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Operators on Differential Forms for Lie Transformation Groups

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Abstract. For any Lie group action $S: G\times P \to P$, we introduce $C^\infty(P)$-linear operators $S^i_\bullet$, that transform n-forms $\omega_n \in \mathcal{A}_n(P,V)$ into $(n-i)$-forms $S^i_\bullet\omega_n \in \mathcal{A}_{n-i}(P,\mathrm{Alt}_i(\mathfrak{g},V))$. We compute the exterior derivative of these generated forms and their behavior under interior products with vector fields and Lie differentiation. By combination with Lie algebra valued forms $\theta \in \mathcal{A}_1(P,\mathfrak{g})$ and $\phi_p \in \mathcal{A}_p(P,\mathfrak{g})$, we recover V-valued forms $\omega \circledS \theta \in \mathcal{A}_n(P,V)$, resp., $(\chi^s_n \circledS \theta)\bullet\phi_p$ for $\chi^s_n \in \mathcal{A}_n(P,\mathrm{Hom}(\bigotimes^s \mathfrak{g},V))$ and compute their exterior derivative. The derived formulae play an important role for local evaluations of connections on fiber bundles.

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1. Motivation

Let $P(M,G)$ denote a principal bundle with base manifold $M = \bigcup_{\alpha\in A} U_\alpha$, projection $\pi: P \to M$, Lie group G, right action $R: P\times G \to P$ and local trivializations $\psi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times G$ with local projections $\pi_\alpha = \mathrm{pr}_G \circ \psi_\alpha$. Recall that any connection Γ on P defines horizontal and vertical projections of vector fields, not only on P, but also on every associated fiber bundle $B(M,F,G) = P\times_G F$ with fiber F and left action $L: G\times F \to F$, such that the vertical fields are tangential to the fiber. We thus obtain projections h, v of differential forms via

$$\omega h(\ldots,\mathcal{X}^i,\ldots) := \omega(\ldots,h\mathcal{X}^i,\ldots), \qquad \omega v(\ldots,\mathcal{X}^i,\ldots) := \omega(\ldots,v\mathcal{X}^i,\ldots) \quad (1)$$

for all V-valued forms $\omega \in \mathcal{A}(B,V)$. If we compute the vertical projections locally on the bundle charts $U_\alpha \times F$, we obtain with $L^f: G \to F$ defined by $L^f(g) := L(g,f)$ and its differential $(dL^f)_e: \mathfrak{g} \to T_f(F)$ at the neutral element $e\in G$:

$$(\phi^\alpha v^\alpha)_{(x,f)}(\ldots,(X^i,F^i),\ldots) = \phi^\alpha_{(x,f)}(\ldots,(0,(dL^f)_e \mathrm{A}^\alpha_x(X^i)+F^i),\ldots) \quad (2)$$

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for all $\phi^\alpha \in \mathcal{A}(U_\alpha \times F, V)$ and $(X^i, F^i) \in T_x(U_\alpha) \oplus T_f(F)$. Here $\mathrm{A}^\alpha \in \mathcal{A}_1(U_\alpha, \mathfrak{g})$ mean the LIE algebra valued gauge potentials, which one obtains from the connection 1-form $\omega^\Gamma \in \mathcal{A}_1(P, \mathfrak{g})$ as pullbacks $\mathrm{A}^\alpha := \sigma^\star_{\alpha,e}\omega^\Gamma$ under the local sections $\sigma_{\alpha,e}: U_\alpha \to \pi^{-1}(U_\alpha)$ defined by $\sigma_{\alpha,e}(x) := \psi_\alpha^{-1}(x, e)$.

Instead of (2), we would like to obtain a handier formula for these local projections, which is merely expressed in terms of the involved forms ϕ and A^α. Especially if ϕ is a (pullback of a) form on F, this is indeed possible. Moreover, we will also be able to give a formula for the exterior derivative $d(\phi v)$. Such formulas are quite essential if one tries to combine the DE-RHAM cohomology of a fiber bundle with connections on that bundle.

For any $X \in \mathfrak{g}$, let $\mathcal{L}_X \in \mathcal{D}^1(F)$ denote the induced vector field on F that is given by $(\mathcal{L}_X)_f := (dL^f)_e(X)$ and let $\mathcal{L}: \mathfrak{g} \to \mathcal{D}^1(F)$ denote the LIE algebra antihomomorphism defined hereby. Then $\mathcal{L} \circ \mathrm{A}^\alpha$ maps vector fields on U_α to vector fields on F. For a n-form $\phi \in \mathcal{A}_n(F, V)$, we may decompose $(\mathrm{pr}_F^\star \phi)v^\alpha$ according to (2) into a sum of differential forms $\chi_i \in \mathcal{A}_n(U_\alpha \times F, V)$, $i = 0, \ldots, n$, such that χ_i acts on $n-i$ vertical fields and i horizontal vector fields via $\mathcal{L} \circ \mathrm{A}^\alpha$. The forms χ are obtained from ϕ in two steps: using a product of i maps $\mathcal{L}$, we first transform ϕ into a $(n-i)$-form $L^i_\bullet \phi \in \mathcal{A}_{n-i}(F, \mathrm{Alt}_i(\mathfrak{g}, V))$. Then we combine that form with i factors of A^α in such a way that for any $\mathcal{X} \in \mathcal{D}^1(U_\alpha)$ the maps $\mathrm{pr}^\star_{U_\alpha}[\mathrm{A}^\alpha(\mathcal{X})] \in C^\infty(U_\alpha \times F, \mathfrak{g})$ serve as input for the maps in $\mathrm{Alt}_i(\mathfrak{g}, V)$. The resulting form will be denoted by $[\mathrm{pr}_F^\star(L^i_\bullet \phi)] \bullet (\mathrm{pr}^\star_{U_\alpha} \mathrm{A}^\alpha)$.

Such a construction is possible for all LIE groups that act on a differentiable manifold from the right or the left. Thus we will choose the general framework of LIE transformation groups in the sequel. For notational convenience, we will recall the basic definitions from differential geometry for LIE group actions. Then we introduce the operators $L^i_\bullet$ and $R^i_\bullet$ for a left, resp., right action and compute, in how far they commute with exterior differentiation d of forms, interior products $\imath_\mathcal{X}$ with respect to a vector field $\mathcal{X}$ and LIE differentiation $\mathrm{L}_\mathcal{X}$, which is given by $\mathrm{L}_\mathcal{X} = \imath_\mathcal{X} \circ d + d \circ \imath_\mathcal{X}$. The operator $\bullet$ has already been discussed in detail in [1], thus we only recall its definition and main properties for our purposes. Finally we introduce operators Ⓛ and Ⓡ , [such that $(\mathrm{pr}_F^\star \phi)v^\alpha$ is indeed given by $(\mathrm{pr}_F^\star \phi)$ Ⓛ $(\mathrm{pr}^\star_{U_\alpha} \mathrm{A}^\alpha)$] and compute the exterior derivative of ω_n Ⓛ θ, resp., ω_n Ⓡ θ.

2. Basic definitions

Let us first recall some of the definitions we have already used above, according to HELGASON [3] and KOBAYASHI, NUMIZU [4]. For any vector spaces V and W, $\mathrm{Alt}_p(W, V)$ and $\mathrm{Sym}_p(W, V)$ denote the vector spaces of all alternating, resp., symmetric p-linear maps from W^p to V. For convenience we define $\mathrm{Sym}^\pm(W, V) := \bigoplus_{p=0}^\infty \mathrm{Sym}^\pm_p(W, V)$ by $\mathrm{Sym}^+_p(W, V) := \mathrm{Sym}_p(W, V)$ and $\mathrm{Sym}^-_p(W, V) := \mathrm{Alt}_p(W, V)$.

If $f: M \to N$ is differentiable, we denote the differential of f at $x \in M$ by df_x. We have $[df_x(\mathcal{X}_x)]g = \mathcal{X}_x(g \circ f)$ for all $\mathcal{X}_x \in T_x(M)$, $g \in C^\infty(N)$. If in addition, f is a diffeomorphism then for $\mathcal{X} \in \mathcal{D}^1(M)$ the push-out $f_\star \mathcal{X} \in \mathcal{D}^1(N)$ is defined by $(f_\star \mathcal{X})_{f(x)} = df_x(\mathcal{X}_x)$ for all $x \in M$.

For $\alpha \in \mathcal{A}_r(N, V)$, $r \in \mathbb{N}$ and $X_i \in T_x(M)$, the pullback $f^\star\alpha \in \mathcal{A}_r(M, V)$ is defined by $(f^\star\alpha)_x(X_1, \ldots, X_r) = \alpha_{f(x)}(df_x(X_1), \ldots, df_x(X_r))$. For $\alpha \in C^\infty(N, V)$

we have $f^\star\alpha := \alpha \circ f$, linear extension defines the pullback on $\mathcal{A}(N,V)$. If we insert $\mathcal{A}(M)\otimes V$ into $\mathcal{A}(M,V)$ in a natural way, then obviously $f^\star(\mathcal{A}(N)\otimes V) \subseteq \mathcal{A}(M)\otimes V$. (If V is finite dimensional, we will identify $\mathcal{A}(M)\otimes V$ and $\mathcal{A}(M,V)$.)

Let $\mathcal{T}(V)$ denote the tensor algebra of V. Then every linear map $\Lambda: V \to W$ defines a pullback $\Lambda^\star: \mathrm{Hom}(\mathcal{T}(W),Z) \to \mathrm{Hom}(\mathcal{T}(V),Z)$: for $K \in \mathrm{Hom}(\bigotimes^p W, Z)$, $X_i \in V$ we have $\Lambda^\star K(X_1,\ldots,X_p) = K(\Lambda(X_1),\ldots,\Lambda(X_p))$, so $\Lambda^\star(\mathrm{Sym}^\pm(W,Z)) \subseteq \mathrm{Sym}^\pm(V,Z)$. Λ also defines a push-out $\Lambda_\star: \mathcal{A}(M,V) \to \mathcal{A}(M,W)$ by $\Lambda_\star\omega = \Lambda\circ\omega$. Again $\Lambda_\star(\mathcal{A}(M)\otimes V) \subseteq \mathcal{A}(M)\otimes W$, where we have $\Lambda_\star(\alpha\otimes v) = \alpha\otimes\Lambda(v)$ for all $\alpha \in \mathcal{A}(M)$, $v \in V$. For an example, let $E_j \in W$, $j = 1,\ldots,s$ and let $E_1\otimes\cdots\otimes E_s: \mathrm{Hom}(\bigotimes^s W, V) \to V$ denote the canonical evaluation morphism. For any differential form $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Hom}(\bigotimes^s W, V))$ define $\chi_r^{E_1,\ldots,E_s} \in \mathcal{A}_r(M,V)$ to be the push-out of χ_r^s under this morphism: $\chi_r^{E_1,\ldots,E_s} := (E_1\otimes\cdots\otimes E_s)_\star\chi_r^s$, i. e., for all $x \in M$ and $\mathcal{X}^i \in \mathcal{D}^1(M)$, $i = 1,\ldots,r$,

$$(\chi_r^{E_1,\ldots,E_s})_x(\mathcal{X}_x^1,\ldots,\mathcal{X}_x^r) := (E_1\otimes\cdots\otimes E_s)\circ(\chi_r^s)_x(\mathcal{X}_x^1,\ldots,\mathcal{X}_x^r). \tag{3}$$

In the sequel, G will always mean a LIE group with LIE algebra $\mathfrak{g}$, left and right multiplication $\lambda,\rho: G \to G$ and inversion $\eta: G \to G$. For $S = L, R$ let $S: G\times P \to P$ denote a left, resp., right LIE group action. We identify S with $S: G \to \mathrm{Diff}(P)$. Also for notational convenience, we always write G on the left, even if S denotes a right action. In that case, we put $\mathrm{sgn}(S) := 1$, whereas $\mathrm{sgn}(S) := -1$, if S denotes a left action. Since S is differentiable, all maps $S^p: G \to P$, $p \in P$, resp., $S_g: P \to P$, $g \in G$, defined by $S^p(g) := S_g(p) := S(g,p)$, are differentiable, resp., diffeomorphisms.

G is called a LIE *transformation group* of the manifold P. If P is a vector space and the action is linear, we speak of a *representation* of G, e. g., the adjoint action $\mathrm{Ad}: G \to \mathrm{Gl}(\mathfrak{g})$ is a left representation. The *trivial action* means the natural projection $\mathrm{pr}_P: G\times V \to V$.

An action is *effective* if $S_g = \mathrm{id}_P$ only for $g = e$. In that case G may be thought of as a subgroup of $\mathrm{Diff}(P)$. An action is *free* if (in addition) $S_g(p) = p$ only for $g = e$ for all $p \in P$. Via λ and ρ every LIE group acts freely on itself.

