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Criterion for proper actions on homogeneous spaces of reductive groups

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Abstract. Let M be a manifold, on which a real reductive Lie group G acts transitively. The action of a discrete subgroup Γ on M is not always properly discontinuous. In this paper, we give a criterion for properly discontinuous actions, which generalizes our previous work [6] for an analogous problem in the continuous setting. Furthermore, we introduce the discontinuous dual $\pitchfork(H{:}G)$ of a subset H of G, and prove a duality theorem that each subset H of G is uniquely determined by its discontinuous dual up to multiplication by compact subsets.

1. Introduction

This paper is a continuation of [6]. We shall give a criterion for the properly discontinuous action on a homogeneous space of a real reductive group.

Suppose a discrete group Γ acts on a manifold M properly discontinuously and freely. Then the coset space $\Gamma\backslash M$ carries naturally a manifold structure on the quotient topology. We are particularly interested in the setting where a Lie group G acts on M transitively and Γ is a discrete subgroup of G, so that any local G-invariant structure on M is inherited by the quotient manifold $\Gamma\backslash M$.

Let us formulate our object of study in terms of groups. We denote by H the isotropy subgroup of G at a point o of M. *A Clifford-Klein form* of a homogeneous manifold $M \simeq G/H$ is the double coset space $\Gamma\backslash G/H$ if Γ is a subgroup of a Lie group G acting properly discontinuously and freely on G/H. A typical example is a closed Riemann surface with genus ≥ 2, which is biholomorphic to a Clifford-Klein form of the Poincaré plane $PSL(2,\mathbb{R})/SO(2)$. Clifford-Klein forms have been studied extensively, in particular, for a Riemannian symmetric space G/H, namely, when H is a maximal compact subgroup of a real reductive Lie group G (e.g. [1, 2, 15, 14]). An important remark here is that if the isotropy subgroup H is compact then the action of a torsion free discrete subgroup Γ on G/H is automatically properly discontinuous (see §2.1

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for the definition) and free. Hence, the study of Clifford-Klein forms of G/H with H compact is essentially equivalent to the study of discrete subgroups of G.

In this paper, we deal with a more general setting where H is a noncompact closed subgroup of a real reductive Lie group G. A distinguishing feature in our setting with H noncompact is that a discrete subgroup Γ of G does not always act *properly discontinuously* on G/H so that the double coset space $\Gamma\backslash G/H$ is not necessarily Hausdorff in the quotient topology. For instance, Calabi-Markus proved in [3] that any relativistic spherical space form with dimension ≥ 3 is noncompact and has a finite fundamental group, which is a consequence of the fact that no infinite discrete subgroup of $O(n,1)$ acts properly discontinuously on $O(n,1)/O(n-1,1)$. The Calabi-Markus phenomenon for a homogeneous space G/H has been studied in [3, 18, 19, 20, 12, 16, 6] where both G and H are reductive; and in [7, 9, 13] where G is a solvable Lie group. The method of "the continuous analog" (e.g. [6, 7, 9, 13]) has been a powerful tool in the study of properly discontinuous actions, in particular, for the Calabi-Markus phenomenon. However, for a further study of discontinuous groups and Clifford-Klein forms of homogeneous manifolds, we think it is important to obtain more straightforward methods for properly discontinuous actions without using the continuous analog (see [7], [10] for open problems and the survey of recent results on Clifford-Klein forms of homogeneous manifolds). This is the motivation for this paper. In fact, our result (see Theorem 3.4) plays a key role in the deformation of compact Clifford-Klein forms with indefinite metric in higher dimensions, where the action of the Zariski closure of a discrete subgroup Γ on G/H is not always proper even though that of Γ on G/H is properly discontinuous (see [11]).

Let us now state our results. Suppose that H and L are subsets of a locally compact topological group G. We write $H \sim L$ in G if and only if there exists a compact set S of G such that $L \subset SHS^{-1}$ and $H \subset SLS^{-1}$, where $SHS^{-1} := \{ahb^{-1} \in G : a, b \in S, h \in H\}$. We write $H \pitchfork L$ in G if and only if $SHS^{-1} \cap L$ is relatively compact for any compact set S in G (see Definition 2.1.1). We note that if H is a closed subgroup and Γ is a discrete subgroup of G, then the action of Γ on the homogeneous manifold G/H is properly discontinuous if and only if $H \pitchfork \Gamma$ in G (see Observation 2.1.3). Now, let G be a real reductive linear Lie group and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition. We fix a maximally abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and denote by W_G the Weyl group of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. For a subset L of G we set $\mathfrak{a}(L) := \{X \in \mathfrak{a} : \exp X \in KLK\}$ (see Definition 3.2). Then we shall prove:

Theorem 1.1. (See Theorem 3.4 and Theorem 5.6). *Suppose H, L are subsets of a real reductive linear Lie group G.*

1) $H \pitchfork L$ *in* $G \Leftrightarrow \mathfrak{a}(H) \pitchfork \mathfrak{a}(L)$ *in* $\mathfrak{a}$.
2) $H \sim L$ *in* $G \Leftrightarrow \mathfrak{a}(H) \sim \mathfrak{a}(L)$ *in* $\mathfrak{a}$.

If both H and L are connected closed subgroups which are reductive in G, then the part (1) yields our previous result (see Theorem 4.1 in [6]). The novelty here is that we allow L to be non-reductive, in particular, to be discrete. We note that we could assume one of H or L to be reductive without loss of generality if we replace the triplet (H, G, L) by $(H \times L,\ G \times G,\ \mathrm{diag}(G))$.

The part (2) is proved by a "duality theorem" (see §5) asserting that a subset H of G is determined up to $\sim$ if we know the *discontinuous dual* of H in G (see §5.2), that is, all the subsets $L \subset G$ with the property $H \pitchfork L$ in G.

In §6, we shall give computations of $\mathfrak{a}(H)$ for some typical subsets H of G. For instance, we describe $\mathfrak{a}(\Gamma)$ for an arbitrary cyclic Γ in Example 6.2; and we find a certain nilpotent subgroup N of $G = GL(n, \mathbb{R})$ such that $\mathfrak{a}(N)$ coincides with that of a reductive subgroup $O(p, n-p)$ in Example 6.5.

Theorem 1.1 leads to:

Corollary 1.2. (See Corollary 3.5). *Suppose H is a closed subgroup of a real reductive linear Lie group G. Then the following conditions are equivalent:*
(1.2.1) *The action of a discrete subgroup Γ on a homogeneous space G/H is properly discontinuous.*
(1.2.2) *For any compact subset V of $\mathfrak{a}$, $(\mathfrak{a}(\Gamma) + V) \cap \mathfrak{a}(H)$ is compact.*

Here is a necessary condition for the Calabi-Markus phenomenon to occur:

Corollary 1.3. *Suppose G is a real reductive linear Lie group and H is a closed subgroup. If $\mathfrak{a}(H)$ is contained in a proper cone of $\mathfrak{a}$, then there exists an infinite discrete subgroup Γ of G acting freely and properly discontinuously on G/H.*

Let us consider the reductive case, namely, H is a closed subgroup which is stable under a Cartan involution of G. Then $\mathrm{rank}_{\mathbb{R}} H < \mathrm{rank}_{\mathbb{R}} G$ if and only if $\mathfrak{a}(H) \neq \mathfrak{a}$, which is equivalent to the condition that $\mathfrak{a}(H)$ is contained in a proper cone of $\mathfrak{a}$. Hence, Corollary 1.3 in the reductive case means that if $\mathrm{rank}_{\mathbb{R}} H < \mathrm{rank}_{\mathbb{R}} G$ then there exists an infinite discrete subgroup Γ of G acting freely and properly discontinuously on G/H. This was the non-trivial implication of Corollary 4.4 in [6] (the criterion for the Calabi-Markus phenomenon in the reductive case).

