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On Hyperbolic Cones and Mixed Symmetric Spaces

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Abstract. A real semisimple Lie algebra $\mathfrak{g}$ admits a Cartan involution, θ, for which the corresponding eigenspace decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ has the property that all operators $\operatorname{ad} X$, $X\in\mathfrak{p}$ are diagonalizable over $\mathbb{R}$. We call such elements hyperbolic, and the elements $X\in\mathfrak{k}$ are elliptic in the sense that $\operatorname{ad} X$ is semisimple with purely imaginary eigenvalues. The pairs $(\mathfrak{g},\theta)$ are examples of symmetric Lie algebras, i.e., Lie algebras endowed with an involutive automorphism, such that the -1-eigenspace of θ contains only hyperbolic elements. Let $(\mathfrak{g},\tau)$ be a symmetric Lie algebra and $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ the corresponding eigenspace decomposition for τ. The existence of "enough" hyperbolic elements in $\mathfrak{q}$ is important for the structural analysis of symmetric Lie algebras in terms of root decompositions with respect to abelian subspaces of $\mathfrak{q}$ consisting of hyperbolic elements. We study the convexity properties of the action of $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$ on the space $\mathfrak{q}$. The key role will be played by those invariant convex subsets of $\mathfrak{q}$ whose interior points are hyperbolic.

Introduction

It was a fundamental observation of Cartan's that each real semisimple Lie algebra $\mathfrak{g}$ admits an involutive automorphism, nowadays called Cartan involution, θ for which the corresponding eigenspace decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ has the property that all operators $\operatorname{ad} X$, $X\in\mathfrak{p}$ are diagonalizable over $\mathbb{R}$, we call such elements hyperbolic, and the elements $X\in\mathfrak{k}$ are elliptic in the sense that $\operatorname{ad} X$ is semisimple with purely imaginary eigenvalues. In this sense the Cartan involutions are the basic tool to separate hyperbolic from elliptic elements and this is why they play such a crucial role in the structure and representation theory of semisimple real groups. The pairs $(\mathfrak{g},\theta)$ are examples of symmetric Lie algebras, i.e., Lie algebras endowed with an involutive automorphism, such that the -1-eigenspace of θ contains only hyperbolic elements. It even can be shown that, up to adding central factors, this property characterizes the Cartan involutions.

Let $(\mathfrak{g},\tau)$ be a symmetric Lie algebra and $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ the corresponding eigenspace decomposition for τ. The existence of "enough" hyperbolic elements

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in $\mathfrak{q}$ is important in many contexts. For Cartan decompositions it is crucial for the restricted root decomposition of semisimple real Lie algebras, and hence for the whole structure theory of these algebras. There are other important classes of symmetric Lie algebras where the set $\mathfrak{q}_{\rm hyp}$ of the hyperbolic elements in $\mathfrak{q}$ still has interior points but is different from all of $\mathfrak{q}$. If $(\mathfrak{g}, \tau)$ is a non-compactly causal symmetric (NCC) Lie algebra in the sense of [8], then $\mathfrak{q}$ contains open convex cones which are invariant under the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ of inner automorphisms of $\mathfrak{g}$ generated by $e^{\mathrm{ad}\,\mathfrak{h}}$ and which consist entirely of hyperbolic elements. In the last years this class of reductive symmetric Lie algebras and the associated symmetric spaces has become a topic of very active research spreading in more and more areas. For a survey of the state of the art we refer to [8] and the literature cited there.

On the other hand there have been attempts to push this theory further to symmetric Lie algebras which are not necessarily semisimple or reductive. The simplest type (called the complex type) is where $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ is a complexification and τ is complex conjugation. Among these symmetric Lie algebras those for which $\mathfrak{h}$ contains an open invariant convex cone W consisting of elliptic elements play a crucuial role (cf. [14], [17], [18]). Then $iW \subseteq \mathfrak{q} = i\mathfrak{h}$ is an open cone consisting of hyperbolic elements so that, in the special case of reductive Lie algebras, we obtains on the one hand the non-compactly causal spaces of complex type and, if we allow $W = \mathfrak{h}$, also the Riemannian symmetric spaces coming from Cartan involutions of complex symmetric Lie algebras. For the associated symmetric spaces of complex type and the reductive spaces mentioned above one nowadays has a quite well developed picture of the harmonic analysis (holomorphic representations: [15], [16]; spherical functions [2], [6]; Hardy spaces [9], [12]) and the invariant complex analysis (invariant Stein domains and plurisubharmonic functions [18]).

The next step in this program is to pass from Lie algebras of complex type $(\mathfrak{h}_{\mathbb{C}}, \tau)$ to the general case. The main problem one has to face here is to find the appropriate class of symmetric Lie algebras which is general enough to encorporate all the cases mentioned above such as the mixed complex type case, the non-compactly causal spaces, and also the Riemannian symmetric spaces. Our main objective in this paper is to describe and develop the structure theory and convex geometry of such a class of symmetric Lie algebras. We have tried to keep the exposition as self-contained as possible. As far as the structure theory of symmetric spaces is concerned we use not much more than [3] for well known facts on the structure of Riemannian symmetric spaces.

We now give a short overview over the contents of this paper:

Section I starts with a collection of structural results concerning Levi decompositions and Cartan decompositions which are invariant under some compact group of automorphisms. Then we introduce the basic notions concerning symmetric Lie algebras $(\mathfrak{g}, \tau)$.

The key notions in our structural analysis of symmetric Lie algebras are those of hyperbolic and elliptic elements. Looking for large subspaces of hyperbolic elements, we are led to the notion of a maximal abelian subspace of $\mathfrak{q}$ consisting of hyperbolic elements (always denoted $\mathfrak{a}$) and a maximal hyperbolic Lie triple system $\mathfrak{p}$ of $\mathfrak{q}$, i.e., $[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]] \subseteq \mathfrak{p}$. One has similar notions for elliptic replaced by hyperbolic. The main results of Section II are Theorem II.8 and Corollary II.9 stating that maximal elliptic and hyperbolic Lie triple systems are always conjugate under the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) = \langle e^{\mathrm{ad}\,\mathfrak{h}}\rangle$ acting naturally on $\mathfrak{q}$. From that Lie algebraic result one easily deduces that all maximal compact subspaces of a symmetric space which contain a given point are conjugate under the isotropy group of this point. This generalizes the well known theorem that all maximal compact subgroups of a connected Lie group are conjugate under inner automorphisms.

In Section III we turn to a closer study of hyperbolic elements and their orbits under the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$. We show that all maximal hyperbolic abelian subspaces $\mathfrak{a}$ of $\mathfrak{q}$ are conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ and deduce that the set $\mathfrak{q}_{\mathrm{hyp}}$ of hyperbolic elements in $\mathfrak{q}$ has non-empty interior if and only if $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$ (Theorem III.3). These observations imply in particular that $\mathfrak{q}_{\mathrm{hyp}} = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).\mathfrak{a}$, i.e., that each hyperbolic element is conjugate to an element in $\mathfrak{a}$. Now the question arises how the orbit $\mathcal{O}_X$ of such an element X intersects $\mathfrak{a}$. The surprisingly simple answer to this question is given in Theorem III.10 saying that for $X \in \mathfrak{a}$ the orbit $\mathcal{O}_X$ intersects $\mathfrak{a}$ in the orbit of X under the Weyl group of a maximal hyperbolic Lie triple system $\mathfrak{p}$ containing $\mathfrak{a}$. This result is obtained by showing that the Weyl group of $\mathfrak{a}$ is not bigger than the Weyl group of the Riemannian symmetric Lie algebra generated by $\mathfrak{p}$.

We have seen in Section III that the size of the set $\mathfrak{q}_{\mathrm{hyp}}$ is related to the subspaces $\mathfrak{a}$ in the sense that it has interior points if and only if $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$. From now on we always assume this. In Section IV we consider the root decomposition of the Lie algebra $\mathfrak{g}$ with respect to a subspace $\mathfrak{a}$. It turns out that each root vector Z generates an at most three dimensional τ-invariant subalgebra:

- (R) The Riemannian type, where $\mathfrak{g}(Z) \cong \mathfrak{sl}(2,\mathbb{R})$ endowed with the Cartan involution. This is the type which exclusively occurs for Riemannian symmetric Lie algebras.
- (SR) The semi-Riemannian type, where $\mathfrak{g}(Z) \cong \mathfrak{sl}(2,\mathbb{R})$ endowed with the involution corresponding to $\mathfrak{h} = \mathfrak{so}(1,1)$.
- (A) The abelian type, where $\mathfrak{g}(Z) \cong \mathbb{R}^2$.
- (N) The nilpotent type, where $\mathfrak{g}(Z)$ is isomorphic to the three dimensional Heisenberg algebra.

In semisimple symmetric Lie algebras only the first three types occur and the occurence of (N) is a “solvable” phenomenon. It is quite illuminating that the type of the test algebra $\mathfrak{g}(Z)$ is, up to the distinction between type (A) and (N), characterized by the sign of the quadratic form $\kappa_\tau(Z) := \mathrm{tr}\,\big(\mathrm{ad}\,Z\,\mathrm{ad}\,\tau(Z)\big)$ (cf. Proposition IV.7). We conclude Section IV with the discussion of some examples which display the different types of behaviour that can occur.

Section V is dedicated to a discussion of a class of symmetric Lie algebras that we call quasihermitian and which are characterized by the property that for a maximal hyperbolic Lie triple system $\mathfrak{p}$ the centralizer of its center in $\mathfrak{q}$ is not bigger than $\mathfrak{p}$.

In Section VI we come to the subject proper of this paper, the convexity properties of the action of $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ on the space $\mathfrak{q}$. The key role will be played by those invariant convex subsets of $\mathfrak{q}$ whose interior points are hyperbolic. We call such invariant convex sets hyperbolic. The main point of this section is that the existence of hyperbolic invariant convex sets has significant consequences for the structure of the symmetric Lie algebra $(\mathfrak{g}, \tau)$. In particular we show that this implies that $(\mathfrak{g}, \tau)$ is quasihermitian, and, whenever pointed cones with these properties exist, also that it has strong cone potential (cf. Definition V.1(g)).

Having already dealt with the quasihermitian Lie algebras in Section V, we turn to the Lie algebras with (strong) cone potential in Section VII. We show that it imposes quite restrictive conditions on the structure of the root decomposition of $\mathfrak{g}$. The most crucial results are the Short String Theorem (Theorem VII.18) and its consequences. We also describe a method to construct interesting examples of mixed symmetric Lie algebras having all of the properties mentioned above. A Lie algebra which displays many features of the theory is the symmetric Jacobi algebra $\mathfrak{g} = \mathfrak{h}_n \rtimes \mathfrak{sp}(n, \mathbb{R})$, where $\mathfrak{h}_n$ is a $(2n+1)$-dimensional Heisenberg algebra, $\mathfrak{sp}(n, \mathbb{R})$ acts naturally on it, and both are endowed with compatible involutions turning $\mathfrak{g}$ into a quasihermitian symmetric Lie algebra with strong cone potential (cf. Example VII.17).

For many applications concerning analysis on symmetric spaces and in particular phenomena related to analytic continuation aspects it is important to understand quite well the embedding of the symmetric Lie algebra $\mathfrak{g}$ into its complexification $\mathfrak{g}_{\mathbb{C}}$ which is endowed with the antilinear extension $\hat{\tau}$ of τ. This corresponds to an embedding $\mathfrak{q} \to \hat{\mathfrak{q}} := \mathfrak{q} + i\mathfrak{h} = i\mathfrak{g}^c$, where $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ is the dual symmetric Lie algebra. We call $\mathfrak{g}_{\mathbb{C}}$, resp. $\hat{\mathfrak{q}}$, the canonical extension of $\mathfrak{g}$, resp. $\mathfrak{q}$. The Inheritage Theorem (Theorem VIII.1) states that whenever a symmetric Lie algebra $(\mathfrak{g}, \tau)$ is quasihermitian and the centralizer $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{h}$ is compactly embedded, then $(\mathfrak{g}_{\mathbb{C}}, \hat{\tau})$ is also quasihermitian. This result is a rather important tool because it makes many results that have been proved for the special case of symmetric Lie algebras where τ is complex conjugation available in the general context. In the remainder of Section VIII we apply this method to derive convexity theorems which describe the (convex hull of) projections of hyperbolic orbits in $\mathfrak{q}$ onto $\mathfrak{a}$ along the complementary subspace $[\mathfrak{a}, \mathfrak{h}]$.

Section IX is almost entirely devoted to the characterization of those symmetric Lie algebras for which there exists an open convex invariant cone in $\mathfrak{q}_{\rm hyp}$. In this case there always exists finitely many maximal cones $W_{\max}^0$ having this property. The main difficulty is to show that the interior of the cone $W_{\max}$ consists of hyperbolic elements which is not at all evident from the definition.

In Section X we finally use the aforementioned convexity theorems to obtain a characterization of invariant subsets of $\mathfrak{q}$ whose interior consists of hyperbolic elements by their intersections with $\mathfrak{a}$. These results can in particular be used to obtain a classification of the invariant hyperbolic cones in $\mathfrak{q}$ by their intersections with $\mathfrak{a}$. From this characterization we derive a particularly interesting result saying that in the natural setup all invariant hyperbolic convex subsets $C \subseteq \mathfrak{q}$ can be extended to invariant convex hyperbolic subsets $\hat{C} \subseteq \hat{\mathfrak{q}}$ satisfying $\hat{C} \cap \mathfrak{q} = C$. We expect that these results have many applications in the further investigation of related Hardy spaces, spherical functions, invariant plurisubharmonic functions, Stein domains, and representations on spaces of holomorphic functions.

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List of symbols

$\mathfrak{a}$	maximal abelian hyperbolic subspace in $\mathfrak{q}$ (Definition I.7(b))
$\hat{\mathfrak{a}}$	maximal abelian hyperbolic subspace in $\hat{\mathfrak{q}}$ extending $\mathfrak{a}$
$\check{\alpha}$	coroot of $\alpha \in \Delta$ (Definition V.1(e))
A_α	representing element of $\alpha \in \Delta$ in $\mathfrak{a}$ (Proposition IV.7(iv),(v))
$B(C)$	lower bounded linear functionals on C (Definition VI.2(a))
$C_{\max}$	maximal cone in $\mathfrak{a}$ (Definition V.1(d))
$C_{\min}$	minimal cone in $\mathfrak{a}$ (Definition V.1(d))
$C_{\min,\mathrm{p}}$	minimal semisimple cone in $\mathfrak{a}$ (Definition V.1(d))
$C_{\min,\mathrm{z}}$	minimal central cone in $\mathfrak{a}$ (Definition V.1(d))
Δ	root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$ (Theorem IV.1)
$\hat{\Delta}$	root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\hat{\mathfrak{a}}$
Δ^+	system of positive roots in Δ
$\check{\Delta}$	coroots of Δ (Definition V.1(e))
Δ_k	compact roots in Δ (Definition V.1(a))
Δ_n	non-compact roots in Δ (Definition V.1(a))
Δ_p	non-compact semisimple roots in Δ(Definition V.1(a))
Δ_r	solvable roots in Δ (Definition IV.4)
Δ_s	semisimple roots in Δ (Definition IV.4)
$(\mathfrak{g}, \tau)$	finite dimensional real symmetric Lie algebra (Definition I.6(a))
$\mathfrak{g}^c$	$= \mathfrak{h} + i\mathfrak{q}$ (c-dual of $\mathfrak{g}$)
$\mathfrak{g}^\alpha$	root space associated to the root $\alpha \in \Delta$ (Theorem IV.1)
$\mathfrak{h}$	1-eigenspace of τ (Definition I.6(a))
$\hat{\mathfrak{h}}$	$= \mathfrak{g}^c$ (1-eigenspace of $\hat{\tau}$)
$H(C)$	edge of C (Definition VI.2(b))
$\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{b})$	group of inner automorphisms of $\mathfrak{g}$ generated by $e^{\mathrm{ad}\,\mathfrak{b}}$, where $\mathfrak{b} \subseteq \mathfrak{g}$
κ	Cartan-Killing form of $\mathfrak{g}$
κ_τ	$= \kappa(\cdot, \tau(\cdot))$ (Proposition IV.7(vi))
$\lim C$	limit cone of C (Definition VI.2(b))
$\mathfrak{n}$	maximal nilpotent ideal in $\mathfrak{g}$
$\mathfrak{n}_{\mathfrak{b}}(\mathfrak{c})$	normalizer of $\mathfrak{c}$ in $\mathfrak{b}$
$N_B(\mathfrak{c})$	normalizer of $\mathfrak{c}$ in $B \subseteq \mathrm{Aut}(\mathfrak{g})$
$\mathfrak{q}$	-1-eigenspace of τ (Definition I.6(a))
$\hat{\mathfrak{q}}$	$= i\mathfrak{g}^c$ (-1-eigenspace of $\hat{\tau}$)
$\mathfrak{p}$	maximal hyperbolic Lie triple system in $\mathfrak{q}$ (Definition I.7(c))
$\hat{\mathfrak{p}}$	maximal hyperbolic Lie triple system in $\hat{\mathfrak{q}}$
$\mathfrak{q}_{\mathrm{ell}}$	elliptic elements in $\mathfrak{q}$ (Definiton I.7(a))
$\mathfrak{q}_{\mathrm{hyp}}$	hyperbolic elements in $\mathfrak{q}$ (Definition I.7(a))

$\mathfrak{r}$ radical of $\mathfrak{g}$
$\mathfrak{r}^\alpha$ solvable part of $\mathfrak{g}^\alpha$ (Definition IV.4)
$\mathfrak{s}$ Levi-complement in $\mathfrak{g}$
$\mathfrak{s}^\alpha$ semisimple part of $\mathfrak{g}^\alpha$ (Definition IV.4)
$\mathfrak{t}$ compactly embedded Cartan subalgebra of $\mathfrak{g}^c$ (Proposition VII.10)
θ Cartan-involution of $\mathfrak{s}$
$\mathfrak{t}_\mathfrak{h}$ Cartan subalgebra of $\mathfrak{h}^0$ (Proposition VII.10)
$\hat\tau$ complex antilinear extension of τ to $\mathfrak{g}_\mathbb{C}$
$\mathcal{W}$ Weyl group of $(\mathfrak{g}, \tau, \mathfrak{a})$ (Definition III.9)
$\hat{\mathcal{W}}$ Weyl group of $(\mathfrak{g}_\mathbb{C}, \hat\tau, \hat{\mathfrak{a}})$
$W_{\max}$ maximal hyperbolic cone in $\mathfrak{q}$ (Chapter IX)
$\hat W_{\max}$ maximal hyperbolic cone in $\hat{\mathfrak{q}}$ (Chapter X)
$\mathfrak{z}(\mathfrak{b})$ $= \mathfrak{z}_\mathfrak{b}(\mathfrak{b})$ (center of $\mathfrak{b}$)
$\mathfrak{z}_\mathfrak{b}(\mathfrak{c})$ centralizer of $\mathfrak{c}$ in $\mathfrak{b}$
$Z_B(\mathfrak{c})$ centralizer of $\mathfrak{c}$ in $B \subseteq \mathrm{Aut}(\mathfrak{g})$

Conventions: Subscripts denote intesections, for instance $\mathfrak{r}_\mathfrak{q} = \mathfrak{r} \cap \mathfrak{q}$ etc. and for a subspace $\mathfrak{b} \subseteq \mathfrak{g}$ we write $\mathfrak{b}^0 := \mathfrak{z}_\mathfrak{b}(\mathfrak{a})$ and denote by $\mathfrak{b}_L$ the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{b}$.

I. Preliminaries on symmetric Lie algebras

Invariant Levi complements

We start with some information that will be crucial to obtain suitable Levi decompositions of Lie groups.

Lemma I.1. *Let V be a finite dimensional vector space over a field $\mathbb{K}$ with $\mathrm{char}\,\mathbb{K} = 0$ and $G \subseteq \mathrm{Gl}(V)$ an algebraic subgroup such that V is a semisimple G-module. Then G is reductive.*

Proof. According to the Levi decomposition $G = G_u \rtimes L$, where L is reductive ([10, Th. VIII.4.3]), we only have to show that the unipotent radical G_u is reduced to $\{\mathbf{1}\}$. Since V is a direct sum of simple modules, it suffices to show that G_u acts trivially on simple modules.

So let $U \subseteq V$ be a simple submodule and put $W := \{v \in U : (\forall g \in G_u) g.v = v\}$. Then W is non-zero if V is non-zero because G_u is unipotent. Moreover W is invariant under G because G_u is normal. Hence $W = U$ and the proof is complete. ∎

Proposition I.2. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over the field $\mathbb{K}$ with $\mathrm{char}\,\mathbb{K} = 0$. Let $A \subseteq \mathrm{Aut}(\mathfrak{g})$ be a subgroup such that $\mathfrak{g}$ is a semisimple A-module. Then there exists an A-invariant Levi complement in $\mathfrak{g}$.*

Proof. First we recall the Levi decomposition $G = G_u \rtimes L$ of the algebraic group $G = \mathrm{Aut}(\mathfrak{g})$, where G_u is the unipotent radical and L is a reductive subgroup. By passing to the Zariski closure of A, we may w.l.o.g. assume that A is algebraic. Then A is a reductive subgroup of G (Lemma I.1), hence conjugate to a subgroup of L ([10, Th. VIII.4.3]) and we therefore may assume that $A \subseteq L$. So it suffices to find an L-invariant Levi complement.

Let $\mathfrak{s} \subseteq \mathfrak{g}$ be a fixed Levi complement. Then $\mathfrak{g}$ is a semisimple $\operatorname{ad}\mathfrak{s}$-module. Hence $\operatorname{ad}\mathfrak{s} \subseteq \operatorname{der}(\mathfrak{g})$ is conjugate to a subalgebra of $\mathfrak{l} := \mathbf{L}(L)$ and we may w.l.o.g. assume that $\operatorname{ad}\mathfrak{s} \subseteq \mathfrak{l}$.

The subalgebra $\operatorname{ad}\mathfrak{g} \subseteq \operatorname{der}(\mathfrak{g})$ is an L-invariant ideal. Hence $(\operatorname{ad}\mathfrak{g}) \cap \mathfrak{l}$ is an L-invariant ideal in $\mathfrak{l}$ containing $\operatorname{ad}\mathfrak{s}$. We conclude that $(\operatorname{ad}\mathfrak{g}) \cap \mathfrak{l}$ is a reductive subalgebra of $\operatorname{ad}\mathfrak{g}$. Since $\operatorname{ad}\mathfrak{s}$ is a Levi complement in $\operatorname{ad}\mathfrak{g}$, it follows that $\operatorname{ad}\mathfrak{s} = [(\operatorname{ad}\mathfrak{g}) \cap \mathfrak{l}, (\operatorname{ad}\mathfrak{g}) \cap \mathfrak{l}]$. Thus $\operatorname{ad}\mathfrak{s}$ is also an L-invariant ideal of $\mathfrak{l}$. Now $\operatorname{ad}^{-1}(\operatorname{ad}\mathfrak{s}) = \mathfrak{s} + \mathfrak{z}$ must be L-invariant. Therefore $\mathfrak{s} = [\mathfrak{s} + \mathfrak{z}, \mathfrak{s} + \mathfrak{z}]$ is L-invariant. ■

We obtain the following result as a corollary (cf. [11, App. 9.4]):

Corollary I.3. *If $\mathfrak{g}$ is a real Lie algebra and $K \subseteq \operatorname{Aut}(\mathfrak{g})$ a compact group of automorphisms of $\mathfrak{g}$, then there exists a K-invariant Levi complement.* ■

Corollary I.4. *If τ is an involutive automorphism of the Lie algebra $\mathfrak{g}$ over a field of characteristic 0. Then there exists a τ-invariant Levi complement.* ■

Invariant Cartan decompositions

Proposition I.5. *Let $\mathfrak{g}$ be a semisimple real Lie algebra and $U \subseteq \operatorname{Aut}(\mathfrak{g})$ a compact subgroup. Then the following assertions hold:*

- *(i) There exists a U-invariant Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The corresponding Cartan involution θ commutes with U.*
- *(ii) If θ and θ' are two Cartan involutions commuting with U, then there exists $X \in \mathfrak{g}$ with $\theta(X) = -X$ and $[X, \operatorname{Ad}(U)] = \{0\}$ such that $\theta' = e^{\operatorname{ad} X}\theta e^{-\operatorname{ad} X}$.*
- *(iii) For each involutive automorphism τ of $\mathfrak{g}$ there exists a Cartan involution θ commuting with τ. If θ' is another Cartan involution commuting with τ, then there exists $X \in \mathfrak{g}$ with $\theta(X) = -X$ and $\tau(X) = X$ such that $\theta' = e^{\operatorname{ad} X}\theta e^{-\operatorname{ad} X}$.*

Proof. (i) Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and θ the corresponding Cartan involution. Then $\operatorname{Aut}(\mathfrak{g}) = \operatorname{Aut}(\mathfrak{g})^\theta e^{\operatorname{ad}\mathfrak{p}}$ is a Cartan decomposition of the real Lie group $\operatorname{Aut}(\mathfrak{g})$ which is semisimple and, since it is an algebraic group over $\mathbb{R}$, it has at most finitely many connected components. Now $M = \operatorname{Aut}(\mathfrak{g})/\operatorname{Aut}(\mathfrak{g})^\theta$ is the associated Riemannian symmetric space which in turn can be identified with the set of all Cartan decomposition of $\mathfrak{g}$. In view of [3, Th. VI.2.1], the group U has a fixed point in M, hence commutes with a Cartan involution and therefore leaves the corresponding Cartan decomposition invariant.
(ii) In view of (i), we may w.l.o.g. assume that $U \subseteq \operatorname{Aut}(\mathfrak{g})^\theta$. Then the set of all Cartan involutions commuting with U can be written as $e^{\operatorname{ad}\mathfrak{a}}.\theta$, where $\mathfrak{a} = \{X \in \mathfrak{p} \colon (\forall \gamma \in U)\gamma.X = X\}$. This proves the assertion.
(iii) This is a special case of (i) and (ii). ■

Symmetric Lie algebras

Definition I.6. (a) A *symmetric Lie algebra* is a pair $(\mathfrak{g}, \tau)$, where τ is an involutive automorphism of $\mathfrak{g}$. A *symmetric Lie group* is defined analogously. Note that if (G, τ) is a symmetric Lie group, then $\big(\mathfrak{g}, d\tau(\mathbf{1})\big)$ is a symmetric Lie algebra. If $H \subseteq G^\tau$ is an open subgroup, then G/H is called a *symmetric space* associated to the symmetric Lie algebra $(\mathfrak{g}, \tau)$.

If $(\mathfrak{g}, \tau)$ is a symmetric Lie algebra, then we write

$$\mathfrak{h} = \mathfrak{g}^\tau = \{X \in \mathfrak{g} \colon \tau(X) = X\} \quad \text{and} \quad \mathfrak{q} = \{X \in \mathfrak{g} \colon \tau(X) = -X\}$$

and note that $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ is a direct vector space decomposition. For a τ-invariant subspace $\mathfrak{b} \subseteq \mathfrak{g}$ we will always write $\mathfrak{b}_\mathfrak{h} := \mathfrak{b} \cap \mathfrak{h}$ and $\mathfrak{b}_\mathfrak{q} := \mathfrak{b} \cap \mathfrak{q}$, so that we have $\mathfrak{b} = \mathfrak{b}_\mathfrak{h} \oplus \mathfrak{b}_\mathfrak{q}$.

(b) For each symmetric Lie algebra $(\mathfrak{g}, \tau)$ the subspace $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$ of the complexification $\mathfrak{g}_\mathbb{C}$ is also a symmetric Lie algebra with respect to the involution obtained by restricting the complex linear extension of τ.

(c) A symmetric Lie algebra $(\mathfrak{g}, \tau)$ is called *effective* if $\mathfrak{h}$ does not contain any non-zero ideal of $\mathfrak{g}$. Note that $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$ does not imply that $(\mathfrak{g}, \tau)$ is effective because $([\mathfrak{h}, \mathfrak{h}] + i\mathfrak{h}, \sigma)$, where $\mathfrak{h}$ is a two-step nilpotent Lie algebra and σ is complex conjugation provides a counterexample.

(d) A symmetric Lie algebra $(\mathfrak{g}, \tau)$ is said to be *irreducible* if each non-zero τ-invariant ideal of $\mathfrak{g}$ coincides with $\mathfrak{g}$.

Note that this implies in particular that $(\mathfrak{g}, \tau)$ is effective. Since $\mathfrak{a} = \mathfrak{a}_\mathfrak{h} + \mathfrak{a}_\mathfrak{q} \mapsto \mathfrak{a}^c := \mathfrak{a}_\mathfrak{h} + i\mathfrak{a}_\mathfrak{q}$ defines a one-to-one correspondence between the τ-invariant ideals of $\mathfrak{g}$ and $\mathfrak{g}^c$, we see that $(\mathfrak{g}^c, \tau)$ is irreducible if and only if $(\mathfrak{g}, \tau)$ is irreducible.

There are three basic types of irreducible symmetric Lie algebras. If $\mathfrak{g}$ is not semisimple, then it must be abelian, and therefore $\mathfrak{g} = \mathfrak{q} \cong \mathbb{R}$. If $\mathfrak{g}$ is semisimple, then either $\mathfrak{g}$ is simple or $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is simple and $\tau(X, Y) = (Y, X)$. ■

Definition I.7. Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra with the corresponding decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$.

(a) An element $X \in \mathfrak{g}$ is called *hyperbolic* if $\operatorname{ad} X$ is diagonalizable over $\mathbb{R}$. We write $\mathfrak{q}_{\rm hyp}$ for the set of all hyperbolic elements in $\mathfrak{q}$. An element $X \in \mathfrak{g}$ is called *elliptic* if $\operatorname{ad} X$ is semisimple with purely imaginary spectrum. We write $\mathfrak{q}_{\rm ell}$ for the set of elliptic elements in $\mathfrak{q}$. We note that if $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ is the dual symmetric Lie algebra, then $i\mathfrak{q}_{\rm hyp} = (i\mathfrak{q})_{\rm ell}$ and vice versa.

(b) An abelian subspace $\mathfrak{a} \subseteq \mathfrak{q}$ is called *maximal hyperbolic abelian*, resp., *maximal elliptic abelian* if $\mathfrak{a}$ consists of hyperbolic, resp., elliptic elements and is maximal with respect to this property.

(c) A subspace $\mathfrak{a} \subseteq \mathfrak{q}$ is called a *Lie triple system* if $\big[\mathfrak{a}, [\mathfrak{a}, \mathfrak{a}]\big] \subseteq \mathfrak{a}$. Note that this means that the space $\mathfrak{a}_L := \mathfrak{a} + [\mathfrak{a}, \mathfrak{a}]$ is a subalgebra of $\mathfrak{g}$. A Lie triple system

$\mathfrak{a} \subseteq \mathfrak{q}$ will be called *hyperbolic*, resp., *elliptic* if it consists of hyperbolic, resp., elliptic elements.
(d) For a subalgebra $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ we write $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{a}) := \langle e^{\mathrm{ad}\,\mathfrak{a}}\rangle$ for the group of inner automorphisms of $\mathfrak{g}$ generated by $e^{\mathrm{ad}\,\mathfrak{a}}$.
(e) A subalgebra $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is said to be *compactly embedded* if $\mathfrak{a}$ consists of elliptic elements which is equivalent to the condition that the closure of $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{a})$ is compact (cf. [4, 2.6]).
(f) A *compactly embedded Cartan subalgebra* $\mathfrak{t}$ of $\mathfrak{g}$ is a compactly embedded subalgebra which in addition is maximal abelian. Note that this implies in particular that $\mathfrak{t}$ is self-normalizing, hence a Cartan subalgebra of $\mathfrak{g}$. A *toral Cartan subalgebra* is a subalgebra consisting of hyperbolic elements which in addition is maximal abelian. ■

The following lemma clarifies the meaning of the effectiveness assumption for a symmetric Lie algebra $(\mathfrak{g}, \tau)$.

