r = -\frac{(p-1)d}{2}$, whereas the condition $m_r = -\frac{pd}{2}$ (or $m_r < -\frac{d(r-1)}{2}$ in case $p = r$) has been shown to be necessary. By induction we eventually get $dR_\mu - \rho_\mu(.)d\nu_p = 0$, completing the proof of theorem (4.1). ∎

5. An example

It seems in general quite hard to find explicit expressions for the measure R_μ. These measures are known when μ has dimension 1 (see [17]). Here we want to discuss a vector-valued case, where however, computations are easy because of the fact that the representation μ stays irreducible when restricted to the maximal compact subgroup K of L (see also [9]).

Let $H = H_r$ be the real vector space of $r \times r$ Hermitian matrices, and define the Jordan product to be $x.y = \frac{1}{2}(xy + yx)$, which turns H into a Euclidean Jordan algebra for the standard inner product $\mathrm{tr}\, xy$. The cone Ω is the cone of positive-definite matrices, the group L may be identified with $\mathbb{R}^+ \times \mathrm{SL}(r,\mathbb{C})$, where $\mathrm{SL}(r,\mathbb{C})$ acts by $l.x = lxl^*$ $(l \in \mathrm{SL}(r,\mathbb{C}), x \in H)$, its maximal compact subgroup K is $\mathrm{SU}(r)$ and for $x, y \in H$, $P(x)y = xyx$. As for a Jordan frame, the natural choice is

$$c_i = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, 1 \le i \le r \ ,$$

where 1 stands in the i th row and column. The corresponding Cartan subspace is

$$\mathfrak{a} = \left\{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_r \end{pmatrix} \quad , \quad a_1, a_2, \ldots, a_r \in \mathbb{R} \right\} \quad .$$

The dimension of H is $n = r^2$, and $d = 2$.

Let μ be a (finite-dimensional) representation of L of the form $l \mapsto \det(l)^m \nu(l)$, where m is an integer and ν is a *holomorphic* representation of $\mathrm{SL}(r,\mathbb{C})$, (for short we say μ is holomorphic) and still denote by μ its dominant weight $\mu = (m_1, m_2, \dots, m_r)$, where for $1 \le i \le r, m_i \in \mathbb{Z}$, and $m_1 \le m_2 \le \dots \le m_r$. Notice that the weights of the representation are *complex* linear forms on $\mathfrak{a}^{\mathbb{C}}$, and so are determined by their restrictions to $\mathfrak{a}$.

Theorem 5.1. *Let μ as above. The kernel $\mathbb{Q}_\mu$ is of positive definite type if and only if either :*

$$m_r < -(r-1) \tag{5.i}$$

$$m_{p+1} = m_{p+2} = \dots = m_r = -p \quad . \tag{5.ii}$$

Proof. In the first case, the corresponding measure R_μ is supported in $\overline{\Omega}$, whereas it is supported in $\overline{\mathcal{O}}_p$ in the second case. These measures are made explicit in due course of the proof.

Sticking to notations used in section **4**, first consider the functional equation for the regular orbit Ω. Observe that E_μ must commute with $\mu(l)$, when $l \in \mathrm{SU}(r)$. But by assumption μ is a holomorphic representation and so is still irreducible when restricted to $\mathrm{SU}(r)$. By Schur's lemma this implies the fact that E_μ must be a multiple of the identity. But now this forces the equality $\rho_\mu(x) = \mu(P(x))^{-1}$, for all $x \in \Omega$, up to a positive scalar. As the positivity condition is clearly satisfied, it remains to check the integrability condition. To this end, define

$$\mathcal{W}_\mu = \left\{ \xi \in \mathcal{V}_\mu | \int_\Omega e^{-\operatorname{tr} v} (\rho_\mu(v)\xi, \xi) d^*v \ < +\infty \right\}$$

Clearly by Schwarz inquality, $\mathcal{W}_\mu$ is a vector subspace, and it is invariant under K. As the restriction of μ to K is irreducible, $\mathcal{W}_\mu$ is 0 or $\mathcal{V}_\mu$, but $\mathcal{W}_\mu = \{0\}$ would imply $E_\mu = 0$. So $\mathcal{W}_\mu = \mathcal{V}_\mu$. So it suffices to check the integrability condition for, say, a highest weight vector. As the integrability condition for a highest weight vector was already tested in the general case, this finishes this case.