The results of this paper were announced in the fifth workshop on Lie groups at Tottori and in the Lie group and representation theory seminar at University of Tokyo in September 1994 (see also [10] §2.11 and §3.9). After completing this work, the author received a preprint of Benoist about a result similar to Theorem 3.4. His proof is different from ours and he also found some applications such as the non-existence theorem of compact Clifford-Klein forms of $SO(4n, \mathbb{C})/SO(4n-1, \mathbb{C})$. (See Y. Benoist, *Actions propres sur les espaces homogènes réductifs*, Preprint.)

2. Preliminary Results

Suppose that H and L are subsets of a locally compact topological group G.

Definition 2.1.1. We denote by $H \sim L$ in G the existence of a compact subset S of G such that $L \subset SHS^{-1}$ and $H \subset SLS^{-1}$. Then it is easy to see that the relation $H \sim L$ in G defines an equivalence relation. That is, the relation $\sim$ is reflexive, symmetric and transitive. We say the pair (H, L) is *proper* in G, denoted by $H \pitchfork L$ in G, if and only if $SHS^{-1} \cap L$ is relatively compact for any compact subset S in G.

Example 2.1.2. Let G be a finite dimensional vector space, and H and L subspaces of G. Then $H \pitchfork L$ in G if and only if $H \cap L = \{0\}$, and $H \sim L$ in G if and only if $H = L$.

The above definitions are motivated by the following:

Observation 2.1.3. *Let H and L be closed subgroups of G, and Γ a discrete subgroup of G.*
1) *The action of L on a homogeneous space G/H is proper if and only if $L \pitchfork H$ in G.*
2) *The action of Γ on a homogeneous space G/H is properly discontinuous if and only if $\Gamma \pitchfork H$ in G.*

Here we recall the definition of proper actions and properly discontinuous actions: Suppose that a locally compact topological group L acts continuously on a locally compact Hausdorff space X. This action is called *proper* if and only if $L_S := \{g \in L : g \cdot S \cap S \neq \emptyset\}$ is compact for every compact subset S in X. The action is called *properly discontinuous* if and only if L is discrete and acts properly on X.

We record some elementary properties of the relations $\sim$, $\pitchfork$:

Lemma 2.2. *Suppose G is a locally compact topological group and that H, H', L are its subsets.*
1) *$H \pitchfork L$ in G if and only if $L \pitchfork H$ in G.*
2) *If $H \sim H'$ and if $H \pitchfork L$ in G, then $H' \pitchfork L$ in G.*
3) *$H \pitchfork L$ in G if and only if $H \pitchfork \overline{L}$ in G. Here $\overline{L}$ denotes the closure of L in G.*

Proof. 1) Let S be a compact subset of G. If $H \pitchfork L$ in G, then $SHS^{-1} \cap L$ is relatively compact for each compact subset S in G. Therefore $H \cap S^{-1}LS \subset S^{-1}(SHS^{-1} \cap L)S$ is also relatively compact. Hence $L \pitchfork H$ in G.
2) Take a compact subset $T \subset G$ such that $H' \subset THT^{-1}$. If $H \pitchfork L$, then $L \cap SH'S^{-1} \subset L \cap (ST)H(ST)^{-1}$ is relatively compact, whence $H' \pitchfork L$.
3) Take a compact neighborhood V of the identity e in G. Then $\overline{L} \subset LV^{-1}$ because $xV \cap L \neq \emptyset$ for each $x \in \overline{L}$. Suppose $H \pitchfork L$ in G. Then for any compact subset $S \subset G$, $S\overline{L}S^{-1} \cap H \subset SLV^{-1}S^{-1} \cap H \subset (SV)L(SV)^{-1} \cap H$ is relatively compact. Hence $H \pitchfork \overline{L}$ in G. ∎

2.3. The following lemma is immediate from Definition 2.1.1:

Lemma 2.3. *Suppose G' is a closed subgroup of a locally compact topological group G. Let H and L be subsets of G'.*
1) *If $H \sim L$ in G', then $H \sim L$ in G.*
2) *If $H \pitchfork L$ in G, then $H \pitchfork L$ in G'.*

3. Criterion for proper actions

Hereafter, we suppose that G is a real reductive linear Lie group. We assume G is contained in a connected complex Lie group $G_{\mathbb{C}}$ (G is not necessarily connected, e.g. $G = GL(n,\mathbb{R})$). We fix a maximal compact subgroup K of G. Let θ be the corresponding Cartan involution of G and we write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. We fix a maximally abelian subspace $\mathfrak{a}_G \equiv \mathfrak{a} \subset \mathfrak{p}$. All such subspaces are mutually conjugate by K. The dimension $\dim \mathfrak{a}$ is called the *real rank* of G, denoted by $\mathrm{rank}_{\mathbb{R}} G$. We write A for the connected subgroup of G having the Lie algebra $\mathfrak{a}$. Let M be the centralizer of $\mathfrak{a}$ in K, M' the normalizer of $\mathfrak{a}$ in K. Then the finite group $W_G := M'/M$ acts effectively on $\mathfrak{a}$ (also on A) as the Weyl group associated to the restricted root system $\Sigma(\mathfrak{g},\mathfrak{a})$ (see §4.1). We have a Cartan decomposition

$$G = KAK. \tag{3.1.1}$$

In (3.1.1), there is a unique element $a(g) \in A$ up to conjugation by W_G such that $g \in Ka(g)K$ for each $g \in G$.

Definition 3.2. For each subset L of G, we define:

$$\begin{aligned} A(L) &:= A \cap KLK = \{w \cdot a(g) : w \in W_G,\ g \in L\} \subset A, \\ \mathfrak{a}(L) &:= \log A(L) \subset \mathfrak{a}. \end{aligned}$$

Here $\log\colon A \to \mathfrak{a}$ is the inverse of the diffeomorphism $\exp\colon \mathfrak{a} \to A$.

We remark that the above notation is essentially the same with that in [6] (see also [8]) in the special case where L is a θ-stable connected subgroup of G (see Example 6.1). (To be more precise, they coincide as subsets of the set of equivalence classes $\mathfrak{a}/W_G$, the quotient of $\mathfrak{a}$ by the action of the Weyl group W_G.)