Lemma I.8. *The largest ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$ coincides with the kernel of the representation $\mathrm{ad}_{\mathfrak{q}}$ of $\mathfrak{h}$ on $\mathfrak{q}$.*

Proof. Let $\mathfrak{j} \subseteq \mathfrak{h}$ denote the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$. Then $[\mathfrak{j}, \mathfrak{q}] \subseteq \mathfrak{j} \cap \mathfrak{q} \subseteq \mathfrak{h} \cap \mathfrak{q} = \{0\}$. On the other hand $\ker \mathrm{ad}_{\mathfrak{q}} \subseteq \mathfrak{h}$ is an ideal of $\mathfrak{h}$ with $[\mathfrak{q}, \ker \mathrm{ad}_{\mathfrak{q}}] = \{0\}$. Therefore $\ker \mathrm{ad}_{\mathfrak{q}}$ is an ideal of $\mathfrak{g}$. ■

The following observation will be of crucial use in the remainder of this paper.

Lemma I.9. (i) *If $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{q}$ are hyperbolic (elliptic) subspaces with $[\mathfrak{a}, \mathfrak{b}] = \{0\}$, then $\mathfrak{a} + \mathfrak{b}$ is hyperbolic (elliptic).*
(ii) *Let s be a semisimple Lie algebra and $\rho: \mathfrak{s} \to \mathrm{End}(V)$ a finite dimensional representation. Then the following assertions hold:*

- *(a) For each Cartan involution θ of $\mathfrak{s}$ there exists a scalar product on V such that $\rho(\theta.X) = -\rho(X)^\top$ for all $X \in \mathfrak{s}$.*
- *(b) If $X \in \mathfrak{s}$ is hyperbolic or elliptic, then the same holds for $\rho(X)$.*

Proof. (i) In view of the duality of $\mathfrak{g}$ and $\mathfrak{g}^c$, it suffices to prove the assertion for the hyperbolic case, where we have to show that for $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$ the sum is hyperbolic. But this follows from the fact that two diagonalizable operators that commute can be diagonalized simultaneously.
(ii) We may w.l.o.g. assume that V is a complex $\mathfrak{s}$-module, so that we obtain an extension of ρ to a representation of the complexification $\mathfrak{s}_{\mathbb{C}}$.
(a) If $\mathfrak{s} = \mathfrak{s}_{\mathfrak{k}} \oplus \mathfrak{s}_{\mathfrak{p}}$ is the Cartan decomposition corresponding to the Cartan involution θ, then $\mathfrak{u} := \mathfrak{s}_{\mathfrak{k}} + i\mathfrak{s}_{\mathfrak{p}}$ is a compact real form of $\mathfrak{s}_{\mathbb{C}}$. Now Weyl's unitary trick shows that there exists a scalar product on V for which the operators in $\rho(\mathfrak{u})$ are skew-Hermitian. Then the operators in $\rho(\mathfrak{s}_{\mathfrak{k}})$ are skew-symmetric and those in $\rho(\mathfrak{s}_{\mathfrak{p}})$ are symmetric. This proves (a).
(b) Since $X \in \mathfrak{s}_{\mathbb{C}}$ is hyperbolic if and only if iX is elliptic, it suffices to prove that $\rho(X)$ is elliptic whenever $X \in \mathfrak{s}_{\mathbb{C}}$ is so. Let $X \in \mathfrak{s}_{\mathbb{C}}$ be elliptic. Then X is contained in a maximal compactly embedded subalgebra $\mathfrak{u} \subseteq \mathfrak{s}_{\mathbb{C}}$ which therefore is a compact real form, i.e., $\mathfrak{s}_{\mathbb{C}} = \mathfrak{u} + i\mathfrak{u}$ is a Cartan decomposition. Therefore

(a) applies and we see that, with respect to a certain scalar product on V, the operators in $\rho(\mathfrak{u})$ are skew-Hermitian, hence elliptic. ■

Lemma I.10. *If $(\mathfrak{g}, \tau) = (\mathfrak{h}_{\mathbb{C}}, \sigma)$, where σ is complex conjugation, then $\mathfrak{g}^c \cong \mathfrak{h} \oplus \mathfrak{h}$, where $\tau(X, Y) = (Y, X)$.*

Proof. We identify $\mathfrak{g}_{\mathbb{C}} = (\mathfrak{h}_{\mathbb{C}})_{\mathbb{C}}$ with $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$. We claim that $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{h}_{\mathbb{C}} \oplus \overline{\mathfrak{h}_{\mathbb{C}}}$, where $\overline{\mathfrak{h}_{\mathbb{C}}}$ denotes the Lie algebra $\mathfrak{h}_{\mathbb{C}}$ endowed with the opposite complex structure. In fact, we define two mappings

$$\eta_{\pm}\colon \mathfrak{h}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}, \quad X \mapsto \tfrac{1}{2}(1 \otimes X \pm i \otimes iX).$$

Then

$$\begin{aligned}
[\eta_{\pm}(X), \eta_{\pm}(Y)] &= \tfrac{1}{4}[1 \otimes X \pm i \otimes iX, 1 \otimes Y \pm i \otimes iY] \\
&= \tfrac{1}{4}(1 \otimes [X, Y] \pm i \otimes i[X, Y] \pm i \otimes i[X, Y] - 1 \otimes [iX, iY]) \\
&= \tfrac{1}{2}(1 \otimes [X, Y] \pm i \otimes i[X, Y]) = \eta_{\pm}([X, Y])
\end{aligned}$$

shows that the mappings $\eta_{\pm}$ are Lie algebra homomorphisms. It is clear that both are injective and that their images intersect in $\{0\}$. Moreover,

$$\begin{aligned}
[\eta_{+}(X), \eta_{-}(Y)] &= \tfrac{1}{2}[1 \otimes X + i \otimes iX, 1 \otimes Y - i \otimes iY] \\
&= \tfrac{1}{2}(1 \otimes [X, Y] - i \otimes i[X, Y] + i \otimes i[X, Y] + 1 \otimes [iX, iY]) = 0.
\end{aligned}$$

Hence

$$\mathfrak{g}_{\mathbb{C}} \cong \eta_{-}(\mathfrak{h}_{\mathbb{C}}) \oplus \eta_{+}(\mathfrak{h}_{\mathbb{C}}) \cong \mathfrak{h}_{\mathbb{C}} \oplus \overline{\mathfrak{h}_{\mathbb{C}}}$$

because η_{+} is complex antilinear and η_{-} is complex linear. It is clear that in this representation the complex conjugation $\sigma_{\mathfrak{g}}$ with respect to $\mathfrak{g}$ acts by $\sigma_{\mathfrak{g}}(X, Y) = (Y, X)$, the complex linear extension τ of the involution τ on $\mathfrak{h}_{\mathbb{C}}$ by $\tau(X, Y) = (\tau.Y, \tau.X)$, and the complex conjugation $\sigma = \tau \circ \sigma_{\mathfrak{g}}$ with respect to $\mathfrak{g}^c$ by $\sigma(X, Y) = (\tau.X, \tau.Y)$. The fixed point set of this involution is

$$\mathfrak{g}^c \cong \{(X, Y)\colon X, Y \in \mathfrak{h}\} = \mathfrak{h} \oplus \mathfrak{h},$$

and the corresponding involution maps (X, Y) to (Y, X). ■

II. Maximal elliptic and hyperbolic Lie triple systems

In this section we will show that maximal elliptic and hyperbolic Lie triple systems are conjugate under the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$. From that we will conclude in particular that all maximal compact symmetric subspaces of a general symmetric space which contain a given point x are conjugate under the stabilizer of x.

In the following we call a symmetric Lie algebra $(\mathfrak{g}, \tau)$ *non-compactly Riemannian* if $\mathfrak{g}$ is reductive, $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{q}$, and $\tau|_{[\mathfrak{g},\mathfrak{g}]}$ is a Cartan involution.

Proposition II.1. *If* $\mathfrak{q}$ *consists of hyperbolic elements and* $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$ *, then* $(\mathfrak{g}, \tau)$ *is non-compactly Riemannian.*

Proof. First we recall that $[\mathfrak{g}, \mathfrak{r}]$ is a nilpotent ideal of $\mathfrak{g}$ which implies that

$$[\mathfrak{g}, \mathfrak{r}] \cap \mathfrak{q} = [\mathfrak{g}, \mathfrak{r}] \cap \mathfrak{q}_{\mathrm{hyp}} \subseteq \mathfrak{z}(\mathfrak{g}).$$

Let $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$ be a τ-invariant Levi decomposition (Corollary I.4). Then $\mathfrak{q} = \mathfrak{r}_\mathfrak{q} + \mathfrak{s}_\mathfrak{q}$. Let $X \in \mathfrak{r}_\mathfrak{q}$. Then

$$(\operatorname{ad} X)^2.\mathfrak{q} \subseteq \mathfrak{r}_\mathfrak{q} \cap [\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{z}(\mathfrak{g})$$

implies that $(\operatorname{ad} X)^3.\mathfrak{q} = \{0\}$ and therefore that $(\operatorname{ad} X)^4 = 0$. Since $\operatorname{ad} X$ is semisimple, it follows that $\operatorname{ad} X = 0$, i.e., that $X \in \mathfrak{z}(\mathfrak{g})$. Now we have $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] = [\mathfrak{r}_\mathfrak{q} + \mathfrak{s}_\mathfrak{q}, \mathfrak{r}_\mathfrak{q} + \mathfrak{s}_\mathfrak{q}] = [\mathfrak{s}_\mathfrak{q}, \mathfrak{s}_\mathfrak{q}] \subseteq \mathfrak{s}$ and therefore $\mathfrak{r} = \mathfrak{r}_\mathfrak{q} \subseteq \mathfrak{z}(\mathfrak{g})$. This means that $\mathfrak{g}$ is reductive and that $\mathfrak{s}$ is the commutator algebra of $\mathfrak{g}$.

Let θ be a Cartan-involution of $\mathfrak{s}$ commuting with τ and $\mathfrak{s} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition. Then the fact that $\mathfrak{q}$ consists of hyperbolic elements means that $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{q}$, and finally $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$ shows that $(\mathfrak{g}, \tau)$ is in fact a Riemannian symmetric Lie algebra. ∎

Corollary II.2. *If* $\mathfrak{q}$ *consists of elliptic elements and* $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$ *, then* $\mathfrak{g}$ *is a compact Lie algebra and* $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{q}$ *.*

Proof. Let $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$ denote the dual symmetric Lie algebra. Then $\mathfrak{g}^c$ satisfies the assumptions of Proposition II.1, hence is Riemannian symmetric. We write $\mathfrak{g}^c = \mathfrak{z}(\mathfrak{g}^c) \oplus \mathfrak{s}^c$, where $\mathfrak{s}^c$ is the commutator algebra. Then $\mathfrak{s}^c = \mathfrak{s}^c_\mathfrak{h} + \mathfrak{s}^c_\mathfrak{q} = \mathfrak{k}^c + \mathfrak{p}^c$ is a Cartan decomposition of $\mathfrak{s}^c$ and hence

$$\mathfrak{g} = i\mathfrak{z}(\mathfrak{g}^c) \oplus (\mathfrak{k}^c + i\mathfrak{p}^c)$$

is a compact Lie algebra. ∎

Corollary II.3. *If* $\mathfrak{a} \subseteq \mathfrak{q}$ *is an elliptic Lie triple system, then the subalgebra* $\mathfrak{a}_L = \mathfrak{a} + [\mathfrak{a}, \mathfrak{a}]$ *is compactly embedded in* $\mathfrak{g}$ *.*

Proof. First we use Corollary II.2 to see that $\mathfrak{a}_L$ is a compact Lie algebra with $\mathfrak{z}(\mathfrak{a}_L) \subseteq \mathfrak{a}$. Then the fact that $[\mathfrak{a}_L, \mathfrak{a}_L]$ is compact and semisimple implies that it is compactly embedded, hence that $\mathfrak{a}_L = \mathfrak{z}(\mathfrak{a}_L) \oplus [\mathfrak{a}_L, \mathfrak{a}_L]$ is compactly embedded. ∎

Next we will show that maximal elliptic Lie triple systems in $\mathfrak{q}$ are conjugate under the group $\operatorname{Inn}_\mathfrak{g}(\mathfrak{h})$. The proof will be by induction over the dimension of $\mathfrak{g}$. Therefore we first collect some preparatory lemmas.

Lemma II.4. *If* $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$ *,* $\mathfrak{r}$ *is the radical of* $\mathfrak{g}$ *,* $\mathfrak{g}/\mathfrak{r}$ *is a compact Lie algebra and* $\mathfrak{a} \subseteq \mathfrak{q}$ *is a maximal elliptic Lie triple system, then the commutator algebra of* $\mathfrak{a}_L$ *is a Levi complement in* $\mathfrak{g}$ *.*

Proof. First we choose a Levi complement $\mathfrak{s} \subseteq \mathfrak{g}$ which is simultaneously invariant under τ and $\mathfrak{a}_L$. Such a Levi complement exists because the group

$\langle e^{\operatorname{ad} \mathfrak{a}_L}, \tau\rangle \subseteq \operatorname{Aut}(\mathfrak{g})$ has compact closure (Corollary I.3). Then the normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{s})$ of $\mathfrak{s}$ in $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ which contains $\mathfrak{s}$ as an ideal, hence $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{n}_{\mathfrak{r}}(\mathfrak{s}) \oplus \mathfrak{s}$, because $[\mathfrak{n}_{\mathfrak{r}}(\mathfrak{s}), \mathfrak{s}] \subseteq \mathfrak{r} \cap \mathfrak{s} = \{0\}$. Now $\mathfrak{a}_L \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{s})$ follows from our construction of $\mathfrak{s}$. Let $p\colon \mathfrak{n}_{\mathfrak{g}}(\mathfrak{s}) \to \mathfrak{n}_{\mathfrak{r}}(\mathfrak{s})$ denote the canonical projection and note that this is a homomorphism of Lie algebras which commutes with τ because the invariance of $\mathfrak{s}$ and $\mathfrak{r}$ implies the invariance of $\mathfrak{n}_{\mathfrak{r}}(\mathfrak{s})$ under τ. Let $X \in \mathfrak{a}_L$. Then $0 = [p(X), X - p(X)] = [p(X), X]$ implies that $p(X) = X + \big(p(X) - X\big)$ is elliptic because $X - p(X) \in \mathfrak{s}$ is elliptic. Therefore $p(\mathfrak{a}_L)$ is a subalgebra of the solvable Lie algebra $\mathfrak{n}_{\mathfrak{r}}(\mathfrak{s})$ consisting of elliptic elements, hence abelian. Now $p(\mathfrak{a}_L) \oplus \mathfrak{s}$ is compactly embedded and contains $\mathfrak{a}_L$, so that the maximality of $\mathfrak{a}$ implies that $\mathfrak{a} = \mathfrak{s}_{\mathfrak{q}} \oplus \mathfrak{a}_{\mathfrak{r}}$ is adapted to the decomposition of $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{s})$. Finally $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$ and the surjectivity of the quotient homomorphism $\mathfrak{g} \to \mathfrak{s}$ yields $[\mathfrak{a}, \mathfrak{a}] = [\mathfrak{s}_{\mathfrak{q}}, \mathfrak{s}_{\mathfrak{q}}] = \mathfrak{s}_{\mathfrak{h}}$ which in turn implies that $\mathfrak{s} = [\mathfrak{a}_L, \mathfrak{a}_L]$. ■

Lemma II.5. *Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra such that $\mathfrak{g} = \mathfrak{v} + \mathfrak{k}$, where $\mathfrak{v}$ is a τ-invariant abelian ideal and $\mathfrak{k}$ is τ-invariant and compactly embedded. Then two maximal elliptic Lie triple system in $\mathfrak{q}$ are conjugate under $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$.*

Proof. Since $\mathfrak{k}$ is compactly embedded, $\mathfrak{g}$ is a semisimple $\mathfrak{k}$-module and therefore $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) \oplus [\mathfrak{k}, \mathfrak{g}]$, where $[\mathfrak{k}, \mathfrak{g}] = [\mathfrak{k}, \mathfrak{v}] + [\mathfrak{k}, \mathfrak{k}] = [\mathfrak{g}, \mathfrak{g}]$ follows from $[\mathfrak{v}, \mathfrak{v}] = \{0\}$. Thus $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$ is a subalgebra complementary to $[\mathfrak{g}, \mathfrak{g}]$ and therefore central, i.e., $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) = \mathfrak{z}(\mathfrak{g})$. So $\mathfrak{g}$ decomposes as a direct sum of the ideals $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. Since both ideals are τ-invariant, the projections of $\mathfrak{k}$ and $\mathfrak{v}$ on $[\mathfrak{g}, \mathfrak{g}]$ still satisfy the assumptions of the lemma, and the assertion is trivial for abelian Lie algebras, it suffices to prove the assertion under the additional assumption that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) = \{0\}$.

Since $\mathfrak{k} \cap \mathfrak{v}$ is a compactly embedded subalgebra which is contained in a nilpotent ideal, it must be central and therefore trivial. Thus $\mathfrak{g} = \mathfrak{v} \rtimes \mathfrak{k}$ is a semidirect sum. Let $G = \mathfrak{v} \rtimes K$ denote a simply connected group corresponding to $\mathfrak{g}$ and $\widetilde{G} = G \rtimes \{\mathbf{1}, \tau\}$. Then the natural affine action of G on $\mathfrak{v}$ defined by

$$(v, k).x := k.x + v$$

together with the restriction of τ to $\mathfrak{v}$ defines an action of $\widetilde{G}$ on $\mathfrak{v}$ by affine mappings.

Let $\mathfrak{a} \subseteq \mathfrak{q}$ be a maximal elliptic Lie triple system and $\mathfrak{a}_L \subseteq \mathfrak{g}$ the corresponding Lie algebra which is compactly embedded (Corollary II.3). Let $\widetilde{\mathfrak{k}} \supseteq \mathfrak{k}$ be maximal compactly embedded. Then $\widetilde{\mathfrak{k}} = \mathfrak{k} \oplus (\widetilde{\mathfrak{k}} \cap \mathfrak{v})$ and $\widetilde{\mathfrak{k}} \cap \mathfrak{v} \subseteq \mathfrak{z}(\mathfrak{g}) = \{0\}$ follows as above. Thus $\widetilde{\mathfrak{k}} = \mathfrak{k}$ and $\mathfrak{k}$ is maximal compactly embedded in $\mathfrak{g}$. Therefore $\mathfrak{a}_L$ is conjugate to a subalgebra of $\mathfrak{k}$ (cf. [4, 2.6]). From that we conclude that the closure U of the image of $\exp(\mathfrak{a}_L) \rtimes \{\mathbf{1}, \tau\}$ in $\operatorname{Aff}(\mathfrak{v})$ is a compact group. Therefore U has a fixed point $x \in \mathfrak{v}$. Then $\tau(x) = x$ implies $x \in \mathfrak{v}_{\mathfrak{h}}$ and moreover

$$(-x, 0)U(x, 0).0 = \{0\}$$

entails that $e^{-\operatorname{ad} x}.\mathfrak{a}_L \subseteq \mathfrak{k}$, hence that $e^{-\operatorname{ad} x}.\mathfrak{a} = \mathfrak{k}_{\mathfrak{q}}$ by maximality of $\mathfrak{a}$. This proves that any maximal elliptic Lie triple system in $\mathfrak{q}$ is conjugate to $\mathfrak{k}_{\mathfrak{q}}$ and therefore that two maximal elliptic Lie triple system are conjugate under $e^{\operatorname{ad} \mathfrak{v}_{\mathfrak{h}}}$. ■

Proposition II.6. *Let*

$$\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{q} \quad \text{and} \quad \mathfrak{a} \subseteq \widetilde{\mathfrak{a}} \subseteq \widetilde{\mathfrak{p}} \subseteq \widetilde{\mathfrak{q}},$$

where $\mathfrak{p} \subseteq \mathfrak{q}$ is a maximal hyperbolic Lie triple system, $\mathfrak{a} \subseteq \mathfrak{p}$ is maximal abelian, $\widetilde{\mathfrak{q}} \subseteq \mathfrak{q}$ is a Lie triple system, $\widetilde{\mathfrak{p}} \subseteq \widetilde{\mathfrak{q}}$ is maximal hyperbolic with respect to $\widetilde{\mathfrak{q}}_L$, and $\widetilde{\mathfrak{a}}$ is maximal abelian in $\widetilde{\mathfrak{p}}$. Then the following assertions hold:

(i) *If $\mathfrak{p} \subseteq \widetilde{\mathfrak{p}}$, then $[\widetilde{\mathfrak{p}}_L, \widetilde{\mathfrak{p}}_L] = [\mathfrak{p}_L, \mathfrak{p}_L]$.*

(ii) *$\mathfrak{a} = \big(\mathfrak{a} \cap \mathfrak{z}(\widetilde{\mathfrak{p}})\big) \oplus \big(\mathfrak{a} \cap \big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]\big)$, where $\mathfrak{a} \cap \mathfrak{z}(\widetilde{\mathfrak{p}}) = \mathfrak{z}(\widetilde{\mathfrak{p}}) \cap \mathfrak{q}_{\rm hyp}$ and $\mathfrak{a} \cap \big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]$ is maximal abelian in $\big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]$.*

The same assertions hold with "elliptic" instead of "hyperbolic".

Proof. If (i) and (ii) hold, then the same assertions for elliptic Lie triple systems follow by applying (i) and (ii) to the dual symmetric Lie algebra $\mathfrak{g}^c$.

(i) Let $\mathfrak{s} := [\widetilde{\mathfrak{p}}_L, \widetilde{\mathfrak{p}}_L] = [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}] + \big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]$ denote the commutator algebra in $\widetilde{\mathfrak{p}}_L$ which is a semisimple Lie algebra invariant under τ (Proposition II.1). Then $\widetilde{\mathfrak{p}}_L = \mathfrak{z} \oplus \mathfrak{s}$, where $\mathfrak{z} = \mathfrak{z}(\widetilde{\mathfrak{p}})$ denotes the center of $\widetilde{\mathfrak{p}}$.

Let $p: \widetilde{\mathfrak{p}} \to \mathfrak{z}$ denote the projection along $\mathfrak{s}_\mathfrak{q}$. For $X \in \mathfrak{p}$ we then have $X = X_z + X_s$ according to the above decomposition. Now X_s is contained in the semisimple Lie algebra $\mathfrak{s}$, where it is hyperbolic, hence is hyperbolic in $\mathfrak{g}$ (Lemma I.9(i)(b)). Therefore $X_z = X - X_s$ is hyperbolic because $[X, X_s] = [X_z, X_s] = 0$. This proves that $p(\mathfrak{p}) + \mathfrak{p}$ is a hyperbolic Lie triple system so that $p(\mathfrak{p}) \subseteq \mathfrak{p}$ follows from the maximality of $\mathfrak{p}$, whence $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{z}) \oplus (\mathfrak{p} \cap \mathfrak{s})$.

Next the hyperbolicity of $(\mathfrak{a} \cap \mathfrak{z}) \oplus \mathfrak{s}_\mathfrak{q}$ and the maximality of $\mathfrak{p}$ yield $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{z}) \oplus \mathfrak{s}_\mathfrak{q}$. Hence $\widetilde{\mathfrak{p}} \subseteq \mathfrak{p} + \mathfrak{z}$ and therefore $[\mathfrak{p}, \mathfrak{p}] = [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]$ as well as $\big[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]\big] = \big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]$. This proves (i).

(ii) Since $\mathfrak{a} \subseteq \widetilde{\mathfrak{p}}$, the maximality of $\mathfrak{a}$ and a similar argument as in (i) show that $p(\mathfrak{a}) \subseteq \mathfrak{a}$, hence that $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{z}) \oplus (\mathfrak{a} \cap \mathfrak{s})$. The maximality of $\mathfrak{a}$ further implies that $\mathfrak{a} \cap \mathfrak{s}$ is maximal abelian in the semisimple hyperbolic Lie triple system $\mathfrak{s}_\mathfrak{q}$. Furthermore $\mathfrak{z} \cap \mathfrak{q}_{\rm hyp}$ is a vector space containing $\mathfrak{a}$, so that $\mathfrak{a} = \mathfrak{z} \cap \mathfrak{q}_{\rm hyp}$ follows from the maximality of $\mathfrak{a}$. ■

In the following we write $\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ for the *Cartan-Killing form* of $\mathfrak{g}$.

Lemma II.7. *Let $(\mathfrak{g}, \tau)$ be a semisimple symmetric Lie algebra. Then the following assertions hold:*

(i) *If $V \subseteq \mathfrak{q}$ is an $\mathfrak{h}$-invariant subspace and $V^\perp = \{X \in \mathfrak{q}: (\forall Y \in V) \kappa(X, Y) = 0\}$ is the orthogonal space with respect to the Cartan-Killing form, then $[V, V^\perp] = \{0\}$. Moreover, if the restriction of κ to V is non-degenerate, then V_L is an ideal of $\mathfrak{g}$.*

(ii) *If $(\mathfrak{g}, \tau)$ is irreducible and $\mathfrak{q}$ is not irreducible as an $\mathfrak{h}$-module, then the following assertions hold:*

(a) *$\mathfrak{q}$ splits into two irreducible components $\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{q}^-$ such that $\theta(\mathfrak{q}^+) = \mathfrak{q}^-$ holds for any Cartan involution θ commuting with τ.*

(b) *The submodules $\mathfrak{q}^\pm$ are isotropic for the Cartan-Killing form and abelian subalgebras of $\mathfrak{g}$.*

(c) *The subalgebras $\mathfrak{h} + \mathfrak{q}^\pm$ of $\mathfrak{g}$ are maximal parabolic.*

Proof. First we claim that the Cartan-Killing form is non-degenerate on $\mathfrak{h}$ and $\mathfrak{q}$. In fact, we choose a Cartan involution θ commuting with τ (Proposition I.5). Then $\mathfrak{h}$ and $\mathfrak{q}$ are θ-invariant subspaces of $\mathfrak{g}$. If $E \subseteq \mathfrak{g}$ is θ-invariant and $E = E^+ \oplus E^-$ is the θ-eigenspace decomposition, then the invariance of κ under θ implies that E^+ and E^- are orthogonal. Moreover, κ is negative definite on E^+ and positive definite on E^-, hence κ is non-degenerate on E. This applies in particular to $\mathfrak{h}$ and $\mathfrak{q}$.
(i) We have

$$\kappa([V^\perp, V], \mathfrak{h}) = \kappa(V^\perp, [V, \mathfrak{h}]) \subseteq \kappa(V^\perp, V) = \{0\}.$$

Since the restriction of κ to $\mathfrak{h}$ is non-degenerate, we conclude that $[V, V^\perp] = \{0\}$. It follows in particular that V_L is an $\mathfrak{h}$-invariant subalgebra of $\mathfrak{g}$ satisfying $[V^\perp, V_L] = \{0\}$. If the restriction of κ to V is non-degenerate, then $\mathfrak{q} = V \oplus V^\perp$, showing that $[\mathfrak{q}, V_L] \subseteq V_L$, hence that V_L is an ideal.
(ii) (a), (b): If $V \subseteq \mathfrak{q}$ is an irreducible $\mathfrak{h}$-submodule which is not isotropic, then the restriction of the Cartan-Killing form κ to V is non-degenerate and so (i) implies that V_L is a τ-invariant ideal of $\mathfrak{g}$. Hence the irreducibility of $(\mathfrak{g}, \tau)$ yields $\mathfrak{q} = V$. This contradiction shows that each irreducible submodule of $\mathfrak{q}$ must be isotropic. Moreover, if $\mathfrak{q}^+$ is such a submodule, the subspace $\mathfrak{q}^+ + \theta(\mathfrak{q}^+)$ is a non-degenerate $\mathfrak{h}$-submodule, hence coincides with $\mathfrak{q}$. To complete the proof, we only have to note that $(\mathfrak{q}^+)^\perp = \mathfrak{q}^+$ follows from the fact that $\mathfrak{q}^+$ is isotropic and of half the dimension of $\mathfrak{q}$.
(c) In view of (b), we have

$$[\mathfrak{q}^+, \mathfrak{q}^-] \subseteq \mathfrak{h}, \quad [\mathfrak{q}^+, \mathfrak{h}] \subseteq \mathfrak{q}^+ \quad \text{and} \quad [\mathfrak{q}^+, \mathfrak{q}^+] = \{0\}.$$

Therefore $\mathfrak{g}$ is a nilpotent $\mathfrak{q}^+$-module and the subalgebra $\mathfrak{b} := \mathfrak{h} + \mathfrak{q}^+$ is not reductive in $\mathfrak{g}$. Since it is a maximal subalgebra, it must be parabolic (cf. [1, Ch. 8, §10, Cor. 1 de Th. 2]). ∎

Theorem II.8. *Any two maximal elliptic Lie triple systems in* $\mathfrak{q}$ *are conjugate under* $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$.

Proof. We prove the assertion by induction over the dimension of $\mathfrak{g}$. If $\dim \mathfrak{g} = 0$, then the assertion is trivial.

Suppose that $\dim \mathfrak{g} > 0$. First we assume that $\mathfrak{g}$ is semisimple. In addition, we may assume that $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$ because $\mathfrak{q}_L = \mathfrak{q} + [\mathfrak{q}, \mathfrak{q}]$ is an ideal of $\mathfrak{g}$ which is complemented by an ideal contained in $\mathfrak{h}$ (Lemma II.7(i)). Hence an element $X \in \mathfrak{q}$ is elliptic in $\mathfrak{g}$ if and only if it is elliptic in $\mathfrak{q}_L$.