$\omega \in \mathcal{A}(P,V)$ is called G*-invariant* or simply *invariant* if $S_g^\star\omega = \omega$ for all $g \in G$. Denote their set by $\mathcal{A}(P,V)_{\mathrm{inv}}$. Analoguosly for any subgroup $H < G$, we define $\mathcal{A}(P,V)_{H-\mathrm{inv}}$ to be the set of H-invariant forms, i. e., those forms that are invariant under the restriction of S onto $H\times P$. Especially we will use this notation for G_e-invariant forms, where G_e is the connected component of $e \in G$.

$\mathcal{A}(P)_{\mathrm{inv}}$ and $\mathcal{A}(P)_{\mathrm{inv}}\otimes V$ are graded subalgebras of $\mathcal{A}(P)$, resp., $\mathcal{A}(P)\otimes V$ (whenever a wedge product $\wedge_V$ of V-valued forms is given by a bilinear mapping $m: V\times V \to V$), with $d(\mathcal{A}(P)_{\mathrm{inv}}) \subseteq \mathcal{A}(P)_{\mathrm{inv}}$. Analogous statements hold for $\mathcal{A}(P)_{H-\mathrm{inv}}$ and $\mathcal{A}(P)_{H-\mathrm{inv}}\otimes V$, which are modules of $\mathcal{A}(P)_{\mathrm{inv}}$. Obviously $\mathcal{A}(P)_{\mathrm{inv}} \subseteq \mathcal{A}(P)_{H-\mathrm{inv}}$ and $\mathcal{A}(P,V)_{\mathrm{inv}} \subseteq \mathcal{A}(P,V)_{H-\mathrm{inv}}$ for any subgroup $H < G$.

Lemma 2.1. *If $S: G\times P \to P$ is a* LIE *group action then $S_\star: G\times\mathcal{D}^1(P) \to \mathcal{D}^1(P)$, $S^\star\circ\eta: G\times\mathcal{A}(P,V) \to \mathcal{A}(P,V)$, $S': G\times\mathcal{A}(P,\mathrm{Hom}(\mathcal{T}(\mathfrak{g}),V)) \to \mathcal{A}(P,\mathrm{Hom}(\mathcal{T}(\mathfrak{g}),V))$ and $S'': G\times\mathcal{A}(P,\mathfrak{g}) \to \mathcal{A}(P,\mathfrak{g})$ defined by*

$$(S_\star)_g(\mathcal{X}) \;:=\; (S_g)_\star\mathcal{X} \qquad \textit{for all} \quad \mathcal{X} \in \mathcal{D}^1(P),$$

$$\begin{aligned}
(S^\star \circ \eta)_g(\omega) &:= (S_{g^{-1}})^\star \omega \quad \textit{for all} \quad \omega \in \mathcal{A}(P,V),\\
S'_g(\chi) &:= (S_{g^{-1}})^\star (\mathrm{Ad}(g^{\mathrm{sgn}(S)})^\star)_\star \chi \quad \textit{for all} \quad \chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}),V)) \quad \textit{and}\\
S''_g(\varphi) &:= (S_{g^{-1}})^\star \,\mathrm{Ad}(g^{-\mathrm{sgn}(S)})_\star \varphi \quad \textit{for all} \quad \varphi \in \mathcal{A}(P,\mathfrak{g}),
\end{aligned}$$

are all representations of G on the same side.

Let S, S' be two actions of G on spaces X, resp., X' on the same side. A mapping $f\colon X \to X'$ is called *G-equivariant*, if

$$\begin{array}{ccc}
G \times X & \xrightarrow{\quad S \quad} & X\\
\Big\downarrow \mathrm{id}\times f & & \Big\downarrow f\\
G \times X' & \xrightarrow{\quad S' \quad} & X'
\end{array}$$

commutes, i. e., if $S'(g, f(x)) = f(S(g,x))$ for all $x \in X$ and $g \in G$.

If S is a LIE group action on P and R is a right representation on W, then a differential form $\omega \in \mathcal{A}(P,W)$ is called *G-equivariant*, if $S_g^\star \chi = R(g^{\mathrm{sgn}(S)})_\star \chi$ for all $g \in G$ (resp., if $S_g^\star \chi = L(g^{-\mathrm{sgn}(S)})_\star \chi$ for a left representation L). Thus — referring to the right representation $\mathrm{Ad}^\star$ on $W = \mathrm{Hom}(\mathcal{T}(\mathfrak{g}),V)$ — we call $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}),V))$ G-equivariant, if χ is invariant under S'. Analogously, $\varphi \in \mathcal{A}(P,\mathfrak{g})$ will be called G-equivariant if φ is invariant under S''. We denote the set of equivariant forms by $\mathcal{A}(P,W)_{\mathrm{equiv}}$. It is a module over $\mathcal{A}(P)_{\mathrm{inv}}$.

If G is compact with HAAR measure μ we have projections onto invariant and G-equivariant forms defined in the following way:

$$\begin{aligned}
\omega_{\mathrm{inv}} &:= \int_G S_g^\star \omega \, d\mu(g) && \text{for all } \omega \in \mathcal{A}(P,V),\\
\chi_{\mathrm{equiv}} &:= \int_G (\mathrm{Ad}(g^{-\mathrm{sgn}(S)})^\star)_\star S_g^\star \chi \, d\mu(g) && \text{for all } \chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}),V)),\\
\varphi_{\mathrm{equiv}} &:= \int_G \mathrm{Ad}(g^{\mathrm{sgn}(S)})_\star S_g^\star \varphi \, d\mu(g) && \text{for all} \quad \varphi \in \mathcal{A}(P,\mathfrak{g}).
\end{aligned}$$

As already introduced in the previous section, every $X \in \mathfrak{g}$ induces a canonical complete vector field $\mathcal{S}_X \in \mathcal{D}^1(P)$ by $(\mathcal{S}_X)_p := (dS^p)_e(X)$, so

$$\begin{aligned}
(\mathcal{S}_X)_p(f) &= (dS^p)_e(X)(f) = \frac{d}{dt} f(S_{e^{tX}}(p))|_{t=0} \quad \text{for all } f \in C^\infty(P),\ p \in P,\\
[\mathcal{S}_X, \mathcal{Y}]_p &= \lim_{t\to 0} \frac{1}{t}\{\mathcal{Y}_p - ((S_{e^{tX}})_\star \mathcal{Y})_p\} = \lim_{t\to 0} \frac{1}{t}\{((S_{e^{-tX}})_\star \mathcal{Y})_p - \mathcal{Y}_p\} \text{ for all } \mathcal{Y} \in \mathcal{D}^1(P).
\end{aligned}$$

$\mathcal{R}\colon \mathfrak{g} \to \mathcal{D}^1(P)$ and $-\mathcal{L}\colon \mathfrak{g} \to \mathcal{D}^1(P)$ are LIE algebra homomorphisms and

$$\begin{aligned}
[\mathcal{R}_X, \mathcal{R}_Y] &= \mathcal{R}_{[X,Y]}, \quad [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[Y,X]} = -\mathcal{L}_{[X,Y]} \quad \text{for all} \quad X, Y \in \mathfrak{g},\\
(R_{g^{-1}})_\star \mathcal{R}_X &= \mathcal{R}_{\mathrm{Ad}(g)X}, \quad (L_g)_\star \mathcal{L}_X = \mathcal{L}_{\mathrm{Ad}(g)X} \quad \text{for all} \quad g \in G,\ X \in \mathfrak{g}.
\end{aligned}$$

Obviously $\mathcal{S} = \mathcal{L}, \mathcal{R}$ only depends on the restriction of S onto $G_e \times P$. For all forms $\omega \in \mathcal{A}(P) \otimes V$ we have $\mathrm{L}_{\mathcal{S}_X}\omega = [\frac{d}{dt}((S_{e^{tX}})^\star \omega)|_{t=0}]$ and for all $X, Y \in \mathfrak{g}$:

$$[\mathrm{L}_{\mathcal{S}_X}, \mathrm{L}_{\mathcal{S}_Y}] = \mathrm{L}_{[\mathcal{S}_X, \mathcal{S}_Y]} = \mathrm{sgn}(S)\mathrm{L}_{\mathcal{S}_{[X,Y]}}, \quad [\mathrm{L}_{\mathcal{S}_X}, d] = 0, \tag{4}$$

$$[\mathrm{L}_{\mathcal{S}_X}, \imath_{\mathcal{S}_Y}] = \imath_{[\mathcal{S}_X, \mathcal{S}_Y]} = \mathrm{sgn}(S)\imath_{\mathcal{S}_{[Y,X]}}, \quad \mathrm{L}_{\mathcal{S}_X} = \imath_{\mathcal{S}_X} \circ d + d \circ \imath_{\mathcal{S}_X}. \tag{5}$$

We call a differential form $\omega \in \mathcal{A}(P)$ $\mathfrak{g}$-*invariant* if $\mathrm{L}_{\mathcal{S}_X}\omega = 0$ for all $X \in \mathfrak{g}$. Analogously, ω will be called *horizontal* if $\imath_{\mathcal{S}_X}\omega = 0$ for all $X \in \mathfrak{g}$. Denote their sets by $\mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}}$, resp., $\mathcal{A}(P)h$ and let $\mathcal{A}(P)h_{\mathfrak{g}-\mathrm{inv}} := \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \cap \mathcal{A}(P)h$.

The notion of "horizontal" forms is due to the fact that for a principal bundle $P(M,G)$, the horizontal forms in the sense of (1) are exactly those forms ω with $\imath_{\mathcal{R}_X}\omega = 0$ for all $X \in \mathfrak{g}$ with respect to the free right action R on P.

Since $\imath_{\mathcal{X}}$ and $\mathrm{L}_{\mathcal{X}}$ are (skew-)derivations of $\mathcal{A}(P)$ and $G_e = \langle \exp \mathfrak{g} \rangle$, we get:

Lemma 2.2. *$\mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}}$, $\mathcal{A}(P)h$ and $\mathcal{A}(P)h_{\mathfrak{g}-\mathrm{inv}}$ are graded subalgebras of $\mathcal{A}(P)$ with $d(\mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}}) \subseteq \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}}$ and $d(\mathcal{A}(P)h_{\mathfrak{g}-\mathrm{inv}}) \subseteq \mathcal{A}(P)h_{\mathfrak{g}-\mathrm{inv}}$. Analogous statements hold for $\mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V$ and $\wedge_V$, etc.*

$\mathcal{A}(P)_{\mathrm{inv}} \otimes V \subseteq \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V = \mathcal{A}(P)_{G_e-\mathrm{inv}} \otimes V$ for every vector space V. If G is connected then $\mathcal{A}(P)_{\mathrm{inv}} \otimes V = \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V$.

Lemma 2.3. *$\mathcal{S}\colon \mathfrak{g} \to \mathcal{D}^1(P)$ induces a G-equivariant $C^\infty(P)$-module homomorphism $\mathcal{S}'\colon C^\infty(P,\mathfrak{g}) \to C^\infty(P)\mathcal{S}(\mathfrak{g}) \subseteq \mathcal{D}^1(P)$ (with respect to S'' and $S_\star$). If G acts effectively on P then $\mathcal{S}$ is injective. If G acts freely on P then even $(dS^p)_e$ is injective for all $p \in P$, thus $X \neq 0$ yields $(\mathcal{S}_X)_p \neq 0$ for all $p \in P$; for every basis $\{E_i\}_{i,\ldots,\dim \mathfrak{g}}$ for $\mathfrak{g}$, $\{\mathcal{S}_{E_i}\}_{i,\ldots,\dim \mathfrak{g}}$ is then a basis for the free $C^\infty(P)$-module $C^\infty(P)\mathcal{S}(\mathfrak{g})$ and the induced $\mathcal{S}'$ is an isomorphism of free $C^\infty(P)$-modules.*

Proof. Assume that G acts effectively. Let $X \in \mathfrak{g}$ and suppose $(\mathcal{S}_X)_p(f) = 0$ for all $f \in C^\infty(P)$ and all $p \in P$. For $p = S(e^{sX}, p')$ this yields $\frac{d}{dt} f(S_{e^{(t+s)X}}(p'))|_{t=0} = \frac{d}{dt} f(S_{e^{tX}}(p'))|_{t=s} = 0$ for all $f \in C^\infty(P)$, $p' \in P$ and $s \in \mathbb{R}$. Thus $S(e^{tX}, p') = p'$ for all $p' \in P$ and $t \in \mathbb{R}$, and thus $X = 0$ since S is effective. Analogously for a free action, one proves injectivity of $(dS^p)_e$ for all $p \in P$ using $(\mathcal{S}_X)_{S(e^{sX},p)} = dS_{e^{sX}}(\mathcal{S}_X)_p$. But then all S_{E_i} are independent over $C^\infty(P)$, since they are independent for all $p \in P$. ∎

Finally we need the notion of $\mathfrak{g}$-equivariant forms. Just as $\mathrm{Ad}\colon G \to \mathrm{Gl}(\mathfrak{g})$ induces the representation $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$ with $\mathrm{ad}(X)(Y) = [X,Y]$, every representation $S\colon G \to \mathrm{Gl}(W)$ of a LIE group G induces a representation $s = dS_e\colon \mathfrak{g} \to \mathrm{gl}(W)$ of $\mathfrak{g}$ such that $S \circ \exp X = e^{sX}$ for all $X \in \mathfrak{g}$. We will identify s with the corresponding bilinear mapping $s\colon \mathfrak{g} \times W \to W$ given by $s(X,w) := s_X(w) := (dS^w)_e(X)$. From this point of view, $\mathcal{R}$ and $\mathcal{L}\colon \mathfrak{g} \to \mathcal{D}^1(P) = \mathrm{der}\, C^\infty(P)$ are the (infinite dimensional) representations induced by the LIE group representations $R^\star$ and $L^\star\colon G \to \mathrm{Aut}(C^\infty(P))$.