We list some elementary properties of $A(L)$ and $\mathfrak{a}(L)$ which are immediate from their definition:

Lemma 3.3. *Suppose L is a subset of G.*
1) *$\mathfrak{a}(L)$ is a W_G-invariant set.*
2) *$L \sim A(L)$ in G.*
3) *L is a closed set in G if and only if $\mathfrak{a}(L)$ is closed in $\mathfrak{a}$.*
4) *If $L' \subset L$ are subsets of G such that $\overline{L'} = L$, then $\overline{A(L')} = A(L)$.*
5) *If L', L are subsets of G such that $KL'K = KLK$, then $A(L') = A(L)$.*

Proof. (1), (3) and (5) are obvious.
(2) follows from $L \subset KA(L)K$ and $A(L) \subset KLK$.
Now we prove (4). As $A(L)$ is closed by (3) and as $A(L') \subset A(L)$, we have $\overline{A(L')} \subset A(L)$. Conversely, as $L' = KA(L')K$, we have $\overline{L'} = \overline{KA(L')K} = K\overline{A(L')}K$. Hence, $A(L) \subset KLK = K\overline{L'}K = K\overline{A(L')}K$, and we have $A(L) \subset \overline{A(L')}$. ∎

Theorem 3.4. *Let H and L be subsets of a real reductive linear Lie group G. Then the following four conditions are equivalent:*

(3.4.1) $H \pitchfork L$ *in* G.

(3.4.1)′ $A(H) \pitchfork A(L)$ *in* G.

(3.4.2) $\mathfrak{a}(H) \pitchfork \mathfrak{a}(L)$ *in* $\mathfrak{a}$.

(3.4.2)′ $A(H) \pitchfork A(L)$ *in* A.

Strategy of Proof. The equivalence (1) $\Leftrightarrow$ (1)′ follows from Lemma 3.3 (2) and Lemma 2.2. The equivalence (2) $\Leftrightarrow$ (2)′ is trivial, because $\exp\colon \mathfrak{a} \simeq A$ is an isomorphism of Lie groups. The implication (1)′ $\Rightarrow$ (2)′ follows from Lemma 2.3. The only non-trivial implication (2) $\Rightarrow$ (1)′ will be proved after a sequence of lemmas in §4.

Corollary 3.5. *Let G be a real reductive linear Lie group, and H, L closed subgroups of G. Then the following conditions are equivalent:*

(3.5.1) *The action of L on a homogeneous space G/H is proper.*

(3.5.2) *For any compact subset V of $\mathfrak{a}$, $(\mathfrak{a}(L)+V)\cap\mathfrak{a}(H)$ is compact.*

Moreover, if L is a discrete subgroup of G, then (3.5.2) is equivalent to

(3.5.1)′ *The action of L on a homogeneous space G/H is properly discontinuous.*

4. Proof of Theorem 3.4

In this section we give a proof of the non-trivial implication (2)′ $\Rightarrow$ (1)′ in Theorem 3.4. The idea of its proof is based on our previous technique in [6]. In particular, the first half of this section parallels [6] §3. However, in order to obtain sharper results such as Lemma 4.8, we need to deal with $\mathfrak{a}$ rather than with the unit sphere in $\mathfrak{a}$ and we have changed some notation of [6] (e.g. $\frac{Y_n}{\|Y_n\|}$ here corresponds to Y_n in [6] §3). So, we have decided to give a self-contained proof here for the convenience of the reader.

Let G be a real reductive linear Lie group. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition, and $\mathfrak{z}(\mathfrak{g})$ the center of $\mathfrak{g}$. We define ideals of $\mathfrak{g}$ by $\mathfrak{g}' := [\mathfrak{g},\mathfrak{g}] + (\mathfrak{z}(\mathfrak{g})\cap\mathfrak{k})$, $\mathfrak{c}_{\mathfrak{g}} := \mathfrak{z}(\mathfrak{g})\cap\mathfrak{p}$. Then $\mathfrak{g} = \mathfrak{g}'\oplus\mathfrak{c}_{\mathfrak{g}}$. Let $C_G := \exp(\mathfrak{c}_{\mathfrak{g}})\,(\subset G)$, and let G' be the (unique) normal subgroup of G with Lie algebra $\mathfrak{g}'$ such that G' meets each connected component of G. Then G is isomorphic to the direct product group $G'\times C_G$ ([17] Ch.1, §1.5). According to this decomposition, we write $g = g'g^c \in G \simeq G'\times C_G$, $X = X' + X^c \in \mathfrak{g} \simeq \mathfrak{g}'\oplus\mathfrak{c}_{\mathfrak{g}}$. Analogous notation is used for other reductive subgroups. The simultaneous diagonalization of $\operatorname{ad}(\mathfrak{a})|_{\mathfrak{g}}$ gives the direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}(\mathfrak{a};0) + \sum_{\alpha\in\Sigma}\mathfrak{g}(\mathfrak{a};\alpha). \tag{4.1.1}$$

Here

$$\mathfrak{g}(\mathfrak{a};\alpha) := \{X\in\mathfrak{g};\quad [H,X] = \alpha(H)X \quad \text{for any} \quad H\in\mathfrak{a}\} \quad \text{for} \quad \alpha\in\mathfrak{a}^*,$$
$$\Sigma \equiv \Sigma(\mathfrak{g},\mathfrak{a}) := \{\alpha\in\mathfrak{a}^*;\quad \mathfrak{g}(\mathfrak{a};\alpha)\neq 0\}\setminus\{0\}.$$

For each $\alpha \in \Sigma \cup \{0\}$, we define the projection with respect to the direct sum (4.1.1) by

$$p_\alpha : \mathfrak{g} \to \mathfrak{g}(\mathfrak{a};\alpha). \tag{4.1.2}$$

For $Y \in \mathfrak{a}$ we define a parabolic subalgebra of $\mathfrak{g}$ by

$$\mathfrak{p}(Y) = \mathfrak{n}(Y) + \mathfrak{l}(Y) := \sum_{\alpha(Y)>0} \mathfrak{g}(\mathfrak{a};\alpha) + \sum_{\alpha(Y)=0} \mathfrak{g}(\mathfrak{a};\alpha).$$

The corresponding parabolic subgroup of G is given by

$$\begin{aligned} P(Y) &:= N_G(\mathfrak{p}(Y)) \equiv \{g \in G : \operatorname{Ad}(g)\mathfrak{p}(Y) = \mathfrak{p}(Y)\} \\ &= N(Y) \cdot L(Y) \quad \text{(Levi decomposition)}, \end{aligned}$$

where $N(Y) = \exp(\mathfrak{n}(Y))$ and $L(Y) = P(Y) \cap \theta P(Y) = Z_G(Y)$ because G is contained in a connected complex reductive group $G_{\mathbb{C}}$.