Let $\mathfrak{k}_\mathfrak{q} \subseteq \mathfrak{q}$ be a maximal elliptic Lie triple system. Then $(\mathfrak{k}_\mathfrak{q})_L$ is a compactly embedded subalgebra (Corollary II.3). Let $U := \overline{\langle e^{\operatorname{ad} \mathfrak{k}_\mathfrak{q}}, \tau \rangle} \subseteq \mathrm{Aut}(\mathfrak{g})$. Then U is a compact subgroup and Proposition I.5 implies that there exists a Cartan involution θ of $\mathfrak{g}$ commuting with U. This means that θ commutes with τ and that $(\mathfrak{k}_\mathfrak{q})_L \subseteq \mathfrak{k}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the corresponding Cartan decomposition of $\mathfrak{g}$. Now the maximality of $\mathfrak{k}_\mathfrak{q}$ implies that $\mathfrak{k}_\mathfrak{q} = \mathfrak{k} \cap \mathfrak{q}$. Let $(\mathfrak{k}_\mathfrak{q})^\perp \subseteq \mathfrak{q}$ denote the orthogonal subspace with respect to the Cartan-Killing form. Then $(\mathfrak{k}_\mathfrak{q})^\perp = \mathfrak{p}_\mathfrak{q}$ and therefore $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$ implies that

$$\mathfrak{k} = \mathfrak{k}_\mathfrak{q} + [\mathfrak{k}_\mathfrak{q}, \mathfrak{k}_\mathfrak{q}] + [\mathfrak{p}_\mathfrak{q}, \mathfrak{p}_\mathfrak{q}] = (\mathfrak{k}_\mathfrak{q})_L + [(\mathfrak{k}_\mathfrak{q})^\perp, (\mathfrak{k}_\mathfrak{q})^\perp]. \tag{2.1}$$

This means that $\mathfrak{k}$ and therefore θ can be reconstructed from $\mathfrak{k}_{\mathfrak{q}}$ via (2.1). Now the fact that two Cartan involutions commuting with τ are conjugate under the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ (Proposition I.5(iii)) implies that two maximal elliptic Lie triple systems in $\mathfrak{q}$ are conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$.

Now we turn to the general case. If $\mathfrak{g}$ is not semisimple, then the radical $\mathfrak{r}$ of $\mathfrak{g}$ is non-zero, hence the maximal nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ is non-zero and therefore its center $\mathfrak{z}_{\mathfrak{n}}$ is also non-zero. The ideals $\mathfrak{r}$, $\mathfrak{n}$ and $\mathfrak{z}_{\mathfrak{n}}$ of $\mathfrak{g}$ are invariant under each automorphism of $\mathfrak{g}$, hence in particular under τ. Let $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{z}_{\mathfrak{n}}$ and τ_1 the involution induced by τ on the quotient algebra $\mathfrak{g}_1$. We write $\pi\colon \mathfrak{g} \to \mathfrak{g}_1$ for the quotient morphism. Then

$$\pi(\mathfrak{h}) = \mathfrak{h}_1 \quad \text{and} \quad \pi(\mathfrak{q}) = \mathfrak{q}_1.$$

Let $\mathfrak{k}, \mathfrak{b} \subseteq \mathfrak{q}$ be maximal elliptic Lie triple systems. Then the subspaces $\pi(\mathfrak{k})$ and $\pi(\mathfrak{b})$ of $\mathfrak{q}_1$ are elliptic Lie triple systems, hence contained in maximal elliptic Lie triple systems $\mathfrak{k}_1$ and $\mathfrak{b}_1$ of $\mathfrak{q}_1$. In view of our induction hypothesis, there exists $h_1 \in \mathrm{Inn}_{\mathfrak{g}_1}(\mathfrak{h}_1)$ with $h_1.\mathfrak{k}_1 = \mathfrak{b}_1$.

The surjective homomorphism $\pi\colon \mathfrak{g} \to \mathfrak{g}_1$ induces a surjective homomorphism $q\colon \mathrm{Inn}\,\mathfrak{g} \to \mathrm{Inn}\,\mathfrak{g}_1$ defined by

$$\pi \circ \gamma = q(\gamma) \circ \pi$$

for $\gamma \in \mathrm{Inn}\,\mathfrak{g}$ and we conclude from $\pi \circ \tau = \tau_1 \circ \pi$ that $q\big(\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})\big) = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_1)$. Hence there exists $h \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ with $q(h) = h_1$.

Let $\widetilde{\mathfrak{k}} := \pi^{-1}(\mathfrak{k}_1) \cap \mathfrak{q}$ and $\widetilde{\mathfrak{b}} := \pi^{-1}(\mathfrak{b}_1) \cap \mathfrak{q}$ and note that these spaces are Lie triple systems. Then $\pi(h.\widetilde{\mathfrak{k}}) = h_1.\mathfrak{k}_1 = \mathfrak{b}_1 = \pi(\widetilde{\mathfrak{b}})$ and hence $h.\widetilde{\mathfrak{k}} = \widetilde{\mathfrak{b}}$. After replacing $\mathfrak{k}$ by $h.\mathfrak{k}$ we may therefore assume that $\widetilde{\mathfrak{k}} = \widetilde{\mathfrak{b}}$.

We note that the fact that π intertwines τ and τ_1 implies that $\tau(\widetilde{\mathfrak{k}}) = \widetilde{\mathfrak{k}}$. Let $\mathfrak{k}' \supseteq \mathfrak{k}$ be a maximal elliptic Lie triple system in $\widetilde{\mathfrak{k}}_{\mathfrak{q}}$ with respect to $\widetilde{\mathfrak{k}}_L$. Then the fact that $\widetilde{\mathfrak{k}}_L$ is an abelian extension of the compact Lie algebra $(\mathfrak{k}_1)_L$ implies that $\widetilde{\mathfrak{k}}_L$ is compact modulo its radical. Therefore Lemma II.4 shows that $[\mathfrak{k}'_L, \mathfrak{k}'_L]$ is a Levi complement in $\widetilde{\mathfrak{k}}_L$. Moreover, the fact that $\mathfrak{k}'_L$ is a compact Lie algebra implies that $[\mathfrak{k}'_L, \mathfrak{k}'_L] = [\mathfrak{k}_L, \mathfrak{k}_L]$ because the elliptic version of Proposition II.6(i) applies with $\widetilde{\mathfrak{q}} := \widetilde{\mathfrak{p}} := \mathfrak{k}'$.

Let $\mathfrak{v} := \mathfrak{z}_{\mathfrak{n}} \cap \widetilde{\mathfrak{k}}_L = (\ker \pi) \cap \widetilde{\mathfrak{k}}_L$. Then $\mathfrak{v}$ is an abelian ideal of $\widetilde{\mathfrak{k}}_L$ which is invariant under τ. Now $\pi([\mathfrak{k}_L, \mathfrak{k}_L]) = \pi([\mathfrak{b}_L, \mathfrak{b}_L])$ is the unique Levi subalgebra of $(\mathfrak{k}_1)_L$. Therefore $[\mathfrak{b}_L, \mathfrak{b}_L] \subseteq \mathfrak{v} + [\mathfrak{k}_L, \mathfrak{k}_L]$. In view of Lemma II.5, we may w.l.o.g. assume that $[\mathfrak{k}_L, \mathfrak{k}_L] = [\mathfrak{b}_L, \mathfrak{b}_L]$. Then $\mathfrak{z}(\mathfrak{k})$ and $\mathfrak{z}(\mathfrak{b})$ are contained in the τ-invariant solvable subalgebra $\mathfrak{c} := \mathfrak{z}_{\widetilde{\mathfrak{k}}_L}([\mathfrak{k}_L, \mathfrak{k}_L])$ of $\mathfrak{g}$. The Lie algebra $\mathfrak{c}$ is an abelian extension of the abelian algebra $\mathfrak{z}(\mathfrak{k}_1)$, hence solvable. Let $\mathfrak{k}_{\mathfrak{h}}$, resp., $\mathfrak{b}_{\mathfrak{h}}$ be Cartan subalgebras of $\mathfrak{c}$ containing $\mathfrak{z}(\mathfrak{k})$, resp., $\mathfrak{z}(\mathfrak{b})$ (cf. [1, Ch. 7]). Then there exists $Z \in [\mathfrak{c}, \mathfrak{c}]$ with $e^{\mathrm{ad}\,Z}.\mathfrak{k}_{\mathfrak{h}} = \mathfrak{h}_{\mathfrak{b}}$ (cf. [1, Ch. 7]). The fact that $\pi(\mathfrak{c})$ is abelian implies that $[\mathfrak{c}, \mathfrak{c}] \subseteq \mathfrak{z}_{\mathfrak{n}}$ and in particular that $(\mathrm{ad}\,Z)^2 = 0$ for $Z \in [\mathfrak{c}, \mathfrak{c}]$. Let $Z = Z_{\mathfrak{h}} + Z_{\mathfrak{q}}$ according to the τ-eigenspace decomposition of $\mathfrak{z}_{\mathfrak{n}}$ and note that

$$e^{\mathrm{ad}\,Z}.\mathfrak{z}(\mathfrak{k}) = (\mathbf{1} + \mathrm{ad}\,Z_{\mathfrak{h}} + \mathrm{ad}\,Z_{\mathfrak{q}}).\mathfrak{z}(\mathfrak{k}) \subseteq \mathfrak{h}_{\mathfrak{b}} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{b}).$$

For $X \in \mathfrak{z}(\mathfrak{k})$ and $Y \in \mathfrak{b}$ we therefore have

$$0 = [e^{\operatorname{ad} Z}.X, Y] = \underbrace{[X + [Z_{\mathfrak{h}}, X], Y]}_{\in \mathfrak{h}} + \underbrace{[[Z_{\mathfrak{q}}, X], Y]}_{\in \mathfrak{q}}.$$

So both summands have to vanish and consequently $e^{\operatorname{ad} Z_{\mathfrak{h}}}.\mathfrak{z}(\mathfrak{k}) \subseteq \mathfrak{z}_{\mathfrak{q}}(\mathfrak{b})$. Now $\mathfrak{b} + e^{\operatorname{ad} Z_{\mathfrak{h}}}.\mathfrak{z}(\mathfrak{k}) = \mathfrak{b} + e^{\operatorname{ad} Z_{\mathfrak{h}}}.\mathfrak{k}$ ist an elliptic Lie triple subsystem of $\mathfrak{q}$ and maximality implies that $\mathfrak{b} = e^{\operatorname{ad} Z_{\mathfrak{h}}}.\mathfrak{k}$. This proves that $\mathfrak{k}$ and $\mathfrak{b}$ are conjugate under $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$. ■

Corollary II.9. *Two maximal hyperbolic Lie triple systems in $\mathfrak{q}$ are conjugate under* $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$.

Proof. This follows by applying Theorem II.8 to the dual symmetric Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$. ■

Let G/H be a symmetric space associated to the symmetric Lie algebra $(\mathfrak{g}, \tau)$ and (G, τ) a corresponding connected symmetric group. The *exponential function* of G/H is defined by

$$\operatorname{Exp}\colon \mathfrak{q} \to G/H, \quad X \mapsto \exp(X)H.$$

An important consequence of the conjugacy of the maximal elliptic Lie triple systems in $\mathfrak{q}$ is the conjugacy of the maximal compact symmetric subspaces of an associated symmetric space G/H which will follow from the following auxiliary proposition.

Proposition II.10. *If G/H is a symmetric space associated to the symmetric Lie algebra $(\mathfrak{g}, \tau)$ and $\mathfrak{q}$ consists of elliptic elements, then*

$$\{X \in \mathfrak{q}\colon \overline{\operatorname{Exp}(\mathbb{R}X)} \text{ is compact }\}$$

is a Lie triple subsystem of $\mathfrak{q}$.

Proof. Since the analytic subgroup of G generated by $\exp(\mathfrak{q} + [\mathfrak{q}, \mathfrak{q}])$ still acts transitively on G/H, we may w.l.o.g. assume that $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$. If $\mathfrak{q}$ is elliptic, then Corollary II.2 says that $\mathfrak{g}$ is a reductive Lie algebra and $\mathfrak{q} = \mathfrak{z}(\mathfrak{g}) + \mathfrak{k}_{\mathfrak{q}}$, where $\mathfrak{k} = [\mathfrak{g}, \mathfrak{g}]$ is the semisimple compact commutator algebra. Let $\mathfrak{t} \subseteq \mathfrak{z}$ be the Lie algebra of a maximal torus T in $Z(G)$. Then $\operatorname{Exp}(\mathfrak{t} + \mathfrak{k}_{\mathfrak{q}}) \subseteq G/H$ is compact and for $X \in \mathfrak{q}$ the conditions $X \in \mathfrak{t} + \mathfrak{k}_{\mathfrak{q}}$ and the relative compactness of $\operatorname{Exp}(\mathbb{R}X)$ are equivalent. Now the assertion follows from the fact that $\mathfrak{t} + \mathfrak{k}_{\mathfrak{q}}$ is a Lie triple system. ■

Theorem II.11. *If G/H is a symmetric space associated to the symmetric Lie algebra $(\mathfrak{g}, \tau)$ and $\mathfrak{k}, \mathfrak{b} \subseteq \mathfrak{q}$ are maximal Lie triple systems with the property that* $\operatorname{Exp}(\mathfrak{k})$ *and* $\operatorname{Exp}(\mathfrak{b})$ *are compact subsets of G/H, then there exists $h \in \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$ with $h.\mathfrak{k} = \mathfrak{b}$. If, in addition, $\mathfrak{q}$ is elliptic, then $\mathfrak{k} = \mathfrak{b}$.*

Proof. We may w.l.o.g. assume that $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$. Then $\mathfrak{k}$ and $\mathfrak{b}$ are elliptic Lie triple systems because the quadratic representation $q\colon G/H \to G, gH \mapsto g\tau(g)^{-1}$

maps the compact sets $\operatorname{Exp}\mathfrak{k}$, resp. $\operatorname{Exp}\mathfrak{b}$, onto $\exp(\mathfrak{k})$, resp. $\exp(\mathfrak{b})$, and we see that both are compact subsets of G. Therefore Theorem II.8 implies that we may w.l.o.g. assume that $\mathfrak{q}$ is elliptic. Then Proposition II.10 shows that

$$\mathfrak{c} := \{X \in \mathfrak{q}\colon \overline{\operatorname{Exp}(\mathbb{R}X)} \ \text{ is compact }\}$$

is a Lie triple system in $\mathfrak{q}$ which contains $\mathfrak{k}$ and $\mathfrak{b}$. Therefore $\mathfrak{c} = \mathfrak{k} = \mathfrak{b}$ by maximality. This proves the second assertion and therefore also the first one. ∎

III. Hyperbolic elements and their orbits

In this section we turn to the study of the set $\mathfrak{q}_{\rm hyp}$ of hyperbolic elements in $\mathfrak{q}$. First we will investigate the interior of the set $\mathfrak{q}_{\rm hyp}$ and derive useful characterizations of the elements in its interior. From that we will conclude that this set has interior points if and only if $\mathfrak{q}$ contains maximal hyperbolic abelian subspaces which are in addition maximal abelian, and this will pave the way to the root decompositions which will be discussed in Section IV.

Lemma III.1. *Let $X \in \mathfrak{q}_{\rm hyp}$, $V \subseteq \mathfrak{q}$ be a subspace containing X, and*

$$\Psi\colon \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}) \times V \to \mathfrak{q}, \quad (h,v) \mapsto h.v.$$

Then $d\Psi(h,X)$ is surjective if and only if

$$\mathfrak{z}_{\mathfrak{q}}(X) \subseteq V + [X,\mathfrak{h}].$$

Proof. We have

$$d\Psi(h,X).(d\lambda_h(\mathbf{1}).\operatorname{ad} Z, Y) = h.[Z,X] + h.Y = h.([Z,X]+Y).$$

Therefore the linear mapping $d\Psi(h,X)\colon T_h(\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})) \times V \to \mathfrak{q}$ is surjective if and only if $[X,\mathfrak{h}] + V = \mathfrak{q}$. Since $\mathfrak{g}$ is a semisimple $\mathbb{R}X$-module, we have $\mathfrak{g} = [X,\mathfrak{g}] \oplus \mathfrak{z}_{\mathfrak{g}}(X)$ and hence that $\mathfrak{q} = [X,\mathfrak{h}] + \mathfrak{z}_{\mathfrak{q}}(X)$. Therefore $[X,\mathfrak{h}] + V = \mathfrak{q}$ is equivalent to $\mathfrak{z}_{\mathfrak{q}}(X) \subseteq V + [X,\mathfrak{h}]$. ∎

Proposition III.2. *For $X \in \mathfrak{q}_{\rm hyp}$ the following are equivalent:*

(1) $X \in \operatorname{int}\mathfrak{q}_{\rm hyp}$.

(2) $\mathfrak{z}_{\mathfrak{q}}(X)$ *is a hyperbolic Lie triple system.*

Proof. (1) "⇒" (2): From $\mathfrak{z}_{\mathfrak{q}}(X) = \mathfrak{q} \cap \mathfrak{z}_{\mathfrak{g}}(X)$ and the fact that $\mathfrak{z}_{\mathfrak{g}}(X)$ is a subalgebra of $\mathfrak{g}$, it follows that $\mathfrak{z}_{\mathfrak{q}}(X)$ is a Lie triple system. Suppose that $X \in \operatorname{int}\mathfrak{q}_{\rm hyp}$ and let $Y \in \mathfrak{z}_{\mathfrak{q}}(X)$. Then there exists a $t > 0$ with $X + tY \in \mathfrak{q}_{\rm hyp}$. Now $[X, X+tY] = 0$ implies that $tY = (tY+X) - X$ is hyperbolic, hence that Y is hyperbolic. This proves that $\mathfrak{z}_{\mathfrak{q}}(X)$ is a hyperbolic Lie triple system.
(2) "⇒" (1): First we note that Lemma III.1 implies that for

$$\Psi\colon \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}) \times \mathfrak{z}_{\mathfrak{q}}(X) \to \mathfrak{q}, \quad (h,v) \mapsto h.v$$

the differential $d\Psi(\mathbf{1},X)$ is surjective. Therefore the implicit function theorem implies that $X = \Psi(\mathbf{1},X) \in \operatorname{int}\Psi\big(\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}), \mathfrak{z}_{\mathfrak{q}}(X)\big) \subseteq \operatorname{int}\mathfrak{q}_{\rm hyp}$. ∎

Theorem III.3. *Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be a symmetric Lie algebra. Then the following assertions hold:*

(i) *All maximal hyperbolic abelian subspaces $\mathfrak{a} \subseteq \mathfrak{q}$ are conjugate under* $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$.

(ii) *If $\mathfrak{a} \subseteq \mathfrak{q}$ is maximal hyperbolic abelian, then $\mathfrak{q}_{\mathrm{hyp}} = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).\mathfrak{a}$.*

(iii) $\mathrm{int}\,\mathfrak{q}_{\mathrm{hyp}} \neq \emptyset$ *if and only if $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$.*

Proof. (i) Let $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{q}$ be maximal hyperbolic abelian subspaces and $\widetilde{\mathfrak{a}}$, resp., $\widetilde{\mathfrak{b}}$ maximal hyperbolic Lie triple systems in $\mathfrak{q}$ containing $\mathfrak{a}$, resp., $\mathfrak{b}$. Then $\widetilde{\mathfrak{a}}$ and $\widetilde{\mathfrak{b}}$ are conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ (Corollary II.9), hence we may w.l.o.g. assume that $\widetilde{\mathfrak{a}} = \widetilde{\mathfrak{b}}$. Then $\mathfrak{a}$ and $\mathfrak{b}$ are nothing but maximal abelian subspaces of $\widetilde{\mathfrak{a}}$.

According to Proposition II.1, $\widetilde{\mathfrak{a}}_L$ is a Riemannian symmetric Lie algebra with $\mathfrak{z}(\widetilde{\mathfrak{a}}_L) \subseteq \mathfrak{q}$. Therefore the conjugacy of $\mathfrak{a}$ and $\mathfrak{b}$ follows from [3, Lemma V.6.3].

(ii) Let $X \in \mathfrak{q}_{\mathrm{hyp}}$ and $\widetilde{\mathfrak{a}} \subseteq \mathfrak{q}$ be maximal hyperbolic abelian with $X \in \widetilde{\mathfrak{a}}$. Then we use (i) to find $h \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ with $h.\widetilde{\mathfrak{a}} = \mathfrak{a}$. Then $h.X \in \mathfrak{a}$ and therefore $\mathfrak{q}_{\mathrm{hyp}} = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).\mathfrak{a}$.

(iii) We consider the mapping

$$\Psi\colon \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) \times \mathfrak{a} \to \mathfrak{q}, \quad (h, Y) \mapsto h.Y.$$

Then Lemma III.1 shows that the linear mapping $d\Psi(h, Y_0)\colon T_h(\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})) \times \mathfrak{a} \to \mathfrak{q}$ is surjective if and only if $\mathfrak{z}_{\mathfrak{q}}(Y_0) \subseteq \mathfrak{a} + [Y_0, \mathfrak{h}]$. Since $\mathfrak{q} = [Y_0, \mathfrak{h}] \oplus \mathfrak{z}_{\mathfrak{q}}(Y_0)$, this holds if and only if $\mathfrak{a} = \mathfrak{z}_{\mathfrak{q}}(Y_0)$.

If $\mathfrak{a}$ is not maximal abelian in $\mathfrak{q}$, then $\mathfrak{z}_{\mathfrak{q}}(Y_0)$ is always strictly bigger than $\mathfrak{a}$. Therefore Sard's Theorem implies that the image $\mathfrak{q}_{\mathrm{hyp}}$ of Ψ contains no interior points.

If, conversely, $\mathfrak{a}$ is maximal abelian, then $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{a}) = \mathfrak{a}$ and there exists $Y_0 \in \mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{z}_{\mathfrak{q}}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{q}}(Y_0)$. Then Proposition III.2 shows that $Y_0 \in \mathrm{int}\,\mathfrak{q}_{\mathrm{hyp}}$ and therefore that the latter set is non-empty. ∎

Lemma III.4. *Let $\mathfrak{p} \subseteq \mathfrak{q}$ be maximal hyperbolic Lie triple system and $\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian. Then $\mathfrak{a}$ is a maximal hyperbolic abelian subspace of $\mathfrak{q}$.*

Proof. Let $\widetilde{\mathfrak{a}} \supseteq \mathfrak{a}$ be maximal hyperbolic abelian and $\widetilde{\mathfrak{p}} \supseteq \widetilde{\mathfrak{a}}$ a maximal hyperbolic Lie triple system. Then there exists $h \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ with $h.\widetilde{\mathfrak{p}} = \mathfrak{p}$ (Corollary II.9). Hence $h.\widetilde{\mathfrak{a}}$ is a maximal abelian subspace of $\mathfrak{p}$. Now the fact that the symmetric Lie algebra $\mathfrak{p}_L$ is Riemannian entails that all maximal abelian subspaces of $\mathfrak{p}$ have the same dimension (cf. [3, Lemma V.6.3]). Therefore $\dim \widetilde{\mathfrak{a}} = \dim \mathfrak{a}$ implies $\mathfrak{a} = \widetilde{\mathfrak{a}}$, i.e., that $\mathfrak{a}$ is maximal hyperbolic abelian in $\mathfrak{q}$. ∎

Since we know at this point that all maximal hyperbolic abelian subspaces of $\mathfrak{q}$ are conjugate, the following specific constructions of maximal hyperbolic Lie triple systems give some important additional information on the location of maximal hyperbolic Lie triple systems with respect to certain nice Levi decompositions.

Proposition III.5. *Let $\mathfrak{r}$ denote the radical of $\mathfrak{g}$ and $\mathfrak{r} = \mathfrak{r}_{\mathfrak{h}} + \mathfrak{r}_{\mathfrak{q}}$ its τ-eigenspace decomposition. Let $\mathfrak{a}_{\mathfrak{r}} \subseteq \mathfrak{r}_{\mathfrak{q}}$ be a subspace which is maximal with* *respect to the property of being abelian and hyperbolic with respect to* $\mathfrak{g}$ *and* $\mathfrak{s} \subseteq \mathfrak{g}$ *a Levi complement which is invariant under* τ *and* $\mathfrak{a}_{\mathfrak{r}}$. *Then the following assertions hold:*

(i) *For any maximal hyperbolic abelian subspace* $\mathfrak{a}_{\mathfrak{s}} \subseteq \mathfrak{s}_{\mathfrak{q}}$ *the space* $\mathfrak{a} := \mathfrak{a}_{\mathfrak{r}} \oplus \mathfrak{a}_{\mathfrak{s}}$ *is maximal hyperbolic.*

(ii) *For any maximal hyperbolic abelian subspace* $\mathfrak{a} \subseteq \mathfrak{q}$ *the intersection* $\mathfrak{a} \cap \mathfrak{r}$ *is maximal hyperbolic in* $\mathfrak{r}_{\mathfrak{q}}$ *with respect to* $\mathfrak{g}$. *All these subspaces are conjugate under* $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$.

(iii) $[\mathfrak{a}_{\mathfrak{r}}, \mathfrak{s}] = \{0\}$.

(iv) *If* θ *is a Cartan involution of* $\mathfrak{s}$ *commuting with* $\tau|_{\mathfrak{s}}$ *and* $\mathfrak{s} = \mathfrak{s}_{\mathfrak{k}} \oplus \mathfrak{s}_{\mathfrak{p}}$ *the corresponding Cartan decomposition, then* $\mathfrak{p} := \mathfrak{a}_{\mathfrak{r}} \oplus (\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}$ *is a maximal hyperbolic Lie triple system in* $\mathfrak{q}$. *Moreover* $\mathfrak{a} \subseteq \mathfrak{p}$ *is maximal abelian if and only if* $\mathfrak{a} = \mathfrak{a}_{\mathfrak{r}} \oplus \mathfrak{a}_{\mathfrak{s}}$, *where* $\mathfrak{a}_{\mathfrak{s}} \subseteq (\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}$ *is maximal abelian.*

(v) *The correspondence in* (iv) *sets up a one-to-one correspondence between Cartan involutions on the ideal* $(\mathfrak{s}_{\mathfrak{q}})_L$ *of* $\mathfrak{s}$ *commuting with* τ *and maximal hyperbolic Lie triple systems in* $\mathfrak{q}$ *containing* $\mathfrak{a}_{\mathfrak{r}}$.

(vi) *If* $\mathfrak{n}$ *is the nilradical of* $\mathfrak{g}$, *then* $\mathfrak{r}_{\mathfrak{h}} \subseteq \mathfrak{z}_{\mathfrak{r}_{\mathfrak{h}}}(\mathfrak{a}) + \mathfrak{n}_{\mathfrak{h}}$.

Proof. (i) First we note that $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{r}}) + [\mathfrak{a}_{\mathfrak{r}}, \mathfrak{g}] \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{r}}) + \mathfrak{r}$ implies that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{r}})$ contains a Levi complement of $\mathfrak{g}$. Since $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{r}})$ is in addition τ-invariant, we find a τ-invariant Levi complement $\mathfrak{s} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{r}})$ (Corollary I.4). We pick a maximal hyperbolic abelian subspace $\mathfrak{a}_{\mathfrak{s}} \subseteq \mathfrak{s}_{\mathfrak{q}}$ and set $\mathfrak{a} := \mathfrak{a}_{\mathfrak{r}} + \mathfrak{a}_{\mathfrak{s}}$. Since both summands commute, this subspace is hyperbolic. Let $\widetilde{\mathfrak{a}} \supseteq \mathfrak{a}$ be maximal hyperbolic abelian and write $\pi\colon \mathfrak{g} \to \mathfrak{s} \cong \mathfrak{g}/\mathfrak{r}$ for the canonical projection. Then $\pi(\widetilde{\mathfrak{a}}) \subseteq \mathfrak{s}_{\mathfrak{q}}$ is hyperbolic and because it contains $\mathfrak{a}_{\mathfrak{s}}$, we even see that $\mathfrak{a}_{\mathfrak{s}} = \pi(\widetilde{\mathfrak{a}})$. Thus $\widetilde{\mathfrak{a}} \subseteq \mathfrak{r} + \mathfrak{a}_{\mathfrak{s}}$ and therefore $\widetilde{\mathfrak{a}} = (\mathfrak{r} \cap \widetilde{\mathfrak{a}}) + \mathfrak{a}_{\mathfrak{s}}$. Then $\mathfrak{r} \cap \widetilde{\mathfrak{a}}$ is hyperbolic in $\mathfrak{r}_{\mathfrak{q}}$ and therefore $\mathfrak{r} \cap \widetilde{\mathfrak{a}} = \mathfrak{a}_{\mathfrak{r}}$ follows from the maximality of $\mathfrak{a}_{\mathfrak{r}}$. This proves that $\widetilde{\mathfrak{a}} = \mathfrak{a}$, i.e., $\mathfrak{a}$ is maximal hyperbolic abelian in $\mathfrak{q}$.
(ii) These two assertions follow from (i) and the fact that all maximal hyperbolic abelian subspaces are conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ (Theorem III.3(i)).
(iii) If $\mathfrak{s} \subseteq \mathfrak{g}$ is an $\mathfrak{a}_{\mathfrak{r}}$-invariant Levi complement, then $[\mathfrak{a}_{\mathfrak{r}}, \mathfrak{s}] \subseteq \mathfrak{r} \cap \mathfrak{s} = \{0\}$.
(iv) Since $\mathfrak{a}_{\mathfrak{r}}$ commutes with $\mathfrak{s}$, it is clear that $\mathfrak{p}$ is a hyperbolic Lie triple system. To see that it is maximal, let $\widetilde{\mathfrak{p}} \supseteq \mathfrak{p}$ be a hyperbolic Lie triple system. First we note that the elements in $(\mathfrak{s}_{\mathfrak{k}})_{\mathfrak{q}}$ are elliptic, so that $(\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}$ is maximal hyperbolic in $\mathfrak{s}_{\mathfrak{q}}$. Hence projecting $\widetilde{\mathfrak{p}}$ along $\mathfrak{r}_{\mathfrak{q}}$ into $\mathfrak{s}_{\mathfrak{q}}$ shows that $\widetilde{\mathfrak{p}} \subseteq \mathfrak{r}_{\mathfrak{q}} + (\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}$ and thus

$$\widetilde{\mathfrak{p}} = (\widetilde{\mathfrak{p}} \cap \mathfrak{r}_{\mathfrak{q}}) \oplus (\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}.$$

Since the hyperbolic Lie triple system $\widetilde{\mathfrak{p}} \cap \mathfrak{r}_{\mathfrak{q}}$ is contained in a solvable Lie algebra, it is abelian (cf. Proposition II.1), and so the maximality of $\mathfrak{a}_{\mathfrak{r}}$ gives $\widetilde{\mathfrak{p}} \cap \mathfrak{r}_{\mathfrak{q}} = \mathfrak{a}_{\mathfrak{r}}$. This shows that $\widetilde{\mathfrak{p}} = \mathfrak{p}$, i.e., that $\mathfrak{p}$ is maximal.