Let $\mathrm{ad}^\star\colon \mathfrak{g} \times \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V) \to \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V)$ denote the bilinear mapping induced by $\mathrm{Ad}^\star$. Then for $X, X_i \in \mathfrak{g}$, $p \in \mathbb{N}_0$ and $K \in \mathrm{Hom}(\bigotimes^p \mathfrak{g}, V)$, we have

$$(\mathrm{ad}^\star_X K)(X_1, \ldots, X_p) = \sum_{i=1}^{p} K(X_1, \ldots, [X, X_i], \ldots, X_p). \tag{6}$$

If S is a LIE group action of G on P and $S'\colon G \to \mathrm{Gl}(W)$ is a representation, then a differential form $\chi \in \mathcal{A}(P) \otimes W$ will be called $\mathfrak{g}$-*equivariant* if

$$\mathrm{L}_{\mathcal{S}_X}\chi = \mathrm{sgn}(S)\,\mathrm{sgn}(S')\, s'(X)_\star \chi \qquad \text{for all} \quad X \in \mathfrak{g}. \tag{7}$$

Thus $\chi \in \mathcal{A}(P) \otimes \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V)$ is $\mathfrak{g}$-equivariant if $\mathrm{L}_{\mathcal{S}_X}\chi = \mathrm{sgn}(S)(\mathrm{ad}^\star_X)_\star \chi$ for all $X \in \mathfrak{g}$. We will denote the vector space of $\mathfrak{g}$-equivariant forms by $\mathcal{A}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes W$. Analogously to Lemma 2.2 we obtain:

Lemma 2.4. *$\mathcal{A}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes W$ is a $\mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}}$-module with $d(\mathcal{A}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes W) \subseteq \mathcal{A}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes W$ and $\mathcal{A}(P)_{\mathrm{equiv}} \otimes W \subseteq \mathcal{A}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes W = \mathcal{A}(P)_{G_e-\mathrm{equiv}} \otimes W$ for all vector spaces W. If G is connected then $\mathcal{A}(P)_{\mathrm{equiv}} \otimes W = \mathcal{A}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes W$.*

It is an elementary result in differential geometry (e. g., cf. [4, p. 34]) that

$$(L_{\mathcal{X}}\omega)(\mathcal{P}^1, \dots, \mathcal{P}^n) = \mathcal{X}(\omega(\mathcal{P}^1, \dots, \mathcal{P}^n)) - \sum_{i=1}^{n} \omega(\mathcal{P}^1, \dots, [\mathcal{X}, \mathcal{P}^i], \dots, \mathcal{P}^n)$$

for all $\mathcal{X}, \mathcal{P}^i \in \mathcal{D}^1(P)$ and $\omega \in \mathcal{A}(P) \otimes V$. This yields the following corollaries:

Corollary 2.5. *If $\omega \in \mathcal{A}_n(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V$, then for all $X \in \mathfrak{g}$ and $\mathcal{P}^i \in \mathcal{D}^1(P)$*

$$\mathcal{S}_X(\omega(\mathcal{P}^1, \dots, \mathcal{P}^n)) = \sum_{i=1}^{n} \omega(\mathcal{P}^1, \dots, [\mathcal{S}_X, \mathcal{P}^i], \dots, \mathcal{P}^n).$$

Corollary 2.6. *Let $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ be $\mathfrak{g}$-equivariant. Then for all $p \in P$, $\mathcal{P}^i \in \mathcal{D}^1(P)$ and $X, E_i \in \mathfrak{g}$:*

$$\begin{aligned}
&(\mathrm{L}_{\mathcal{S}_X}\chi_n^s)(\mathcal{P}^1, \dots, \mathcal{P}^n)(p)(E_1 \otimes \dots \otimes E_s) = \\
&= \Big\{\mathcal{S}_X(\chi_n^s(\mathcal{P}^1, \dots, \mathcal{P}^n)) - \sum_{i=1}^{n} \chi_n^s(\mathcal{P}^1, \dots, [\mathcal{S}_X, \mathcal{P}^i], \dots, \mathcal{P}^n)\Big\}(p)(E_1 \otimes \dots \otimes E_s) \\
&= \mathrm{sgn}(S) \sum_{j=1}^{s} \chi_n^s(\mathcal{P}^1, \dots, \mathcal{P}^n)(p)(E_1 \otimes \dots \otimes E_{j-1} \otimes [X, E_j] \otimes E_{j+1} \otimes \dots \otimes E_s).
\end{aligned}$$

3. Mapping invariant forms onto equivariant forms

Now everything is prepared for the definition of the operators $S_\bullet^i$ which map V-valued forms onto $\mathrm{Alt}_i(\mathfrak{g}, V)$-valued forms:

Definition 3.1. Let S be a LIE group action of G on P and $\omega_n \in \mathcal{A}_n(P, V)$. We define $S_\bullet^i \omega_n \in \mathcal{A}_{n-i}(P, \mathrm{Alt}_i(\mathfrak{g}, V))$, $i \le n$, for all $\mathcal{P}^j \in \mathcal{D}^1(P)$, $E_k \in \mathfrak{g}$ and $p \in P$ by

$$[(S_\bullet^i \omega_n)(\mathcal{P}^1, \dots, \mathcal{P}^{n-i})(p)](E_1, \dots, E_i) := \tfrac{n!}{(n-i)!}\omega_n(\mathcal{S}^1, \dots, \mathcal{S}^i, \mathcal{P}^1, \dots, \mathcal{P}^{n-i})(p) \in V,$$

where $\mathcal{S}^i := \mathcal{S}_{E_i}$. Thus $S_\bullet^i \omega_n \in \mathcal{A}_{n-i}(P) \otimes \mathrm{Alt}_i(\mathfrak{g}, V)$ if $\omega_n \in \mathcal{A}_n(P) \otimes V$. For $i > n$ we put $S_\bullet^i \omega_n = 0$.

The factor $\frac{n!}{(n-i)!}$ is inherited from the definition of the interior product with vector fields: recall $(S_\bullet^i \omega_n)_{n-i}^{E_1, \dots, E_i} \in \mathcal{A}_{n-i}(P, V)$ for $E_k \in \mathfrak{g}$ from (3), then we have

$$(S_\bullet^i \omega_n)_{n-i}^{E_1, \dots, E_i} = (\imath_{\mathcal{S}^i} \circ \dots \circ \imath_{\mathcal{S}^1})\omega_n. \tag{8}$$

If $\{E_k\}$ is a base for $\mathfrak{g}$, we obtain for $\omega \in \mathcal{A}_n(P)$ and $v \in V$:

$$[S_\bullet^i(\omega \otimes v)](\mathcal{P}^1, \dots, \mathcal{P}^{n-i}) = \tfrac{n!}{(n-i)!} \sum_{k_1 < \dots < k_i} \omega(\mathcal{S}^{k_1}, \dots, \mathcal{S}^{k_i}, \mathcal{P}^1, \dots, \mathcal{P}^{n-i}) \otimes [(E_{k_1} \wedge \dots \wedge E_{k_i}) \mapsto v].$$

Lemma 3.2. *For all* $i \leq n$, $S^i_\bullet \colon \mathcal{A}_n(P,V) \to \mathcal{A}_{n-i}(P, \mathrm{Alt}_i(\mathfrak{g},V))$ *is* $C^\infty(P)$*-linear. For* $\omega_n \in \mathcal{A}_n(P,V)$, $\chi^s_n \in \mathcal{A}_n(P, \mathrm{Alt}_s(\mathfrak{g},V))$ *and* $i+j \leq n$, *we have*

$$\begin{aligned} S^0_\bullet \omega_n &= \omega_n, \qquad (S^n_\bullet \omega_n)(p) = n!\,[(S^p)^\star \omega_n]_e \quad \text{for all} \quad p \in P, & (9)\\ S^i_\bullet(\Lambda_\star \omega_n) &= \Lambda_\star(S^i_\bullet \omega_n) \qquad \text{for all} \quad \Lambda \in \mathrm{Hom}(V,W), & (10)\\ S^\star_g(S^i_\bullet \omega_n) &= (\mathrm{Ad}(g^{\mathrm{sgn}(S)})^\star)_\star [S^i_\bullet(S^\star_g \omega_n)], \qquad \text{thus} & (11)\\ S^\star_g(S^i_\bullet \chi^s_n) &= (\mathrm{Ad}(g^{\mathrm{sgn}(S)})^\star)_\star (S^i_\bullet \chi^s_n), \quad \text{if} \quad S^\star_g \chi^s_n = (\mathrm{Ad}(g^{\mathrm{sgn}(S)})^\star)_\star \chi^s_n. & (12) \end{aligned}$$

Let $f^{i,j} \colon \mathrm{Alt}_{i+j}(\mathfrak{g},V) \hookrightarrow \mathrm{Alt}_i(\mathfrak{g}, \mathrm{Alt}_j(\mathfrak{g},V))$ *denote the injection defined by*

$$[f^{i,j}(a)](E_1,\dots,E_i)(F_1,\dots,F_j) := a(E_1,\dots,E_i,F_1,\dots,F_j) \ \text{for} \ a \in \mathrm{Alt}_{i+j}(\mathfrak{g},V).$$

Then

$$f^{i,j}_\star(S^{i+j}_\bullet \omega_n) = (-1)^{ij} S^i_\bullet(S^j_\bullet \omega_n). \tag{13}$$

Lemma 3.3. *For all* $i \leq n$ *we have:*

$$\begin{aligned} S^i_\bullet(\mathcal{A}_n(P,V)_{\mathrm{inv}}) &\subseteq \mathcal{A}_{n-i}(P, \mathrm{Alt}_i(\mathfrak{g},V))_{\mathrm{equiv}},\\ S^i_\bullet(\mathcal{A}_n(P)_{\mathrm{inv}} \otimes V) &\subseteq \mathcal{A}_{n-i}(P)_{\mathrm{equiv}} \otimes \mathrm{Alt}_i(\mathfrak{g},V),\\ S^i_\bullet(\mathcal{A}_n(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V) &\subseteq \mathcal{A}_{n-i}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes \mathrm{Alt}_i(\mathfrak{g},V). \end{aligned}$$

Proof. (12) yields that $S^i_\bullet \omega_n$ is G-equivariant if ω_n is invariant under S. Now the operators $S^i_\bullet$ only depend on the restriction of S to $G_e \times P$. Thus Lemmas 2.2 and 2.4 prove that $S^i_\bullet \omega_n$ is $\mathfrak{g}$-equivariant if ω_n is $\mathfrak{g}$-invariant. ∎

Let us compute in how far the operators $S^i_\bullet$ commute with the exterior differentiation, interior products with vector fields and Lie differentiation.