Suppose $\mathfrak{a}_1$, $\mathfrak{a}_2$ are W_G-invariant subsets of $\mathfrak{a}$, $A_j := \exp \mathfrak{a}_j \subset A$ $(j = 1, 2)$. If $A_1 \not\pitchfork A_2$ in G, namely, if there exists a compact subset S of G such that $SA_1S^{-1} \cap A_2$ is not relatively compact, then we find sequences

$$a_n, b_n \in S, \quad Y_n \in \mathfrak{a}_1, \quad Z_n \in \mathfrak{a}_2 \quad (n \in \mathbb{N})$$

such that $a_n \exp(Y_n) b_n^{-1} = \exp(Z_n)$, and $\{Z_n : n \in \mathbb{N}\}$ is not relatively compact. We fix a closed positive Weyl chamber $\overline{\mathfrak{a}_+}$, i.e., the closure of a connected component of the subset $\mathfrak{a}' := \{H \in \mathfrak{a} : \alpha(H) \neq 0 \text{ for any } \alpha \in \Sigma\}$, and a W_G-invariant norm $|\cdot|$ on $\mathfrak{a}$. Replacing a_n, b_n, Y_n, Z_n by appropriate subsequences and taking conjugation by W_G (or by representatives in M') if necessary, we have the following setting:

Setting 4.2. There are sequences $a_n, b_n \in G; Y_n, Z_n \in \mathfrak{a}$ $(n \in \mathbb{N})$, and there are $a, b \in G$, $Y, Z \in \overline{\mathfrak{a}_+} \setminus \{0\}$ such that

(4.2.1) $Y_n \in \mathfrak{a}_1, \quad Z_n \in \mathfrak{a}_2 \quad (n \in \mathbb{N}),$

(4.2.2) $a_n = \exp(Z_n)\, b_n \exp(-Y_n),$

(4.2.3) $\displaystyle\lim_{n\to\infty} |Y_n| = \lim_{n\to\infty} |Z_n| = \infty, \quad \lim_{n\to\infty} \frac{Y_n}{|Y_n|} = Y, \quad \lim_{n\to\infty} \frac{Z_n}{|Z_n|} = Z,$

(4.2.4) $\displaystyle\lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} b_n = b.$

For each $\alpha, \beta \in \Sigma(\mathfrak{g}, \mathfrak{a})$ and for any $X_\alpha \in \mathfrak{g}(\mathfrak{a};\alpha)$, the equation (4.2.2) gives rise to a formula:

$$p_\beta\left(\operatorname{Ad}(a_n)X_\alpha\right) = \exp\left(\beta(Z_n) - \alpha(Y_n)\right) p_\beta\left(\operatorname{Ad}(b_n)X_\alpha\right). \tag{4.2.5}$$

This formula will play a crucial role later in the reduction from a non-commutative setting to an abelian setting.

Now we explain the remaining part of the proof of Theorem 3.4. We shall prove later in Lemma 4.8 that $T := \{Y_n - Z_n : n \in \mathbb{N}\} \subset \mathfrak{a}$ is a bounded subset. This means that $\mathfrak{a}_1 \cap (\mathfrak{a}_2 + T)$ contains an unbounded set $\{Y_n : n \in \mathbb{N}\}$. Hence, $\mathfrak{a}_1 \not\pitchfork \mathfrak{a}_2$, which shows (2) $\Rightarrow$ (1)$'$ of Theorem 3.4. The rest of this section is devoted to the proof of Lemma 4.8.

Lemma 4.3. *In the setting (4.2), the sequence* $\frac{|Z_n|}{|Y_n|}$ *is bounded away from* 0 *and* ∞*.*

Proof. Assume there were a subsequence $n(k)$, $k = 1, 2, \dots$ such that

$$\lim_{k\to\infty} \frac{|Z_{n(k)}|}{|Y_{n(k)}|} = \infty.$$

We take $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a})$ such that $\beta(Z) > 0$. For each $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \cup \{0\}$, we have

$$\lim_{k\to\infty} \exp\big(\beta(Z_{n(k)}) - \alpha(Y_{n(k)})\big) = \infty. \tag{4.3.1}$$

Because the left side of (4.2.5) is a convergent sequence, (4.3.1) leads to

$$p_\beta\left(\mathrm{Ad}(b)X_\alpha\right) = \lim_{k\to\infty} p_\beta\left(\mathrm{Ad}(b_{n(k)})X_\alpha\right) = 0.$$

Since this equation holds for any $X_\alpha \in \mathfrak{g}(\mathfrak{a}; \alpha)$ and for any $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \cup \{0\}$, we have $p_\beta(\mathrm{Ad}(b)\ \mathfrak{g}) = \{0\}$, which contradicts the fact that $\mathrm{Ad}(b)\ \mathfrak{g} = \mathfrak{g}$. Thus, we have proved $\sup_n \frac{|Z_n|}{|Y_n|} < \infty$. Because the role of Y_n and Z_n is symmetric, we have similarly $\sup_n \frac{|Y_n|}{|Z_n|} < \infty$. Hence $\frac{|Z_n|}{|Y_n|}$ is bounded away from 0 and ∞. ■

Suppose we are in the setting of (4.2). It follows from Lemma 4.3 that we can find a subsequence $n(k)$ such that $\lim_{k\to\infty} \frac{|Z_{n(k)}|}{|Y_{n(k)}|} = C$ for some positive constant C (we shall show that C is equal to 1 in the proof of Lemma 4.6).

Lemma 4.4. *In the setting as above,*

$$\mathrm{Ad}(b)\left(\sum_{\alpha(Y)\le t} \mathfrak{g}(\mathfrak{a}; \alpha)\right) = \sum_{C\beta(Z)\le t} \mathfrak{g}(\mathfrak{a}; \beta) \quad \textit{for any} \quad t \in \mathbb{R}. \tag{4.4.1}$$

Proof. If $C\beta(Z) > \alpha(Y)$, then we have $\lim_{k\to\infty} \exp\big(\beta(Z_{n(k)}) - \alpha(Y_{n(k)})\big) = \infty$. This implies $p_\beta\left(\mathrm{Ad}(b)X_\alpha\right) = 0$ from (4.2.5). Therefore we have

$$\mathrm{Ad}(b)\,(X_\alpha) \in \sum_{C\beta(Z)\le t} \mathfrak{g}(\mathfrak{a}; \beta),$$

if $X_\alpha \in \mathfrak{g}(\mathfrak{a}; \alpha)$ such that $\alpha(Y) \le t$. Hence we have shown (4.4.1). ■

We recall a basic structural result of a reductive Lie algebra.

Lemma 4.5. *If* $b \in G$ *and* X, $X' \in \overline{\mathfrak{a}_+}$ *satisfy*

$$\mathrm{Ad}(b)\left(\sum_{\alpha(X)\le t} \mathfrak{g}(\mathfrak{a}; \alpha)\right) = \sum_{\alpha(X')\le t} \mathfrak{g}(\mathfrak{a}; \alpha) \quad \textit{for any} \quad t \in \mathbb{R}, \tag{4.5.1}$$

then $b \in P(-X) = P(-X')$ *and* $X - X' \in \mathfrak{c}_{\mathfrak{g}}$ *(see §4.1 for the definition).*

Proof. The first claim $b \in P(-X) = P(-X')$ follows from standard properties of parabolic subalgebras (see [17] Ch. 1), if we substitute $t = 0$ in (4.5.1).