For the second part of the assertion we first observe that $\mathfrak{a}_{\mathfrak{r}} \subseteq \mathfrak{z}(\mathfrak{p})$, so that any maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ contains $\mathfrak{a}_{\mathfrak{r}}$. Now the assertion is immediate.
(v) We only have to show that $\mathfrak{p}$ determines the Cartan involution on $(\mathfrak{s}_{\mathfrak{q}})_L$ uniquely. The subalgebra $\mathfrak{p}^c := i(\mathfrak{p} \cap \mathfrak{s}) + [\mathfrak{p}, \mathfrak{p}]$ of $\mathfrak{s}^c = \mathfrak{s}_{\mathfrak{h}} + i\mathfrak{s}_{\mathfrak{q}}$ is compactly embedded, so that Proposition I.5 implies the existence of a Cartan involution $\widetilde{\theta}$

of $\mathfrak{s}^c$ commuting with τ such that $i(\mathfrak{p}\cap\mathfrak{s}) \subseteq \mathfrak{s}^c_{\mathfrak{k}}$ and therefore $i(\mathfrak{p}\cap\mathfrak{s}) \subseteq i\mathfrak{s}_{\mathfrak{q}}\cap\mathfrak{s}^c_{\mathfrak{k}}$. Then

$$\mathfrak{s}_{\mathfrak{k}} = (\mathfrak{s}_{\mathfrak{h}}\cap\mathfrak{s}^c_{\mathfrak{k}}) \oplus (\mathfrak{s}_{\mathfrak{q}}\cap i\mathfrak{s}^c_{\mathfrak{p}}) \quad \text{and} \quad \mathfrak{s}_{\mathfrak{p}} = (\mathfrak{s}_{\mathfrak{h}}\cap\mathfrak{s}^c_{\mathfrak{p}}) \oplus (\mathfrak{s}_{\mathfrak{q}}\cap i\mathfrak{s}^c_{\mathfrak{k}})$$

defines a Cartan involution of $\mathfrak{s}$ with $\mathfrak{p}\cap\mathfrak{s}\subseteq\mathfrak{s}_{\mathfrak{q}}\cap\mathfrak{s}_{\mathfrak{p}}$. In view of the maximality of $\mathfrak{p}$, this implies that

$$\mathfrak{p} = \mathfrak{a}_{\mathfrak{r}} \oplus (\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}.$$

Then $(\mathfrak{s}_{\mathfrak{k}})_{\mathfrak{q}} = \mathfrak{s}_{\mathfrak{q}}\cap(\mathfrak{p}\cap\mathfrak{s})^\perp$ and

$$[\mathfrak{s}_{\mathfrak{q}},\mathfrak{s}_{\mathfrak{q}}]\cap\mathfrak{s}_{\mathfrak{k}} = [(\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}},(\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}] + [(\mathfrak{s}_{\mathfrak{k}})_{\mathfrak{q}},(\mathfrak{s}_{\mathfrak{k}})_{\mathfrak{q}}]$$

shows that $\mathfrak{p}$ determines the Cartan decomposition of $(\mathfrak{s}_{\mathfrak{q}})_L$ uniquely.
(vi) Since $\mathfrak{r}$ is a semisimple $\mathfrak{a}$-module, we have $\mathfrak{r} = \mathfrak{z}_{\mathfrak{r}}(\mathfrak{a})\oplus[\mathfrak{a},\mathfrak{r}]$. From the invariance of this decomposition under τ we further conclude that $\mathfrak{r}_{\mathfrak{h}} = \mathfrak{z}_{\mathfrak{r}_{\mathfrak{h}}}(\mathfrak{a})\oplus[\mathfrak{a},\mathfrak{r}_{\mathfrak{q}}]$. Hence the assertion follows from $[\mathfrak{a},\mathfrak{r}]\subseteq[\mathfrak{g},\mathfrak{r}]\subseteq\mathfrak{n}$. ■

In the following we simply write $\mathfrak{b}^0$ instead of $\mathfrak{z}_{\mathfrak{b}}(\mathfrak{a})$ for the centralizer of $\mathfrak{a}$ in the subspace $\mathfrak{b}$ of $\mathfrak{g}$.

Lemma III.6. *For a maximal hyperbolic Lie triple system $\mathfrak{p}\subseteq\mathfrak{q}$ and a maximal abelian subspace $\mathfrak{a}\subseteq\mathfrak{p}$ we have*

$$N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})}(\mathfrak{a}) = N_{\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{p},\mathfrak{p}])}(\mathfrak{a})\,\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}^0)$$

and

$$Z_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})}(\mathfrak{a}) = Z_{\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{p},\mathfrak{p}])}(\mathfrak{a})\,\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}^0).$$

Proof. First we note that, in view of Proposition III.5 and the conjugacy of all maximal hyperbolic Lie triple systems, there exists a maximal hyperbolic abelian subspace $\mathfrak{a}_{\mathfrak{r}}\subseteq\mathfrak{r}_{\mathfrak{q}}$ with respect to $\mathfrak{g}$, a τ-$\mathfrak{a}_{\mathfrak{r}}$-invariant Levi decomposition $\mathfrak{g}=\mathfrak{r}\rtimes\mathfrak{s}$, and a Cartan decomposition $\mathfrak{s}=\mathfrak{s}_{\mathfrak{k}}\oplus\mathfrak{s}_{\mathfrak{p}}$ which is invariant under τ such that $\mathfrak{p}=\mathfrak{a}_{\mathfrak{r}}\oplus(\mathfrak{s}_{\mathfrak{q}}\cap\mathfrak{s}_{\mathfrak{p}})$.

We first show that

$$N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})}(\mathfrak{a}) = N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s}_{\mathfrak{h}})}(\mathfrak{a})\,\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{r}^0_{\mathfrak{h}}). \tag{3.1}$$

The inclusion "$\supseteq$" is obvious. For the converse let $h\in N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})}(\mathfrak{a})$. In view of $\mathfrak{h}=\mathfrak{r}_{\mathfrak{h}}+\mathfrak{s}_{\mathfrak{h}}=\mathfrak{n}_{\mathfrak{h}}+\mathfrak{r}^0_{\mathfrak{h}}+\mathfrak{s}_{\mathfrak{h}}$ (Proposition III.5(vi)), we can write h as $h=snz$, where $s\in\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s}_{\mathfrak{h}})$, $n\in\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{n}_{\mathfrak{h}})$ and $z\in\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{r}^0_{\mathfrak{h}})$.

We denote the projection onto $\mathfrak{s}$ along $\mathfrak{r}$ by $p_{\mathfrak{s}}\colon\mathfrak{g}\to\mathfrak{s}$. Since $p_{\mathfrak{s}}(r.Y)=Y_s$ holds for every $Y\in\mathfrak{a}$ and $r\in\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{r})$, we see that $s\in N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s}_{\mathfrak{h}})}(\mathfrak{a}_{\mathfrak{s}})\subseteq N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s}_{\mathfrak{h}})}(\mathfrak{a})$ because $[\mathfrak{a}_{\mathfrak{r}},\mathfrak{s}]=\{0\}$. Hence $n\in N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{n}_{\mathfrak{h}})}(\mathfrak{a})$. The nilpotency of the subalgebra $\mathfrak{n}_{\mathfrak{h}}$ implies the existence of an element $Y\in\mathfrak{n}_{\mathfrak{h}}$ with $n=e^{\mathrm{ad}\,Y}$, and further the nilpotency of $\mathrm{ad}\,Y$ entails that $\mathrm{ad}\,Y=\log e^{\mathrm{ad}\,Y}$ preserves $\mathfrak{a}$. Now we use the semisimplicity of the $\mathfrak{a}$-module $\mathfrak{g}$ to see that

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$$

because any $\mathfrak{a}$-submodule $\mathfrak{m}$ of $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$ complementary to $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ satisfies $[\mathfrak{a},\mathfrak{m}] \subseteq \mathfrak{a}\cap\mathfrak{m} = \{0\}$. This proves that $Y \in \mathfrak{r}_{\mathfrak{h}}^0$. Hence $h \in s\,\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{r}_{\mathfrak{h}}^0)$, which establishes (3.1).

Next we show that

$$N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s}_{\mathfrak{h}})}(\mathfrak{a}) = N_{\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{p},\mathfrak{p}])}(\mathfrak{a})\,\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s}_{\mathfrak{h}}^0). \tag{3.2}$$

Let $\mathfrak{s}_{\mathfrak{h}} = \mathfrak{k}_{\mathfrak{h}} \oplus \mathfrak{p}_{\mathfrak{h}}$ the Cartan decomposition according to $\theta\,|_{\mathfrak{s}_{\mathfrak{h}}}$. Applying [21, Lemma 1.1.3.7] with respect to θ, we see that

$$N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s}_{\mathfrak{h}})}(\mathfrak{a}) = N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k}_{\mathfrak{h}})}(\mathfrak{a})e^{\mathrm{ad}\,\mathfrak{p}_{\mathfrak{h}}^0}. \tag{3.3}$$

Since $(\mathfrak{p}\cap\mathfrak{s})_L = (\mathfrak{p}\cap\mathfrak{s})\oplus[\mathfrak{p},\mathfrak{p}]$ is an ideal of the reductive subalgebra $\mathfrak{k}_{\mathfrak{h}}\oplus(\mathfrak{s}_{\mathfrak{q}}\cap\mathfrak{s}_{\mathfrak{p}}) = \mathfrak{k}_{\mathfrak{h}}\oplus(\mathfrak{p}\cap\mathfrak{s})$ (cf. Lemma II.7(i)), it follows in particular that $[\mathfrak{p},\mathfrak{p}]$ is an ideal of $\mathfrak{k}_{\mathfrak{h}}$ so that $\mathfrak{k}_{\mathfrak{h}} = [\mathfrak{p},\mathfrak{p}]\oplus[\mathfrak{p},\mathfrak{p}]^{\perp}$, where $[\mathfrak{p},\mathfrak{p}]^{\perp} \subseteq \mathfrak{z}_{\mathfrak{k}_{\mathfrak{h}}}(\mathfrak{p}) \subseteq \mathfrak{k}_{\mathfrak{h}}^0$. We conclude that

$$N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k}_{\mathfrak{h}})}(\mathfrak{a}) = N_{\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{p},\mathfrak{p}])}(\mathfrak{a})\,\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k}_{\mathfrak{h}}^0). \tag{3.4}$$

Hence (3.2) follows from (3.3), and (3.4).

Combining (3.1) and (3.2) implies the first statement of the lemma. The second assertion is immediate from the first one. ■

Proposition III.7. *Two maximal hyperbolic Lie triple systems in $\mathfrak{q}$ containing a maximal hyperbolic abelian subspace $\mathfrak{a}$ are conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}^0)$.*

Proof. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be maximal hyperbolic Lie triple systems containing $\mathfrak{a}$. Let $\gamma \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ such that $\gamma.\mathfrak{p}' = \mathfrak{p}$ (Corollary II.8). Then $\mathfrak{a}$ and $\gamma.\mathfrak{a}$ are maximal abelian subspaces in $\mathfrak{p}$, hence conjugate under $\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{p},\mathfrak{p}])$ (Theorem III.3). Thus we may assume that $\gamma.\mathfrak{a} = \mathfrak{a}$, i.e., $\gamma \in N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})}(\mathfrak{a})$.

Now Lemma III.6 applies and yields $\gamma = \sigma\eta$, where $\sigma \in \mathrm{Inn}_{\mathfrak{g}}([\mathfrak{p},\mathfrak{p}])$ and $\eta \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}^0)$. Obviously σ stabilizes $\mathfrak{p}$ and therefore $\eta.\mathfrak{p}' = \mathfrak{p}$, which proves the proposition. ■

For symmetric Lie algebras of the type $(\mathfrak{h}_{\mathbb{C}},\sigma)$, where σ is complex conjugation, we call these of *complex type*, Proposition III.7 can be used to obtain a new proof of the following results which plays a crucial role for Lie algebras with compactly embedded Cartan subalgebras (cf. [4, 3.13]).

Corollary III.8. *Let $\mathfrak{g}$ be a Lie algebra with compactly embedded Cartan subalgebra $\mathfrak{t}$. Then $\mathfrak{t}$ is contained in a unique maximal compactly embedded subalgebra $\mathfrak{k}$. Moreover, $i\mathfrak{k}$ is a maximal hyperbolic Lie triple system in $i\mathfrak{g}$ for the symmetric Lie algebra $(\mathfrak{g}_{\mathbb{C}},\sigma)$, where σ denotes complex conjugation.*

Proof. We consider the symmetric Lie algebra $(\mathfrak{g}_{\mathbb{C}},\sigma)$ with σ complex conjugation. Then $\mathfrak{a} := i\mathfrak{t}$ is a maximal hyperbolic abelian subspace in $\mathfrak{q} = i\mathfrak{g}$. If $\mathfrak{k} \supseteq \mathfrak{t}$ is a maximal compactly embedded subalgebra, then $\mathfrak{p} := i\mathfrak{k}$ is a hyperbolic Lie triple system in $\mathfrak{q}$ containing $\mathfrak{a}$. We claim that $\mathfrak{p}$ is maximal.

To see this, we only have show that one maximal hyperbolic Lie triple systems $\mathfrak{p}'$ in $\mathfrak{q}$ the subspace $i\mathfrak{p}'$ of $\mathfrak{g}$ is a subalgebra. Then the conjugacy result in Corollary II.8 implies that this holds for all maximal hyperbolic Lie triple systems. So observe that $\mathfrak{a}_\mathfrak{r} := i(\mathfrak{t} \cap \mathfrak{r})$ is a maximal hyperbolic abelian subspace in $i\mathfrak{r}$ with respect to $\mathfrak{g}_\mathbb{C}$ because, according to Theorem III.3, all compactly embedded Cartan subalgebras of $\mathfrak{g}$ are conjugate. Let $\mathfrak{g}_\mathbb{C} = \mathfrak{r}_\mathbb{C} \rtimes \mathfrak{s}_\mathbb{C}$ be an $\mathfrak{a}_\mathfrak{r}$-$\sigma$-invariant Levi decomposition, i.e., $\mathfrak{r}_\mathbb{C}$ and $\mathfrak{s}_\mathbb{C}$ are complexifications of $\mathfrak{r}$ and $\mathfrak{s}$ for a Levi decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ of $\mathfrak{g}$. We know from Proposition III.5 how to construct from this data a maximal hyperbolic subspace of $\mathfrak{q}$ by taking

$$\mathfrak{p}' := \mathfrak{a}_\mathfrak{r} \oplus \big((\mathfrak{s}_\mathbb{C})_\mathfrak{p} \cap i\mathfrak{s}\big),$$

where $\mathfrak{s}_\mathbb{C} = (\mathfrak{s}_\mathbb{C})_\mathfrak{k} \oplus (\mathfrak{s}_\mathbb{C})_\mathfrak{p}$ is a Cartan decomposition, i.e., $\mathfrak{u} := (\mathfrak{s}_\mathbb{C})_\mathfrak{k}$ is a compact real form of $\mathfrak{s}_\mathbb{C}$ and $(\mathfrak{s}_\mathbb{C})_\mathfrak{p} = i\mathfrak{u}$. Hence $\mathfrak{p}' = \mathfrak{a}_\mathfrak{r} \oplus i(\mathfrak{u} \cap \mathfrak{s})$ implies that $i\mathfrak{p}'$ is a subalgebra of $\mathfrak{g}$ and therefore $\mathfrak{p} = i\mathfrak{k}$ is maximal.

Now the assertion follows from $\mathfrak{h}^0 = \mathfrak{t} \subseteq \mathfrak{k}$ and Proposition III.7. ∎

Definition III.9. We define the *Weyl group* $\mathcal{W}$ of $(\mathfrak{g}, \tau, \mathfrak{a})$ by

$$\mathcal{W} := N_{\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})}(\mathfrak{a}) / Z_{\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})}(\mathfrak{a})$$

and recall from Lemma III.6 that

$$\mathcal{W} \cong N_{\mathrm{Inn}_\mathfrak{g}([\mathfrak{p},\mathfrak{p}])}(\mathfrak{a}) / Z_{\mathrm{Inn}_\mathfrak{g}([\mathfrak{p},\mathfrak{p}])}(\mathfrak{a})$$

holds for any maximal hyperbolic Lie triple system $\mathfrak{p}$ containing $\mathfrak{a}$. This shows in particular that $\mathcal{W}$ is isomorphic to the Weyl group of the Riemannian symmetric Lie algebra $\mathfrak{p}_L$ and that $\mathcal{W}$ is trivial whenever $\mathfrak{g}$ is solvable. ∎

Next we apply the information we have on the normalizer of a maximal hyperbolic abelian subspace to get a nice description of the intersection of $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$-orbits in $\mathfrak{q}_{\mathrm{hyp}}$ with $\mathfrak{a}$ (cf. also Theorem III.3).

Theorem III.10. *Let $\mathfrak{p}$ be a maximal hyperbolic Lie triple system in $\mathfrak{q}$, $\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian, and $X \in \mathfrak{a}$. Then the orbit $\mathcal{O}_X := \mathrm{Inn}_\mathfrak{g}(\mathfrak{h}).X$ intersects $\mathfrak{a}$ in an orbit of the Weyl group $\mathcal{W}$.*

Proof. Let $\mathfrak{g}^X = \ker \mathrm{ad}\, X$ denote the centralizer of X in $\mathfrak{g}$ and note that this subalgebra is τ-invariant and contains $\mathfrak{a}$. Let $\mathfrak{p}^X := \mathfrak{z}_\mathfrak{p}(X)$ and $\widetilde{\mathfrak{p}} \supseteq \mathfrak{p}^X$ be a maximal hyperbolic Lie triple system in $\mathfrak{q}^X := \mathfrak{z}_\mathfrak{q}(X)$ with respect to $\mathfrak{q}_L^X$. Then Proposition II.6(ii) applied with $\widetilde{\mathfrak{q}} = \mathfrak{q}^X$ yields

$$\mathfrak{a} = \big(\mathfrak{a} \cap \mathfrak{z}(\widetilde{\mathfrak{p}})\big) \oplus \big(\mathfrak{a} \cap \big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]\big) \quad \text{and} \quad \mathfrak{a} \cap \mathfrak{z}(\widetilde{\mathfrak{p}}) = \mathfrak{z}(\widetilde{\mathfrak{p}}) \cap \mathfrak{q}_{\mathrm{hyp}}, \tag{3.5}$$

where $\mathfrak{a} \cap \big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]$ is maximal abelian in $\big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]$.

Now let $h \in \mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$ with $h.X \in \mathfrak{a}$. Then $\mathfrak{b} := h^{-1}.\mathfrak{a} \subseteq \mathfrak{z}_\mathfrak{q}(X)$ is a hyperbolic subspace. Hence there exists $h_1 \in \langle e^{\mathrm{ad}\, \mathfrak{h}^X} \rangle$ with $h_1.\mathfrak{b} \subseteq \widetilde{\mathfrak{p}}$ (Corollary II.9). Since $hh_1^{-1}.X = h.X$, we may therefore w.l.o.g. assume that $\mathfrak{b} \subseteq \widetilde{\mathfrak{p}}$. Now (3.5) applies to $\mathfrak{b}$ as well and asserts that

$$\mathfrak{b} \cap \mathfrak{z}(\widetilde{\mathfrak{p}}) = \mathfrak{z}(\widetilde{\mathfrak{p}}) \cap \mathfrak{q}_{\mathrm{hyp}} = \mathfrak{a} \cap \mathfrak{z}(\widetilde{\mathfrak{p}}).$$

Let $\mathfrak{a}_0 := \mathfrak{a} \cap \mathfrak{z}(\widetilde{\mathfrak{p}})$, put $\mathfrak{a}_1 := \mathfrak{a} \cap \big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]$, and note that this is a maximal abelian subspace in $\big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]$. Defining $\mathfrak{b}_1$ similarly, we obtain

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \quad \text{and} \quad \mathfrak{b} = \mathfrak{a}_0 \oplus \mathfrak{b}_1,$$

where $\mathfrak{b}_1$ is also maximal abelian in $\big[\widetilde{\mathfrak{p}}, [\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]\big]$. Next we use [3, Lemma V.6.3] to find $h_2 \in \exp[\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}]$ with $h_2.\mathfrak{b}_1 = \mathfrak{a}_1$. Replacing h by hh_2^{-1}, we may now w.l.o.g. assume that $h.\mathfrak{a} = \mathfrak{a}$. So we have shown that

$$\mathcal{O}_X \cap \mathfrak{a} = N_{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})}(\mathfrak{a}).X.$$

Finally we use Lemma III.6 to obtain

$$\mathcal{O}_X \cap \mathfrak{a} = N_{\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{p},\mathfrak{p}])}(\mathfrak{a})\,\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}^0).X = N_{\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{p},\mathfrak{p}])}(\mathfrak{a}).X = \mathcal{W}.X. \qquad \blacksquare$$

For later reference we also record the following lemma.

Lemma III.11. *Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra, $\mathfrak{j} \subseteq \mathfrak{g}$ a τ-invariant ideal, and $\pi\colon \mathfrak{g} \to \mathfrak{g}_1 := \mathfrak{g}/\mathfrak{j}$ the quotient map. Write $\mathfrak{b}_1 := \pi(\mathfrak{b})$ for a subspace $\mathfrak{b} \subseteq \mathfrak{g}$. Then τ induces a symmetric structure τ_1 on $\mathfrak{g}_1$ and the eigenspace decomposition of $\mathfrak{g}_1$ w.r.t. τ_1 is given by $\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \mathfrak{q}_1$. Moreover, if $\mathfrak{p} \subseteq \mathfrak{q}$ is a maximal hyperbolic Lie triple system and $\mathfrak{a} \subseteq \mathfrak{p}$ is maximal abelian in $\mathfrak{p}$ and $\mathfrak{q}$, then $\mathfrak{p}_1$ is a maximal hyperbolic Lie triple system in $\mathfrak{q}_1$, and $\mathfrak{a}_1$ is maximal abelian in $\mathfrak{p}_1$ and $\mathfrak{q}_1$.*

Proof. The existence of τ_1 and the corresponding decomposition $\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \mathfrak{q}_1$ is trivial.

Writing $\mathfrak{q}$ as $\mathfrak{q} = \mathfrak{a} \oplus [\mathfrak{a}, \mathfrak{h}]$, we see that $\mathfrak{q}_1 = \mathfrak{a}_1 \oplus [\mathfrak{a}_1, \mathfrak{h}_1]$ and therefore that $\mathfrak{a}_1$ is maximal abelian in $\mathfrak{q}_1$ because $\mathfrak{g}_1$ is a semisimple $\mathfrak{a}_1$-module. The same argument proves that $\mathfrak{a}_1$ is maximal abelian in $\mathfrak{p}_1$.

Therefore it only remains to show that $\mathfrak{p}_1$ is a maximal hyperbolic Lie triple system. For that we may w.l.o.g. assume that $\mathfrak{p} = \mathfrak{a}_{\mathfrak{r}} \oplus \mathfrak{p}_{\mathfrak{s}}$ is constructed as in Proposition III.5(iv) (cf. Theorem III.3), where $\mathfrak{s}$ is a τ-$\mathfrak{a}$-invariant Levi complement, $\mathfrak{s} = \mathfrak{s}_{\mathfrak{k}} \oplus \mathfrak{s}_{\mathfrak{p}}$ is a Cartan decomposition of $\mathfrak{s}$ commuting with τ, and $\mathfrak{p}_{\mathfrak{s}} = (\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}$. Then the ideal $\mathfrak{j}$ decomposes as $\mathfrak{j} = (\mathfrak{j} \cap \mathfrak{r}) \rtimes (\mathfrak{j} \cap \mathfrak{s})$ (cf. [1, Ch. 1, §6, no. 8, Cor. 4]) and $\mathfrak{j} \cap \mathfrak{s}$ is invariant under θ because ideals of semisimple Lie algebras are invariant under Cartan involutions. Choosing a complementary ideal $\mathfrak{s}_0$ for $\mathfrak{j} \cap \mathfrak{s}$ in $\mathfrak{s}$, we see that $\mathfrak{s}_1 = (\mathfrak{s}_{\mathfrak{k}})_1 \oplus (\mathfrak{s}_{\mathfrak{p}})_1$ is a Cartan decomposition of $\mathfrak{s}_1$ invariant under τ_1, and that $(\mathfrak{p}_{\mathfrak{s}})_1 = \pi(\mathfrak{p}_{\mathfrak{s}}) = (\mathfrak{s}_{\mathfrak{p}})_1 \cap (\mathfrak{s}_{\mathfrak{q}})_1$. Since $\mathfrak{p}_1 = (\mathfrak{a}_{\mathfrak{r}})_1 \oplus (\mathfrak{p}_{\mathfrak{s}})_1$, in view of Proposition III.5(iv), it suffices to show that $(\mathfrak{a}_{\mathfrak{r}})_1$ is maximal hyperbolic abelian in $\mathfrak{r}$ with respect to $\mathfrak{g}_1$. Since $\mathfrak{a}_1$ is a maximal hyperbolic abelian subspace of $\mathfrak{q}_1$ and $(\mathfrak{a}_{\mathfrak{r}})_1 = \mathfrak{r}_1 \cap \mathfrak{a}_1$, this follows from Proposition III.5(ii). ■

Example III.12. Let $\mathfrak{h}$ be a 2-step nilpotent Lie algebra and $(\mathfrak{g}, \tau) := (\mathfrak{h}_{\mathbb{C}}, \sigma)$, where σ denotes complex conjugation. Then $\mathfrak{a} = i\mathfrak{z}(\mathfrak{h})$ is a maximal hyperbolic abelian subspace in $\mathfrak{q} = i\mathfrak{h}$ which is not at the same time maximal abelian in $\mathfrak{q}$. The image of $\mathfrak{a}$ in $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is $\{0\}$ but since $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is abelian, it is not maximally hyperbolic in $\mathfrak{q}$. This shows that the additional assumption that $\mathfrak{a}$ is also maximal abelian in $\mathfrak{q}$ is crucial in Lemma III.11. ■

IV. Root decompositions

In this section $(\mathfrak{g}, \tau)$ denotes a symmetric Lie algebra. Here we analyze the root decomposition of $\mathfrak{g}$ with respect to an abelian hyperbolic subspace $\mathfrak{a}$ of $\mathfrak{q}$ which is, in addition, maximal abelian in $\mathfrak{q}$. The latter condition is important to ensure that $\mathfrak{a}$ is big enough so that the corresponding root decomposition carries significant information on the structure of the whole Lie algebra.

Theorem IV.1. *Let $\mathfrak{a} \subseteq \mathfrak{q}$ be a maximal hyperbolic abelian subspace which is maximal abelian in $\mathfrak{q}$. For a linear functional $\alpha \in \mathfrak{a}^*$ we set*

$$\mathfrak{g}^\alpha := \{X \in \mathfrak{g} : (\forall Y \in \mathfrak{a})[Y, X] = \alpha(Y)X\}$$

and

$$\Delta := \Delta(\mathfrak{g}, \mathfrak{a}) := \{\alpha \in \mathfrak{a}^* \setminus \{0\} : \mathfrak{g}^\alpha \neq \{0\}\}.$$

Then the following assertions hold:

(i) $\mathfrak{g} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha$.

(ii) $\tau(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$.

(iii) *Let $\alpha \neq 0$, $Z \in \mathfrak{g}^\alpha$, and write $\mathfrak{g}(Z) = \mathrm{span}\{Z, \tau(Z), [Z, \tau(Z)]\}$ for the τ-invariant subalgebra of $\mathfrak{g}$ generated by Z. Then $[Z, \tau(Z)] \in \mathfrak{a}$, $\mathfrak{h}(Z) = \mathbb{R}\big(Z + \tau(X)\big)$ is one-dimensional, and there are four possibilities:*

(SR) $\alpha([Z, \tau(Z)]) > 0$. *Then $\mathfrak{g}(Z) \cong \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{h}(Z) \cong \mathfrak{so}(1,1)$ and $\mathfrak{q}(Z)$ is neither elliptic nor hyperbolic. The pair $\big(\mathfrak{g}(Z), \mathfrak{h}(Z)\big)$ is semi-Riemannian.*

(R) $\alpha([Z, \tau(Z)]) < 0$. *Then $\mathfrak{g}(Z) \cong \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{h}(Z) \cong \mathfrak{so}(2, \mathbb{R})$ and $\mathfrak{q}(Z)$ is hyperbolic. The pair $\big(\mathfrak{g}(Z), \mathfrak{h}(Z)\big)$ is Riemannian.*

(N) $\alpha([Z, \tau(Z)]) = 0$ *and* $[Z, \tau(Z)] \neq 0$. *Then $\mathfrak{g}(Z)$ is isomorphic to the three-dimensional Heisenberg algebra, and for every $E \in \mathfrak{a}$ with $\alpha(E) \neq 0$ the algebra $\mathfrak{g}(Z, E) := \mathfrak{g}(Z) \rtimes \mathbb{R}E$ is a four dimensional solvable symmetric Lie algebra.*

(A) $[Z, \tau(Z)] = 0$. *Then $\mathfrak{g}(Z) \cong \mathbb{R}^2$ and for every $E \in \mathfrak{a}$ with $\alpha(E) \neq \{0\}$ the algebra $\mathfrak{g}(Z, E) = \mathfrak{g}(Z) \rtimes \mathbb{R}E$ is a three dimensional solvable Lie algebra.*

Proof. (i) The algebra $\mathrm{ad}\,\mathfrak{a} \subseteq \mathfrak{gl}(\mathfrak{g})$ is abelian and consists of diagonalizable elements. Hence this set permits a simultaneous diagonalization. This proves (i).

(ii) For $E \in \mathfrak{a}$ and $Z \in \mathfrak{g}^\alpha$ we have

$$[E, \tau(Z)] = \tau([\tau(E), Z]) = -\alpha(E)\tau(Z).$$

Therefore $\tau(Z) \in \mathfrak{g}^{-\alpha}$.

(iii) Let $H_Z := [Z, \tau(Z)]$. Then, in view of (ii), $H_Z \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \subseteq \mathfrak{g}^0 = \mathfrak{z}_\mathfrak{g}(\mathfrak{a})$ and

$$\tau(H_Z) = \tau([Z, \tau(Z)]) = [\tau(Z), Z] = -H_Z$$

it follows that $H_Z \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{q} = \mathfrak{a}$. After rescaling Z, we may w.l.o.g. assume that $\alpha(H_Z) \in \{2, 0, -2\}$.
(SR) If $\alpha([Z, \tau(Z)]) > 0$, then $\alpha(H_Z) = 2$ so that

$$[H_Z, Z] = 2Z, \quad [H_Z, \tau(Z)] = -2\tau(Z), \quad H_Z = [Z, \tau(Z)]$$

implies that $\mathfrak{g}(Z) \cong \mathfrak{sl}(2, \mathbb{R})$, where $\mathfrak{h}(Z) = \mathbb{R}\big(Z + \tau(Z)\big)$ corresponds to $\mathfrak{so}(1, 1)$.
(R) If $\alpha([Z, \tau(Z)]) < 0$, then $\alpha(H_Z) = -2$ so that

$$[H_Z, Z] = -2Z, \quad [H_Z, \tau(Z)] = 2\tau(Z), \quad H_Z = [Z, \tau(Z)]$$

implies that $\mathfrak{g}(Z) \cong \mathfrak{sl}(2, \mathbb{R})$, where $\mathfrak{h}(Z) = \mathbb{R}\big(Z + \tau(Z)\big)$ corresponds to $\mathfrak{so}(2)$.
(N) If $\alpha([Z, \tau(Z)]) = 0$ and $[Z, \tau(Z)] \neq 0$, then H_Z is central in $\mathfrak{g}(Z)$ and $H_Z = [Z, \tau(Z)]$ implies that $\mathfrak{g}(Z) \cong \mathfrak{h}_1$, the three dimensional Heisenberg algebra.
(A) If $[Z, \tau(Z)] = 0$, then $\mathfrak{g}(Z) \cong \mathbb{R}^2$.

The remaining assertions are easy consequence of these computations. ■

Definition IV.2. In the preceding theorem we have seen that for an element $Z \in \mathfrak{g}^\alpha$ there are four possibilities. In the following we will say that Z is *of semi-Riemannian type* (SR), *Riemannian type* (R), *nilpotent type* (N), and *abelian type* (A), whenever the corresponding case in Theorem IV.1 occurs. ■

Before we turn to the structure of the root decomposition we discuss some basic examples.