Lemma 3.4. *Let* S *be a* Lie *group action of* G *on* P. *For all* $\omega_n \in \mathcal{A}_n(P) \otimes V$, $i \leq n+1$ *and* $E_k \in \mathfrak{g}$ *we have* $\{S^i_\bullet(d\omega_n) - (-1)^i d(S^i_\bullet \omega_n)\}^{E_1,\dots,E_i}_{n+1-i} =$

$$= -\sum_{j=1}^{i} (-1)^j \Big\{ [S^{i-1}_\bullet(\mathrm{L}_{\mathcal{S}^j} \omega_n)]^{E_1,\dots,\widehat{E_j},\dots,E_i}_{n+1-i} + \mathrm{sgn}(S) \sum_{k=j+1}^{i} (S^{i-1}_\bullet \omega_n)^{E_1,\dots,\widehat{E_j},\dots,[E_j,E_k],\dots,E_i}_{n+1-i} \Big\}$$

$$= -\sum_{j=1}^{i} (-1)^j \Big\{ [\mathrm{L}_{\mathcal{S}^j}(S^{i-1}_\bullet \omega_n)]^{E_1,\dots,\widehat{E_j},\dots,E_i}_{n+1-i} - \mathrm{sgn}(S) \sum_{k=j+1}^{i} (S^{i-1}_\bullet \omega_n)^{E_1,\dots,\widehat{E_j},\dots,[E_j,E_k],\dots,E_i}_{n+1-i} \Big\},$$

where $\widehat{\ }$ *indicates that the term is omitted.*

Proof. From (8), the fact that d commutes with the push-outs $(E_1 \otimes \cdots \otimes E_i)_\star$ and the identities (5) we get by induction:

$$\{S^i_\bullet(d\omega_n) - (-1)^i d(S^i_\bullet \omega_n)\}^{E_1,\dots,E_i}_{n+1-i} = -\sum_{j=1}^{i} (-1)^j (\imath_{\mathcal{S}^i} \circ \cdots \circ \mathrm{L}_{\mathcal{S}^j} \circ \cdots \circ \imath_{\mathcal{S}^1}) \omega_n$$

$$= -\sum_{j=1}^{i} (-1)^j (\cdots \circ \widehat{\imath_{\mathcal{S}^j}} \circ \cdots)(\mathrm{L}_{\mathcal{S}^j} \omega_n) - \sum_{j=1}^{i} (-1)^j \sum_{k=1}^{j-1} (\imath_{\mathcal{S}^i} \circ \cdots \circ \widehat{\imath_{\mathcal{S}^j}} \circ \cdots \circ \imath_{[\mathcal{S}^j,\mathcal{S}^k]} \circ \cdots \circ \imath_{\mathcal{S}^1}) \omega_n.$$

Interchanging j and k in the last sum and $[\mathcal{S}^j, \mathcal{S}^k] = \mathrm{sgn}(S) \mathcal{S}_{[E_j,E_k]}$ yield the first equation. The second is proved analogously. ∎

If $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ is $\mathfrak{g}$-equivariant, Corollary 2.6 yields

$$[S_\bullet^{i-1}(\mathrm{L}_{\mathcal{S}^j}\chi_n^s)]_{n+1-i}^{E_1,\dots,\widehat{E_j},\dots,E_{i+s}} = \mathrm{sgn}(S)\sum_{k=1}^{s}(S_\bullet^{i-1}\chi_n^s)^{E_1,\dots,\widehat{E_j},\dots,[E_j,E_{i+k}],\dots,E_{i+s}},$$

(we again identify $\mathrm{Hom}(\bigotimes^{i+s}\mathfrak{g}, V)$ and $\mathrm{Hom}(\bigotimes^i \mathfrak{g}, \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)))$. We obtain:

Corollary 3.5. *For $\mathfrak{g}$-equivariant $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ and $i \le n+1$,*

$$\begin{aligned}\{[S_\bullet^i(d\chi_n^s)] - (-1)^i d(S_\bullet^i\chi_n^s)\}_{n+1-i}^{E_1,\dots,E_{i+s}} &= \\ = -\,\mathrm{sgn}(S)\sum_{j=1}^{i}\sum_{k=j+1}^{i+s}(-1)^j(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{E_1,\dots,\widehat{E_j},\dots,[E_j,E_k],\dots,E_{i+s}}.\end{aligned}$$

Thus for $\mathfrak{g}$-invariant ω_n, $d\omega_n = 0$ yields $d(S_\bullet\omega_n) = 0$, too.

Analogously one proves using $\imath_{\mathcal{X}} \circ \imath_{\mathcal{Y}} = -\imath_{\mathcal{X}} \circ \imath_{\mathcal{Y}}$ and $[L_{\mathcal{X}}, \imath_{\mathcal{Y}}] = \imath_{[\mathcal{X},\mathcal{Y}]}$:

Lemma 3.6. *For all $\omega_n \in \mathcal{A}_n(P) \otimes V$, $\mathcal{X} \in \mathcal{D}^1(P)$, $E_k \in \mathfrak{g}$, and $i \le n$,*

$$\begin{aligned}[S_\bullet^i(\imath_{\mathcal{X}}\omega_n)]_{n-1-i}^{E_1,\dots,E_i} &= (-1)^i[\imath_{\mathcal{X}}(S_\bullet^i\omega_n)]_{n-1-i}^{E_1,\dots,E_i}, \\ [S_\bullet^i(\mathrm{L}_{\mathcal{X}}\omega_n) - \mathrm{L}_{\mathcal{X}}(S_\bullet^i\omega_n)]_{n-i}^{E_1,\dots,E_i} &= \sum_{j=1}^{i}(-1)^j[S_\bullet^{i-1}(\imath_{[\mathcal{X},\mathcal{S}^j]}\omega_n)]_{n-i}^{E_1,\dots,\widehat{E_j},\dots,E_i}.\end{aligned}$$

If $\mathcal{X} = \mathcal{S}_X$ with $X \in \mathfrak{g}$, we get $[S_\bullet^i(\imath_{\mathcal{S}_X}\omega_n)]_{n-1-i}^{E_1,\dots,E_i} = (S_\bullet^{i+1}\omega_n)_{n-1-i}^{X,E_1,\dots,E_i}$,

$$[S_\bullet^i(\mathrm{L}_{\mathcal{S}_X}\omega_n) - \mathrm{L}_{\mathcal{S}_X}(S_\bullet^i\omega_n)]_{n-i}^{E_1,\dots,E_i} = -\,\mathrm{sgn}(S)\sum_{j=1}^{i}(S_\bullet^i\omega_n)_{n-i}^{E_1,\dots,[X,E_j],\dots,E_i}.$$

4. Mapping equivariant forms onto invariant forms

For our purposes we also need operators in the opposite direction, that produce V-valued forms from $\mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V)$-valued and $\mathfrak{g}$-valued forms. This can be done in a very general way and does not require a LIE group action (cf. [1]). Given forms $\chi_r^s \in \mathcal{A}_r(P, \mathrm{Hom}(\bigotimes^s W, V)$ and $\phi_p = \sum_{i=1}^m \phi^i \otimes E_i \in \mathcal{A}_p(P) \otimes W$ with $p, r, s-1 \in \mathbb{N}_0$, we define a V-valued form $\chi_r^s \bullet \phi_p$ in the following way [recall (3)]:

$$\chi_r^s \bullet \phi_p = \sum_{i_1,\dots,i_s=1}^{m} \chi_r^{E_{i_1},\dots,E_{i_s}} \wedge \phi^{i_1} \wedge \dots \wedge \phi^{i_s} \in \mathcal{A}(P, V). \tag{14}$$

Thus if $\chi_r^s \in \mathcal{A}_r(P) \otimes \mathrm{Hom}(\bigotimes^s W, V)$ then also $\chi_r^s \bullet \phi_p \in \mathcal{A}_{r+sp}(P) \otimes V$. Linear extension defines the operator $\bullet$ for $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(W), V))$. Note that if $\chi_r^s \in \mathcal{A}_r(P, \mathrm{Sym}_s^\varsigma(W, V)$, $\varsigma = \pm$, with $s > 1$ and $\varsigma(-1)^p = -1$ then $\chi_r^s \bullet \phi_p = 0$.

Since $\bullet$ behaves well under pullbacks and push-outs, one easily proves that $\bullet$ maps equivariant forms onto invariant forms (cf. [1, Lemma 7.1]):

Lemma 4.1. *Let $S\colon G \times P \to P$ be a LIE group action and $L\colon G \to \mathrm{Gl}(W)$ be a left representation. If $\varphi_r \in \mathcal{A}_r(P, W)$ and $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(W), V))$ are equivariant (i. e., $S_g^\star\varphi_r = L(g^{-\mathrm{sgn}(S)})_\star\varphi_r$ and $S_g^\star\chi = (L(g^{\mathrm{sgn}(S)})^\star)_\star\chi$ for all $g \in G$), then $\chi \bullet \varphi_r$ is invariant. E. g., if $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))_{\mathrm{equiv}}$ and $\varphi_r \in \mathcal{A}_r(P, \mathfrak{g})_{\mathrm{equiv}}$ then $\chi \bullet \varphi_r$ is invariant.*

Analogously, if χ and φ_r are both $\mathfrak{g}$-equivariant then $\chi \bullet \varphi_r$ is $\mathfrak{g}$-invariant.

We are interested especially in the case where $\chi_n^s = S_\bullet^s\omega_n$ and we combine both operators in the following form:

Definition 4.2. Let S be a LIE group action of G on P. Then for $\omega_n \in \mathcal{A}_n(P,V)$ and $\theta \in \mathcal{A}_1(P,\mathfrak{g})$ we define

$$\omega_n \circledS \theta := \sum_{i=0}^{n} \frac{(-1)^{i(n-i)}}{i!} (S^i_\bullet \omega_n) \bullet \theta \in \mathcal{A}_n(P,V).$$

Analogously, for $f\colon M \to P$ and $\theta \in \mathcal{A}_1(M,\mathfrak{g})$, resp., linear $\Lambda\colon V \to W$ we write

$$(f^\star\omega_n) \circledS \theta := \sum_{i=0}^{n} \frac{(-1)^{i(n-i)}}{i!} f^\star(S^i_\bullet \omega_n) \bullet \theta \in \mathcal{A}_n(M,V), \quad \text{resp.},$$

$$(\Lambda_\star\omega_n) \circledS \theta := \sum_{i=0}^{n} \frac{(-1)^{i(n-i)}}{i!} \Lambda_\star[(S^i_\bullet \omega_n) \bullet \theta] \in \mathcal{A}_n(P,W), \quad \text{etc.}$$

Linear extension defines $\omega \circledS \theta$ for $\omega \in \mathcal{A}(P,V)$.

Due to Lemmas 3.3 and 4.1, $\omega \circledS \theta$ is ($\mathfrak{g}$-)invariant if ω is ($\mathfrak{g}$-)invariant and θ is ($\mathfrak{g}$-)equivariant. $\omega \circledS \theta$ is the differential form that one obtains if ω_p, $p \in P$, does not act on the tangent vectors $\mathcal{X}^i_p \in T_p(P)$ themselves but on $\mathcal{X}^i_p + (dS^p)_e\theta_p(\mathcal{X}^i_p)$. Indeed, we have:

Lemma 4.3. *Let $p \in P$ and $\mathcal{X}^i \in \mathcal{D}^1(P)$. Then*

$$(\omega \circledS \theta)_p(\ldots,\mathcal{X}^i_p,\ldots) = \omega_p(\ldots,\mathcal{X}^i_p + (dS^p)_e\theta_p(\mathcal{X}^i_p),\ldots). \tag{15}$$

Proof. Let $\omega \in \mathcal{A}_n(P,V)$. Then $\omega_p(\ldots,\mathcal{X}^i_p + (dS^p)_e\theta_p(\mathcal{X}^i_p),\ldots) =$

$$= \sum_{i=0}^{n} \binom{n}{i} \sum_{\rho\in S_n} \tfrac{(-1)^\rho}{n!}\, \omega_p((dS^p)_e\theta_p\mathcal{X}^{\rho(1)}_p,\ldots,(dS^p)_e\theta_p\mathcal{X}^{\rho(i)}_p,\mathcal{X}^{\rho(i+1)}_p,\ldots,\mathcal{X}^{\rho(n)}_p)$$

$$= \sum_{i=0}^{n} \tfrac{1}{i!} \sum_{\rho\in S_n} \tfrac{(-1)^\rho}{n!}\, (S^i_\bullet\omega)_p(\mathcal{X}^{\rho(i+1)}_p,\ldots,\mathcal{X}^{\rho(n)}_p)[\theta_p\mathcal{X}^{\rho(1)}_p,\ldots,\theta_p\mathcal{X}^{\rho(i)}_p]$$

$$= \sum_{i=0}^{n} \tfrac{(-1)^{i(n-i)}}{i!} [(S^i_\bullet\omega)\bullet\theta](\mathcal{X}^1_p,\ldots,\mathcal{X}^n_p) = (\omega \circledS \theta)_p(\mathcal{X}^1_p,\ldots,\mathcal{X}^n_p). \qquad \blacksquare$$

Lemma 4.4. *For all $\omega \in \mathcal{A}(P,V)$, $\phi \in \mathcal{A}_1(P,\mathfrak{g})$ and horizontal $\theta \in \mathcal{A}_1(P,\mathfrak{g})$,*

$$[(\omega \circledS \theta) \circledS \phi] = \omega \circledS (\theta + \phi). \tag{16}$$