Then it follows from $b \in P(-X)$ and from (4.5.1) that

(4.5.2)
$$\sum_{\alpha(X') \le t} \mathfrak{g}(\mathfrak{a};\alpha) = \mathrm{Ad}(b)\left(\sum_{\alpha(X)\le t} \mathfrak{g}(\mathfrak{a};\alpha)\right) \subset \sum_{\alpha(X)\le t} \mathfrak{g}(\mathfrak{a};\alpha) \quad \text{for any} \quad t \in \mathbb{R}.$$

We fix $\beta \in \Sigma(\mathfrak{g};\mathfrak{a})$ and $\varepsilon > 0$ and put $t := \beta(X) - \varepsilon$. Applying the projection p_β to (4.5.2), we have $\beta(X) - \varepsilon < \beta(X')$ because the right side vanishes. Because $\varepsilon > 0$ is arbitrary, The converse inequality $\beta(X) \ge \beta(X')$ is proved in a similar way by using the equality (4.5.1) multiplied by $\mathrm{Ad}(b^{-1})$. Therefore $\beta(X) = \beta(X')$ for any $\beta \in \Sigma(\mathfrak{g},\mathfrak{a})$. Hence $X - X' \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{a} = \mathfrak{c}_\mathfrak{g}$. ∎

Lemma 4.6. *In the setting (4.2), we have $b \in P(-Z)$, $a \in P(Z)$ and $Y = Z$.*

Proof. The first statement $b \in P(-Z)$ is deduced from Lemmas 4.4 and 4.5. Similarly, we have $a \in P(Z)$. Now, let us prove $Y = Z$. Let C be an accumulation point of $\frac{|Z_n|}{|Y_n|}$. We take a subsequence $n(k)$ such that $\lim_{k\to\infty} |\frac{Z_{n(k)}}{Y_{n(k)}}| = C$. It follows from Lemma 4.4 and Lemma 4.5 that

$$Y - CZ \in \mathfrak{c}_\mathfrak{g}. \tag{4.6.1}$$

According to the decomposition $G \simeq G' \times C_G$, $g \mapsto g'g^c$, we have from (4.2.2) that

$$a_{n(k)}{}^c = \exp(Z_{n(k)}{}^c)\, b_{n(k)}{}^c \, \exp(-Y_{n(k)}{}^c) \in C_G.$$

Hence $\exp(Z_{n(k)}{}^c - Y_{n(k)}{}^c) = a_{n(k)}{}^c (b_{n(k)}{}^c)^{-1}$ is a convergent sequence. By (4.2.3), we have

$$CZ - Y = \lim_{k\to\infty} \left(\frac{Z_{n(k)}}{\frac{1}{C}|Z_{n(k)}|} - \frac{Y_{n(k)}}{|Y_{n(k)}|} \right) = \lim_{k\to\infty} \frac{Z_{n(k)} - Y_{n(k)}}{|Y_{n(k)}|},$$

and then

$$(CZ - Y)^c = \lim_{k\to\infty} \frac{Z_{n(k)}{}^c - Y_{n(k)}{}^c}{|Y_{n(k)}|} = 0$$

because $\lim_{k\to\infty} |Y_{n(k)}| = \infty$ and because $Z_{n(k)}{}^c - Y_{n(k)}{}^c$ converges. Combined with (4.6.1), we have $CZ - Y = 0$. Because $|Y| = |Z| = 1$ and $Y, Z \in \overline{\mathfrak{a}_+}$, we have $C = 1$. Hence, $Y = Z$. ∎

Now, we prove a rank reduction lemma.

Lemma 4.7. *In the setting (4.2), assume that there are sequences $a_n, b_n \in G$, $Y_n, Z_n \in \mathfrak{a}$ satisfying (4.2.1), (4.2.2), (4.2.3), and (4.2.4). Then we can find sequences A_n, B_n in $L(Z)$ and elements A, B in $L(Z)$ that satisfy (4.2.2) and (4.2.4) if we replace a_n, b_n, a, b by A_n, B_n, A, B, respectively.*

Proof. First we note that $b \in P(-Z) \subset P(-Z)N(Z)$ by Lemma 4.6. Because $P(-Z)N(Z)$ is an open set of G, $b_n \in P(-Z)N(Z)$ if n is large enough. According to the diffeomorphism

$$N(-Z) \times N(Z) \times L(Z) \simeq P(-Z)N(Z) \subset G,$$

we write $b_n = B_n^- B_n^+ B_n$. The point here is that each sequence $B_n^\pm \in N(\pm Z)$, $B_n \in L(Z)$ converges as $n \to \infty$, respectively. We put

$$A_n^\pm := \exp(Z_n) B_n^\pm \exp(-Z_n), \quad A_n := \exp(Z_n) B_n \exp(-Y_n).$$

Then $\lim_{n\to\infty} A_n^- = 1$ because $B_n^- \in N(-Z)$ are bounded and because

$$\lim_{n\to\infty} \frac{Z_n}{|Z_n|} = Z \quad \text{and} \quad \lim_{n\to\infty} |Z_n| = \infty.$$

It follows from (4.2.2) and the definitions of $A^\pm$, $B^\pm$, A_n and B_n that

$$\left(A_n^-\right)^{-1} a_n = \left(A_n^-\right)^{-1} \exp(Z_n) B_n^- B_n^+ B_n \exp(-Y_n) = \left(A_n^-\right)^{-1} A_n^- A_n^+ A_n = A_n^+ A_n.$$

The first term converges to a as $n \to \infty$. Therefore the last term $A_n^+ A_n \in N(Z)L(Z)$ also converges as $n \to \infty$. Because $N(Z) \times L(Z) \simeq P(Z)$ is a homeomorphism, $A_n^+ \in N(Z)$ and $A_n \in L(Z)$ converge, respectively. In particular,

$$A_n = \exp(Z_n) B_n \exp(-Y_n)$$

converges. Hence the conditions (4.2.1), (4.2.2), (4.2.3) and (4.2.4) are also satisfied if we replace a_n, $b_n \in G$ by A_n, $B_n \in L(Z)$. ■

Let us complete the proof of Theorem 3.4 (2)⇒(1)$'$. What we want to prove now is:

Lemma 4.8. *In the setting of (4.2), the sequence $Y_n - Z_n$ forms a bounded subset in $\mathfrak{a}$.*

Proof. We proceed by induction on $\operatorname{rank}_{\mathbb{R}} G$. If $\operatorname{rank}_{\mathbb{R}} G = 0$, that is, if $\mathfrak{a} = 0$, there is nothing to prove. Assume now that we have proved the Lemma for real reductive groups with real rank $< r$. Suppose the setting (4.2) is satisfied for G with $\operatorname{rank}_{\mathbb{R}} G = r$. It follows from Lemma 4.7 that we find sequences A_n, $B_n \in L(Z)$, $Y_n \in \mathfrak{a}_1$, $Z_n \in \mathfrak{a}_2$ satisfying the setting (4.2). We write the Lie group $L(Z)$ which is a reductive subgroup, for simplicity, as L. Likewise we write $\mathfrak{l} := \mathfrak{l}(Z)$. With similar notation as in §4.1, we have

$$\mathfrak{l} = \mathfrak{l}' \oplus \mathfrak{c}_{\mathfrak{l}}, \quad L \simeq L' \times C_L. \tag{4.8.1}$$

According to the decomposition (4.8.1), we write $Y_n = Y_n' + Y_n^c$, $Z_n = Z_n' + Z_n^c \in \mathfrak{l}(Z)$, $A_n = A_n' A_n^c$, $B_n = B_n' B_n^c \in L(Z)$, respectively, with Y_n', $Z_n' \in \mathfrak{l}'$, Y_n^c, $Z_n^c \in \mathfrak{c}_{\mathfrak{l}}$, A_n', $B_n' \in L'$ and A_n^c, $B_n^c \in C_L$. Here we note that A_n', $B_n' \in L'$

and A_n^c, $B_n^c \in C_L$ are convergent sequences, respectively. Then the equation $A_n = \exp(Z_n)B_n\exp(-Y_n)$ is decomposed as

$$\begin{aligned} A_n' &= \exp(Z_n')B_n'\exp(-Y_n') \quad \in L', \\ A_n^c &= \exp(Z_n^c)B_n^c\exp(-Y_n^c) \quad \in C_L. \end{aligned}$$