Example IV.3. (a) Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0$ where $\mathfrak{g}_0$ is a Lie algebra and τ is given by $\tau(X, Y) = (Y, X)$. Then we get

$$\mathfrak{h} = \{(X, X) : X \in \mathfrak{g}_0\} \quad \text{and} \quad \mathfrak{q} = \{(X, -X) : X \in \mathfrak{g}_0\}.$$

We observe that an element $(X, -X) \in \mathfrak{q}$ is hyperbolic if and only if X is a hyperbolic element of the Lie algebra $\mathfrak{g}_0$. Therefore there exists a hyperbolic subspace $\mathfrak{a} \subseteq \mathfrak{q}$ which is maximal abelian in $\mathfrak{q}$ if and only if $\mathfrak{g}_0$ possesses a Cartan subalgebra $\mathfrak{a}_0$ consisting of hyperbolic elements, i.e., if $\mathfrak{g}_0$ has a toral Cartan subalgebra. If $\mathfrak{g}_0$ is semisimple, this means that $\mathfrak{g}_0$ is a normal real form of its complexification.

Applying Theorem III.3 to this situation, we see that all maximal abelian hyperbolic subspaces of a Lie algebra $\mathfrak{g}_0$ are conjugate under inner automorphisms and in particular that two toral Cartan subalgebras are conjugate whenever such Cartan subalgebras exist.

Let $\mathfrak{a}_0 \subseteq \mathfrak{g}_0$ be a toral Cartan subalgebra. Then the subspace $\mathfrak{a} = \{(X, -X) : X \in \mathfrak{a}_0\}$ is hyperbolic and maximal abelian in $\mathfrak{q}$. If $\Delta_0 := \Delta(\mathfrak{a}_0, \mathfrak{g}_0)$ is the root system of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$, then the root spaces for $\mathfrak{a}$ are given by

$$\mathfrak{g}^\alpha := (\mathfrak{g}_0^{\alpha_0} \times \{0\}) \oplus (\{0\} \times \mathfrak{g}_0^{-\alpha_0}), \quad \alpha_0 \in \Delta_0,$$

where $\alpha(X, -X) = \alpha_0(X)$ is the corresponding root. Let $Z = (Z_{\alpha_0}, Z_{-\alpha_0}) \in \mathfrak{g}^\alpha$. Then

$$[Z, \tau(Z)] = ([Z_{\alpha_0}, Z_{-\alpha_0}], -[Z_{\alpha_0}, Z_{-\alpha_0}]) \quad \text{and} \quad \alpha\big([Z, \tau(Z)]\big) = \alpha_0([Z_{\alpha_0}, Z_{-\alpha_0}]).$$

Suppose that $\mathfrak{g}_0$ is semisimple with Cartan involution θ satisfying $\mathfrak{a}_0 \subseteq \mathfrak{p}$. Then Theorem IV.1 implies that $\alpha_0\big([Z_{\alpha_0}, \theta(Z_{\alpha_0})]\big) < 0$. Thus, putting $Z_{-\alpha_0} = 0$ or $Z_{-\alpha_0} = \pm\theta(Z_{\alpha_0})$, we obtain root vectors Z of the types (SR), (R) and (A) in each root space $\mathfrak{g}^\alpha$.
(b) Let $\mathfrak{h}$ be a Lie algebra and $(\mathfrak{g}, \tau) := (\mathfrak{h}_\mathbb{C}, \sigma)$, where σ denotes complex conjugation. In this case $\mathfrak{q} = i\mathfrak{h}$. An element $X \in \mathfrak{q}$ is hyperbolic if and only if X is an elliptic element of the Lie algebra $\mathfrak{h}$. Therefore a hyperbolic subspace $\mathfrak{a} \subseteq \mathfrak{q}$ is maximal abelian in $\mathfrak{q}$ if and only if $i\mathfrak{a}$ is a compactly embedded Cartan subalgebra of $\mathfrak{h}$.

Using Theorem III.3 again, we regain the fact that all maximal abelian compactly embedded subalgebras of a Lie algebra $\mathfrak{h}$ are conjugate under inner automorphisms and in particular that two compactly embedded Cartan subalgebras are conjugate whenever they exist.

Let $\mathfrak{t} \subseteq \mathfrak{h}$ be a compactly embedded Cartan subalgebra. Then the subspace $\mathfrak{a} := i\mathfrak{t} \subseteq \mathfrak{q}$ is hyperbolic and maximal abelian in $\mathfrak{q}$ and the root systems $\Delta(\mathfrak{g}, \mathfrak{a})$ and $\Delta(\mathfrak{h}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ coincide. For $Z \in \mathfrak{g}^\alpha = \mathfrak{h}_\mathbb{C}^\alpha$ we have $[Z, \tau(Z)] = [Z, \overline{Z}]$ so the four types from Theorem IV.1 correspond to the four types of root spaces considered in [5, Ch. 7]. ■

Having introduced the general setup for the root decompositions, we now turn to their relations to well chosen Levi decompositions of $\mathfrak{g}$.

Definition IV.4. If $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ is an $\mathfrak{a}$-τ-invariant Levi decomposition, then we say that a root $\alpha \in \Delta$ is *semisimple* if $\mathfrak{s}^\alpha \neq \{0\}$ and *solvable* if $\mathfrak{g}^\alpha \subseteq \mathfrak{r}$. We write Δ_s for the set of semisimple roots and Δ_r for the set of solvable roots. ■

Lemma IV.5. *For an $\mathfrak{a}$-τ-invariant Levi decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ the following assertions hold:*

(i) $\mathfrak{g}^\alpha = \mathfrak{r}^\alpha \oplus \mathfrak{s}^\alpha$, *where* $\mathfrak{r}^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{r}$ *and* $\mathfrak{s}^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{s}$.

(ii) $\Delta_s^\perp = \mathfrak{a}_\mathfrak{r}$.

(iii) $\mathfrak{g} \subseteq \mathfrak{q}_L + \mathfrak{h}^0$.

Proof. (i) Since $\mathfrak{r}$ and $\mathfrak{s}$ are invariant under $\mathfrak{a}$, both subspaces of $\mathfrak{g}$ decompose according to the root decomposition of Theorem IV.1. Therefore each root space $\mathfrak{g}^\alpha$ can be written as $\mathfrak{g}^\alpha = (\mathfrak{g}^\alpha \cap \mathfrak{r}) \oplus (\mathfrak{g}^\alpha \cap \mathfrak{s})$, whence the assertion.
(ii) If $\alpha \in \Delta_s$, then $\mathfrak{s}^\alpha \neq \{0\}$, so that $[\mathfrak{a}_\mathfrak{r}, \mathfrak{s}^\alpha] \subseteq \mathfrak{r} \cap \mathfrak{s}^\alpha = \{0\}$ implies that $\alpha|_{\mathfrak{a}_\mathfrak{r}} = 0$.

If, conversely, $X \in \Delta_s^\perp$, then $X \in \mathfrak{z}_\mathfrak{a}(\mathfrak{s}) \subseteq \mathfrak{a}_\mathfrak{r}$, because each element in $\mathfrak{z}_\mathfrak{g}(\mathfrak{s})$ is mapped by the natural homomorphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{r} \cong \mathfrak{s}$ onto a central element of $\mathfrak{s}$, hence to 0.
(iii) According to the root space decomposition, we have $\mathfrak{g} = \mathfrak{h}^0 \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^\alpha$. Therefore $\mathfrak{a} \subseteq \mathfrak{q}$ and $\mathfrak{g}^\alpha \subseteq \mathfrak{q}_L$ imply that $\mathfrak{g} \subseteq \mathfrak{q}_L + \mathfrak{h}^0$. ■

We recall our notation κ for the Cartan-Killing form of $\mathfrak{g}$ and write $m_\alpha := \dim \mathfrak{g}^\alpha$ for the *multiplicity of the root* $\alpha \in \Delta$.

Definition IV.6. Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra and V a $\mathfrak{g}$-module carrying a bilinear form ϕ. The form ϕ is called τ*-covariant*, if

$$\phi(X.v, w) = -\phi(v, \tau(X).w)$$

holds for all $X \in \mathfrak{g}$ and $v, w \in V$. ∎

Proposition IV.7. *If $\mathfrak{s}$ is τ-$\mathfrak{a}$-invariant Levi complement and $\kappa_{\mathfrak{a}}$ denotes the restriction of the Cartan-Killing form to $\mathfrak{a}$, then the following assertions hold:*

(i) $\kappa_{\mathfrak{a}}(X,Y) = \sum_{\alpha\in\Delta} m_\alpha \alpha(X)\alpha(Y)$ *and* $\kappa_{\mathfrak{a}}(X,X) \geq 0$ *for* $X, Y \in \mathfrak{a}$.

(ii) $\mathfrak{a}^{\perp_{\kappa_\mathfrak{a}}} = \mathfrak{a}_\mathfrak{z} := \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{a}$, $\mathfrak{a}_\mathfrak{r}^{\perp_{\kappa_\mathfrak{a}}} = \mathfrak{a}_\mathfrak{z} \oplus \mathfrak{a}_\mathfrak{s}$, *and* $\mathfrak{a}_\mathfrak{s}^{\perp_{\kappa_\mathfrak{a}}} = \mathfrak{a}_\mathfrak{r}$.

(iii) *If we define* $\phi\colon \mathfrak{a} \to \mathfrak{a}^*$ *by* $\phi(X)(Y) := \kappa(X,Y)$, *then*

$$\phi(X) = \sum_{\alpha\in\Delta} m_\alpha \alpha(X)\alpha,$$

$\ker\phi = \mathfrak{a}_\mathfrak{z}$, *and* $\phi(\mathfrak{a}) = \operatorname{span}\Delta$. *The prescription* $\langle\phi(X),\phi(Y)\rangle := \kappa(X,Y)$ *defines a scalar product on* $\operatorname{span}\Delta$.

(iv) *For each* $\alpha \in \Delta_s$ *there exists a unique element* $A_\alpha \in \mathfrak{a}_\mathfrak{s}$ *with* $\phi(A_\alpha) = \alpha$ *which, in addition, satisfies* $\alpha(A_\alpha) > 0$.

(v) *If* $Z \in \mathfrak{g}^\alpha$, *then* $[Z,\tau(Z)] \in \mathfrak{a}_\mathfrak{z}$ *for* $\alpha \in \Delta_r$, *and*

$$[Z,\tau(Z)] \in \kappa\big(Z,\tau(Z)\big)A_\alpha + \mathfrak{a}_\mathfrak{z}$$

for $\alpha \in \Delta_s$. *If, in addition,* $Z \in \mathfrak{s}^\alpha$, *then*

$$[Z,\tau(Z)] = \kappa\big(Z,\tau(Z)\big)A_\alpha.$$

(vi) *The form defined by* $\kappa_\tau(X,Y) := \kappa(X,\tau.Y)$ *is a symmetric* τ*-covariant form on* $\mathfrak{g}$. *Moreover, on each root space* $\mathfrak{g}^\alpha$ *the degenerate subspace coincides with* $\mathfrak{r}^\alpha$, *and an element* $X \in \mathfrak{g}^\alpha$ *is*

(a) *of type* (SR) *if* $\kappa_\tau(X) > 0$,

(b) *of type* (R) *if* $\kappa_\tau(X) < 0$, *and*

(c) *of type* (N) *or* (A) *if* $\kappa_\tau(X) = 0$.

Proof. (i) This is an immediate consequence of the definition of the Cartan-Killing form.

(ii) That $\mathfrak{a}^{\perp_{\kappa_\mathfrak{a}}} = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{a}$ follows directly from the explicit formula in (i) and the observation that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{a} = \bigcap_{\alpha\in\Delta} \ker\alpha$.

Next we recall from [1, Ch. 1, §5.5, Prop. 5(b)] that $\mathfrak{r} = [\mathfrak{g},\mathfrak{g}]^{\perp_\kappa}$. From that we conclude that $\mathfrak{a}_\mathfrak{r} \subseteq \mathfrak{a}_\mathfrak{s}^{\perp_{\kappa_\mathfrak{a}}}$, so that the equality follows from the non-degeneracy of $\kappa_\mathfrak{a}$ on $\mathfrak{a}_\mathfrak{s}$. Thus we get that

$$\mathfrak{a}_\mathfrak{r}^{\perp_{\kappa_\mathfrak{a}}} = (\mathfrak{a}_\mathfrak{s}^{\perp_{\kappa_\mathfrak{a}}})^{\perp_{\kappa_\mathfrak{a}}} = \mathfrak{a}_\mathfrak{s} \oplus \mathfrak{a}_\mathfrak{z}.$$

(iii) The formula for ϕ follows by rewriting (i). From that we observe that $\ker\phi = \mathfrak{a}_\mathfrak{z}$ and $\phi(\mathfrak{a}) = \operatorname{span}\Delta$. In view of (i), it now is clear that $\langle\phi(X),\phi(Y)\rangle := \kappa(X,Y)$ defines a scalar product on $\operatorname{span}\Delta$.

(iv) Let $\alpha \in \Delta_s$ and $X \in \phi^{-1}(\alpha)$. Then $\kappa(X,\mathfrak{a}_\mathfrak{r}) = \alpha(\mathfrak{a}_\mathfrak{r}) = 0$ (cf. Lemma IV.5) implies that $X \in \mathfrak{a}_\mathfrak{r}^{\perp_{\kappa_\mathfrak{a}}} = \mathfrak{a}_\mathfrak{s} \oplus \mathfrak{a}_\mathfrak{z}$. Writing $X = X_s + X_z$ with $X_z \in \mathfrak{a}_\mathfrak{z}$ and

$X_s \in \mathfrak{a}_\mathfrak{s}$, we see that $X_s \in \mathfrak{a}_\mathfrak{s} \cap \phi^{-1}(\alpha) \neq \emptyset$. Now the uniqueness of A_α follows from the injectivity of $\phi|_{\mathfrak{a}_\mathfrak{s}}$. Finally we observe that

$$\alpha(A_\alpha) = \kappa(A_\alpha, A_\alpha) = \sum_{\beta \in \Delta} \beta(A_\alpha)^2 > 0.$$

(v) Let $Z \in \mathfrak{g}^\alpha$. If $\alpha \in \Delta_r$, then $\mathfrak{g}^\alpha = \mathfrak{r}^\alpha$ and therefore $[Z, \tau(Z)] \in \mathfrak{a}_\mathfrak{r} \cap [\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{a}_\mathfrak{z}$ because for each element X in $[\mathfrak{r}, \mathfrak{r}]$ the operator $\operatorname{ad} X$ is nilpotent, so that the simultaneous semisimplicity implies that it is central.

If $\alpha \in \Delta_s$ and A_α is chosen as in (iv), then we have for $Y \in \mathfrak{a}$:

$$\begin{aligned} \phi\big([Z, \tau(Z)]\big)(Y) &= \kappa\big([Z, \tau(Z)], Y\big) = \kappa\big(Z, [\tau(Z), Y]\big) \\ &= \alpha(Y)\kappa\big(Z, \tau(Z)\big) = \kappa\big(Z, \tau(Z)\big)\phi(A_\alpha)(Y) \end{aligned}$$

which entails that

$$[Z, \tau(Z)] \in \phi^{-1}\big(\kappa(Z, \tau(Z))A_\alpha\big) = \kappa\big(Z, \tau(Z)\big)A_\alpha + \mathfrak{a}_\mathfrak{z}.$$

If, in addition, $Z \in \mathfrak{s}^\alpha$, then $A_\alpha \in \mathfrak{a}_\mathfrak{s}$ entails $[Z, \tau(Z)] = \kappa\big(Z, \tau(Z)\big)A_\alpha$.

(vi) First we note that the invariance of the Cartan-Killing form κ under τ and $\tau^2 = \mathrm{id}_\mathfrak{g}$ imply that $\kappa_\tau(X, Y) := \kappa(X, \tau.Y)$ defines a τ-covariant symmetric bilinear form on $\mathfrak{g}$.

We recall from the proof of (ii) that $\mathfrak{r} = [\mathfrak{g}, \mathfrak{g}]^{\perp_\kappa}$ which, in view of $\mathfrak{g}^\alpha \subseteq [\mathfrak{g}, \mathfrak{g}]$, shows that $\kappa_\tau(\mathfrak{r}^\alpha, \mathfrak{g}^\alpha) = \{0\}$. Since κ_τ is τ-covariant, $\operatorname{ad} \mathfrak{a}$ acts by κ_τ-symmetric operators, hence $\kappa_\tau(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = \{0\}$ for $\alpha + \beta \neq 0$. For any $\mathfrak{a}$-τ-invariant Levi complement $\mathfrak{s}$ we have $\kappa(\mathfrak{r}, \mathfrak{s}) \subseteq \kappa(\mathfrak{r}, [\mathfrak{g}, \mathfrak{g}]) = \{0\}$ and hence $\mathfrak{s}^\perp \cap \mathfrak{s} \subseteq \mathfrak{g}^\perp \cap \mathfrak{s} \subseteq \mathfrak{r} \cap \mathfrak{s} = \{0\}$. Thus κ_τ restricted to $\mathfrak{s}$ is non-degenerate and the orthogonality of the root spaces implies that κ_τ is non-degenerate on $\mathfrak{s}^\alpha$.

Now let $Z \in \mathfrak{g}^\alpha$. If $\alpha \in \Delta_\mathfrak{r}$, then $\mathfrak{g}^\alpha \subseteq \mathfrak{r}$, κ_τ vanishes on $\mathfrak{g}^\alpha$, and since $\mathfrak{r}$ contains no semsisimple subalgebras, Z is of type (N) or (A). If $\alpha \in \Delta_s$, then (v) shows that

$$\alpha([Z, \tau.Z]) = \alpha(A_\alpha)\kappa_\tau(Z).$$

Therefore the assertion follows from $\alpha(A_\alpha) > 0$. ■

Proposition IV.8. *Let $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ be a τ-$\mathfrak{a}$-invariant Levi decomposition and θ a Cartan involution of $\mathfrak{s}$ commuting with τ. Then the following assertions hold:*

(i) *The involution $\tau^a := \theta\tau|_\mathfrak{s}$ preserves the root spaces $\mathfrak{s}^\alpha$ of $\mathfrak{a}$ and the quadratic form κ_τ is positive definite on the -1-eigenspace and negative definite on the 1-eigenspace of τ^a in $\mathfrak{s}^\alpha$.*

(ii) *Each element $Z \in \mathfrak{s}^\alpha$ is of type* (SR), (R) *or* (A). *If the types* (SR) *and* (R) *occur in $\mathfrak{s}^\alpha$, then type* (A) *also occurs.*

(iii) *Each root space $\mathfrak{g}^\alpha$, $\alpha \in \Delta_s$ contains an element of type* (SR) *or* (R).

(iv) *In $\mathfrak{r}^\alpha$ only the types* (N) *and* (A) *may occur.*

Proof. (i) Since $\mathfrak{a}_\mathfrak{r}$ commutes with $\mathfrak{s}$ and σ leaves the elements of $\mathfrak{a}_\mathfrak{s}$ pointwise fixed, it preserves the root spaces $\mathfrak{s}^\alpha$. For $Z \in \mathfrak{s}^\alpha$ we write $Z =$

$Z_+ + Z_-$ according to the σ-eigenspace decomposition. Note that the invariance of κ_τ under σ implies that the components are orthogonal with respect to κ_τ. Therefore

$$\begin{aligned}\kappa_\tau(Z) &= \kappa\big(Z,\tau(Z)\big) = \kappa\big(Z_+ + Z_-, \theta(Z_+) - \theta(Z_-)\big)\\ &= \underbrace{\kappa\big(Z_+,\theta(Z_+)\big)}_{\leq 0} - \underbrace{\kappa\big(Z_-,\theta(Z_-)\big)}_{\geq 0}\end{aligned}$$

follows from the fact that the form $\kappa_\theta(Z) := \kappa\big(Z,\theta(Z)\big)$ is negative definite on $\mathfrak{s}$ which in turns follows from the fact that for $\theta.X = X$ the element X is elliptic and non-central, and for $\theta.X = -X$ the element X is hyperbolic and non-central (cf. Lemma I.9(ii)).

(ii) Since for $Z \in \mathfrak{s}^\alpha$ we have $[Z,\tau(Z)] = \kappa\big(Z,\tau(Z)\big)A_\alpha$, the assertion follows from Proposition IV.7(vi) if we note that the occurence of type (R) and (SR) means that κ_τ is indefinite on $\mathfrak{s}^\alpha$ which implies the existence of isotropic vectors and therefore of elements of type (A).

(iii), (iv) In view of Proposition IV.7(vi), the form κ_τ is non-zero on $\mathfrak{g}^\alpha$ if and only if $\alpha \in \Delta_s$. Therefore the assertions follow from Proposition IV.7(vi). ■

Lemma IV.9. *If $\alpha \in \Delta_s$ with $\mathfrak{g}^\alpha = \mathfrak{s}^\alpha$ and κ_τ is definite on $\mathfrak{g}^\alpha$, then $\mathfrak{g}^\alpha$ is an irreducible $\mathfrak{h}^0$-module.*

Proof. Let $0 \neq Z \in \mathfrak{g}^\alpha$ and note that, in view of Proposition IV.8, our assumption implies that the subalgebra spanned by Z, $\tau(Z)$ and $[Z,\tau(Z)]$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Thus $\mathfrak{sl}_2$-theory, applied to the module $\sum_{n\in\mathbb{Z}} \mathfrak{g}^{n\alpha}$ shows that $\operatorname{ad} Z\colon \mathfrak{g}^0 \to \mathfrak{g}^\alpha$ is surjective. Therefore $\mathfrak{g}^\alpha = [Z,\mathfrak{a}] + [Z,\mathfrak{h}^0] = \mathbb{R}Z + [Z,\mathfrak{h}^0]$. From that we obtain that the $\mathfrak{h}^0$-submodule of $\mathfrak{g}^\alpha$ generated by Z coincides with $\mathfrak{g}^\alpha$. Since $Z \in \mathfrak{g}^\alpha$ was arbitrary, the irreducibility follows. ■

Example IV.10. (a) Let $\mathfrak{g} = (\mathfrak{h}_1 \oplus V) \rtimes \mathbb{R}H$, where $\mathfrak{h}_1 = \operatorname{span}\{P,Q,Z\}$ is the three dimensional Heisenberg algebra with $[P,Q] = Z \in \mathfrak{z}(\mathfrak{h}_1)$ on which H acts as by

$$H.P = P, \quad H.Q = -Q, \quad \text{and} \quad H.Z = 0,$$

and $V = \mathbb{R}P' \oplus \mathbb{R}Q'$ on which H acts in the same way. Let

$$\mathfrak{h} := \mathbb{R}(P+Q) \oplus \mathbb{R}(P'+Q'), \quad \text{and} \quad \mathfrak{q} := \mathbb{R}Z \oplus \mathbb{R}(P-Q) \oplus \mathbb{R}(P'-Q') \oplus \mathbb{R}H.$$

Since $[\mathfrak{q},\mathfrak{q}] \subseteq \mathfrak{h}$, $[\mathfrak{q},\mathfrak{h}] \subseteq \mathfrak{q}$ and $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$, we obtain an involutive automorphism τ of $\mathfrak{g}$ by $\tau|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$ and $\tau|_{\mathfrak{q}} = -\mathrm{id}_{\mathfrak{q}}$.

Now $\mathfrak{a} = \mathfrak{z}(\mathfrak{g}) \oplus \mathbb{R}H$ is a hyperbolic subspace which is maximal abelian in $\mathfrak{q}$, and the root system is given by $\Delta = \Delta_r = \{\pm\alpha\}$, where $\alpha(H) = 1$. Moreover

$$\mathfrak{g}^\alpha = \operatorname{span}\{P,P'\}, \qquad \mathfrak{g}^{-\alpha} = \operatorname{span}\{Q,Q'\},$$

and $\tau(P) = Q$, $\tau(P') = Q'$. Hence type (N) as well as type (A) occurs in $\mathfrak{g}^\alpha$.

(b) Let $\mathfrak{g} = \mathfrak{h}_2 \rtimes \mathbb{R}H$, where $\mathfrak{h}_2 = \operatorname{span}\{P,Q,P',Q',Z\}$ is the five dimensional Heisenberg algebra with $[P,Q] = [P',Q'] = Z \in \mathfrak{z}(\mathfrak{h}_2)$. The action of H is defined by

$$H.P = P, \quad H.Q = -Q, \quad H.P' = -P', \quad H.Q' = Q', \quad \text{and} \quad H.Z = 0.$$

We put

$$\mathfrak{h} := \mathbb{R}(P - P') \oplus \mathbb{R}(Q + Q') \quad \text{and} \quad \mathfrak{q} := \mathbb{R}Z \oplus \mathbb{R}(P + P') \oplus \mathbb{R}(Q - Q') \oplus \mathbb{R}H.$$

As in the preceding example, the prescriptions $\tau|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$ and $\tau|_{\mathfrak{q}} = -\mathrm{id}_{\mathfrak{q}}$ define an involutive automorphism of $\mathfrak{g}$. We obtain $\Delta = \{\pm\alpha\}$, where $\alpha(H) = 1$, and the root spaces are given by

$$\mathfrak{g}^{\alpha} = \mathrm{span}\{P, Q'\} \quad \text{and} \quad \mathfrak{g}^{-\alpha} = \mathrm{span}\{Q, P'\}.$$

From $\tau(P) = -P'$ and $\tau(Q) = Q'$, we get

$$[\lambda P + \mu Q', \tau(\lambda P + \mu Q')] = [\lambda P + \mu Q', -\lambda P' + \mu Q)] = 2\lambda\mu Z.$$

Thus the set of type (A)-elements in $\mathfrak{g}^{\alpha}$ is not a subspace.
(c) We give an example of a symmetric simple Lie algebra in which the root spaces are all of mixed type. So let us take $\mathfrak{s} = \mathfrak{so}(3,3)$ and τ conjugation by $\mathrm{diag}(-1,1,1,1,1,1)$. Then $\mathfrak{a} = \mathbb{R}(E_{1,6} + E_{6,1})$ is a hyperbolic subspace in $\mathfrak{q}$ which is also maximal abelian in $\mathfrak{q}$. Further, the root system is given by $\Delta = \{\pm\alpha\}$, where $\alpha(E_{1,6} + E_{6,1}) = 1$. For $j = 2, 3$, we set

$$X_j = E_{1,j} - E_{j,1} + E_{6,j} + E_{j,6},$$

while for $j = 4, 5$, we put

$$X_j = E_{1,j} + E_{j,1} + E_{6,j} - E_{j,6}.$$

Then it is easy to check that $\mathfrak{s}^{\alpha} = \mathrm{span}\{X_2, X_3, X_4, X_5\}$, and another simple computation shows that the form κ_τ on $\mathfrak{s}^{\alpha}$ has signature $(1,1,-1,-1)$. Hence in $\mathfrak{s}^{\alpha}$ all types (SR), (R) and (A) occur.

This example also shows that the subspace $\mathfrak{a}$ might be contained in several maximal hyperbolic Lie triple systems. In fact, we have $\mathfrak{h}^0 \cong \mathfrak{so}(2,2)$ and $[\mathfrak{h}^0, (\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}] \not\subseteq (\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}$, where the Cartan involution is given by $\theta(X) = -X^\top$. Therefore all the conjugates of $(\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}$ under the group $\mathrm{Inn}_{\mathfrak{s}}(\mathfrak{h}^0)$ are maximal hyperbolic Lie triple systems containing $\mathfrak{a}$ (cf. Proposition III.7). To make this more explicit, let $E \subseteq \mathfrak{s}^{\alpha}$ be a two-dimensional plane on which the scalar product κ_τ is negative definite. Then $\mathfrak{a} + \{\tau(X) - X \colon X \in E\}$ is a maximal hyperbolic Lie triple system in $\mathfrak{q}$ containing $\mathfrak{a}$ (cf. Proposition IV.8). Since there exists a continuous family of such subspaces E, this $(\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}$ is not unique. ■

Example IV.11. Let $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R})$ with the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with the commutator relations

$$[H, U] = 2T, \quad [H, T] = 2U, \quad \text{and} \quad [U, T] = 2H.$$

We consider the involution on $\mathfrak{s}$ of type (SR) which corresponds to the decomposition

$$\mathfrak{s}_{\mathfrak{h}} = \mathbb{R}T, \quad \text{and} \quad \mathfrak{s}_{\mathfrak{q}} = \mathbb{R}H + \mathbb{R}U.$$

Let $\mathfrak{r}$ be the three dimensional $\mathfrak{s}$-module with the basis (H', U', T') corresponding to the identification with $\mathfrak{s}$ endowed with the adjoint representation. We form the Lie algebra $\mathfrak{g} := \mathfrak{r} \rtimes \mathfrak{s}$ and put

$$\mathfrak{r}_{\mathfrak{h}} := \operatorname{span}\{H', U'\}, \quad \mathfrak{r}_{\mathfrak{q}} := \mathbb{R}T', \quad \mathfrak{h} := \mathfrak{r}_{\mathfrak{h}} + \mathfrak{s}_{\mathfrak{h}} \quad \text{and} \quad \mathfrak{q} := \mathfrak{r}_{\mathfrak{q}} + \mathfrak{s}_{\mathfrak{q}}.$$

Then $[\mathfrak{s}_{\mathfrak{h}}, \mathfrak{r}_{\mathfrak{h}}] \subseteq \mathfrak{r}_{\mathfrak{h}}$, $[\mathfrak{s}_{\mathfrak{h}}, \mathfrak{r}_{\mathfrak{q}}] \subseteq \mathfrak{r}_{\mathfrak{q}}$, $[\mathfrak{s}_{\mathfrak{q}}, \mathfrak{r}_{\mathfrak{q}}] \subseteq \mathfrak{r}_{\mathfrak{h}}$, and $[\mathfrak{s}_{\mathfrak{q}}, \mathfrak{r}_{\mathfrak{h}}] \subseteq \mathfrak{r}_{\mathfrak{q}}$ imply that $\tau|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$ and $\tau|_{\mathfrak{q}} := -\mathrm{id}_{\mathfrak{q}}$ defines an automorphism of $\mathfrak{g}$.

The subspace $\mathfrak{a} := \mathbb{R}H \subseteq \mathfrak{q}$ is maximal abelian, hyperbolic, and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a} + \mathbb{R}H'$. The corresponding roots are given by $\Delta = \Delta_s = \{\pm\alpha\}$, $\alpha(H) = 2$, and

$$\mathfrak{g}^\alpha = \operatorname{span}\{U + T, U' + T'\}, \quad \mathfrak{r}^\alpha = \mathbb{R}(U' + T'), \quad \text{and} \quad \mathfrak{s}^\alpha = \mathbb{R}(U + T).$$

Since the form κ_τ is positive definite on $\mathfrak{s}^\alpha$ and degenerate on $\mathfrak{r}^\alpha$ (cf. Proposition IV.7(vi)), the elements in $\mathfrak{r}^\alpha$ are of type (A) and the other elements are of type (SR).