Proof. If θ is horizontal then $\theta_p(dS^p)_e = 0$ for all $p \in P$. Thus for all vector fields $\mathcal{X}^i$, Lemma 4.3 yields $[(\omega \circledS \theta) \circledS \phi]_p(\ldots,\mathcal{X}^i_p,\ldots) = \omega_p(\ldots,\mathcal{X}^i_p + (dS^p)_e\phi_p(\mathcal{X}^i_p) + (dS^p)_e\theta_p(\mathcal{X}^i_p) + (dS^p)_e\theta_p(dS^p)_e\phi_p(\mathcal{X}^i_p),\ldots) = \omega_p(\ldots,\mathcal{X}^i_p + (dS^p)_e(\phi_p + \theta_p)(\mathcal{X}^i_p),\ldots) = [\omega \circledS (\theta+\phi)]_p(\ldots,\mathcal{X}^i_p,\ldots)$. $\blacksquare$

Again for $S = L, R$, let $\Theta^S \in \mathcal{A}_1(G,\mathfrak{g})$ denote the left, resp., right canonical 1-form on the LIE group G that is given by $\Theta^S = \psi^S(\mathrm{id}_\mathfrak{g})$, where $\psi^S\colon \mathrm{Alt}(\mathfrak{g},V) \to \mathcal{A}(G,V)_{\mathrm{inv}}$ (invariance with respect to left, resp., right multiplication) means the isomorphism that is inverse to the evaluation at e; i. e., $\Theta^L_g(\mathcal{X}_g) = d\lambda_{g^{-1}}(\mathcal{X}_g)$ and $\Theta^R_g(\mathcal{X}_g) = d\rho_{g^{-1}}(\mathcal{X}_g)$ for all $g \in G$ and $\mathcal{X}_g \in T_g(G)$. We thus have $\Theta^R = \mathrm{Ad} \bullet \Theta^L$, i. e., $\Theta^R_g = \mathrm{Ad}(g) \circ \Theta^L_g$ for all $g \in G$. If $f\colon M \to G$ is differentiable, $f^\star\Theta^R =$

$(\mathrm{Ad}\circ f)\bullet f^\star\Theta^L$ and $f^\star\Theta^L=(\mathrm{Ad}\circ f^{-1})\bullet f^\star\Theta^R$, where $f^{-1}:=\eta\circ f\colon M\to G$. Also with the constant map $1\in C^\infty(G)$:

$$\psi^S(K)=(1\otimes K)\bullet\Theta^S\in\mathcal{A}(G,V)_{\mathrm{inv}}\qquad\text{for all}\quad K\in\mathrm{Alt}(\mathfrak{g},V).\tag{17}$$

Let us first give an application of Ⓢ. Suppose S is a LIE group action of G on P and $\omega\in\mathcal{A}_n(P,V)_{\mathrm{inv}}$. For any differentiable $g\colon M\to G$ and $f\colon M\to P$ one would like to compute $[S\circ(g,f)]^\star\omega$. Then in order to split this form into its portions that belong to $f^\star$, resp., $g^\star$, one needs Ⓢ. In fact, the following holds:

Theorem 4.5. *Let S be a LIE group action of G on P, S' a representation of G on V on the same side and $\omega\in\mathcal{A}(P,V)$ be equivariant. If $g\colon M\to G$ and $f\colon M\to P$ are differentiable, then $S'_g\in\mathcal{A}(M,\mathrm{Gl}(W))$ and*

$$[S\circ(g,f)]^\star\omega=S'_g\bullet(f^\star\omega \circledS g^\star\Theta^S_G)=(S'_g\bullet f^\star\omega)\circledS g^\star\Theta^S_G.\tag{18}$$

Proof. Let $\mathcal{X}^i\in\mathcal{D}^1(M)$ and $x\in M$. Then $\{[S\circ(g,f)]^\star\omega\}_x(\ldots,\mathcal{X}^i_x,\ldots)=$

$$\begin{aligned}
&=\ \omega_{S(g(x),f(x))}(\ldots,[(dS_{g(x)})_{f(x)}df_x+(dS^{f(x)})_{g(x)}dg_x]\mathcal{X}^i_x,\ldots)\\
&=\ (S^\star_{g(x)}\omega)_{f(x)}(\ldots,df_x\mathcal{X}^i_x+d(S_{g^{-1}(x)}\circ S^{f(x)})_{g(x)}dg_x\mathcal{X}^i_x,\ldots)\\
&=\ S'_{g(x)}\circ[\omega_{f(x)}(\ldots,df_x\mathcal{X}^i_x+(dS^{f(x)})_e(g^\star\Theta^S_G)_x\mathcal{X}^i_x,\ldots)]\\
&=\ S'_{g(x)}\circ[(f^\star\omega\circledS g^\star\Theta^S_G)_x(\ldots,\mathcal{X}^i_x,\ldots)]=[S'_g\bullet(f^\star\omega\circledS g^\star\Theta^S_G)]_x(\ldots,\mathcal{X}^i_x,\ldots).\quad\blacksquare
\end{aligned}$$

Corollary 4.6. *If S be a LIE group action of G on P and $\omega\in\mathcal{A}(P,V)$ is invariant then for any differentiable $g\colon M\to G$ and $f\colon M\to P$*

$$[S\circ(g,f)]^\star\omega=f^\star\omega\circledS g^\star\Theta^S_G.\tag{19}$$

Suppose that under the conditions of Theorem 4.5, $(S^p)^\star\omega_n$ is independent of $p\in P$. Then $(S^p)^\star\omega_n\in\mathcal{A}_n(G,V)$ is invariant: $(L^p)^\star\omega=\psi^R(K)$, resp., $(R^p)^\star\omega=\psi^L(K)$ for a $K\in\mathrm{Alt}_n(\mathfrak{g},V)$. Moreover, we find $(S'_g)_\star K=\mathrm{Ad}(g^{-\operatorname{sgn}S})^\star K$, so for the $i=n$ term in the definition of Ⓢ in Theorem 4.5 we get from (9) with $-S:=R$ for $S=L$, and vice versa:

$$S'_g\bullet[f^\star(S^n_\bullet\omega_n)\bullet g^\star\Theta^S_G]\ =\ g^\star\psi^{-S}(K).$$

The $i=0$ term reads $S'_g\bullet f^\star\omega$, so for $\omega\in\mathcal{A}_1(G,\mathfrak{g})$ we obtain

Corollary 4.7. *Let $L,R\colon G\times P\to P$ be a left, resp., right action of G on P and $f\colon M\to P$ and $g\colon M\to G$ be differentiable; $K\in\mathrm{Alt}_1(\mathfrak{g},\mathfrak{g})$ be invertible and $\omega\in\mathcal{A}_1(P,\mathfrak{g})$. Then $K(\mathrm{Ad}\circ g)K^{-1}\in\mathcal{A}_0(M,\mathrm{Alt}_1(\mathfrak{g},\mathfrak{g}))$ and we have*

1. *If $(L^p)^\star\omega=\psi^R(K)$ and $L^\star_c\omega=K\,\mathrm{Ad}(c)K^{-1}\circ\omega$ for all $p\in P$, $c\in G$, then*

$$[L\circ(g,f)]^\star\omega=K(\mathrm{Ad}\circ g)K^{-1}\bullet f^\star\omega+g^\star\psi^R(K).$$

2. *If $(R^p)^\star\omega=\psi^L(K)$ and $R^\star_c\omega=K\mathrm{Ad}(c^{-1})K^{-1}\circ\omega$ for all $p\in P$, $c\in G$, then*

$$[R\circ(g,f)]^\star\omega=K(\mathrm{Ad}\circ g^{-1})K^{-1}\bullet f^\star\omega+g^\star\psi^L(K).$$

For $P = G$ and $L = \lambda$, resp., $R = \rho$, Corollary 4.7 gives a proof for the formulas for $(f \cdot g)^\star \Theta^S$ and $(f^{-1})^\star \Theta^S$: put $K = \mathrm{id}_{\mathfrak{g}}$ and observe that $(f \cdot f^{-1})^\star = e^\star = 0$, where $e\colon M \to \{e\} \subseteq G$ is the constant map onto the neutral element. Then Corollary 4.7 yields:

$$\begin{aligned}
(f \cdot g)^\star \Theta^L &= (\mathrm{Ad} \circ g^{-1}) \bullet f^\star \Theta^L + g^\star \Theta^L, \\
(f \cdot g)^\star \Theta^R &= f^\star \Theta^R + (\mathrm{Ad} \circ f) \bullet g^\star \Theta^R, \\
(f^{-1})^\star \Theta^L &= -(\mathrm{Ad} \circ f) \bullet f^\star \Theta^L \;=\; -f^\star \Theta^R, \\
(f^{-1})^\star \Theta^R &= -(\mathrm{Ad} \circ f^{-1}) \bullet f^\star \Theta^R \;=\; -f^\star \Theta^L.
\end{aligned}$$

We already stated another application of $\circledS$ in Section 1.: if Γ is a connection on a principal bundle $P(M,G)$ and $B(M,F,G)$ is an associated fiber bundle with a left Lie group action $L\colon G \times F \to F$ (we also use L for the natural extensions to the bundle charts $U_\alpha \times F$), then for any $\phi \in \mathcal{A}(F,V)$, the local vertical projections of $\mathrm{pr}_F^\star \phi \in \mathcal{A}(U_\alpha \times F)$ are given by

$$(\mathrm{pr}_F^\star \phi) v^\alpha = (\mathrm{pr}_F^\star \phi) \;\textcircled{L}\; (\mathrm{pr}_{U_\alpha}^\star \mathrm{A}^\alpha). \tag{20}$$

This follows immediately from Lemma 4.3 and $v^\alpha_{(x,f)}(X,Y) = (0, Y + (dL^f)_e A^\alpha_x(X))$ for all $(X,Y) \in T_x(U_\alpha) \oplus T_f(F)$. If ϕ is invariant, then one easily computes that

$$L_g^\star[(\mathrm{pr}_F^\star \phi) v^\alpha] = (\mathrm{pr}_F^\star \phi) \;\textcircled{L}\; [(\mathrm{pr}_{U_\alpha}^\star(\mathrm{Ad}(g^{-1})_\star \mathrm{A}^\alpha)]. \tag{21}$$

Let $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ and $T_{\beta\alpha} := (\psi_\beta|_{U_{\alpha\beta}})^{-1} \circ (\psi_\alpha|_{U_{\alpha\beta}})$ denote the maps for the change of bundle charts. If $g_{\beta\alpha}\colon U_{\alpha\beta} \to G$ are the transition functions, then the maps $T_{\beta\alpha}$ are given by

$$T_{\beta\alpha} = (\mathrm{pr}_{U_{\alpha\beta}}, L \circ (g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}}, \mathrm{pr}_F)) = L \circ (g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}}, \mathrm{id}_{U_{\alpha\beta} \times F}). \tag{22}$$

For computations on fiber bundles one needs to know how differential forms transform under such a change of bundle charts, e. g., in order to check whether a collection of local forms defines a global form. In view of this question we obtain from (19) for $\phi \in \mathcal{A}(F,V)_{\mathrm{inv}}$:

$$T_{\beta\alpha}^\star(\mathrm{pr}_F^\star \phi) = (\mathrm{pr}_F^\star \phi) \;\textcircled{L}\; (g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}})^\star \Theta^L. \tag{23}$$

Recall from the theory of connections that the gauge potentials A^α transform according to $\mathrm{A}^\alpha = (\mathrm{Ad} \circ g_{\alpha\beta}) \bullet \mathrm{A}^\beta + g_{\beta\alpha}^\star \Theta^L$, where we omitted the restriction to $U_{\alpha\beta}$. In fact this is a consequence of Corollary 4.7.2 for $\omega = \omega^\Gamma$, $K = \mathrm{id}_{\mathfrak{g}}$ and $f = \sigma_{\beta,e}$ because $\sigma_{\alpha,e} = R \circ (g_{\beta\alpha}, \sigma_{\beta,e})$. Further observe that in view of Lemma 4.4, $(g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}})^\star \Theta^L$ and $(\mathrm{pr}_{U_\alpha}^\star \mathrm{A}^\alpha)$ are both horizontal. Although due to (21) for invariant $\phi \in \mathcal{A}(F,V)$, the vertical form $(\mathrm{pr}_F^\star \phi) v^\alpha$ needs not be invariant and thus Corollary 4.6 does not apply, one quickly checks tracing the proof of Theorem 4.5, that

$$\begin{aligned}
T_{\beta\alpha}^\star[(\mathrm{pr}_F^\star \phi) v^\beta] &= [(\mathrm{pr}_F^\star \phi) \;\textcircled{L}\; [(\mathrm{pr}_{U_{\alpha\beta}}^\star(\mathrm{Ad}(g_{\alpha\beta})_\star \mathrm{A}^\beta)]] \;\textcircled{L}\; (g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}})^\star \Theta^L \\
&= [(\mathrm{pr}_F^\star \phi) \;\textcircled{L}\; [\mathrm{pr}_{U_{\alpha\beta}}^\star((\mathrm{Ad} \circ g_{\alpha\beta}) \bullet \mathrm{A}^\beta + g_{\beta\alpha}^\star \Theta^L)] = (\mathrm{pr}_F^\star \phi) v^\alpha.
\end{aligned}$$