Then the set $\{Y_n^c - Z_n^c : n \in \mathbb{N}\}$ is a bounded set in $\mathfrak{c}_{\mathfrak{l}}$ because C_L is an abelian Lie group and because both of the sequences A_n^c, B_n^c converge as $n \to \infty$. Also the set $\{Y_n' - Z_n' : n \in \mathbb{N}\}$ is a bounded set in $\mathfrak{l}'$ from the inductive assumption because $\operatorname{rank}_{\mathbb{R}} L' = \operatorname{rank}_{\mathbb{R}} L - \dim \mathfrak{c}_{\mathfrak{l}} < \operatorname{rank}_{\mathbb{R}} L = \operatorname{rank}_{\mathbb{R}} G = r$. Therefore, $\{Y_n - Z_n : n \in \mathbb{N}\} = \{(Y_n' + Y_n^c) - (Z_n' + Z_n^c) : n \in \mathbb{N}\}$ is also a bounded set in $\mathfrak{a}$, proving the Lemma. ■

5. Duality

Consider the following duality problems:

Question 5.1.

1) *Is a subset H determined up to the equivalence relation $\sim$ if one knows all the subsets L with the property $L \pitchfork H$ in G?*
2) *Is a subgroup H determined up to the equivalence relation $\sim$ if one knows all the subgroups L with the property $L \pitchfork H$ in G?*

In this section we give an affirmative answer to Question 5.1(1). The proof also leads us to affirmative answers to the following natural questions as we shall see in Theorem 5.6 (cf. Theorem 3.4 and Observation 2.1.3):

i) Does $\mathfrak{a}(H) \sim \mathfrak{a}(H')$ in $\mathfrak{a}$ if $H \sim H'$ in G?
ii) Does $\mathfrak{a}(H) \sim \mathfrak{a}(gHg^{-1})$ in $\mathfrak{a}$ if $g \in G$?

Let $\mathcal{P}(G)$ denote the totality of subsets of a locally compact topological group G. For $H \in \mathcal{P}(G)$, we define the *discontinuous dual* of H by

$$\pitchfork(H : G) := \{L \subset G : L \pitchfork H \quad \text{in} \quad G\} \subset \mathcal{P}(G). \tag{5.2.1}$$

Suppose $\mathfrak{a}$ is a finite dimensional vector space over $\mathbb{R}$ and $\mathfrak{a}_1$ is a subset. We write the multiplication of $\mathfrak{a}$ in an additive way. Then $\pitchfork(\mathfrak{a}_1 : \mathfrak{a})$ is given by

$$\begin{aligned} \pitchfork(\mathfrak{a}_1 : \mathfrak{a}) &= \{\mathfrak{a}_2 \subset \mathfrak{a} : \mathfrak{a}_1 \pitchfork \mathfrak{a}_2 \quad \text{in} \quad \mathfrak{a}\} \\ &= \{\mathfrak{a}_2 \subset \mathfrak{a} : \mathfrak{a}_1 \cap (\mathfrak{a}_2 + V) \quad \text{is relatively compact} \\ &\qquad \text{for any compact subset} \quad V \subset \mathfrak{a}\}. \end{aligned}$$

If a finite group W acts linearly on $\mathfrak{a}$, we set

$$\begin{aligned} \mathcal{P}(\mathfrak{a})^W &:= \{\mathfrak{a}' \in \mathcal{P}(\mathfrak{a}) : \mathfrak{a}' \quad \text{is stable under the action of } W\}, \\ \pitchfork(\mathfrak{a}_1 : \mathfrak{a})^W &:= \pitchfork(\mathfrak{a}_1 : \mathfrak{a}) \cap \mathcal{P}(\mathfrak{a})^W. \end{aligned}$$

The following Lemma is obvious from the definition of the pitchfork symbol $\pitchfork(\ :\ )$ and from Lemma 2.2 (3):

Lemma 5.3.1. *Suppose* $\mathfrak{a}_1 \in \mathcal{P}(\mathfrak{a})^W$. *Then* $\mathfrak{a}_2 \in \pitchfork(\mathfrak{a}_1 : \mathfrak{a})$ *if and only if* $W \cdot \overline{\mathfrak{a}_2} \in \pitchfork(\mathfrak{a}_1 : \mathfrak{a})^W$.

Lemma 5.3.2. *Retain the above setting. Suppose* W *is a finite group acting linearly on* $\mathfrak{a}$.
1) *Let* $\mathfrak{a}_1, \mathfrak{a}_1' \in \mathcal{P}(\mathfrak{a})$. *Then* $\pitchfork(\mathfrak{a}_1 : \mathfrak{a}) = \pitchfork(\mathfrak{a}_1' : \mathfrak{a})$ *if and only if* $\mathfrak{a}_1 \sim \mathfrak{a}_1'$ *in* $\mathfrak{a}$.
2) *Let* $\mathfrak{a}_1, \mathfrak{a}_1' \in \mathcal{P}(\mathfrak{a})^W$. *Then* $\pitchfork(\mathfrak{a}_1 : \mathfrak{a})^W = \pitchfork(\mathfrak{a}_1' : \mathfrak{a})^W$ *if and only if* $\mathfrak{a}_1 \sim \mathfrak{a}_1'$ *in* $\mathfrak{a}$.

Proof. 1) If $\mathfrak{a}_1 \sim \mathfrak{a}_1'$ in $\mathfrak{a}$, then $\pitchfork(\mathfrak{a}_1 : \mathfrak{a}) = \pitchfork(\mathfrak{a}_1' : \mathfrak{a})$ by Lemma 2.2 (2). Let us prove the converse implication. Assume $\mathfrak{a}_1 \not\sim \mathfrak{a}_1'$ in $\mathfrak{a}$. Let us show $\pitchfork(\mathfrak{a}_1 : \mathfrak{a}) \neq \pitchfork(\mathfrak{a}_1' : \mathfrak{a})$. Without loss of generality, we may assume $\mathfrak{a}_1 \not\subset \mathfrak{a}_1' + B_n$ for all $n \in \mathbb{N}$, where $B_n = \{X \in \mathfrak{a} : |X| < n\}$. Then we find $x_n \in \mathfrak{a}_1$ such that $\inf_{y \in \mathfrak{a}_1'} |x_n - y| \geq n$ for each $n \in \mathbb{N}$. We set $\mathfrak{a}_2 := \{x_j \in \mathfrak{a}_1 : j \in \mathbb{N}\}$. Then we have

$$\mathfrak{a}_2 \cap (B_n + \mathfrak{a}_1' + (-B_n)) \subset \{x_j : 1 \leq j \leq 2n\}.$$

Therefore $\mathfrak{a}_2 \in \pitchfork(\mathfrak{a}_1' : \mathfrak{a})$. On the other hand, $\mathfrak{a}_2 \notin \pitchfork(\mathfrak{a}_1 : \mathfrak{a})$ because $\mathfrak{a}_2$ is a noncompact subset of $\mathfrak{a}_1$. Hence $\pitchfork(\mathfrak{a}_1 : \mathfrak{a}) \neq \pitchfork(\mathfrak{a}_1' : \mathfrak{a})$.
2) The proof is similar. We only need to replace the definition of $\mathfrak{a}_2$ in the above proof by $\mathfrak{a}_2 := \{w \cdot x_j : w \in W,\ j \in \mathbb{N}\} \in \mathcal{P}(\mathfrak{a})^W$. ■

Now we return to our setting where G is a real reductive linear Lie group. Retain the notation of §3.1.