The space $\mathfrak{q} = \operatorname{span}\{H, U, T'\}$ is three dimensional, and to see the action of $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$ on this space, we note the the matrices of the elements of $\mathfrak{h}$ with respect to the basis (T', H, U) are given by

$$\operatorname{ad}_{\mathfrak{q}} T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}, \quad \operatorname{ad}_{\mathfrak{q}} H' = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\operatorname{ad}_{\mathfrak{q}} U' = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From that for

$$W_{\max} := \mathbb{R}T' + \mathbb{R}^+(H + U) + \mathbb{R}^+(H - U)$$

the set $\mathfrak{q}_{\mathrm{hyp}} = \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}).\mathfrak{a}$ is given by

$$\{0\} \cup W_{\max}^0 \cup -W_{\max}^0.$$

Note that $W_{\max} \cap \mathfrak{a} = \mathbb{R}^+ H$. The orbits in $W_{\max}^0$ are surfaces of the type $\mathbb{R}T' + e^{\operatorname{ad} \mathfrak{s}_{\mathfrak{h}}} \lambda H$, $\lambda > 0$. It follows in particular that their convex hulls always contain affine lines.

For the coadjoint action on $\mathfrak{q}^*$ we obtain the matrices

$$\operatorname{ad}_{\mathfrak{q}}^* T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}, \quad \operatorname{ad}_{\mathfrak{q}}^* H' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

and

$$\operatorname{ad}^*_{\mathfrak q} U' = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the orbit of each functional not vanishing on T' is a two-dimensional plane parallel to $(T')^\perp$. In the plane $(T')^\perp$ the orbits meeting $\mathfrak a^* \cong \{T', U\}^\perp$ are hyperbolas. Therefore the only orbits in $W^\star_{\max}$ which are closed and have a pointed convex hull are those through $\mathfrak a^*$. ∎

V. The Weyl group and quasihermitian symmetric Lie algebras

In this section $(\mathfrak g, \tau)$ denotes a symmetric Lie algebra and $\mathfrak a \subseteq \mathfrak q$ is a maximal hyperbolic abelian subspace which is maximal abelian in $\mathfrak q$.

Definition V.1. Let $\mathfrak p \supseteq \mathfrak a$ be a maximal hyperbolic Lie triple system.
(a) We say that a root $\alpha \in \Delta$ is *compact* if $\mathfrak g^\alpha \cap \mathfrak p_L \neq \emptyset$ and *non-compact* otherwise. We write $\Delta_k = \Delta(\mathfrak p_L, \mathfrak a)$, resp. Δ_n, for the set of compact, resp. non-compact, roots. Note that, in view of Proposition III.7, the set Δ_k does not depend on the choice of $\mathfrak p$ because all maximal hyperbolic Lie triple systems containing $\mathfrak a$ are conjugate under the group $\operatorname{Inn}_{\mathfrak g}(\mathfrak h_0)$ which preserves the root spaces.

If $\mathfrak p$ is related to a τ-$\mathfrak a$-invariant Levi complement $\mathfrak s$ as in Proposition III.5(iv), then it is clear that

$$\Delta_k \subseteq \Delta_s \quad \text{and} \quad \Delta_r \subseteq \Delta_n.$$

The roots in $\Delta_p := \Delta_n \cap \Delta_s$ are called the *non-compact semisimple roots.*
(b) The root system Δ is called *split* if all root vectors $X \in \mathfrak g^\alpha$, $\alpha \in \Delta_k$ are of Riemannian type (R).
(c) A positive system Δ^+ is called $\mathfrak p$*-adapted* if Δ_n^+ is invariant under the Weyl group.
(d) If C is a subset of the finite dimensional real vector space V, then we write $\operatorname{cone}(C)$ for the smallest closed convex cone containing C. For a positive system Δ_n^+ of non-compact roots, we consider the following cones:

$$C_{\min,\mathrm p} := C_{\min,\mathrm p}(\Delta_n^+) := \operatorname{cone}(\{[X,\tau(X)] : X \in \mathfrak g^\alpha, \alpha \in \Delta_p^+\}) \subseteq \mathfrak a,$$

$$C_{\min,\mathrm z} := C_{\min,\mathrm z}(\Delta_n^+) := \operatorname{cone}(\{[X,\tau(X)] : X \in \mathfrak r^\alpha, \alpha \in \Delta^+\}) \subseteq \mathfrak a \cap \mathfrak n = \mathfrak z(\mathfrak g)_{\mathfrak q},$$

$C_{\min} = C_{\min}(\Delta_n^+) = \overline{C_{\min,\mathrm p} + C_{\min,\mathrm z}}$, and

$$C_{\max} := C_{\max}(\Delta_n^+) = \{X \in \mathfrak a : (\forall \alpha \in \Delta_n)\alpha(X) \geq 0\}.$$

(e) For every $\alpha \in \Delta_s$ we set $\check\alpha := \frac{2A_\alpha}{\langle \alpha, \alpha\rangle}$ and accordingly $\check\Delta_s := \{\check\alpha : \alpha \in \Delta_s\}$. Note that $\alpha(\check\alpha) = 2$ for all $\alpha \in \Delta_s$ and, more generally, that

$$\beta(\check\alpha) = \frac{2\beta(A_\alpha)}{\langle\alpha,\alpha\rangle} = \frac{2\phi(A_\beta)(A_\alpha)}{\langle\alpha,\alpha\rangle} = \frac{2\kappa(A_\beta, A_\alpha)}{\langle\alpha,\alpha\rangle} = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$$

for all $\beta \in \operatorname{span} \Delta = \phi(\mathfrak{a})$.
(f) We say that $(\mathfrak{g}, \tau)$ has *cone potential*, if no non-zero $Z \in \mathfrak{g}^\alpha$, $\alpha \in \Delta$ is of type (A).
(g) We say that $(\mathfrak{g}, \tau)$ has *strict cone potential*, if it has cone potential and for each $\beta \in \Delta$ there is an element $\alpha \in \mathfrak{a}_\mathfrak{z}^*$ such that

$$\alpha\big([X, \tau(X)]\big) > 0 \qquad \text{for all} \quad 0 \neq X \in \mathfrak{r}^\beta. \tag{5.1}$$

If, in addition, there exists a $\mathfrak{p}$-adapted positive system Δ^+ such that (5.1) holds simultaneously for all $\beta \in \Delta^+$, then we say that $(\mathfrak{g}, \tau)$ has *strong cone potential.*■

Proposition V.2. *Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra, $\mathfrak{p} \subseteq \mathfrak{q}$ a maximal hyperbolic Lie triple system and $\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian. Then the following assertions hold:*

(i) *The Weyl group $\mathcal{W}$ is isomorphic to the finite group generated by the reflections*

$$s_\beta \colon \mathfrak{a} \to \mathfrak{a}, \quad X \mapsto X - \beta(X)\check{\beta}$$

for $\beta \in \Delta_k$.

(ii) *The center of $\mathfrak{p}$ is given by the fixed point set of the Weyl group $\mathcal{W}$, i.e.*

$$\mathfrak{z}(\mathfrak{p}) = \mathfrak{a}^{\mathcal{W}} = \{X \in \mathfrak{a} \colon (\forall \alpha \in \Delta_k)\ \alpha(X) = 0\}.$$

It follows in particular that this space is independent of the choice of $\mathfrak{p}$.

(iii) *If $\mathcal{W}_s \subseteq \mathrm{Gl}(\mathfrak{a})$ is the group generated by the reflections s_α, $\alpha \in \Delta_s$, then $\mathcal{W}_s$ is finite and $\mathcal{W}_s.\Delta = \Delta$, where the action of $\mathcal{W}_s$ on $\mathfrak{a}^*$ is given by $s_\beta(\alpha) = \alpha - \alpha(\check{\beta})\beta$.*

Proof. (i) In view of Definition III.9, this is a statement on Riemannian symmetric Lie algebras and therefore follows from [3, Cor. VII.2.1].
(ii) Since $\mathfrak{z}(\mathfrak{p}) \subseteq \mathfrak{a}$ holds for any maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, the assertion follows immediate from the definition of the compact roots and (i).
(iii) Let $\mathfrak{a}_\mathfrak{r} = \mathfrak{a} \cap \mathfrak{r}$, and $\mathfrak{s}$ a τ-$\mathfrak{a}_\mathfrak{r}$-invariant Levi complement. Then we may w.l.o.g. assume that $\mathfrak{a} = \mathfrak{a}_\mathfrak{r} \oplus \mathfrak{a}_\mathfrak{s}$, where $\mathfrak{a}_\mathfrak{s} = \mathfrak{a} \cap \mathfrak{s}_\mathfrak{q}$ (cf. Proposition III.5(i),(ii)). Now $\mathfrak{a}_\mathfrak{r} = \Delta_s^\perp$ (Lemma IV.5(ii)) shows that $\mathfrak{a}_\mathfrak{r} = \mathfrak{a}^{\mathcal{W}_s}$ is the fixed point set of $\mathcal{W}_s$.

Let $\alpha \in \Delta$, $\beta \in \Delta_s$, and $0 \neq Z_\beta \in \mathfrak{g}^\beta$ be an element of type (SR) or (R) (Proposition IV.7(vi)) and note that this implies that $\mathfrak{g}(Z) \cong \mathfrak{sl}(2, \mathbb{R})$. We consider the $\mathfrak{g}(Z)$-module

$$V := \sum_{n \in \mathbb{Z}} \mathfrak{g}^{\alpha + n\beta}.$$

Then $(\alpha + n\beta)(\check{\beta}) = \alpha(\check{\beta}) + 2n$ shows that the spaces $\mathfrak{g}^{\alpha+n\beta}$ are the $\check{\beta}$-eigenspaces in V. Therefore the representations theory of $\mathfrak{sl}(2, \mathbb{R})$ shows that whenever $\alpha - p\beta$ and $\alpha + q\beta$ are roots, the same holds for all $n \in \{-p, \ldots, q\}$. Moreover, if p and q are maximal, then

$$\alpha(\check{\beta}) - 2p = (\alpha - p\beta)(\check{\beta}) = -(\alpha + q\beta)(\check{\beta}) = -\alpha(\check{\beta}) - 2q$$

([1, Ch. 8, §1, no. 3]). This proves that $\alpha(\check{\beta}) = p - q$, and hence that

$$s_\beta.\alpha = \alpha - \alpha(\check{\beta})\beta = \alpha + (q-p)\beta \in \Delta.$$

Thus Δ is invariant under s_β, and therefore under $\mathcal{W}_s$.

To see that $\mathcal{W}_s$ is finite, we note that $\Delta_s^\perp = \mathfrak{a}_\mathfrak{r}$ implies that $\mathfrak{a}_\mathfrak{s}^* = \operatorname{span} \Delta_s$. Now the fact that $\mathcal{W}_s$ acts effectively on $\mathfrak{a}_\mathfrak{s}$ and the invariance of the finite set Δ_s imply that $\mathcal{W}_s$ is finite. ∎

Corollary V.3. *Let $\alpha \in \Delta$ and $\beta \in \Delta_s$. Then the β-string containing α has the form $\alpha + n\beta$, $-p \leq n \leq q$, where $p, q \geq 0$. Furthermore $p - q = \alpha(\check{\beta})$. Moreover, we have*

(i) $[\mathfrak{g}^\beta, \mathfrak{g}^\alpha] = \mathfrak{g}^{\alpha+\beta}$ *whenever* $\alpha(\check{\beta}) \geq -1$.

(ii) $[\mathfrak{g}^\beta, \mathfrak{g}^\alpha] \neq \{0\}$ *if and only if* $\alpha + \beta \in \Delta$.

(iii) $[\mathfrak{g}^\beta, \mathfrak{g}^\alpha] = \mathfrak{g}^{\alpha+\beta}$ *whenever* $\alpha, \beta \in \Delta_s$ *and* $\alpha \neq -\beta$.

Proof. The first part has been shown in the proof of Proposition V.2(iii).
(i), (ii) This follows from the representation theory of $\mathfrak{sl}(2,\mathbb{R})$ which shows that $[Z, \mathfrak{g}^\alpha] = \mathfrak{g}^{\alpha+\beta}$ whenever $\alpha(\check{\beta}) \geq -1$ and $[Z, \mathfrak{g}^\alpha] \neq \{0\}$ whenever $\alpha(\check{\beta}) \leq -1$ (cf. [1, Ch. 8, §1, no. 3]).
(iii) (cf. [20]) Let $\alpha, \beta \in \Delta_s$ with $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \neq \mathfrak{g}^{\alpha+\beta}$. Then (i) implies that

$$\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} = \alpha(\check{\beta}) \leq -2 \quad \text{and} \quad \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = \beta(\check{\alpha}) \leq -2.$$

Multiplication of both inequalities yields $\langle\alpha,\beta\rangle^2 \geq \langle\alpha,\alpha\rangle\langle\beta,\beta\rangle$, so that equality follows from the Cauchy-Schwarz inequality. We conclude that $\beta = \lambda\alpha$ and obtain $\beta(\check{\alpha}) = 2\lambda \leq -2$ as well as $\alpha(\check{\beta}) = \frac{2}{\lambda} \leq -2$, hence that $\lambda = -1$, i.e., $\alpha = -\beta$. ∎

Proposition V.4. *Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra, $\mathfrak{p} \subseteq \mathfrak{q}$ a maximal hyperbolic Lie triple system and $\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian. Then for a positive system Δ^+ the following are equivalent:*

(1) *The system Δ^+ is $\mathfrak{p}$-adapted.*

(2) *The cone $C_{\max}$ is $\mathcal{W}$-invariant.*

(3) $C_{\max}^0 \cap \mathfrak{z}(\mathfrak{p}) \neq \emptyset$.

(4) $(\Delta_n^+ + \Delta_k) \cap \Delta \subseteq \Delta_n^+$.

(5) *If*

$$\mathfrak{m} = \bigoplus_{\alpha \in \Delta_k \cup \{0\}} \mathfrak{g}^\alpha \quad \textit{and} \quad \mathfrak{p}^+ = \bigoplus_{\alpha \in \Delta_n^+} \mathfrak{g}^\alpha,$$

then $[\mathfrak{m}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+$.

Proof. (1) ⇒ (2): It is clear that the $\mathcal{W}$-invariance of Δ_n^+ implies the $\mathcal{W}$-invariance of $C_{\max}$.
(2) ⇒ (3): If $C_{\max}$ is invariant under $\mathcal{W}$ and $X \in C_{\max}^0$, then $X_0 := \sum_{\gamma \in \mathcal{W}} \gamma.X$ is still contained in $C_{\max}^0$ and, in addition, $\mathcal{W}$-invariant. Therefore each compact root vanishes on X_0 and thus $X \in \mathfrak{z}(\mathfrak{p})$ (Proposition V.2(ii)).

(3) $\Rightarrow$ (4): Let $X_0 \in C^0_{\max} \cap \mathfrak{z}(\mathfrak{p})$. Then $\Delta_n^+ = \{\alpha \in \Delta\colon \alpha(X) > 0\}$ because all compact roots vanish on X. Hence $\alpha \in \Delta_n^+$, $\beta \in \Delta_k$ implies that $(\alpha+\beta)(X) = \alpha(X) > 0$, and therefore that $\alpha + \beta \in \Delta_n^+$ whenever it is a root.
(4) $\Rightarrow$ (5): This is an immediate consequence of $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha+\beta}$ and (4).
(5) $\Rightarrow$ (1): Let $\alpha \in \Delta_k$ and $Z \in \mathfrak{g}^\alpha$ be of type (R). Then $\mathfrak{g}(Z) \subseteq \mathfrak{m}$, and (5) implies that $[\mathfrak{g}(Z), \mathfrak{p}^+] \subseteq \mathfrak{p}^+$. Thus $\mathfrak{p}^+$ is a module of the subalgebra $\mathfrak{g}(Z) \cong \mathfrak{sl}(2,\mathbb{R})$, and the invariance of Δ_n^+, the corresponding set of weights, under the reflection s_α follows from $\mathfrak{sl}(2,\mathbb{R})$-representation theory. We conclude that $\mathcal{W}.\Delta_n^+ = \Delta_n^+$. ∎

Definition V.5. Let $(\mathfrak{g},\tau)$ a symmetric Lie algebra and $\mathfrak{p} \subseteq \mathfrak{q}$ a maximal hyperbolic Lie triple system.
(i) We call $(\mathfrak{g},\tau)$ *quasihermitian*, if $\mathfrak{z}_\mathfrak{q}\big(\mathfrak{z}(\mathfrak{p})\big) = \mathfrak{p}$, where $\mathfrak{z}(\mathfrak{p}) = \{X \in \mathfrak{p}\colon [X,\mathfrak{p}] = \{0\}\}$.
(ii) If $(\mathfrak{g},\tau)$ is semisimple and irreducible, then we call $(\mathfrak{g},\tau)$
(NCR) *non-compactly Riemannian*, if $\mathfrak{g}$ is non-compact and τ is a Cartan involution.
(NCC) *non-compactly causal*, if $(\mathfrak{g},\tau)$ is quasihermitian and $\mathfrak{z}(\mathfrak{p}) \neq \{0\}$.
(CC) *compactly causal*, if $(\mathfrak{g}^c,\tau)$ is (NCC).
(CT) of *Cayley type*, if it is both (CC) and (NCC). ∎

Note that $(\mathfrak{g},\tau)$ is quasihermitian if and only if it is (NCR) or (NCC). The three dimensional simple Lie algebra $(\mathfrak{s},\tau)$ of type (R) is (NCR) and if it is of type (SR), then it is (CT) (cf. Theorem IV.1).

Let $(\mathfrak{g},\tau)$ be a semisimple symmetric Lie algebra and θ a Cartan involution commuting with τ. Denote by $\mathfrak{g} = \mathfrak{g}_\mathfrak{k} \oplus \mathfrak{g}_\mathfrak{p}$ the corresponding Cartan decomposition. The prescription $\tau^a = \theta \circ \tau$ defines an involution and we call $(\mathfrak{g},\tau^a)$ an *associated symmetric Lie algebra.* Note that the eigenspace decomposition of $\mathfrak{g}$ according to τ^a is given by

$$\mathfrak{g} = \mathfrak{h}^a \oplus \mathfrak{q}^a, \qquad \text{where} \qquad \mathfrak{h}^a = (\mathfrak{g}_\mathfrak{k})_\mathfrak{h} \oplus (\mathfrak{g}_\mathfrak{p})_\mathfrak{q}, \quad \mathfrak{q}^a = (\mathfrak{g}_\mathfrak{k})_\mathfrak{q} \oplus (\mathfrak{g}_\mathfrak{p})_\mathfrak{h}.$$

Proposition V.6. *If $(\mathfrak{g},\tau)$ is irreducible semisimple such that $\mathfrak{z}(\mathfrak{p}) \neq \{0\}$, then the following assertions hold:*

(i) *The symmetric Lie algebra $(\mathfrak{g},\tau)$ is one of the following two types:*
(1) *The c-dual $\mathfrak{g}^c$ is simple and hermitian with $\mathfrak{z}(\mathfrak{k}^c) \subseteq \mathfrak{q}^c$.*
(2) *The subalgebra $\mathfrak{h}$ is simple hermitian and $(\mathfrak{g},\tau) \cong (\mathfrak{h}_\mathbb{C},\sigma)$, where σ denotes complex conjugation.*
It follows in particular that $\mathfrak{g}$ is simple.

(ii) *There exists an up to sign unique element $H \in \mathfrak{z}(\mathfrak{p})$ with $\mathfrak{z}(\mathfrak{p}) = \mathbb{R}H$,* $\mathrm{Spec}(\operatorname{ad} H) = \{2,0,-2\}$ *and $\mathfrak{z}_\mathfrak{q}(H) = \mathfrak{p}$. It follows in particular that $(\mathfrak{g},\tau)$ is quasihermitian, i.e.,* (NCC) *and that H defines a triangular decomposition of $\mathfrak{g}$:*

$$\mathfrak{g} = \mathfrak{g}(\operatorname{ad} H; -2) \oplus \mathfrak{g}(\operatorname{ad} H; 0) \oplus \mathfrak{g}(\operatorname{ad} H; 2).$$

(iii) *The involution τ^a is given by $\tau^a = e^{i\frac{\pi}{2}\operatorname{ad} H}$.*

Proof. (cf. [8, Lemma 1.3.5, Th. 1.3.8]) (i), (ii) First we choose a Cartan decomposition $\mathfrak{g} = \mathfrak{g}_\mathfrak{k} \oplus \mathfrak{g}_\mathfrak{p}$ such that $\mathfrak{p} = (\mathfrak{g}_\mathfrak{p})_\mathfrak{q}$ (Proposition III.5(iv)). Then the fact that $\mathfrak{p}_L \subseteq \mathfrak{h}^a$ is an ideal (Lemma II.7(i)) implies that $\mathfrak{z}(\mathfrak{p}) \subseteq \mathfrak{z}(\mathfrak{h}^a)$. On the other hand $\mathfrak{k}^c := (\mathfrak{g}_\mathfrak{k})_\mathfrak{h} \oplus i(\mathfrak{g}_\mathfrak{p})_\mathfrak{q} \subseteq \mathfrak{g}^c$ is a maximal compactly embedded subalgebra.

As already observed in Definition I.6(d), there are two possibilities:
(1) $\mathfrak{g}^c$ is simple. Then the fact that $\mathfrak{z}(\mathfrak{k}^c) \neq \{0\}$ implies that $\mathfrak{g}^c$ is a hermitian Lie algebra with $\mathfrak{z}(\mathfrak{k}^c) \subseteq i\mathfrak{q}$.
(2) $\mathfrak{g}^c \cong \mathfrak{h} \oplus \mathfrak{h}$ and $\mathfrak{h}$ is simple. Then $\mathfrak{k}^c \cong (\mathfrak{g}_\mathfrak{k})_\mathfrak{h} \oplus (\mathfrak{g}_\mathfrak{k})_\mathfrak{h}$, so that the same argument as above shows that $\mathfrak{h}$ is simple hermitian. Moreover $\mathfrak{g} \cong (\mathfrak{h}_\mathbb{C}, \sigma)$ (Lemma I.10), so that $\mathfrak{p} = i(\mathfrak{g}_\mathfrak{k})_\mathfrak{h}$.

If $\mathfrak{g}$ is not simple, then $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{h}$ and $\mathfrak{g}^c \cong \mathfrak{h}_\mathbb{C}$ (cf. Lemma I.11). Therefore the fact that a simple hermitian Lie algebra is never complex implies that $\mathfrak{g}$ is simple.

In both cases we see that $\mathfrak{z}(\mathfrak{p})$ is one-dimensional. Since the spectrum of an element in the center of a maximal compactly embedded subalgebra of a hermitian simple Lie algebra consists of $\{0, ci, -ci\}$ (cf. [3, Prop. VIII.6.2]), we can find $H \in \mathfrak{z}(\mathfrak{p})$ with $\operatorname{Spec}(\operatorname{ad} H) = \{-2, 0, 2, \}$. Moreover, the fact that $\mathfrak{z}_{\mathfrak{g}^c}\big(\mathfrak{z}(\mathfrak{k}^c)\big) = \mathfrak{k}^c$ implies that $\mathfrak{z}_\mathfrak{q}(H) = \mathfrak{p}$. This completes the proof.
(iii) This is an immediate consequence of (ii). ∎

The preceding result can be sharpened significantly if we assume, in addition, that the Lie algebra under consideration is of Cayley type.

Lemma V.7. *For a symmetric* (CT) *Lie algebra* $(\mathfrak{g}, \tau)$ *the following assertions hold:*

(i) *There are elements* $T \in \mathfrak{z}(\mathfrak{h}) \cap \mathfrak{g}_\mathfrak{p}$ *and* $U \in \mathfrak{z}(\mathfrak{g}_\mathfrak{k})_\mathfrak{q}$ *such that*

$$[H, U] = 2T, \qquad [H, T] = 2U, \qquad [U, T] = 2H.$$

(ii) *The element* T *defines a triangular decomposition*

$$\mathfrak{g} = \mathfrak{g}(\operatorname{ad} T; -2) \oplus \mathfrak{g}(\operatorname{ad} T; 0) \oplus \mathfrak{g}(\operatorname{ad} T; 2),$$

and $\tau = e^{i\frac{\pi}{2} \operatorname{ad} T}$.

(iii) *The Cartan involution is given by* $\theta = e^{\frac{\pi}{2} \operatorname{ad} U}$.

Proof. (i) - (ii) [8, Th. 1.3.11].
(iii) [8, Lemma 1.2.1]. ∎

Lemma V.8. *The symmetric Lie algebra* $(\mathfrak{g}, \tau)$ *is quasihermitian if and only if there exists an element* $X \in \mathfrak{z}(\mathfrak{p})$ *such that* $\mathfrak{z}_\mathfrak{q}(X) = \mathfrak{p}$.

Proof. "$\Leftarrow$": This is trivial.
"$\Rightarrow$": Let $X_0 \in \mathfrak{z}(\mathfrak{p})$ such that no root in $\Delta\big(\mathfrak{g}, \mathfrak{z}(\mathfrak{p})\big)$ vanishes on X_0. Then $\mathfrak{z}_\mathfrak{g}(X_0) = \mathfrak{z}_\mathfrak{g}(\mathfrak{z}(\mathfrak{p}))$ and the implication follows from the τ-invariance of both sides. ∎

Proposition V.9. *Let* $(\mathfrak{g}, \tau)$ *be a quasihermitian symmetric Lie algebra,* $\mathfrak{p}$ *a maximal hyperbolic Lie triple system, and* $\mathfrak{a}$ *maximal abelian in* $\mathfrak{p}$. *Then the following assertions hold:*

(i) *The maximal hyperbolic abelian subspace* $\mathfrak{a}$ *is maximal abelian in* $\mathfrak{q}$.

(ii) *The root system* $\Delta(\mathfrak{g}, \mathfrak{a})$ *is split.*

(iii) *The maximal hyperbolic abelian subspace* $\mathfrak{a}$ *is contained in a unique maximal hyperbolic Lie triple system* $\mathfrak{p}$ *which is given by*

$$\mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta_k} (\mathbf{1} - \tau)(\mathfrak{g}^\alpha).$$

(iv) $\operatorname{int} \mathfrak{q}_{\mathrm{hyp}} \cap \mathfrak{z}(\mathfrak{p}) = \mathfrak{z}(\mathfrak{p}) \setminus \bigcup_{\alpha \in \Delta_n} \ker \alpha.$

(v) *If* $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ *is endowed with the inherited involution* $\tau_{\mathfrak{s}}$, *then* $(\mathfrak{s}, \tau_{\mathfrak{s}})$ *is quasihermitian. This means that the irreducible constituents of* $(\mathfrak{s}, \tau)$ *are either* (NCR) *or* (NCC).

(vi) *There exists a* $\mathfrak{p}$*-adapted positive system.*

(vii) *If* Δ^+ *is a* $\mathfrak{p}$*-adapted positive system, then* $(\Delta_p^+ + \Delta_p^+) \cap \Delta \subseteq \Delta_r^+$ *and* $C_{\min,\mathrm{p}} \subseteq (\Delta_p^+)^\star$.

(viii) *The Lie algebra* $\mathfrak{s}$ *has cone potential.*

Proof. (i) Since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and $(\mathfrak{g}, \tau)$ is quasihermitian, any abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}$ must be contained in $\mathfrak{z}_{\mathfrak{q}}\big(\mathfrak{z}(\mathfrak{p})\big) = \mathfrak{p}$, hence has to coincide with $\mathfrak{a}$. This proves that $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$.

(ii) First we note that each root $\alpha \in \Delta_k$ vanishes on $\mathfrak{z}(\mathfrak{p}) \subseteq \mathfrak{a}$. Therefore $\mathfrak{g}^\alpha \subseteq \mathfrak{z}_{\mathfrak{g}}\big(\mathfrak{z}(\mathfrak{p})\big)$ and hence for each $X \in \mathfrak{g}^\alpha$ the element $X - \tau(X)$ is contained in $\mathfrak{z}_{\mathfrak{q}}\big(\mathfrak{z}(\mathfrak{p})\big) = \mathfrak{p}$. Now the $\mathfrak{a}$-invariance of $\mathfrak{p}_L$ shows that the $\mathfrak{g}^\alpha$-component X of this element is contained in $\mathfrak{p}_L$. Thus $\mathfrak{g}^\alpha = (\mathfrak{p}_L)^\alpha$ and this is what we had to show.

(iii) The proof of (ii) implies that for $\alpha \in \Delta_{\mathfrak{k}}$ we have $\mathfrak{g}^\alpha = (\mathfrak{p}_L)^\alpha$, hence that

$$\mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta_k} (\mathbf{1} - \tau)(\mathfrak{g}^\alpha)$$

which shows that $\mathfrak{p}$ is unique.

(iv) In view of (iii), an element $X \in \mathfrak{z}(\mathfrak{p})$ is contained in $\operatorname{int} \mathfrak{q}_{\mathrm{hyp}}$ if and only if $\mathfrak{z}_{\mathfrak{q}}(X)$ is not bigger than $\mathfrak{p}$ (cf. Proposition III.2), i.e., if and only if no non-compact root vanishes on X. This proves (iv).

(v) In view of Proposition III.5(iv), we may assume that $\mathfrak{s}$ is realized in $\mathfrak{g}$ as a τ-$\mathfrak{a}$-invariant Levi complement and that $\mathfrak{p}$ has been constructed as $\mathfrak{p} = \mathfrak{a}_{\mathfrak{r}} \oplus \mathfrak{p}_{\mathfrak{s}}$, where $\mathfrak{p}_{\mathfrak{s}} = (\mathfrak{s}_{\mathfrak{p}})_{\mathfrak{q}}$ with a Cartan decomposition $\mathfrak{s} = \mathfrak{s}_{\mathfrak{k}} \oplus \mathfrak{s}_{\mathfrak{p}}$ which is invariant under $\tau_{\mathfrak{s}}$. Then the fact that $\mathfrak{a}_{\mathfrak{r}}$ commutes with $\mathfrak{s}$ (Proposition III.5(iii)) shows that $\mathfrak{z}(\mathfrak{p}) = \mathfrak{a}_{\mathfrak{r}} \oplus \mathfrak{z}(\mathfrak{p}_s)$. Hence

$$\mathfrak{z}_{\mathfrak{s}_{\mathfrak{q}}}\big(\mathfrak{z}(\mathfrak{p}_{\mathfrak{s}})\big) = \mathfrak{z}_{\mathfrak{s}_{\mathfrak{q}}}\big(\mathfrak{z}(\mathfrak{p})\big) \subseteq \mathfrak{p} \cap \mathfrak{s}_{\mathfrak{q}} = \mathfrak{p}_{\mathfrak{s}}$$

and so $(\mathfrak{s}, \tau_{\mathfrak{s}})$ is quasihermitian.

If we decompose $(\mathfrak{s}, \tau)$ into a direct sum of irreducible symmetric Lie algebras $(\mathfrak{s}_j, \tau_j)$, it is clear that $\mathfrak{p}$ is adapted to this decomposition. Therefore all irreducible factors are quasihermitian, i.e. (NCR) or (NCC), if and only if $(\mathfrak{s}, \tau)$ is quasihermitian.