We have thus proved

Theorem 4.8. *If $\phi \in \mathcal{A}(F,V)$ is invariant, $\{(\mathrm{pr}_F^\star \phi)v^\alpha \in \mathcal{A}(U_\alpha \times F, V)\}_{\alpha\in A}$, resp., $\{(\pi_\alpha^\star\phi)v^\alpha \in \mathcal{A}(\pi^{-1}(U_\alpha), V)\}_{\alpha\in A}$ defines a global form $\phi v \in \mathcal{A}(B,V)$. If ϕ is invariant and locally vertical, then $\{\pi_\alpha^\star\phi\}_{\alpha\in A}$ is global.*

Generalizations of this theorem to combinations of equivariant differential forms as in Lemma 4.1 are possible. E. g., if $\chi \in \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))$ is equivariant and $\mathrm{F}^\alpha \in \mathcal{A}_2(U_\alpha, \mathfrak{g})$ denote the local gauge fields that are obtained from the curvature 2-form Ω^Γ as $\mathrm{F}^\alpha := \sigma_{\alpha,e}^\star \Omega^\Gamma$, then $\{[(\pi_\alpha^\star\chi)v^\alpha] \bullet (\pi^\star \mathrm{F}^\alpha) \in \mathcal{A}(\pi^{-1}(U_\alpha), V)\}_{\alpha\in A}$ defines a global form $\chi v \bullet F$ on the bundle B, cf. [2]. Locally this form is given by

$$[(\mathrm{pr}_F^\star\, \chi)v^\alpha] \bullet (\mathrm{pr}_{U_\alpha}^\star\, \mathrm{F}^\alpha) = [(\mathrm{pr}_F^\star\, \phi) \circledcirc (\mathrm{pr}_{U_\alpha}^\star\, \mathrm{A}^\alpha)] \bullet (\mathrm{pr}_{U_\alpha}^\star\, \mathrm{F}^\alpha). \tag{24}$$

5. Differentiation of the combined forms

From the previous applications it should be clear that it is important to control interior products, Lie derivatives and above all, the exterior derivatives of the differential forms $\chi_r^s \bullet \phi_p$ and $\omega_n \circledS\, \theta$. E. g., one is interested in the exterior derivative $d(\phi v)$ from Theorem 4.8 if ϕ is closed. Thus the computation of $d(\omega_n \circledS\, \theta)$ and, more generally, of $d[(\chi_n^s \circledS\, \theta) \bullet \phi_p]$ will be the main task of this last section. Unfortunately, the most general formulas turn out to be quite voluminous. For this reason, we will then discuss the important special cases.

We need to generalize $\bullet$ to give formulas for $\chi_r^s \bullet (\phi_p + \psi_p)$ and $d(\chi_r^s \bullet \phi_p)$, cf. [1]. First we observe that $\chi_r^s \in \mathcal{A}_r(P, \mathrm{Sym}_s^\varsigma(\mathfrak{g}, V))$, $\varsigma = \pm$, naturally defines

$$\chi_r^{s';s''} \in \mathcal{A}_r(P, \mathrm{Sym}_{s'}^\varsigma(\mathfrak{g}, \mathrm{Sym}_{s''}^\varsigma(\mathfrak{g}, V))) \quad \text{for all } s', s'' \in \mathbb{N}_0,\ s' + s'' = s. \tag{25}$$

For any such combination of s' and s'', $\chi_r^s \bullet (\phi_p^q + \psi_p^q)$ will contain terms, where s' factors of ϕ_p^q and s'' terms of ψ_p^q serve as input for χ_r^s. In order to cover this situation, we need the following two definitions.

Generally, for $\chi_r^{s';s''} \in \mathcal{A}_r(P, \mathrm{Hom}(\bigotimes^{s'} \mathfrak{g}, \mathrm{Hom}(\bigotimes^{s''} \mathfrak{g}, V)))$, $s', s'' \in \mathbb{N}$, $r \in \mathbb{N}_0$, and any $E_i \in \mathfrak{g}$, $i = 1, \ldots, s'$, we define

$$\chi_r^{E_1,\ldots,E_{s'};s''} := [(E_1 \otimes \cdots \otimes E_{s'})^\star]_\star \chi_r^{s';s''} \in \mathcal{A}_r(P, \mathrm{Hom}(\textstyle\bigotimes^{s''} \mathfrak{g}, V))).$$

[Thus if $\chi_r^{s',s''} \in \mathcal{A}_r(P) \otimes \mathrm{Hom}(\bigotimes^{s'} \mathfrak{g}, \mathrm{Hom}(\bigotimes^{s''} \mathfrak{g}, V))$ then $\chi_r^{E_1,\ldots,E_{s'};s''} \in \mathcal{A}_r(P) \otimes \mathrm{Hom}(\bigotimes^{s''} \mathfrak{g}, Z)$.] For any such differential form $\chi_r^{s';s''}$ and any $\phi_p \in \mathcal{A}_p(P) \otimes \mathfrak{g}$, let $V' := \mathrm{Hom}(\bigotimes^{s''} \mathfrak{g}, Z)$ and $\widetilde{\chi}_r^{s'} := \chi_r^{s';s''} \in \mathcal{A}_r(P, \mathrm{Hom}(\bigotimes^{s'} \mathfrak{g}, V'))$, and define

$$\chi_r^{s';s''} \blacktriangleleft \phi_p := \widetilde{\chi}_r^{s'} \bullet \phi_p \in \mathcal{A}_{r+s'p}(P, \mathrm{Hom}(\textstyle\bigotimes^{s''} \mathfrak{g}, V))).$$

If $\phi_p = \sum\limits_{j=1}^{m} \phi^j \otimes E_j \in \mathcal{A}_p(P) \otimes W$ then we obtain

$$\chi_r^{s';s''} \blacktriangleleft \phi_p = \sum_{j_1,\ldots,j_{s'}=1}^{m} \chi_r^{E_{j_1},\ldots,E_{j_{s'}};s''} \wedge \phi^{j_1} \wedge \cdots \wedge \phi^{j_{s'}}, \tag{26}$$

which shows that $\chi_r^{s';s''} \blacktriangleleft \phi_p \in \mathcal{A}_{r+s'p}(P) \otimes \mathrm{Hom}(\bigotimes^{s''} \mathfrak{g}, V))$ if $\chi_r^{s';s''} \in \mathcal{A}_r(P) \otimes \mathrm{Hom}(\bigotimes^{s'} \mathfrak{g}, \mathrm{Hom}(\bigotimes^{s''} \mathfrak{g}, V))$.

We also introduce generalizations $\binom{s}{k}_{\pm}$ of the ordinary binomial coefficients:

$$\binom{s}{k}_{+} := \binom{s}{k}, \qquad \binom{s}{k}_{-} := \begin{cases} 0, & \text{if } s \text{ even and } k \text{ odd,} \\ \binom{[s/2]}{[k/2]}, & \text{else (for } r \in \mathbb{R},\ [r] := \max_{z\in\mathbb{Z}}\{z \le r\}). \end{cases} \tag{27}$$

Thus $\binom{s}{k}_{\pm} = \binom{s}{s-k}_{\pm}$ as before. Now if $\chi_r^s \in \mathcal{A}_r(P, \mathrm{Sym}_s^{\varsigma}(\mathfrak{g}, V))$ and $\ell := \varsigma(-1)^p$ then $\chi_r^s \bullet (\phi_p + \psi_p)$ can be written as

$$\chi_r^s \bullet (\phi_p + \psi_p) = \sum_{k=0}^{s} \binom{s}{k}_{\ell} (\chi_r^{k;s-k} \blacktriangleleft \phi_p) \bullet \psi_p = \textstyle\sum_{k=0}^{s} \binom{s}{k}_{\ell} (\chi_r^{k;s-k} \blacktriangleleft \psi_p) \bullet \phi_p.$$

If $\chi_r^s \in \mathcal{A}_r(P) \otimes \mathrm{Sym}_s^{\varsigma}(\mathfrak{g}, V)$ we obtain for $d(\chi_r^s \bullet \phi_p)$, $\imath_{\mathcal{X}}(\chi_r^s \bullet \phi_p)$ and $\mathrm{L}_{\mathcal{X}}(\chi_r^s \bullet \phi_p)$:

$$d(\chi_r^s \bullet \phi_p) = (d\chi)_{r+1}^s \bullet \phi_p + (-1)^r \binom{s}{1}_{\ell} [\chi_r^{1;s-1} \blacktriangleleft (d\phi)_{p+1}] \bullet \phi_p, \tag{28}$$

$$\imath_{\mathcal{X}}(\chi_r^s \bullet \phi_p) = (\imath_{\mathcal{X}}\chi)_{r-1}^s \bullet \phi_p + (-1)^r \binom{s}{1}_{\ell} [\chi_r^{1;s-1} \blacktriangleleft (\imath_{\mathcal{X}}\phi)_{p-1}] \bullet \phi_p, \tag{29}$$

$$\mathrm{L}_{\mathcal{X}}(\chi_r^s \bullet \phi_p) = (\mathrm{L}_{\mathcal{X}}\chi)_r^s \bullet \phi_p + \binom{s}{1}_{\ell} [\chi_r^{1;s-1} \blacktriangleleft (\mathrm{L}_{\mathcal{X}}\phi)_p] \bullet \phi_p. \tag{30}$$

Note that whenever $\chi_r^s \bullet \phi_p \neq 0$, $\binom{s}{k}_{\ell} = \binom{s}{k}$. If $\chi_r^s \in \mathcal{A}_r(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$, we get

$$d(\chi_r^s \bullet \phi_p) = (d\chi_r^s) \bullet \phi_p + (-1)^r \sum_{j=0}^{s-1} (-1)^{jp} [(\chi_r^{j;s-j} \blacktriangleleft \phi_p)^{1;s-j-1} \blacktriangleleft d\phi_p] \bullet \phi_p, \tag{31}$$

$$\imath_{\mathcal{X}}(\chi_r^s \bullet \phi_p) = (\imath_{\mathcal{X}}\chi_r^s) \bullet \phi_p + (-1)^r \sum_{j=0}^{s-1} (-1)^{jp} [(\chi_r^{j;s-j} \blacktriangleleft \phi_p)^{1;s-j-1} \blacktriangleleft \imath_{\mathcal{X}}\phi_p] \bullet \phi_p, \tag{32}$$

$$\mathrm{L}_{\mathcal{X}}(\chi_r^s \bullet \phi_p) = (\mathrm{L}_{\mathcal{X}}\chi_r^s) \bullet \phi_p + \sum_{j=0}^{s-1} (-1)^{jp} [(\chi_r^{j;s-j} \blacktriangleleft \phi_p)^{1;s-j-1} \blacktriangleleft \mathrm{L}_{\mathcal{X}}\phi_p] \bullet \phi_p. \tag{33}$$

Now for the operator ⓢ: if θ is horizontal one quickly verifies analogously to the proof of Lemma 4.4 that $\imath_{\mathcal{S}_X}(\omega ⓢ \theta) = (\imath_{\mathcal{S}_X}\omega) ⓢ \theta$. Thus we have:

Lemma 5.1. *If $\chi_r^s \in \mathcal{A}_r(P) \otimes \mathrm{Sym}_s^{\varsigma}(\mathfrak{g}, V)$ and θ is horizontal then*

$$\imath_{\mathcal{S}_X}[(\chi_r^s ⓢ \theta) \bullet \phi_p] = [(\imath_{\mathcal{S}_X}\chi)_{r-1}^s ⓢ \theta] \bullet \phi_p + (-1)^r \binom{s}{1}_{\ell} [(\chi_r^s ⓢ \theta)_r^{1;s-1} \blacktriangleleft (\imath_{\mathcal{S}_X}\phi)_{p-1}] \bullet \phi_p;$$

If $\chi_r^s \in \mathcal{A}_r(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ and both θ and ϕ_p are horizontal then

$$\imath_{\mathcal{S}_X}[(\chi_r^s ⓢ \theta) \bullet \phi_p] = [(\imath_{\mathcal{S}_X}\chi)_{r-1}^s ⓢ \theta] \bullet \phi_p.$$