Lemma 5.4. *Let* G *be a real reductive linear Lie group. Then the mapping*

$$\Psi : \mathcal{P}(G) \to \mathcal{P}(\mathfrak{a})^{W_G}, \quad H \mapsto \mathfrak{a}(H)$$

is well-defined and surjective.

Proof. The mapping Ψ is well-defined because of Lemma 3.3 (1). If $\mathfrak{a}' \in \mathcal{P}(\mathfrak{a})^{W_G}$, then we put $H := \exp(\mathfrak{a}') \subset G$. Then we have $\mathfrak{a}(H) = \mathfrak{a}'$, proving the surjectivity of Ψ. ■

Lemma 5.5. *Suppose* H *is a subset of a real reductive linear Lie group* G. *Then we have*

$$\Psi(\pitchfork(H : G)) = \pitchfork(\mathfrak{a}(H) : \mathfrak{a})^{W_G}, \tag{5.5.1}$$

$$\Psi^{-1}\left(\pitchfork(\mathfrak{a}(H) : \mathfrak{a})^{W_G}\right) = \pitchfork(H : G). \tag{5.5.2}$$

Proof. The equivalence (1) ⇔ (2) in Theorem 3.4 implies the following relations:

$$\Psi(\pitchfork(H : G)) \subset \pitchfork(\mathfrak{a}(H) : \mathfrak{a})^{W_G}$$
$$\Psi^{-1}\left(\pitchfork(\mathfrak{a}(H) : \mathfrak{a})^{W_G}\right) \subset \pitchfork(H : G).$$

Then we have $\pitchfork(H : G) \subset \Psi^{-1}\Psi(\pitchfork(H : G)) \subset \Psi^{-1}(\pitchfork(\mathfrak{a}(H) : \mathfrak{a})^{W_G})$, showing (5.5.2). Because $\Psi : \mathcal{P}(G) \to \mathcal{P}(\mathfrak{a})^{W_G}$ is surjective, we have

$$\pitchfork(\mathfrak{a}(H) : \mathfrak{a})^{W_G} = \Psi\Psi^{-1}(\pitchfork(\mathfrak{a}(H) : \mathfrak{a})^{W_G}) \subset \Psi(\pitchfork(H : G)),$$

showing (5.5.1). ■

Now we are ready to prove a duality theorem:

Theorem 5.6. *Let G be a real reductive linear Lie group, and H and H' subsets. Then the following four conditions are equivalent:*

(5.6.1) $H \sim H'$ *in* G.

(5.6.2) $\mathfrak{a}(H) \sim \mathfrak{a}(H')$ *in* $\mathfrak{a}$.

(5.6.3) $\pitchfork(H : G) = \pitchfork(H' : G)$.

(5.6.4) $\pitchfork(\mathfrak{a}(H) : \mathfrak{a})^{W_G} = \pitchfork(\mathfrak{a}(H') : \mathfrak{a})^{W_G}$.

Proof. (2) $\Rightarrow$ (1): Suppose $H, H' \in \mathcal{P}(G)$ satisfy $\mathfrak{a}(H) \sim \mathfrak{a}(H')$ in $\mathfrak{a}$, equivalently, $A(H) \sim A(H')$ in A. Then it follows from Lemma 2.3 (1) that $A(H) \sim A(H')$ in G. Since $H \sim A(H)$ and $H' \sim A(H')$ in G, we have $H \sim H'$ in G.

(1) $\Rightarrow$ (3): Suppose $H \sim H'$ in G. Then we have $\pitchfork(H : G) = \pitchfork(H' : G) \subset \mathcal{P}(G)$ from Lemma 2.2 (2).

(3) $\Rightarrow$ (4): Applying Ψ to (5.6.3), we have $\pitchfork(\mathfrak{a}(H) : \mathfrak{a})^{W_G} = \pitchfork(\mathfrak{a}(H') : \mathfrak{a})^{W_G}$ by Lemma 5.5.

(4) $\Rightarrow$ (2): This part is proved in Lemma 5.3.2. ■

The implication (1) $\Rightarrow$ (2) in Theorem 5.6 gives a uniform estimate of the Cartan decomposition. It is also related to a uniform estimate of eigenvalues of matrices under perturbation, through the transform $g \mapsto g\theta(g^{-1})$, namely, $kak' \mapsto ka^2k^{-1}$ for k, $k' \in K$ and $a \in A$. In this direction, there have been extensive studies by H. Weyl, P. Lax, and so on, with the following prototype of the inequalities due to H. Weyl:

Let A, B be Hermitian matrices with eigenvalues $\alpha_1 \geq \ldots \geq \alpha_n$ and $\beta_1 \geq \ldots \geq \beta_n$, respectively. Then we have

$$\max |\alpha_k - \beta_k| \leq \|A - B\|.$$

We end this section with a clarification of Theorem 3.4 and a part of Theorem 5.6 by the following reformulation:

Let $[\mathcal{P}(G)]$ be the set of equivalence class of $\mathcal{P}(G)$ defined by the equivalence relation in Definition 2.1.1. Similarly, $[\mathcal{P}(\mathfrak{a})^{W_G}]$ denotes $\mathcal{P}(\mathfrak{a})^{W_G}$, where the role of G in Definition 2.1.1 is replaced by an abelian Lie group $\mathfrak{a}$. We fix a subset H of G. Theorem 3.4 and the equivalence of (5.6.1) and (5.6.2) in Theorem 5.6 can be stated in the well-definedness and the bijection of $\overline{\Psi}\colon [\mathcal{P}(G)] \xrightarrow{\sim} [\mathcal{P}(\mathfrak{a})^{W_G}]$ in the following:

The commutative diagram

$$\begin{array}{cccc} \Psi\colon & \mathcal{P}(G) & \twoheadrightarrow & \mathcal{P}(\mathfrak{a})^{W_G} \\ & \cup & & \cup \\ \Psi\colon & \pitchfork(H : G) & \twoheadrightarrow & \pitchfork(\mathfrak{a}(H) : \mathfrak{a}) \end{array}$$

induces

$$\begin{array}{cccc} \overline{\Psi}\colon & [\mathcal{P}(G)] & \xrightarrow{\sim} & [\mathcal{P}(\mathfrak{a})^{W_G}] \\ & \cup & & \cup \\ \overline{\Psi}\colon & [\pitchfork(H : G)] & \xrightarrow{\sim} & [\pitchfork(\mathfrak{a}(H) : \mathfrak{a})^{W_G}]. \end{array}$$

6. Examples

Example 6.1. If H is a subgroup subgroup of G. Assume that there exists a Cartan involution θ of G such that $\theta H = H$. Then H is a reductive subgroup. Let $\mathfrak{a}_H$ be a maximally abelian subspace of $\mathfrak{h} \cap \mathfrak{p}$. We take a maximally abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that $\mathfrak{a}_H \subset \mathfrak{a}$. Then $\mathfrak{a}(H) = W_G \cdot \mathfrak{a}_H$ (cf. [6]).