(vi) Let $X \in \mathfrak{z}(\mathfrak{p})$ with $\mathfrak{p} = \mathfrak{z}_{\mathfrak{q}}(X)$ (Lemma V.8). Then (ii) shows that $\alpha(X) \neq \{0\}$ holds for all $\alpha \in \Delta_n^+$. We choose $X_1 \in \mathfrak{a}$ such that $\operatorname{sgn} \alpha(X_1) = \operatorname{sgn} \alpha(X)$ for all $\alpha \in \Delta_n$ and $\alpha(X_1) \neq 0$ for all $\alpha \in \Delta_k$. Then $\Delta^+ = \{\alpha \in \Delta\colon \alpha(X_1) > 0\}$ is a positive system, and $\Delta_n^+ = \{\alpha \in \Delta\colon \alpha(X) > 0\}$ is $\mathcal{W}$-invariant because X is fixed by $\mathcal{W}$. This proves that Δ^+ is $\mathfrak{p}$-adapted.
(vii) In view of (v), it suffices to prove $(\Delta_p^+ + \Delta_p^+) \cap \Delta = \emptyset$ for (NCC) algebras because this holds trivially for (NCR) algebras. Let H be as in Proposition V.6(ii) and Δ^+ be a $\mathfrak{k}$-adapted positive system. Then Proposition V.4(3) shows that we may w.l.o.g. assume that $H \in C_{\max}^0$. Then $\Delta_p^+ = \{\alpha \in \Delta\colon \alpha(H) = 2\}$, and the assertion follows from the fact that 4 is no eigenvalue of $\operatorname{ad} H$.

To see that $C_{\min,\mathrm{p}} \subseteq (\Delta_p^+)^\star$, let $\alpha, \beta \in \Delta_p^+$. For $Z \in \mathfrak{g}^\beta$ we have $[Z, \tau.Z] \in \kappa(Z, \tau.Z)A_\beta + \mathfrak{a}_\mathfrak{z}$ and hence $\alpha([Z, \tau.Z]) = \kappa(Z, \tau.Z)\alpha(A_\beta)$. Since $\kappa(Z, \tau.Z) \geq 0$ by (ii) and Proposition IV.7(vi), we have to show that $\alpha(\check\beta) \geq 0$. Since $\alpha + \beta$ is no root in $\Delta(\mathfrak{s}, \mathfrak{a}_\mathfrak{s})$, this follows from Corollary V.3.
(viii) We have already seen in (v) that $(\mathfrak{s}, \tau_\mathfrak{s})$ is quasihermitian, hence (ii) implies that $\Delta(\mathfrak{s}, \mathfrak{a}_\mathfrak{s})$ is split. Now the assertion follows from the non-degeneracy of the forms κ_τ on the root spaces $\mathfrak{s}^\alpha$ (cf. Proposition IV.8). ■

Putting all these facts together, we arrive at the following characterization of the quasihermitian symmetric Lie algebras.

Proposition V.10. *The symmetric Lie algebra $(\mathfrak{g}, \tau)$ is quasihermitian if and only if there exists a maximal hyperbolic abelian subspace $\mathfrak{a} \subseteq \mathfrak{q}$ such that*
(1) $\mathfrak{a}$ *is maximal abelian in* $\mathfrak{q}$,
(2) $\Delta(\mathfrak{g}, \mathfrak{a})$ *is split, and*
(3) $\Delta(\mathfrak{g}, \mathfrak{a})$ *contains a* $\mathfrak{p}$*-adapted positive system.*

Proof. The necessity of (1)-(3) follows from Propositon V.9.

Assume that (1)-(3) are satisfied and choose a maximal hyperbolic Lie triple system $\mathfrak{p}$ containing $\mathfrak{a}$. Using Proposition V.4(3), we find $X_0 \in \mathfrak{z}(\mathfrak{p}) \cap C_{\max}^0$. Hence

$$\mathfrak{z}_\mathfrak{q}(X_0) = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta_k} (\mathbf{1} - \tau).\mathfrak{g}^\alpha$$

because $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$. To see that this implies that $\mathfrak{z}_\mathfrak{q}(X_0) = \mathfrak{p}$, it now suffices to show that $\mathfrak{g}^\alpha \subseteq \mathfrak{p}_L$ holds for all $\alpha \in \Delta_k$.

For this we recall from Proposition III.5(iv) that we can realize $\mathfrak{p}$ as $\mathfrak{p} = \mathfrak{a}_\mathfrak{r} \oplus (\mathfrak{s}_\mathfrak{p} \cap \mathfrak{s}_\mathfrak{q})$, where $\mathfrak{a}_\mathfrak{r}$ is maximal hyperbolic abelian in $\mathfrak{r}_\mathfrak{q}$ with respect to $\mathfrak{g}$, $\mathfrak{s}$ is an $\mathfrak{a}_\mathfrak{r}$-$\tau$-invariant Levi complement, and $\mathfrak{s} = \mathfrak{s}_\mathfrak{k} + \mathfrak{s}_\mathfrak{p}$ is a τ-invariant Cartan decomposition defined by the Cartan involution θ.

Let $\alpha \in \Delta_k \subseteq \Delta_s$ and recall that $\mathfrak{s}^\alpha$ is invariant under the involution $\sigma := \tau\theta$ of $\mathfrak{s}$, and that its -1-eigenspace consists of elements of type (SR) (Proposition IV.8). Now the fact that Δ is split shows that $\sigma|_{\mathfrak{s}^\alpha} = \mathrm{id}$, hence that

$$\mathfrak{s}^\alpha \subseteq (\mathbf{1} - \tau).\mathfrak{s}^\alpha + [\mathfrak{a}, (\mathbf{1} - \tau).\mathfrak{s}^\alpha] \subseteq \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}_L.$$

Moreover, in view of Lemma IV.5 and Proposition IV.8(iii), (2) implies that $\mathfrak{g}^\alpha = \mathfrak{s}^\alpha$ and finally that $\mathfrak{g}^\alpha \subseteq \mathfrak{p}_L$. ■

Example V.11. (a) Let $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R})$ with the basis from Example IV.11. We consider the Cartan involution on $\mathfrak{s}$ which corresponds to the decomposition

$$\mathfrak{s}_{\mathfrak{h}} = \mathbb{R}U, \quad \text{and} \quad \mathfrak{s}_{\mathfrak{q}} = \mathbb{R}H + \mathbb{R}T.$$

Let $\mathfrak{r}$ be the three dimensional $\mathfrak{s}$-module with the basis (H', U', T') corresponding to the identification with $\mathfrak{s}$ endowed with the adjoint representation. We form the Lie algebra $\mathfrak{g} := \mathfrak{r} \rtimes \mathfrak{s}$ and put

$$\mathfrak{r}_{\mathfrak{h}} := \operatorname{span}\{H', T'\}, \quad \mathfrak{r}_{\mathfrak{q}} := \mathbb{R}U', \quad \mathfrak{h} := \mathfrak{r}_{\mathfrak{h}} + \mathfrak{s}_{\mathfrak{h}} \quad \text{and} \quad \mathfrak{q} := \mathfrak{r}_{\mathfrak{q}} + \mathfrak{s}_{\mathfrak{q}}.$$

Then, as in Example IV.11, these prescriptions define an involutive automorphism τ of $\mathfrak{g}$.

The subspace $\mathfrak{p} := \mathbb{R}H + \mathbb{R}T \subseteq \mathfrak{q}$ is a maximal hyperbolic Lie triple system, $\mathfrak{a} := \mathbb{R}H \subseteq \mathfrak{p}$ is maximal abelian, and $\mathfrak{z}(\mathfrak{p}) = \{0\}$. Therefore $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{z}(\mathfrak{p})) = \mathfrak{q} \neq \mathfrak{p}$ and we see that $(\mathfrak{g}, \tau)$ is not quasihermitian. Nevertheless, the subspace $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$, $\mathcal{W} \cong \mathbb{Z}^2$, $\Delta = \Delta_k$, and $\Delta_n = \emptyset$, so that each positive system is $\mathfrak{p}$-adapted. Of course Δ is not split because the root spaces $\mathfrak{g}^\alpha = \mathfrak{r}^\alpha \oplus \mathfrak{s}^\alpha$ are 2-dimensional, the form κ_τ is negative semidefinite with one-dimensional degeneracy in $\mathfrak{r}^\alpha$.

(b) Let $\mathfrak{s}$ be a compact Lie algebra and V a real $\mathfrak{s}$-module. Let further $\mathfrak{n} = V \times V \times \mathbb{R}$ denote the Heisenberg algebra with bracket

$$[(v, w, t), (v', w', t')] := (0, 0, \langle v, w'\rangle - \langle v', w\rangle).$$

Let further H denote the operator on $\mathfrak{n}$ given by $H.(v, w, t) = (v, -w, 0)$ and consider the Lie algebra $\mathfrak{g} := \mathfrak{n} \rtimes (\mathfrak{s} \oplus \mathbb{R}H)$. Then

$$\mathfrak{h} = \{(v, v, 0) \colon v \in V\} \rtimes \mathfrak{s} \quad \text{and} \quad \mathfrak{q} = \{(v, -v, t) \colon v \in V, t \in \mathbb{R}\} \rtimes \mathbb{R}H$$

defines an involution τ on $\mathfrak{g}$. The subspace $\mathfrak{a} := \mathfrak{z} \oplus \mathbb{R}H$ with $\mathfrak{z} = \{0\} \times \{0\} \times \mathbb{R}$ is maximal hyperbolic and maximal abelian in $\mathfrak{q}$. Moreover, $\mathfrak{p} = \mathfrak{a}$ is a maximal hyperbolic Lie triple system, and $\Delta = \{\pm\alpha\}$ with $\alpha(H) = 1$, where $\mathfrak{g}^\alpha = V \times \{0\}$. Therefore $(\mathfrak{g}, \tau)$ is quasihermitian, effective with strong cone potential. Nevertheless $\mathfrak{h}^0 = \mathfrak{s}$ is non-trivial.

(c) Let $\mathfrak{g}_0$ be a split semisimple real Lie algebra and $\mathfrak{a}_0 \subseteq \mathfrak{g}_0$ a toral Cartan subalgebra. We consider $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0$ with $\tau(X, Y) = (Y, X)$. Then $\mathfrak{q} = \{(X, -X) \colon X \in \mathfrak{g}_0\}$ and if $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is a Cartan decomposition, then $\mathfrak{p} := \{(X, -X) \colon X \in \mathfrak{p}_0\}$ is a maximal hyperbolic Lie triple system in $\mathfrak{q}$. According to Example IV.3(a), all root spaces are of mixed type, hence $\Delta = \Delta_k$. We conclude in particular that all positive systems are $\mathfrak{p}$-adapted, but that Δ is not split. ■

VI. Convexity properties and invariant convex sets

In this section we come to the subject proper of this paper, the convexity properties of the action of $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$ on the space $\mathfrak{q}$. The key role will be played by those invariant convex subsets having a sufficiently large intersection with

$\mathfrak{a}$. The main point of this section is that the existence of such invariant convex sets has significant consequences for the structure of the symmetric Lie algebra $(\mathfrak{g},\tau)$. In particular we will see that it implies that $(\mathfrak{g},\tau)$ is quasihermitian, and also that it has strong cone potential (cf. Definition V.1(g)).

Throughout this section we assume that $\mathfrak{a}\subseteq\mathfrak{q}$ is a maximal hyperbolic abelian subspace which is in addition maximal abelian in $\mathfrak{q}$.

A key tool in everything that follows is the following lemma which gives precise information on the projections of orbits of elements in $\mathfrak{a}$ with respect to the action of one-parameter subgroups of $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ coming from root vectors.

Lemma VI.1. *For $\alpha\in\Delta$, $X\in\mathfrak{a}$, and $Z\in\mathfrak{g}^\alpha$ we have the following formulas:*

(i)
$$\mathrm{ad}\big(Z+\tau(Z)\big)^m(X) = \begin{cases} X & \text{for } m=0\\ -\alpha(X)\alpha(2[Z,\tau(Z)])^n\big(Z-\tau(Z)\big) & \text{for } m=2n+1\\ -\alpha(X)2\alpha\big(2[Z,\tau(Z)]\big)^n[\tau(Z),Z] & \text{for } m=2(n+1).\end{cases}$$

(ii) *If $p\colon\mathfrak{q}\to\mathfrak{a}$ is the projection along $\mathfrak{q}\cap[\mathfrak{a},\mathfrak{g}]$, then*

$$\begin{aligned} &p\big(e^{\mathrm{ad}(Z+\tau(Z))}.X\big)\\ &=\cosh\big(\mathrm{ad}(Z+\tau(Z))\big).X\\ &=X-\begin{cases}\alpha(X)[\tau(Z),Z] & \text{for } \alpha([Z,\tau(Z)])=0\\ \alpha(X)\frac{\cosh(\sqrt{2\alpha([Z,\tau(Z)])})-1}{\alpha([Z,\tau(Z)])}[\tau(Z),Z] & \text{for } \alpha([Z,\tau(Z)])>0\\ \alpha(X)\frac{\cos(\sqrt{2\alpha([\tau(Z),Z])})-1}{\alpha([Z,\tau(Z)])}[\tau(Z),Z] & \text{for } \alpha([Z,\tau(Z)])<0\end{cases}\end{aligned}$$

and
$$\begin{aligned} &p(e^{\mathbb{R}\,\mathrm{ad}(Z+\tau(Z))}.X)\\ &=p(e^{\mathbb{R}^+\,\mathrm{ad}(Z+\tau(Z))}.X)\\ &=X+\begin{cases}\mathbb{R}^+\alpha(X)[Z,\tau(Z)] & \text{for } \alpha([Z,\tau(Z)])=0\\ \mathbb{R}^+\alpha(X)[Z,\tau(Z)] & \text{for } \alpha([Z,\tau(Z)])>0\\ [0,2]\frac{\alpha(X)}{\alpha([\tau(Z),Z])}[Z,\tau(Z)] & \text{for } \alpha([Z,\tau(Z)])<0.\end{cases}\end{aligned}$$

Proof. (i) We prove the assertion by induction with respect to m. For $m=0$ there is nothing to prove. Suppose that the assertion is true for $m=2n$. If $n=0$, then

$$\mathrm{ad}(Z+\tau(Z))^{m+1}(X)=[Z+\tau(Z),X]=-\alpha(X)(Z-\tau(Z)).$$

If $m=2n>0$, then

$$\begin{aligned}\mathrm{ad}(Z+\tau(Z))^{m+1} &= \mathrm{ad}(Z+\tau(Z))\big(-\alpha(X)2\alpha(2[Z,\tau(Z)])^{n-1}[\tau(Z),Z]\big)\\ &=-\alpha(X)2\alpha(2[Z,\tau(Z)])^{n-1}\big[Z+\tau(Z),[\tau(Z),Z]\big]\\ &=-\alpha(X)\alpha(2[Z,\tau(Z)])^n(Z-\tau(Z)),\end{aligned}$$

and if $m = 2n+1$, then

$$\begin{aligned}\operatorname{ad}(Z+\tau(Z))^{m+1}(X) &= \operatorname{ad}(Z+\tau(Z))\big(-\alpha(X)\alpha(2[Z,\tau(Z)])^n(Z-\tau(Z))\big)\\ &= -\alpha(X)\alpha(2[Z,\tau(Z)])^n 2[\tau(Z),Z].\end{aligned}$$

(ii) Using (i), we see that $\operatorname{ad}(Z+\tau(Z))^{2n+1}.X \in [\mathfrak{a},\mathfrak{g}]\cap\mathfrak{q}$ and $\operatorname{ad}(Z+\tau(Z))^{2n}.X \in \mathfrak{a}$. Thus

$$p(e^{\operatorname{ad}(Z+\tau(Z))}.X) = \cosh\big(\operatorname{ad}(Z+\tau(Z))\big).X.$$

Now the first formula for $\cosh\big(\operatorname{ad}(Z+\tau(Z))\big).X$ follows from (i). For the remaining assertions we distinguish several cases. The case $\alpha([Z,\tau(Z)]) = 0$ is trivial. If $\alpha([Z,\tau(Z)]) > 0$, then we use the surjectivity of the function $\mathbb{R}^+ \to \mathbb{R}^+$, $t \mapsto \cosh(ts)-1$ for $s>0$, and for $\alpha([Z,\tau(Z)]) < 0$ we use $\cos(\mathbb{R})-1 = [-2,0]$. ■

Definition VI.2. Let V be a finite dimensional real vector space and V^* its dual space.
(a) For a subset $C \subseteq V$ we define

$$B(C) := \{\alpha\in V^* : \inf\alpha(C) > -\infty\} \quad \text{and } C^\star := \{\alpha\in V^*: \inf\alpha(C)\geq 0\}.$$

Note that both are convex cones and that $C^\star$ is always closed, whereas $B(C)$ need not be closed. One obtains an instructive example by taking the graph of the exponential function in $V=\mathbb{R}^2$.
(b) For a convex subset $C\subseteq V$ we put

$$\lim C := \{v\in V: v + C\subseteq C\} \quad \text{and} \quad H(C) := \lim(C)\cap -\lim(C).$$

We call $\lim C$ the *limit cone* of C. Note that $\lim C$ is always a convex cone which is closed if C is closed or open (cf. [19, Prop. III.1.5]). If C is open or closed, then the geometric meaning of $H(C)$ is that $c + H(C)$, $c\in C$ are the maximal affine subspaces contained in C. ■

Proposition VI.3. *For $Y\in\mathfrak{a}$ the following assertions hold:*

(i) *If $X\in\mathfrak{g}^\alpha$ is of type* (SR), (A) *or* (N)*, then*

$$Y + \mathbb{R}^+\alpha(Y)[X,\tau(X)] \subseteq \operatorname{conv}(\mathcal{O}_Y).$$

(ii) *If $X\in\mathfrak{g}^\alpha$ is of type* (SR) *and $\mathfrak{g}^\alpha$ also contains elements of type* (R)*, then*

$$Y + \mathbb{R}\alpha(Y)\check\alpha \subseteq \operatorname{conv}(\mathcal{O}_Y).$$

(iii) $C_Y := \operatorname{cone}\big(\{\alpha(Y)[X,\tau(X)]: X \text{ of type (SR), (N)}\}\big) \subseteq \lim\overline{\operatorname{conv}(\mathcal{O}_Y)}$.

Proof. (i) In view of Lemma VI.1(ii), this follows from the observation that for $Z\in\mathfrak{h}$ we have

$$p(e^{\operatorname{ad} Z}.Y) = \cosh(\operatorname{ad} Z).Y = \frac{1}{2}\big(e^{\operatorname{ad} Z}.Y + e^{-\operatorname{ad} Z}.Y\big) \in \operatorname{conv}(\mathcal{O}_Y).$$

(ii) First we note that if $\mathfrak{g}^\alpha$ contains an element of type (R), then the same holds for $\mathfrak{s}^\alpha$ (cf. Proposition IV.7(v)). Hence the reflection in $\ker \alpha$ defined by

$$s_\alpha(X) := X - \alpha(X)\check{\alpha}$$

is an element of the Weyl group $\mathcal{W}$. Now (i) and Proposition IV.7(vi) imply that $Y + \mathbb{R}^+\alpha(Y)\check{\alpha} \subseteq \mathrm{conv}(\mathcal{O}_Y)$ and therefore the invariance of $\mathcal{O}_Y$ under $\mathcal{W}$ yields

$$s_\alpha\big(Y + \mathbb{R}^+\alpha(Y)\check{\alpha}\big) = s_\alpha(Y) - \mathbb{R}^+\alpha(Y)\check{\alpha} \subseteq \mathrm{conv}(\mathcal{O}_Y).$$

If $\alpha(Y) \neq 0$, then the convex hull of these two half-lines coincides with the line $Y + \mathbb{R}\check{\alpha}$, and this proves (ii).
(iii) This is immediate from (i). ■

Next we introduce some notions that will be used to describe the effect of the existence of certain invariant convex sets in $\mathfrak{q}$ on the structure of the Lie algebra $\mathfrak{g}$.

Definition VI.4. Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra.
(a) We call a convex $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant set $C \subseteq \mathfrak{q}$ *hyperbolic* (resp. *elliptic*), if it has non-empty interior and $\mathrm{int}\, C$ consists of hyperbolic (resp. elliptic) elements.
(b) A symmetric Lie algebra $(\mathfrak{g}, \tau)$ is called *admissible* if there exists a closed convex generating hyperbolic invariant subset $C \subseteq \mathfrak{q}$ with $H(C) = \{0\}$. ■

In the following D^0 for a subset D of $\mathfrak{a}$ means the relative interior of D with respect to $\mathfrak{a}$.

Lemma VI.5. (i) *If $C \subseteq \mathfrak{q}$ is an invariant hyperbolic convex subset, then $C^0 = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).(C \cap \mathfrak{a})^0$.*
(ii) *If $C \subseteq \mathfrak{q}$ is an invariant subset such that $C \cap \mathfrak{a}$ has interior points, then C has interior points.*

Proof. (i) Let $C_{\mathfrak{a}} := C \cap \mathfrak{a}$. Then $C^0 \cap \mathfrak{a} \subseteq C_{\mathfrak{a}}^0$, so that, in view of $C^0 \subseteq \mathfrak{q}_{\mathrm{hyp}}$, Theorem III.3(ii) implies that $C^0 = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).(C^0 \cap \mathfrak{a}) \subseteq \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).C_{\mathfrak{a}}^0$. It remains to show that $C_{\mathfrak{a}}^0 \subseteq C^0 \cap \mathfrak{a}$.

So let $X \in C_{\mathfrak{a}}^0$ and $U \subseteq C_{\mathfrak{a}}$ be an open convex subset. If $Y \in U$ satisfies $\mathfrak{z}_{\mathfrak{q}}(Y) = \mathfrak{a}$, i.e., if $\alpha(Y) \neq 0$ holds for all $\alpha \in \Delta$, then Lemma III.1 shows that the mapping

$$\Psi\colon \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) \times \mathfrak{a} \to \mathfrak{q}, \quad (h, Z) \mapsto h.Z$$

has surjective differential in $(\mathbf{1}, Y)$, hence that $\Psi(\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) \times U)$ is a neighborhood of Y in $\mathfrak{q}$. Thus $Y \in C^0$. Since

$$U = \mathrm{conv}(\{Y \in U \colon (\forall \alpha \in \Delta)\alpha(Y) \neq 0\}),$$

it follows that $X \in C^0$.
(ii) If $U \subseteq C \cap \mathfrak{a}$ is an open subset of $\mathfrak{a}$, then the argument above shows that $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).U \subseteq C$ contains an interior point. ■

The following theorem contains some key observations relating invariant hyperbolic sets to the structure of the Lie algebra.

Theorem VI.6. *Let $(\mathfrak{g},\tau)$ be a symmetric Lie algebra, $\mathfrak{p}\subseteq\mathfrak{q}$ a maximal hyperbolic Lie triple system, and $\mathfrak{a}\subseteq\mathfrak{p}$ maximal abelian. Suppose that there exists an invariant hyperbolic closed convex subset $C\subseteq\mathfrak{q}$. Then the following assertions hold:*

(i) *The hyperbolic subspace $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$.*

(ii) *The symmetric Lie algebra $(\mathfrak{g},\tau)$ is quasihermitian and there exists a $\mathfrak{p}$-adapted positive system Δ^+ such that*

$$C_{\min}\subseteq\lim(C\cap\mathfrak{a})\quad \text{and}\quad C\cap\mathfrak{a}\subseteq C_{\max}.$$

In particular we have $C_{\min}\subseteq C_{\max}$.

(iii) *If $H(C)=\{0\}$, then $(\mathfrak{g},\tau)$ has strong cone potential.*

(iv) *If, in addition, $C\neq\mathfrak{q}$, then $\mathfrak{z}(\mathfrak{p})\neq\{0\}$.*

Proof. (i) The existence of C implies that $\mathfrak{q}_{\rm hyp}$ has non-empty interior. Hence (i) follows from Theorem III.3(ii).

(ii) Let $C_{\mathfrak{a}}:=C\cap\mathfrak{a}$ and $X\in C^0\cap\mathfrak{a}$. Then $X_0:=\sum_{\gamma\in\mathcal{W}}\gamma.X\in C^0_{\mathfrak{a}}$ is fixed under $\mathcal{W}$. According to Proposition V.2(ii), we have $X_0\in\mathfrak{z}(\mathfrak{p})$. In view of Lemma VI.5, $X_0\in C^0\subseteq\operatorname{int}\mathfrak{q}_{\rm hyp}$, so that Proposition III.2 implies that $\mathfrak{z}_{\mathfrak{q}}(X_0)$ is a hyperbolic Lie triple system. Clearly we have $\mathfrak{p}\subseteq\mathfrak{z}_{\mathfrak{q}}(X_0)$, so that the maximality of $\mathfrak{p}$ implies that $\mathfrak{p}=\mathfrak{z}_{\mathfrak{q}}(X_0)$, hence that $(\mathfrak{g},\tau)$ is quasihermitian.

We define a positive system of non-compact roots by

$$\Delta_n^+:=\{\alpha\in\Delta_n:\alpha(X_0)>0\}.$$

Since no non-compact root vanishes on an element of $C^0_{\mathfrak{a}}$ (Lemma VI.5, Proposition V.9(iv)), we get $C^0_{\mathfrak{a}}\subseteq C_{\max}$ and hence that $C_{\mathfrak{a}}\subseteq C_{\max}$. Finally Proposition VI.3(iii) implies that $C_{\min}\subseteq\lim C_{\mathfrak{a}}$.

(iii) First we show that $(\mathfrak{g},\tau)$ has cone potential. Let $Z\in\mathfrak{g}^\alpha$ be of type (A). We have to show that $Z=0$. Let $X\in C^0_{\mathfrak{a}}$. Now Lemma VI.1(i) shows that

$$e^{t\,\mathrm{ad}\,(Z+\tau(Z))}.X=X-t\alpha(X)\big(Z-\tau(Z)\big)\in C$$

for all $t\in\mathbb{R}$. Now $H(C)=\{0\}$ implies that $Z-\tau(Z)=0$, hence that $Z=0$, and this proves that $(\mathfrak{g},\tau)$ has cone potential.

Since $\lim(C_{\mathfrak{a}})$ is pointed and contains the cone $C_{\min}$, this cone is also pointed. Thus $(\mathfrak{g},\tau)$ has strong cone potential because each functional $\alpha\in\operatorname{int}\lim(C_{\mathfrak{a}})^\star$ satisfies $\alpha([X,\tau.X])>0$ for all $X\in\mathfrak{g}^\alpha$, $\alpha\in\Delta^+$. The $\mathfrak{p}$-adaptedness of Δ^+ follows from the choice of Δ_n^+ which ensures that it is invariant under the Weyl group.

(iv) If $\mathfrak{z}(\mathfrak{p})=\{0\}$, then the proof of (ii) shows that $0\in C^0$. If C is a cone, then this implies that $C=\mathfrak{q}$. This proves (iv). ∎

Example VI.7. (a) Let $(\mathfrak{s},\tau)=(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}),\tau_1\oplus\tau_2)$, where $\tau_1=\theta$ is of type (R), and τ_2 of type (SR) is given by conjugation by T (cf. Example IV.11). Then $(\mathfrak{s},\tau)$ is quasihermitian and admits $e^{\mathrm{ad}\,\mathfrak{h}}$-invariant pointed generating convex hyperbolic cones. An example is given by

$$C=\{(tT+hH,h'H'+u'U'):0\leq|u'|\leq h',0\leq t^2+h^2\leq(h')^2-(u')^2\}.$$

Note that $\dim\mathfrak{q}=4$ and $\dim\mathfrak{p}=3$. The set C is invariant since the Lorentzian form given in coordinates by $(h)'^2-(u')^2-t^2-u^2$ is invariant under $\mathrm{Inn}_{\mathfrak{s}}(\mathfrak{h})$. Moreover, C is a Lorentzian cone in $\mathfrak{q}$. ∎

The preceding example shows that it is not necassary that the irreducible pieces of $(\mathfrak{s}, \tau)$ are purely (NCC), which in the irreducible case can be shown to be equivalent to the existence of a hyperbolic invariant convex cone (cf. [8]). Below we will give a necessary and sufficient condition for symmetric Lie algebras to have this property.

VII. Symmetric Lie algebras with cone potential

In this chapter we exploit the implications of (strong) cone potential for the structure of a symmetric Lie algebra. We recall from Proposition V.9 that a quasihermitian semisimple symmetric Lie algebra always has cone potential. The following example shows that the converse is not true.

Example VII.1. Let $\mathfrak{g} = \mathfrak{so}(2,n)$, $n \geq 3$ and τ given by conjugation with

$$I_{n,2} = \mathrm{diag}(\underbrace{1,\ldots,1}_{n}, -1, -1).$$

Then $\mathfrak{h} \cong \mathfrak{so}(2, n-2) \oplus \mathfrak{so}(2)$, $\mathfrak{a} = \mathbb{R}X_1 + \mathbb{R}X_2$, where $X_1 = E_{1,n} + E_{n,1}$ and $X_2 = E_{2,n-1} + E_{n-1,2}$, is a hyperbolic subalgebra which is maximal abelian in $\mathfrak{q}$. The root system is given by

$$\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 \pm \alpha_2)\},$$

where $\alpha_i(X_j) = \delta_{ij}$, $i,j \in \{1,2\}$. It is easy to check that $(\mathfrak{g}, \tau)$ has cone potential. In particular $\Delta_k = \{\pm(\alpha_1 \pm \alpha_2)\}$ and $\Delta_p = \{\pm\alpha_1, \pm\alpha_2\}$. The Weyl group $\mathcal{W}$ is isomorphic to the Weyl group of the Riemannian symmetric Lie algebra $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ and is generated by the two orthogonal reflections $s_{\alpha_1+\alpha_2}, s_{\alpha_1-\alpha_2}$. Hence $(\mathfrak{g}, \tau)$ does not have any $\mathfrak{p}$-adapted positive system. This gives an example of a simple symmetric Lie algebra with cone potential which is not quasihermitian. ∎

For any $\mathfrak{a}$-invariant subspace $\mathfrak{b} \subseteq \mathfrak{g}$ we write $\mathfrak{b}^+ := [\mathfrak{a}, \mathfrak{b}]$ for the effective part of $\mathfrak{b}$ and $\mathfrak{b}^0$ for $\mathfrak{z}_{\mathfrak{b}}(\mathfrak{a})$. Note that $\mathfrak{b} = \mathfrak{b}^0 \oplus \mathfrak{b}^+$.

Proposition VII.2. *If $(\mathfrak{g}, \tau)$ has cone potential and $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$, then the following assertions hold:*

(i) *Every τ-invariant abelian ideal $\mathfrak{b} \subseteq \mathfrak{g}$ is contained in $\mathfrak{n}^0$. Moreover, $\mathfrak{b}_{\mathfrak{q}} \subseteq \mathfrak{z}(\mathfrak{g})$ and $\mathfrak{b}_{\mathfrak{h}} \trianglelefteq \mathfrak{g}$.*

(ii) $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{n}^0$. *In particular $[\mathfrak{r}^\alpha, \mathfrak{r}^\beta] = \{0\}$ for $\alpha \neq -\beta$.*

(iii) *If $(\mathfrak{g}, \tau)$ is effective, then*

(a) $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}(\mathfrak{g})$,

(b) $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{q}$, *and*

(c) $[\mathfrak{n}_{\mathfrak{h}}, \mathfrak{n}_{\mathfrak{h}}] = \{0\}$.