Recall $\wedge_V$ from Section 2. and let $\wedge_{\mathfrak{g}}$ denote the exterior product for $\mathfrak{g}$-valued differential forms which is induced by ad: $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Lemma 5.2. *Let $\chi_n^1 \in \mathcal{A}_n(P, \mathrm{Hom}(\mathfrak{g}, V))$ and $\{E_k\}_{k=1,\dots,\dim\mathfrak{g}}$ be a basis for $\mathfrak{g}$. Then for $\theta_q = \sum_{k=1}^{\dim\mathfrak{g}} \theta_q^k \otimes E_k \in \mathcal{A}_q(P, \mathfrak{g})$ and $\phi_p = \sum_{l=1}^{\dim\mathfrak{g}} \phi_p^l \otimes E_l \in \mathcal{A}_p(P, \mathfrak{g})$,*

$$\chi_n^1 \bullet (\theta_q \wedge_{\mathfrak{g}} \phi_p) = \sum_{j=1}^{\dim\mathfrak{g}} \chi_n^{E_j} \wedge (\theta_q \wedge_{\mathfrak{g}} \phi_p)^j = \sum_{k,l=1}^{\dim\mathfrak{g}} \chi_n^{[E_k,E_l]} \wedge \theta_q^k \wedge \phi_p^l. \tag{34}$$

Proof. Let $[E_k,E_l]=\sum_{j=1}^{\dim\mathfrak{g}} c_{kl}^j\,E_j$ with structure constants c_{kl}^j. Then by definition, $\theta_q\wedge_{\mathfrak{g}}\phi_p=\sum_{k,l=1}^{\dim\mathfrak{g}}\theta_q^k\wedge\phi_p^l\otimes[E_k,E_l]=\sum_{j,k,l=1}^{\dim\mathfrak{g}} c_{kl}^j\,\theta_q^k\wedge\phi_p^l\otimes E_j=:\sum_j^{\dim\mathfrak{g}}(\theta_q\wedge_{\mathfrak{g}}\phi_p)^j\otimes E_j$, thus

$$\chi_n^1\bullet(\theta_q\wedge_{\mathfrak{g}}\phi_p)=\sum_{j=1}^{\dim\mathfrak{g}}\chi_n^{E_j}\wedge(\theta_q\wedge_{\mathfrak{g}}\phi_p)^j=\sum_{j,k,l=1}^{\dim\mathfrak{g}}c_{kl}^j\,\chi_n^{E_j}\wedge\theta_q^k\wedge\phi_p^l=\sum_{k,l=1}^{\dim\mathfrak{g}}\chi_n^{\sum_{j=1}^{\dim\mathfrak{g}}c_{kl}^jE_j}\wedge\theta_q^k\wedge\phi_p^l.\ \blacksquare$$

We will divide the computation of the exterior derivatives into two steps:

Proposition 5.3. *Let S be a* LIE *group action of G on P, $\theta_q\in\mathcal{A}_q(P,\mathfrak{g})$, $\phi_p\in\mathcal{A}_p(P,\mathfrak{g})$ and $\chi_n^s\in\mathcal{A}_n(P)\otimes\mathrm{Hom}(\bigotimes^s\mathfrak{g},V)$ $\mathfrak{g}$-equivariant. Then for all $i\le n+1$ with $\ell=(-1)^{q-1}$*

$$\begin{aligned}&\{[d(S_\bullet^i\chi_n^s)-(-1)^iS_\bullet^i(d\chi_n^s)]_{n+1}^{i;s}\blacktriangleleft\theta_q\}^s\bullet\phi_p=\\&=\mathrm{sgn}(S)\Big\{-\tbinom{i}{2}_\ell\{[(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{i-2;s+1}\blacktriangleleft\theta_q]^{1;s}\blacktriangleleft(\theta_q\wedge_{\mathfrak{g}}\theta_q)\}^s\bullet\phi_p+\\&\quad+\tbinom{i}{1}_\ell\sum_{k=1}^{s}(-1)^{qp(k-1)}\{[[(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{i-1;s}\blacktriangleleft\theta_q]^{k-1;s-k+1}\blacktriangleleft\phi_p]^{1;s-k}\blacktriangleleft(\theta_q\wedge_{\mathfrak{g}}\phi_p)\}^{s-k}\bullet\phi_p\Big\}.\end{aligned}$$

Proof. With the notation of the previous lemma, we evaluate the left side using (14). Then by Corollary 3.5,

$$\begin{aligned}&\sum_{l_1,\dots,l_{i+s}}^{\dim\mathfrak{g}}\{d(S_\bullet^i\chi_n^s)-(-1)^iS_\bullet^i(d\chi_n^s)\}_{n+1-i}^{E_{l_1},\dots,E_{l_{i+s}}}\wedge\cdots\wedge\theta_q^{l_i}\wedge\phi_p^{l_{i+1}}\wedge\cdots\wedge\phi_p^{l_{i+s}}=\\
&=\mathrm{sgn}(S)\sum_{j=1}^{i}\sum_{k=j+1}^{i+s}(-1)^{i+j}\sum_{l_1,\dots,l_{i+s}}^{\dim\mathfrak{g}}(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{E_{l_1},\dots,\widehat{E_{l_j}},\dots,[E_{l_j},E_{l_k}],\dots,E_{l_{i+s}}}\wedge\cdots\wedge\theta_q^{l_i}\wedge\phi_p^{l_{i+1}}\wedge\cdots\\
&=-\mathrm{sgn}(S)\sum_{j=1}^{i}\sum_{k=j+1}^{i}\ell^{k-j+1}\sum_{l_1,\dots,l_{i+s}}^{\dim\mathfrak{g}}(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{E_{l_1},\dots,\widehat{E_{l_j}},\dots,\widehat{E_{l_k}},\dots,E_{l_i},[E_{l_j},E_{l_k}];\dots,E_{l_{i+s}}}\wedge\\
&\qquad\wedge\theta_q^{l_1}\wedge\cdots\widehat{\theta_q^{l_j}}\cdots\widehat{\theta_q^{l_k}}\cdots\wedge\theta_q^{l_i}\wedge\theta_q^{l_j}\wedge\theta_q^{l_k}\wedge\phi_p^{l_{i+1}}\wedge\cdots\wedge\phi_p^{l_{i+s}}\\
&\quad+\mathrm{sgn}(S)\sum_{j=1}^{i}\ell^{i-j}\sum_{k=1}^{s}(-1)^{qp(k-1)}\sum_{l_1,\dots,l_{i+s}}^{\dim\mathfrak{g}}(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{E_{l_1},\dots,\widehat{E_{l_j}},\dots,E_{l_i};\dots,[E_{l_j},E_{l_{i+k}}],\dots,E_{l_{i+s}}}\wedge\\
&\qquad\wedge\theta_q^{l_1}\wedge\cdots\widehat{\theta_q^{l_j}}\cdots\wedge\theta_q^{l_i}\wedge\phi_p^{l_{i+1}}\wedge\cdots\wedge\theta_q^{l_j}\wedge\phi_p^{l_{i+k}}\wedge\cdots\wedge\phi_p^{l_{i+s}}\\
&=-\mathrm{sgn}(S)\sum_{j=1}^{i}\sum_{k=j+1}^{i}\ell^{k-j+1}\sum_{l_1,\dots,l_{i+s-1}}^{\dim\mathfrak{g}}(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{E_{l_1},\dots,E_{l_{i-1}};E_{l_i},\dots,E_{l_{i+s}}}\wedge\\
&\qquad\wedge\theta_q^{l_1}\wedge\cdots\wedge\theta_q^{l_{i-2}}\wedge(\theta_q\wedge_{\mathfrak{g}}\theta_q)^{l_{i-1}}\wedge\phi_p^{l_i}\wedge\cdots\wedge\phi_p^{l_{i+s-1}}\\
&\quad+\mathrm{sgn}(S)\sum_{j=1}^{i}\ell^{i-j}\sum_{k=1}^{s}(-1)^{qp(k-1)}\sum_{l_1,\dots,l_{i+s-1}}^{\dim\mathfrak{g}}(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{E_{l_1},\dots,E_{l_{i-1}};E_{l_i},\dots,E_{l_{i+s}}}\wedge\\
&\qquad\wedge\theta_q^{l_1}\wedge\cdots\wedge\theta_q^{l_{i-1}}\wedge\phi_p^{l_i}\wedge\cdots\wedge(\theta_q\wedge_{\mathfrak{g}}\phi_p)^{l_{i+k-1}}\wedge\cdots\wedge\phi_p^{l_{i+s-1}},\end{aligned}$$

by (34). Since $\sum_{j=1}^{i}\sum_{k=j+1}^{i}\ell^{k-j+1}=\binom{i}{2}_\ell$ and $\sum_{j=1}^{i}\ell^{i-j}=\binom{i}{1}_\ell$, all follows from (26). $\blacksquare$

Corollary 5.4. *Suppose* $\theta \in \mathcal{A}_1(P,\mathfrak{g})$ *and* $\chi_n^s \in \mathcal{A}_n(P)\otimes \mathrm{Sym}_s^{\varsigma}(\mathfrak{g},V)$ *in* Proposition 5.3, *then with* $\ell = \varsigma(-1)^p$ *for all* $i \le n+1$

$$\begin{aligned}
&\{[d(S_\bullet^i \chi_n^s)]^{i;s} \blacktriangleleft \theta\}^s \bullet \phi_p - (-1)^i \{[S_\bullet^i (d\chi_n^s)]^{i;s} \blacktriangleleft \theta\}^s \bullet \phi_p = \\
&= -\operatorname{sgn}(S) \tbinom{i}{2} \{[(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{i-2;s+1} \blacktriangleleft \theta]^{1;s} \blacktriangleleft (\theta \wedge_{\mathfrak{g}} \theta)\}^s \bullet \phi_p \\
&\quad + \operatorname{sgn}(S)\, i \tbinom{s}{1}_\ell \{[(S_\bullet^{i-1}\chi_n^s)^{i-1;s} \blacktriangleleft \theta]^{1;s-1} \blacktriangleleft (\theta \wedge_{\mathfrak{g}} \phi_p)\}^{s-1} \bullet \phi_p.
\end{aligned}$$

Proof. This follows from $(-1)^{p(k-1)}[[(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{i-1;s} \blacktriangleleft \theta]^{k-1;s-k+1} \blacktriangleleft \phi_p]^{1;s-k} \blacktriangleleft (\theta \wedge_{\mathfrak{g}} \phi_p) = \ell^{k-1}[[(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{i-1;s} \blacktriangleleft \theta]^{1;s-1} \blacktriangleleft (\theta \wedge_{\mathfrak{g}} \phi_p)]^{k-1;s-k} \blacktriangleleft \phi_p$ and $\sum_{k=1}^s \ell^{k-1} = \binom{s}{1}_\ell$. ∎

Theorem 5.5. *Let* S *be a* LIE *group action of* G *on* P, $\theta \in \mathcal{A}_1(P,\mathfrak{g})$, $\phi_p \in \mathcal{A}_p(P,\mathfrak{g})$ *and* $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Sym}_s^{\varsigma}(\mathfrak{g},V)$ $\mathfrak{g}$*-equivariant. If* $\ell := \varsigma(-1)^p$, *then*

$$\begin{aligned}
&d[(\chi_n^s \circledS \theta) \bullet \phi_p] - [(d\chi_n^s) \circledS \theta] \bullet \phi_p = \\
&= \{[(S_\bullet \chi_n^s) \circledS \theta]^{1;s} \blacktriangleleft (d\theta - \operatorname{sgn}(S)\tfrac{1}{2}\theta \wedge_{\mathfrak{g}} \theta)\}^s \bullet \phi_p \\
&\quad + (-1)^n \tbinom{s}{1}_\ell [(\chi_n^s \circledS \theta)^{1;s-1} \blacktriangleleft (d\phi_p - \operatorname{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \phi_p)]^{s-1} \bullet \phi_p.
\end{aligned}$$