Example 6.2. Let G be a real reductive linear Lie group, and $\Gamma \subset G$ a cyclic group. Then there exist $H, H' \in \mathfrak{a}$ such that

$$\mathfrak{a}(\Gamma) \sim W_G \cdot \left\{ xH + \sinh^{-1}(x)H' : x \in \mathbb{Z} \right\} \quad \text{in} \quad \mathfrak{a}/W_G. \tag{6.2.1}$$

Proof. Let γ be a generator of Γ, $\gamma = \gamma_e \gamma_h \gamma_u$ be its complete multiplicative Jordan decomposition (see [5] Chapter IX), where γ_e, γ_h and γ_u are elliptic, hyperbolic and unipotent, respectively, and all three commute. Because $\{{\gamma_e}^n : n \in \mathbb{Z}\}$ is relatively compact, we have $\Gamma \sim \{{\gamma_h}^n {\gamma_u}^n : n \in \mathbb{Z}\}$ in G. It follows from Theorem 5.6 that

$$\mathfrak{a}(\Gamma) \sim \mathfrak{a}\left(\{{\gamma_h}^n {\gamma_u}^n : n \in \mathbb{Z}\}\right) \quad \text{in} \mathfrak{a}.$$

Thus we may and do assume $\gamma_e = 1$. Let $L := Z_G(\gamma_h)$, which is reductive in G. We note that $\gamma_u \in L$. By a theorem of Jacobson-Morozov, there is a Lie group homomorphism $\psi : SL(2, \mathbb{R}) \to L$ such that $\psi \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \gamma_u$. There is a Cartan involution θ of G such that $\theta\psi(SL(2,\mathbb{R})) = \psi(SL(2,\mathbb{R}))$ (see [5], p.277), and such that $\theta L = L$. We write the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $H \in \mathfrak{p} \cap \mathfrak{l}$ be the unique element such that $\gamma_h = \exp H$. We take a maximally abelian subspace $\mathfrak{a}$ for L which contains $H' := d\psi(\operatorname{diag}(1,-1))$ $(\in d\psi(\mathfrak{sl}(2,\mathbb{R}))$. Since H is contained in the center of $\mathfrak{l}$, we have $H \in \mathfrak{a}$. Then we have

$${\gamma_h}^n {\gamma_u}^n \in (L \cap K) \cdot \exp(nH + \sinh^{-1}(n)H') \cdot (L \cap K).$$

thanks to the following lemma. Hence, we have (6.2.1). ■

Lemma 6.3. *Let $G = SL(2,\mathbb{R})$, $K = SO(2)$. We take a maximally abelian subspace $\mathfrak{a} := \{\operatorname{diag}(y, -y) : y \in \mathbb{R}\} \subset \mathfrak{sl}(2,\mathbb{R})$. Then the Cartan projection of $\begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix} \in G$ amounts to*

$$\mathfrak{a}\left\{\begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix}\right\} = \left\{ \begin{pmatrix} \sinh^{-1} x & 0 \\ 0 & -\sinh^{-1} x \end{pmatrix}, \begin{pmatrix} -\sinh^{-1} x & 0 \\ 0 & \sinh^{-1} x \end{pmatrix} \right\} \subset \mathfrak{a}.$$

Proof. We put $\theta := -\frac{1}{2}\tan^{-1}(\frac{1}{x}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $x \neq 0$. Then we have $2x = -2\cot 2\theta = \tan\theta - \cot\theta$ and so $\exp(\sinh^{-1}(x)) = \tan\theta$. Now, the Lemma is a direct consequence of the following formula

$$\begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \tan\theta & 0 \\ 0 & \cot\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ ■

In the following two examples, we let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $G := GL(m+n, \mathbb{F})$. We fix a maximally abelian subspace

$$\mathfrak{a}_G := \{\mathrm{diag}(a_1, \dots, a_{m+n}) : a_j \in \mathbb{R} \quad (1 \le j \le m+n)\},$$

on which $W_G \simeq \mathfrak{S}_{m+n}$ acts by permutation of coordinates. The following example gives a simpler proof of one of the main results in [4].

Example 6.4 (Friedland). Consider a subgroup $H = GL(m, \mathbb{F}) \simeq GL(m, \mathbb{F}) \times I_n$ of $G = GL(m+n, \mathbb{F})$. Since H is reductive in G, it follows from Example 6.1 that

$$\mathfrak{a}(H) = W_G \cdot \{\mathrm{diag}(a_1, \dots, a_m, 0, \dots, 0) : a_j \in \mathbb{R} \quad (1 \le j \le m)\}.$$

In particular, a discrete subgroup Γ of G acts properly discontinuously on $G/H = GL(m+n, \mathbb{F})/GL(m, \mathbb{F})$ if and only if

$$\mathfrak{a}(\Gamma) \cap \{\mathrm{diag}(a_1, \dots, a_m, b_1, \dots, b_n) : a_j \in \mathbb{R}\, (1 \le j \le m),\ |b_j| \le C\, (1 \le j \le n)\}$$

is a finite set for any $C > 0$.

Example 6.5. Consider a subgroup $N := \exp(\mathfrak{n})$ of $G = GL(m+n, \mathbb{F})$, where

$$\mathfrak{n} := \{(X_{ij}) \in M(m+n, \mathbb{F}) : X_{ij} = 0 \quad \text{if} \quad i \ge m \quad \text{or} \quad j \le m\} \subset \mathfrak{gl}(m+n, \mathbb{F}).$$

Put $k := \min(m, n)$. In order to compute $\mathfrak{a}(N)$, we define a k-dimensional subalgebra by

$$\mathfrak{n}' := \{(X_{ij}) \in \mathfrak{n} : X_{ij} = 0 \quad \text{if} \quad j - i \ne m\},$$

and define subgroups of G by $N' := \exp(\mathfrak{n}')$ and $L_K := U(m; \mathbb{F}) \times U(n; \mathbb{F})$. In view of

$$\mathfrak{n} \simeq M(m, n; \mathbb{F}) = \{AXB \in M(m, n; \mathbb{F}) : A \in U(m; \mathbb{F}),\ X \in \mathfrak{n}',\ B \in U(n; \mathbb{F})\},$$

we have $KNK = KN'K$ and therefore $\mathfrak{a}(N) = \mathfrak{a}(N')$. By using Lemma 6.3, we conclude

$$\mathfrak{a}(N) = W_G \cdot \{\mathrm{diag}(a_1, -a_1, \dots, a_k, -a_k, 0, \dots, 0) : a_j \in \mathbb{R} \quad (1 \le j \le k)\}.$$

Thus, a subgroup Γ of G acts properly discontinuously on G/N if and only if

$$\mathfrak{a}(\Gamma) \cap \left\{\mathrm{diag}(a_1, \dots, a_{2k}, b_1, \dots, b_{|n-m|}) : |a_{2i-1} + a_{2i}| \le C,\ |b_j| \le C\, (\forall i, j)\right\}$$

is a finite set for any $C > 0$. It might be remarkable that $\mathfrak{a}(N)$ (a nilpotent subgroup) coincides with $\mathfrak{a}(U(m, n; \mathbb{F}))$ (a reductive subgroup).

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