Now assume $\mathrm{Supp}(R_\mu) = \overline{\mathcal{O}}_p$, for some $p, 0 \le p \le r-1$. This forces $m_{p+1} = m_{p+2} = \dots = m_r = -p$. As before, let ρ_μ be the density with respect to the relatively invariant measure ν_p, and let $E_\mu = \rho_\mu(e_p)$. Let

$$l = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_r} \end{pmatrix},$$

where $\theta_1 + \dots + \theta_r \equiv 0 \quad mod\ 2\pi$. Observe that $l.e_p = e_p, l^* = l^{-1}$ and $|\det(l)| = 1$. The condition (3.4i) clearly implies that E_μ commutes with all matrices $\mu(l)$, when l is diagonal in $\mathrm{SU}(r)$. So E_μ preserves the weight spaces of

V_μ. Now let ξ_λ be a weight vector corresponding to the weight $\lambda = (l_1, l_2, \dots, l_r)$. Notice from the preceding remark that $E_\mu \xi_\lambda$ is also of weight λ. Let

$$l = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & a_{p+1} & & \\ & & & & \ddots & \\ & & & & & a_r \end{pmatrix},$$

where $a_{p+1}, a_{p+2}, \dots, a_r \in \mathbb{C}^*$, $a_{p+1}a_{p+2}\dots a_r \in \mathbb{R}$. Then $l \in L_p$,

$$E_\mu \xi_\lambda = |a_{p+1}a_{p+2}\dots a_r|^{-2p}|a_{p+1}|^{-2l_{p+1}}|a_{p+2}|^{-2l_{p+2}}\dots|a_r|^{-2l_r}E_\mu\xi_\lambda \ .$$

Hence, if $E_\mu\xi_\lambda \neq 0$, $l_{p+1} = l_{p+2} + \dots = l_r = -p$. Let $\mathcal{W}$ be the sum of all weight spaces with a weight satisfying this condition. $\mathcal{W}$ coincides with the submodule of $\mathcal{V}_\mu$ generated by the highest weight vector ξ_λ under the action of the subgroup

$$H_p = \left\{ \begin{pmatrix} h & 0 \\ 0 & \mathbf{1}_q \end{pmatrix}, h \in \mathrm{SL}(p,\mathbb{C}) \text{ and } q = r - p \right\}.$$

Clearly, $\mathcal{W}$ as H_p module is isomorphic with *the* highest weight module of $\mathrm{SL}(p,\mathbb{C})$ with highest weight $(m_1, m_2, \dots, m_p)$ and in particular is irreducible. Since μ is holomorphic, $\mathcal{W}$ is also irreducible under the action of the maximal compact subgroup K_p of H_p (isomorphic to p)). But E_μ commutes with $\mu(l)$ when l belongs to K_p, so is the identity (up to a scalar) on $\mathcal{W}$. In other terms, E_μ is (up to a positive scalar) the orthogonal projection on $\mathcal{W}$.

Consider now the representation $l \mapsto \det(l)^{-p}\mu(l^t)^{-1}$. Its lowest weight is $(-p - m_1, -p - m_2, \dots, -p - m_p, 0, 0, \dots, 0)$, so this representation can be extended polynomially to the full algebra $M_r(\mathbb{C})$. By checking on each weight vector, one verifies $\tilde{\mu}(e_p) = E_\mu$. By a simple computation using the condition (3.3i), this implies that $\rho_\mu(y) = \tilde{\mu}(y)$, for all $y \in \mathcal{O}_p$. For the integrability condition, one has (with obvious notations)

$$\int_{\mathcal{O}_p} e^{-\operatorname{tr} w}(\tilde{\mu}(w)\xi_\mu|\xi_\mu)d\nu_p(w)$$

$$= \int_0^{+\infty}\dots\int_0^{+\infty}\dots\int_{\mathbb{C}}\dots\int_{\mathbb{C}^{q\times p}} e^{-(a_1^2+\dots+a_p^2)}e^{-\|u\|^2}e^{-\|v\|^2}\dots$$

$$\dots a_1^{2(-m_1-p)}\dots a_p^{2(-m_p-p)}da_1\dots da_p\dots du_{ij}\dots d\overline{u}_{ij}\dots dv\ d\overline{v}\ ,$$

and the last integral converges, as $m_1 \leq m_2 \leq \dots \leq m_p \leq -p$. ∎

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