Proof. By linearity of d and $\bullet$ in its left argument we obtain for the left side

$$\begin{aligned}
&\sum_{i=0}^{n} \tfrac{(-1)^{in-i}}{i!}\, d\{[(S_\bullet^i\chi_n^s)^{i;s} \blacktriangleleft \theta] \bullet \phi_p\} - \sum_{i=0}^{n+1} \tfrac{(-1)^{in}}{i!}\, \{[S_\bullet^i(d\chi_n^s)]^{i;s} \blacktriangleleft \theta\} \bullet \phi_p = \\
&= \sum_{i=0}^{n} \tfrac{(-1)^{in-i}}{i!}\, [d(S_\bullet^i\chi_n^s)^{i;s} \blacktriangleleft \theta] \bullet \phi_p + \tbinom{s}{1}_\ell \sum_{i=0}^{n} \tfrac{(-1)^{in-n-i}}{i!} \{[(S_\bullet^i\chi_n^s)^{i;s} \blacktriangleleft \theta]^{1;s-1} \blacktriangleleft d\phi_p\} \bullet \phi_p \\
&\quad - \sum_{i=1}^{n} \tfrac{(-1)^{in-n-i}}{(i-1)!} \{[(S_\bullet^i\chi_n^s)^{i-1;s+1} \blacktriangleleft \theta]^{1;s} \blacktriangleleft d\theta\} \bullet \phi_p - \sum_{i=0}^{n+1} \tfrac{(-1)^{in}}{i!} \{[S_\bullet^i(d\chi_n^s)]^{i;s} \blacktriangleleft \theta\} \bullet \phi_p
\end{aligned}$$

by (28). With Corollary 5.4 we get

$$\begin{aligned}
&\sum_{i=0}^{n} \tfrac{(-1)^{in-i}}{i!}\, [d(S_\bullet^i\chi_n^s)^{i;s} \blacktriangleleft \theta] \bullet \phi_p - \sum_{i=0}^{n+1} \tfrac{(-1)^{in}}{i!}\, \{[S_\bullet^i(d\chi_n^s)]^{i;s} \blacktriangleleft \theta\} \bullet \phi_p = \\
&= -\sum_{i=2}^{n+1} \tfrac{(-1)^{in-i}}{(i-2)!} \{[(S_\bullet^{i-1}\chi_n^s)^{i-2;s+1} \blacktriangleleft \theta]^{1;s} \blacktriangleleft (\operatorname{sgn}(S)\tfrac{1}{2}\theta \wedge_{\mathfrak{g}} \theta)\} \bullet \phi_p \\
&\quad + \tbinom{s}{1}_\ell \sum_{i=1}^{n+1} \tfrac{(-1)^{in-i}}{(i-1)!} \{[(S_\bullet^{i-1}\chi_n^s)^{i-1;s} \blacktriangleleft \theta]^{1;s-1} \blacktriangleleft (\operatorname{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \phi_p)\} \bullet \phi_p.
\end{aligned}$$

Finally we put all together and use $S_\bullet^{i+1}\chi_n^s = (-1)^i S_\bullet^i(S_\bullet\chi_n^s)$ from (13). ∎

For $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$, the last term in Theorem 5.5 reads

$$\sum_{k=1}^{s} (-1)^{n+p(k-1)} \{[[(\chi_n^s \circledS \theta)^{k-1;s-k+1} \blacktriangleleft \phi_p]^{1;s-k} \blacktriangleleft (d\phi_p - \operatorname{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \phi_p)]^{s-k} \bullet \phi_p\}$$

as a consequence of Proposition 5.3, cf. (31). In any case we get the following

Corollary 5.6. *If S is a* LIE *group action of G on P, $\chi_n^s \in \mathcal{A}_n(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$, and $\theta \in \mathcal{A}_1(P,\mathfrak{g})$, $\phi_p \in \mathcal{A}_p(P,\mathfrak{g})$ with $d\phi_p = \mathrm{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \phi_p$, then*

$$d[(\chi_n^s \circledS \theta) \bullet \phi_p] = [(d\chi_n^s) \circledS \theta] \bullet \phi_p + \{[(S_\bullet \chi_n^s) \circledS \theta]^{1;s} \blacktriangleleft (d\theta - \mathrm{sgn}(S)\tfrac{1}{2}\theta \wedge_{\mathfrak{g}} \theta)\}^s \bullet \phi_p.$$

Now suppose, θ is a pullback of an invariant 1-form on G. Then the MAURER-CARTAN identities $d\Theta^S = \mathrm{sgn}(S)\frac{1}{2}\Theta^S \wedge_{\mathfrak{g}} \Theta^S$ and (17) give

Corollary 5.7. *Let S be a* LIE *group action of G on P, $f\colon P \to G$ differentiable, $K \in \mathrm{End}(\mathfrak{g})$ and $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ $\mathfrak{g}$-equivariant.*

1. *If $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Sym}_s^c(\mathfrak{g}, V)$ and $\phi_p \in \mathcal{A}_p(P,\mathfrak{g})$, then*

$$\begin{aligned} d[(\chi_n^s \circledS f^\star\psi^S(K)) \bullet \phi_p] &= [(d\chi_n^s) \circledS f^\star\psi^S(K)] \bullet \phi_p \\ +(-1)^n \tbinom{s}{1}_\ell \, [(\chi_n^s \circledS f^\star\psi^S(K))^{1;s-1} &\blacktriangleleft (d\phi_p - \mathrm{sgn}(S)\, f^\star\psi^S(K) \wedge_{\mathfrak{g}} \phi_p)]^{s-1} \bullet \phi_p, \\ d[(\chi_n^s \circledS f^\star\Theta^S) \bullet \phi_p] &= [(d\chi_n^s) \circledS f^\star\Theta^S] \bullet \phi_p \\ +(-1)^n \tbinom{s}{1}_\ell \, [(\chi_n^s \circledS f^\star\Theta^S)^{1;s-1} &\blacktriangleleft (d\phi_p - \mathrm{sgn}(S) f^\star\Theta^S \wedge_{\mathfrak{g}} \phi_p)]^{s-1} \bullet \phi_p. \end{aligned}$$

2. *For $\phi_p \in \mathcal{A}_p(P,\mathfrak{g})$ with $d\phi_p = \mathrm{sgn}(S) f^\star\psi^S(K) \wedge_{\mathfrak{g}} \phi_p$, e. g. for $\phi_2 = d(f^\star\psi^S(K))$,*

$$\begin{aligned} d[(\chi_n^s \circledS f^\star\psi^S(K)) \bullet \phi_p] &= [(d\chi_n^s) \circledS f^\star\psi^S(K)] \bullet \phi_p, \\ d[(\chi_n^s \circledS f^\star\Theta^S) \bullet \phi_p] &= [(d\chi_n^s) \circledS f^\star\Theta^S] \bullet \phi_p. \end{aligned}$$

Finally, in the case $s = 0$, Theorem 5.5 yields

Corollary 5.8. *If S is a* LIE *group action of G on P and $\omega_n \in \mathcal{A}_n(P) \otimes V$ is $\mathfrak{g}$-invariant, then for all $\theta \in \mathcal{A}_1(P,\mathfrak{g})$*

$$d(\omega_n \circledS \theta) = (d\omega_n) \circledS \theta \; + \; [(S_\bullet \omega_n) \circledS \theta]^1 \blacktriangleleft (d\theta - \tfrac{1}{2}\,\mathrm{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \theta).$$

For any $f\colon P \to G$, $K \in \mathrm{End}(\mathfrak{g})$, especially $K = \mathrm{id}_{\mathfrak{g}}$, we thus obtain

$$d(\omega_n \circledS f^\star\psi^S(K)) = (d\omega_n) \circledS f^\star\psi^S(K), \qquad d(\omega_n \circledS f^\star\Theta^S) = (d\omega_n) \circledS f^\star\Theta^S.$$

Recall that the gauge fields $\mathrm{F}^\alpha \in \mathcal{A}_2(U_\alpha, \mathfrak{g})$ are given by $\mathrm{F}^\alpha = d\mathrm{A}^\alpha + \frac{1}{2}\mathrm{A}^\alpha \wedge_{\mathfrak{g}} \mathrm{A}^\alpha$. For that reason we are interested especially in the case where $\phi_2 = d\theta - \frac{1}{2}\,\mathrm{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \theta$. Using $\theta \wedge_{\mathfrak{g}} (\theta \wedge_{\mathfrak{g}} \theta) = 0$ one easily checks that this yields $d\phi_2 = \mathrm{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \phi_2$. Thus Corollary 5.6 reads

$$d[(\chi_n^s \circledS \theta) \bullet \phi_2] = [(d\chi_n^s) \circledS \theta] \bullet \phi_2 + [(S_\bullet \chi_n^s) \circledS \theta] \bullet \phi_2.$$

Now $S_\bullet \chi_n^s \in \mathcal{A}_{n-1}(P, \mathrm{Hom}(\mathfrak{g}, \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V))) \cong \mathcal{A}_{n-1}(P, \mathrm{Hom}(\bigotimes^{s+1} \mathfrak{g}, V))$. Since ϕ_2 has even degree, only the symmetric part of $\mathrm{Hom}(\bigotimes^{s+1} \mathfrak{g}, V)$ counts [e.g., confer (14)]. So $[(S_\bullet \chi_n^s) \circledS \theta] \bullet \phi_2 = \mathrm{Sym}_\star[(S_\bullet \chi_n^s) \circledS \theta] \bullet \phi_2 = [\mathrm{Sym}_\star(S_\bullet \chi_n^s) \circledS \theta] \bullet \phi_2$, because $\circledS$ only acts on $\mathcal{A}(P)$ and commutes with any operation on $\mathrm{Hom}(\bigotimes^{s+1} \mathfrak{g}, V)$. This leads to the following definition:

Definition 5.9. For $\chi_n^s \in \mathcal{A}_n(P, \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V))$ and $S\colon G \times P \to P$, we define

$$S_\bullet^\vee \chi_n^s := \mathrm{Sym}_\star(S_\bullet \chi_n^s) \in \mathcal{A}_{n-1}(P, \mathrm{Sym}_{s+1}(\mathfrak{g}, V)).$$

Corollary 5.10. *If S is a* LIE *group action of G on P, $\chi_n^s \in \mathcal{A}_n(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$, $\theta \in \mathcal{A}_1(P, \mathfrak{g})$ and $\phi_2 = d\theta - \frac{1}{2}\,\mathrm{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \theta \in \mathcal{A}_2(P, \mathfrak{g})$, then*

$$d[(\chi_n^s ⓢ \theta) \bullet \phi_2] = [(d\chi_n^s) ⓢ \theta] \bullet \phi_2 + [(S_\bullet^\vee \chi_n^s) ⓢ \theta] \bullet \phi_2.$$

Extend the symmetric product $\vee$ in $\mathrm{Sym}(\mathfrak{g}, \mathbb{R}) \cong \mathrm{S}(\mathfrak{g}^*)$ to $\mathrm{Sym}(\mathfrak{g}, V)$, whenever a bilinear map $\phi\colon V \times V \to V$ is given. Equip $\mathcal{A}(P) \otimes \mathrm{Sym}(\mathfrak{g}, V)$ with the gradation induced by $\mathcal{A}(P)$, then we obtain from (8) since $\imath_{\mathcal{X}}$ is a skew-derivation of degree -1:

Lemma 5.11. *$S_\bullet^\vee$ is a skew-derivation of degree -1 of $\mathcal{A}(P)_{\mathrm{equiv}} \otimes \mathrm{Sym}(\mathfrak{g}, V)$ and $\mathcal{A}(P) \otimes \mathrm{Sym}(\mathfrak{g}, V)$, e. g. for $\alpha_n \in \mathcal{A}_n(P) \otimes \mathrm{Sym}(\mathfrak{g}, V)$ and $\omega \in \mathcal{A}(P) \otimes \mathrm{Sym}(\mathfrak{g}, V)$,*

$$S_\bullet^\vee(\alpha_n \wedge_\vee \omega) = (S_\bullet^\vee \alpha_n) \wedge_\vee \omega + (-1)^n \alpha_n \wedge_\vee (S_\bullet^\vee \omega).$$

In view of our applications to connections on bundles we have thus proved:

Theorem 5.12. *Let Γ be a connection on a principal fiber bundle $P(M, G)$ and let $B(M, F, G)$ be an associated bundle, V any vector space, $\chi_n^s \in \mathcal{A}_n(F) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ be G-equivariant and $\phi_n \in \mathcal{A}_n(F) \otimes V$ be invariant under G. Then*

$$\begin{aligned} d(\chi_n^s v \bullet \mathrm{F}) &= [(d\chi_n^s)v]_{n+1}^s \bullet \mathrm{F} + [(L_\bullet \chi_n^s)v]_{n-1}^{s+1} \bullet \mathrm{F}, \\ &= [(d\chi_n^s)v]_{n+1}^s \bullet \mathrm{F} + [(L_\bullet^\vee \chi_n^s)v]_{n-1}^{s+1} \bullet \mathrm{F}, \\ d(\phi_n v) &= (d\phi_n)v + [(L_\bullet \phi_n)v]_{n-1}^1 \bullet \mathrm{F}. \end{aligned}$$

References

[1] Gross, C., *A generalization of the exterior product of differential forms combining Hom-valued forms*, submitted.

[2] Gross, C., *Connections on fiber bundles and canonical lifts of differential forms*, submitted.

[3] Helgason, S., "Differential Geometry and Symmetric Spaces," Academic Press, 1962.

[4] Kobayashi, S. and Nomizu, K., "Foundations of Differential Geometry," Vol. I, John Wiley & Sons, 1963.

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