(iv) *The subspace $[\mathfrak{a}, \mathfrak{h}]$ of $\mathfrak{q}$ contains no non-zero $\mathfrak{h}$-submodule.*

Proof. (i) (cf. [5, Lemma 7.14]) Since $\mathfrak{b}$ is an ideal, it is $\mathfrak{a}$-invariant. We claim that $[\mathfrak{a}, \mathfrak{b}] = \{0\}$. Suppose that this is false and let $0 \neq Z \in \mathfrak{g}^\alpha \cap \mathfrak{b}$. Since

$\mathfrak{b}$ was supposed to be τ-invariant, we obtain $\mathfrak{g}(Z) \subseteq \mathfrak{b}$. Since $(\mathfrak{g}, \tau)$ has cone potential, $\mathfrak{g}(Z)$ is not abelian, contradicting the assumption that $\mathfrak{b}$ is abelian. Now $\mathfrak{b} = \mathfrak{b}^0$ yields $\mathfrak{b}_{\mathfrak{q}} \subseteq \mathfrak{q}^0 \cap \mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{z}(\mathfrak{g})$ because $\mathfrak{b}$ is a nilpotent ideal. Since $\mathfrak{n} + \mathfrak{b}$ is a nilpotent ideal, we have $\mathfrak{b} \subseteq \mathfrak{n}$.

To see that $\mathfrak{b}_{\mathfrak{h}}$ is an ideal, we first note that

$$[\mathfrak{b}, \mathfrak{g}^\alpha] \subseteq [\mathfrak{g}^0, \mathfrak{g}^\alpha] \cap \mathfrak{b} \subseteq \mathfrak{g}^\alpha \cap \mathfrak{b} = \{0\}$$

implies $\big[\mathfrak{b}, [\mathfrak{a}, \mathfrak{g}]\big] = \{0\}$. Now the assertion follows from the fact that $\mathfrak{b}_{\mathfrak{h}}$ is an ideal in $\mathfrak{h}$ and from $\mathfrak{q} = \mathfrak{a} \oplus [\mathfrak{a}, \mathfrak{h}]$.

(ii) (cf. [5, Th. 7.15]) We prove the assertion by induction on $\dim(\mathfrak{g})$. We denote by $\mathfrak{n}^m$ the elements in the lower central series of $\mathfrak{n}$. Suppose that $\mathfrak{n}^m \neq \{0\}$ for some integer $m \geq 2$. According to [5, Lemma 7.13], the ideal $\mathfrak{b} = \mathfrak{n}^{m-1}$ is abelian. Note that $\mathfrak{b}$ is τ-invariant, so that (i) applies. If $\mathfrak{b}_{\mathfrak{h}} \neq \{0\}$, we consider the symmetric Lie algebra $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{b}_{\mathfrak{h}}$. Let $\pi \colon \mathfrak{g} \to \mathfrak{g}_1$ denote the quotient homomorphism. In view of Lemma III.11, $\pi(\mathfrak{a})$ is a maximal hyperbolic abelian subspace of $\mathfrak{q}_1 := \pi(\mathfrak{q})$. Moreover, $\mathfrak{g}_1$ has cone potential because $\mathfrak{b}_{\mathfrak{h}}$ does not intersect $\mathfrak{a}$ and $\mathfrak{g}$ has cone potential. Hence induction applies and yields the assertion if $\mathfrak{b}_{\mathfrak{h}} \neq \{0\}$.

Thus, in view of (i), we may assume that $\mathfrak{b} = \mathfrak{b}_{\mathfrak{q}}$ is central. But this contradicts $\mathfrak{n}^m \neq \{0\}$. Hence $\mathfrak{n}^1 = [\mathfrak{n}, \mathfrak{n}]$ is abelian. Again (i) applies and shows that $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}_{\mathfrak{n}}(\mathfrak{a})$.

(iii) (a) We can use the same arguments as in the proof of (ii). Here $\mathfrak{b}_{\mathfrak{h}} = \{0\}$ since $(\mathfrak{g}, \tau)$ is effective and $\mathfrak{b}_{\mathfrak{h}}$ is an ideal of $\mathfrak{g}$ (cf. Lemma I.8). Hence the argument above shows that $[\mathfrak{n}, \mathfrak{n}]$ is an abelian ideal. Thus effectivity and (i) give $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}(\mathfrak{g})$.

(b) That $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{q}$ follows from the fact that $\mathfrak{z}(\mathfrak{g})_{\mathfrak{h}}$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$.

(c) In view of (a) and (b), we have $[\mathfrak{n}_{\mathfrak{h}}, \mathfrak{n}_{\mathfrak{h}}] \subseteq [\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{h} \subseteq \mathfrak{z}(\mathfrak{g})_{\mathfrak{h}} = \{0\}$.

(iv) Let $V \subseteq [\mathfrak{a}, \mathfrak{h}]$ be an $\mathfrak{h}$-submodule. Then its orthogonal subspace $V^{\perp_\kappa} \subseteq \mathfrak{q}$ with respect to the Cartan-Killing form is an $\mathfrak{h}$-submodule containing $\mathfrak{a}$, hence contains $\mathfrak{q}_{\mathrm{hyp}} = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).\mathfrak{a}$, and therefore coincides with $\mathfrak{q}$. We conclude that $V \subseteq \mathfrak{q}^{\perp_\kappa}$ and hence that $V \subseteq \mathfrak{g}^{\perp_\kappa} \subseteq \mathfrak{r}$.

If, on the other hand, $\mathfrak{s}$ is an $\mathfrak{a}$-τ-invariant Levi complement, then

$$[\mathfrak{a}, \mathfrak{h}] = [\mathfrak{a}, \mathfrak{s}_{\mathfrak{h}} + \mathfrak{r}_{\mathfrak{h}}] = \underbrace{[\mathfrak{a}, \mathfrak{s}_{\mathfrak{h}}]}_{\subseteq \mathfrak{s}_{\mathfrak{q}}} + \underbrace{[\mathfrak{a}, \mathfrak{r}_{\mathfrak{h}}]}_{\subseteq \mathfrak{r}_{\mathfrak{q}}}.$$

Therefore $V \subseteq [\mathfrak{a}, \mathfrak{r}_{\mathfrak{h}}] = \sum_{\alpha \in \Delta} (\mathbf{1} - \tau).\mathfrak{r}^\alpha$. Let $\beta \in \Delta$ and $X_\beta \in \mathfrak{r}^\beta$. Then (ii) implies that

$$[(\mathbf{1} + \tau).X_\beta, \sum_{\alpha \in \Delta} (\mathbf{1} - \tau).\mathfrak{r}^\alpha] = [(\mathbf{1} + \tau).X_\beta, (\mathbf{1} - \tau).\mathfrak{r}^\beta].$$

Moreover

$$[(\mathbf{1} + \tau).X_\beta, (\mathbf{1} - \tau).X_\beta] = -[X_\beta, \tau.X_\beta] + [\tau.X_\beta, X_\beta] = 2[\tau.X_\beta, X_\beta] \in \mathfrak{a}_{\mathfrak{r}} \setminus \{0\}$$

whenever $X_\beta \neq \{0\}$. For $v \in V$ with $v = \sum_{\alpha\in\Delta} v_\alpha$, $v_\alpha = (\mathbf{1}-\tau).X_\alpha \in (\mathbf{1}-\tau).\mathfrak{r}^\alpha$ we now obtain

$$[(\mathbf{1}+\tau).X_\alpha, v] = 2[\tau.X_\alpha, X_\alpha] \in V \cap \mathfrak{a}_\mathfrak{r}.$$

This proves that $v_\alpha = 0$ for all α, hence $v = 0$ for all $v \in V$, and eventually $V = \{0\}$. ■

Example VII.3. The assertion of (ii) becomes false if $(\mathfrak{g},\tau)$ does not have cone potential. We consider the Lie algebra

$$\mathfrak{g} = \operatorname{span}\{T, X_1, Y_1, X_2, Y_2, X_3, Y_3\}$$

with the non-zero brackets

$$[T, X_k] = kX_k,\ [T, Y_k] = -kY_k,\ [X_1, X_2] = X_3,\ [Y_1, Y_2] = Y_3,$$

$1 \leq k \leq 3$. We define

$$\mathfrak{h} = \mathbb{R}(X_1 + Y_1) \oplus \mathbb{R}(X_2 + Y_2) \oplus \mathbb{R}(X_3 + Y_3),$$

$$\mathfrak{q} = \mathbb{R}T \oplus \mathbb{R}(X_1 - Y_1) \oplus \mathbb{R}(X_2 - Y_2) \oplus \mathbb{R}(X_3 - Y_3).$$

Since $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{h},\mathfrak{q}] \subseteq \mathfrak{q}$ and $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$, the prescriptions $\tau\mid_\mathfrak{h} = \mathrm{id}_\mathfrak{h}$ and $\tau\mid_\mathfrak{q} = -\mathrm{id}_\mathfrak{q}$ define an involution on $\mathfrak{g}$. Further on $\mathfrak{a} = \mathbb{R}T$ is a maximal hyperbolic abelian subspace in $\mathfrak{q}$, which is maximal abelian in $\mathfrak{q}$. An easy calculation yields $[\mathfrak{n},\mathfrak{n}] = \mathbb{R}X_3 \oplus \mathbb{R}Y_3$, hence $[\mathfrak{n},\mathfrak{n}] \not\subseteq \mathfrak{n}^0$. ■

Definition VII.4. Let $(\mathfrak{g},\tau)$ be a symmetric Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a representation. Then we call an operator $A \in \operatorname{End}(V)$ a τ-*intertwiner* if

$$A \circ \rho(X) = \rho(\tau.X) \circ A$$

holds for all $X \in \mathfrak{g}$. ■

The significance of the notion of a τ-intertwiner becomes apparent in the following lemma.

Lemma VII.5. *Let V be a module of the symmetric Lie algebra $(\mathfrak{g},\tau)$ and ϕ a τ-covariant bilinear form on V. Further let σ be a second involution on $\mathfrak{g}$ commuting with τ and $A \in \operatorname{End}(V)$ a σ-intertwiner. Then the following assertions hold:*

(i) *The form ϕ_A defined by $\phi_A(v,w) := \phi(A.v,w)$ is $\tau\sigma$-covariant.*

(ii) *If ϕ is symmetric (skew-symmetric), and A is ϕ-symmetric, then the same holds for ϕ_A, and if A is ϕ-skew-symmetric, then ϕ_A is skew-symmetric (symmetric).*

Proof. (i) For $v, w \in V$ and $X \in \mathfrak{g}$ we have

$$\begin{aligned}\phi_A(X.v,w) &= \phi(AX.v,w) = -\phi\big((\sigma.X)A.v,w\big)\\ &= -\phi\big(A.v,(\tau\sigma.X).w\big) = -\phi_A\big(v,(\tau\sigma.X).w\big).\end{aligned}$$

(ii) This is a simple computation. ■

Lemma VII.6. *Let $\mathcal{A}$ be a convex set of symmetric operators on $\mathbb{R}^n$ satisfying the condition*

$$d := \max\{\operatorname{rank} A : A \in \mathcal{A}\} < n.$$

Then there exists $0 \neq v \in \mathbb{R}^n$ such that

$$\langle A.v, v \rangle = 0 \qquad \textit{for all } A \in \mathcal{A}.$$

Proof. (cf. [13, Prop. II.32]) Let $A_0 \in \mathcal{A}$ be of maximal rank d and $\{v_1, \ldots, v_n\}$ an orthonormal basis of eigenvectors, i.e.

$$A_0.v_i = \begin{cases} \alpha_i v_i & i \leq d, \\ 0 & i \geq d+1. \end{cases}$$

We claim : $\langle A.v_{d+1}, v_{d+1} \rangle = 0$ for all $A \in \mathcal{A}$.

Since all assumptions transfer to the subspace $B := \langle v_1, \ldots, v_{d+1} \rangle$ we may assume that $n = d+1$. Let $0 < \mu < 1$ and $\mu_0 := \frac{1-\mu}{\mu}$. For all $A \in \mathcal{A}$ we consider the expression

$$0 = \det(\mu A + (1-\mu) A_0) = \mu^{d+1} \det(A + \mu_0 A_0) = \mu^{d+1} \big(\mu_0^d \alpha_1 \cdot \ldots \cdot \alpha_d \cdot b_{d+1} + p(\mu_0)\big),$$

where $b_{d+1} = \langle A.v_{d+1}, v_{d+1} \rangle$ and p is a polynomial of degree smaller than $d-1$. The expression above is clearly analytic in μ_0 and the identity theorem for analytic functions implies $b_{d+1} = 0$ since $\alpha_1 \cdot \ldots \cdot \alpha_d \neq 0$. ■

In the following we identify $\mathfrak{q}^*$ with the subspace $\mathfrak{h}^\perp$ in $\mathfrak{g}^*$ and $\mathfrak{a}^*$ with the subspace $[\mathfrak{a}, \mathfrak{h}]^\perp$ of $\mathfrak{q}^*$.

Proposition VII.7. *Let $\mathfrak{n}$ be the nilradical of a symmetric Lie algebra $(\mathfrak{g}, \tau)$ with cone potential. To every $\alpha \in \mathfrak{a}_{\mathfrak{z}}^*$ we associate the skew symmetric bilinear form*

$$\phi^\alpha : \mathfrak{n} \times \mathfrak{n} \to \mathbb{R}; \quad \phi^\alpha(X, Y) := \alpha([X, Y]).$$

Then the following assertions hold:

(i) *The forms ϕ^α are $\mathfrak{g}$-invariant and τ-antiinvariant, i.e., $\phi^\alpha(\tau.X, \tau.Y) = -\phi^\alpha(X, Y)$. Furthermore $\mathfrak{n}^\beta \perp_{\phi^\alpha} \mathfrak{n}^\gamma$ if $\beta \neq -\gamma$.*

(ii) *The bilinear forms*

$$\psi^\alpha : \mathfrak{n} \times \mathfrak{n} \to \mathbb{R}, (X, Y) \mapsto \alpha([\tau.X, Y])$$

are symmetric and τ-covariant.

(iii) *There exists an element $\alpha \in \mathfrak{a}_{\mathfrak{z}}^*$ such that ψ^α is non-degenerate on $\mathfrak{n}^+$. Moreover, the set of these α is open and dense in $\mathfrak{a}_{\mathfrak{z}}^*$.*

Proof. (i) By the Jacobi identity it is sufficient to show that

$$p_{\mathfrak{a}}\big([Z, [X, Y]]\big) = 0 \qquad \text{for } Z \in \mathfrak{g} \text{ and } X, Y \in \mathfrak{n}.$$

But this is an immediate consequence of $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{g}^0$ (cf. Proposition VII.2(ii)) which shows that

$$p_{\mathfrak{a}}\big([\mathfrak{g}, [\mathfrak{n}, \mathfrak{n}]]\big) \subseteq p_{\mathfrak{a}}\big([\mathfrak{g}^0, [\mathfrak{n}, \mathfrak{n}]]\big) \subseteq p_{\mathfrak{a}}([\mathfrak{g}^0, \mathfrak{g}^0]) = \{0\}$$

because $\mathfrak{g}^0 = \mathfrak{h}^0 \oplus \mathfrak{a}$ and $[\mathfrak{g}^0, \mathfrak{g}^0] \subseteq \mathfrak{h}^0$. The τ-antiinvariance of the forms follows immediately from $\tau^*\alpha = -\alpha$. The last assertion is a consequence of Proposition VII.2(ii).
(ii) Since ϕ^α is τ-antiinvariant, τ is ϕ^α-skew-symmetric, so that the form ψ^α is symmetric. Moreover, the $\mathfrak{g}$-invariance of ϕ^α implies that ψ^α is τ-covariant (Lemma VII.5).
(iii) First we prove the assertion for a single $\mathfrak{r}^\beta$ instead of $\mathfrak{n}^+$. We choose a basis of $\mathfrak{r}^\beta$ and thus identify it with $\mathbb{R}^n$. Under this identification the symmetric forms $\psi^\alpha|_{\mathfrak{r}^\beta\times\mathfrak{r}^\beta}$ correspond to symmetric matrices on $\mathbb{R}^n$. Clearly, $\mathcal{A} := \{\psi^\alpha|_{\mathfrak{r}^\beta\times\mathfrak{r}^\beta} : \alpha \in \mathfrak{a}^*\}$ is a vector space of symmetric operators, hence convex. Then Lemma VII.6 shows that if (iii) is false, then there exists $0 \neq X \in \mathfrak{r}^\beta$ with

$$\psi^\alpha(X, X) = 0$$

for all $\alpha \in \mathfrak{a}_\mathfrak{z}^*$, i.e., $[X, \tau(X)] = 0$, contradicting the assumption that $\mathfrak{g}$ has cone potential.

We define

$$M_\beta := \{\alpha \in \mathfrak{a}_\mathfrak{z}^* : \psi^\alpha|_{\mathfrak{r}^\beta\times\mathfrak{r}^\beta} \ \text{ non-degenerate}\}.$$

The observation above shows that M_β is non-empty. Now we see that

$$M_\beta = \{\alpha \in \mathfrak{a}_\mathfrak{z}^* : \det(\psi^\alpha|_{\mathfrak{r}^\beta\times\mathfrak{r}^\beta}) \neq 0\}.$$

Since it is non-empty, the fact that the determinant is a polynomial in the matrix entries shows that M_β is open and dense. Now $M := \bigcap_{\beta\in\Delta_r} M_\beta$ is a finite intersection of open and dense sets, hence is open and dense. This proves (iii) because the elements of M correspond to non-degenerate symmetric forms on $\mathfrak{n}^+$. ■

Proposition VII.8. *If $(\mathfrak{g}, \tau)$ has cone potential and $\alpha \in \Delta_s$, then $\mathfrak{r}^\alpha = \{0\}$.*

Proof. Let $0 \neq Z \in \mathfrak{s}^\alpha$. Then, in view of Proposition IV.8(ii), the fact that $(\mathfrak{g}, \tau)$ has cone potential implies that Z is either of type (SR) or of type (R). Hence the corresponding subalgebra $\mathfrak{g}(Z)$ (cf. Theorem IV.1(iii)) is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

We consider the space

$$V := \sum_{\beta\in\mathbb{R}\alpha} \mathfrak{r}^\beta.$$

Then the fact that $\mathfrak{r}$ is an ideal implies that $[\mathfrak{s}^\alpha, \mathfrak{r}^\beta] \subseteq \mathfrak{r}^{\beta+\alpha}$, hence shows that V is invariant under $\mathfrak{g}(Z)$. Suppose that $\mathfrak{r}^\alpha \neq \{0\}$. Then $\alpha(\check{\alpha}) = 2$ implies that there exists a simple $\mathfrak{g}(Z)$-submodule $W \subseteq V$ of odd dimension intersecting $\mathfrak{r}^\alpha$ non-trivially. We write

$$W = \oplus_{j=-m}^m W^j,$$

where the subspaces $W^j := W \cap \mathfrak{r}^{j\alpha}$ are one-dimensional. Since $\mathfrak{g}(Z)$ is τ-invariant, the subspace $\tau(W)$ of $\mathfrak{r}$ is also a $\mathfrak{g}(Z)$-submodule. Hence the irreducibility of W implies that either $W \cap \tau(W) = \{0\}$ or $W = \tau(W)$.

Case 1: $\tau(W) = W$. Let $0 \neq X_0 \in W^1 = W \cap \mathfrak{r}^\alpha$. Now cone potential and the τ-invariance of W imply the existence of an element $\beta \in \mathfrak{a}^*$ such that $\phi^\beta|_{W \times W}$ is non-degenerate (Proposition VII.7(iii)).

In view of Proposition VII.7(i), the skew-symmetric bilinear form $\phi^\beta \mid_{W \times W}$ is invariant for $\mathfrak{g}(Z)$. Since W is odd-dimensional, this contradicts the fact that an irreducible $\mathfrak{sl}(2,\mathbb{R})$-module has an invariant skew-symmetric bilinear form if and only if it is even dimensional (cf. [1, Ch. 8, §7, no. 5, Prop. 12]).

Case 2: $\tau(W) \cap W = \{0\}$. Then we consider the $\mathfrak{g}(Z)$-submodule $\widetilde{W} := W + \tau(W) \subseteq [\mathfrak{g}(Z), \mathfrak{r}] \subseteq \mathfrak{n}$. Let $0 \neq X_0 \in W^0$. Then $\widetilde{W}^j := W^j + \tau(W^{-j}) \subseteq \mathfrak{r}^{j\alpha}$ and therefore

$$X_0 - \tau(X_0) \in \widetilde{W}^0 \cap \mathfrak{q} \cap \mathfrak{n} \subseteq \mathfrak{a} \cap \mathfrak{n} \subseteq \mathfrak{z}(\mathfrak{g}).$$

We conclude that $X_0 - \tau(X_0)$ and therefore X_0 commutes with $\mathfrak{g}(Z)$, contradicting the fact that W is a non-trivial simple $\mathfrak{g}(Z)$-module. ∎

Corollary VII.9. *If $(\mathfrak{g}, \tau)$ has cone potential, then Δ is split, and*

$$\Delta_r = \{\alpha \in \Delta \colon \mathfrak{g}^\alpha \subseteq \mathfrak{r}\} \quad \text{and} \quad \Delta_s = \{\alpha \in \Delta \colon \mathfrak{g}^\alpha \subseteq \mathfrak{s}\}, \tag{7.1}$$

where $\mathfrak{s}$ in an $\mathfrak{a}$-invariant Levi complement.

Proof. Firstly Proposition VII.8 shows that (7.1) holds. Now this and the fact that $\mathfrak{g}$ has cone potential implies that Δ is split (cf. Proposition IV.8(ii)). ∎

Proposition VII.10. *Suppose that $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$ and that $\mathfrak{h}^0$ is compactly embedded in $\mathfrak{g}$. If $\mathfrak{t}_\mathfrak{h} \subseteq \mathfrak{h}^0$ is a Cartan subalgebra, then $\mathfrak{t} := \mathfrak{t}_\mathfrak{h} + i\mathfrak{a}$ is a compactly embedded Cartan subalgebra of $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$.*

Proof. From the fact that $\mathfrak{h}^0$ is compactly embedded it follows in particular that $\mathfrak{t}_\mathfrak{h}$ is abelian, and hence that $\mathfrak{t}$ is abelian. It is clear that $\mathfrak{t}$ is compactly embedded in $\mathfrak{g}^c$. That it is maximal abelian follows from

$$\mathfrak{z}_{\mathfrak{g}^c}(\mathfrak{t}) = \mathfrak{z}_{\mathfrak{g}^c}(\mathfrak{t}_\mathfrak{h}) \cap \mathfrak{z}_{\mathfrak{g}^c}(i\mathfrak{a}) = \mathfrak{z}_{\mathfrak{h}^0}(\mathfrak{t}_\mathfrak{h}) \oplus i\mathfrak{a} = \mathfrak{t}_\mathfrak{h} \oplus i\mathfrak{a}.$$ ∎

Definition VII.11. Let $\mathfrak{g}$ be a Lie algebra and V a finite dimensional real $\mathfrak{g}$-module, where the module structure is defined by the representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. Then we say that V *is of compact type* if the group $\langle e^{\rho(\mathfrak{g})}\rangle$ has compact closure.

In this sense a subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is compactly embedded if and only if $\mathfrak{g}$ is a $\mathfrak{b}$-module of compact type.

Note that the class of modules of compact type is invariant under taking submodules, quotients, direct sums, tensor products etc. ∎

Proposition VII.12. *If the symmetric Lie algebra $(\mathfrak{g}, \tau)$ has strict cone potential, then the following assertions hold:*

(i) *$\mathfrak{q}_L$ and $\mathfrak{q}$ are $\mathfrak{h}^0$-modules of compact type.*

(ii) *If, in addition, $(\mathfrak{g}, \tau)$ is effective, then $\mathfrak{h}^0$ is compactly embedded and $\mathfrak{n}_\mathfrak{h}^0 = \{0\}$.*

Proof. (i) First we show that each root space $\mathfrak{g}^\alpha$ is an $\mathfrak{h}^0$-module of compact type. In fact, if $\alpha \in \Delta_s$, then, in view of Proposition VII.8, the fact that $\mathfrak{g}$ has cone potential implies that the form κ_τ is definite on $\mathfrak{g}^\alpha$ (Proposition IV.7(vi)). Since this form is preserved by $\mathfrak{h}^0$, the module $\mathfrak{g}^\alpha$ is of compact type.

Now let $\alpha \in \Delta_r$. Then we use the strict cone potential to see that for an appropriate $\beta \in \mathfrak{a}_{\mathfrak{z}}^*$ the form ψ^β on $\mathfrak{g}^\alpha$ is positive definite (cf. Proposition VII.7). Now the invariance of this form under $\mathfrak{h}^0$ (Proposition VII.7(ii)) shows that $\mathfrak{g}^\alpha$ is a module of compact type.

Since the projection $p_\mathfrak{q}: \mathfrak{g} \to \mathfrak{q}, X \mapsto \frac{1}{2}\big(X - \tau(X)\big)$ is $\mathfrak{h}$-equivariant and

$$\mathfrak{q} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Delta} (\mathbf{1} - \tau).\mathfrak{g}^\alpha,$$

we conclude that $\mathfrak{q}$ is an $\mathfrak{h}^0$-module of compact type. Therefore $\mathfrak{q} \otimes \mathfrak{q}$ and hence $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}$ are $\mathfrak{h}^0$-modules of compact type, whence $\mathfrak{q}_L = \mathfrak{q} + [\mathfrak{q}, \mathfrak{q}]$ is an $\mathfrak{h}^0$-module of compact type.

(ii) If $(\mathfrak{g}, \tau)$ is effective, then the representation $\mathrm{ad}_\mathfrak{q}$ of $\mathfrak{h}$ is faithful. Thus we can embed $\mathfrak{h}$ via $\mathrm{ad}_\mathfrak{q}$ in the $\mathfrak{h}^0$-module $\mathfrak{gl}(\mathfrak{q})$ of compact type. It follows that $\mathfrak{h}^0$ is compactly embedded in $\mathfrak{h}$, and therefore that $\mathfrak{h}^0$ is compactly embedded in $\mathfrak{g}$.

Since the subalgebra $\mathfrak{n}_\mathfrak{h}^0$ is on the one hand side is compactly embedded and on the other hand $\mathfrak{n}$ is a nilpotent ideal, so that all operators $\mathrm{ad}\, X$, $X \in \mathfrak{n}$ are nilpotent, it follows that $\mathfrak{n}_\mathfrak{h}^0 = \mathfrak{z}_\mathfrak{h}$. Now the effectiveness implies that $\mathfrak{z}_\mathfrak{h} = \{0\}$. ∎

Corollary VII.13. *If $(\mathfrak{g}, \tau)$ is an effective symmetric Lie algebra with strong cone potential and $\mathfrak{t}_\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}^0$, then $\mathfrak{t} := \mathfrak{t}_\mathfrak{h} + \mathfrak{a}$ is Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{t}^c := \mathfrak{t}_\mathfrak{h} + i\mathfrak{a}$ is a compactly embedded Cartan subalgebra of $\mathfrak{g}^c$.*

Proof. We only have to combine Proposition VII.10 with Proposition VII.12(ii). ∎

Covariant forms on modules

Now we are going to describe the fine structure of quasihermitian Lie algebras with strong cone potential. But first we need some information about covariant forms and (NCC) Lie algebras.

Lemma VII.14. *Let $(\mathfrak{g}, \tau)$ be a semisimple symmetric Lie algebra, θ a Cartan involution commuting with τ, $\tau^a := \theta\tau$, and V a finite dimensional irreducible real $\mathfrak{g}$-module. Let further ϕ be a τ-covariant symmetric (skew-symmetric) bilinear form on V and assume that there exists a non-trivial ϕ-symmetric (skew-symmetric) τ^a-intertwiner. Then each τ-covariant symmetric bilinear form on V is a multiple of ϕ.*

Proof. We prove the assertion for the case where ϕ is symmetric. The other case can be proved in the same way. Let A be a non-zero ϕ-symmetric τ-intertwiner and ρ the representation of $\mathfrak{g}$ defining the module structure on V.

Then the form ϕ_A is symmetric and θ-covariant (Lemma VII.13). Using Weyl's unitary trick, we also find a θ-covariant positive definite form ψ on V (Lemma I.9(ii)). Now we can write $\phi_A = \psi_B$ for a ψ-symmetric operator B and get

$$\psi(B.(X.v), w) = \psi_B(X.v, w) = -\psi_B\big(v, \theta(X).w\big) \\ = -\psi\big(B.v, \theta(X).w\big) = \psi(X.(B.v), w),$$

so that the positive definiteness of ψ implies that $\rho(X)B = B\rho(X)$ for all $X \in \mathfrak{g}$. Since B is ψ-symmetric, it is diagonalizable and all its eigenspaces are invariant under $\mathfrak{g}$. Therefore the irreducibility of V implies that $B = \lambda\mathbf{1}$ with $\lambda \neq 0$. So ϕ_A is a definite form.

If $\widetilde{\phi}$ is also τ-covariant, then $\widetilde{\phi}_A = \mu\phi_A$, and finally $\widetilde{\phi} = \mu\phi$. This completes the proof. ∎

Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite dimensional real representation of a (NCC) Lie algebra $\mathfrak{g}$. Then, according to Lemma I.9(ii), the element $\rho(H)$ is diagonalizable. For every $\mu \in \mathbb{R}$ we denote by V_μ the corresponding eigenspace of $\rho(H)$ and obtain the decomposition

$$V = \bigoplus_{\mu\in\mathbb{R}} V_\mu.$$

Proposition VII.15. *Let* $(\mathfrak{g}, \tau)$ *be a* (NCC) *symmetric Lie algebra and* $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ *a finite dimensional irreducible real representation. Then the following assertions hold:*

(i) *If* $\lambda \in \mathbb{R}$ *is the maximal eigenvalue of* $\rho(H)$*, then* $\operatorname{Spec}\rho(H) \subseteq \lambda - 2\mathbb{N}_0$.

(ii) *There exists an, up to scalar multiple unique, non-trivial* τ*-covariant symmetric bilinear form* ϕ *on* V. *This form can be normalized in such a way that*

$$\phi|_{V_{\lambda-2n}} \quad is \quad \begin{cases} positive\ definite, & if\ n\ is\ even, \\ negative\ definite, & if\ n\ is\ odd. \end{cases}$$

Proof. (i) From the triangular decomposition of $\mathfrak{g}$ (Proposition V.6(ii)) we deduce that

$$M := \bigoplus_{n\in\mathbb{N}_0} V_{\lambda-2n}$$

is a non-trivial submodule of V. Hence irreducibility yields $M = V$.

(ii) By Weyl's unitary trick we find a θ-covariant scalar product ψ on V (Lemma I.9(ii)). Then the operator $\rho(H) - \lambda$ is ψ-symmetric with even eigenvalues. Therefore

$$A := e^{i\frac{\pi}{2}\big(\rho(H)-\lambda\big)}$$

is ψ-symmetric with $A^2 = \mathbf{1}$. Moreover, Proposition V.6(iii) implies that A is a τ^a-intertwiner. Now the form $\phi := \psi_A$ is symmetric and τ-covariant (Lemma VII.5) and Lemma VII.14 shows that such a form is unique up to a scalar multiple. The statement on the signs of ϕ now follows from $A.v = (-1)^n v$ for $v \in V_{\lambda-2n}$. ∎

Suppose that $(\mathfrak{g},\tau)$ is (CT) and $\rho\colon\mathfrak{g}\to\mathfrak{gl}(V)$ is an irreducible finite dimensional real representation. Then Lemma V.7(i) and finite dimensional $\mathfrak{sl}(2,\mathbb{R})$-representation theory imply that $\mathrm{Spec}(T)\subseteq\mathbb{Z}$. In the same way as in Proposition VII.15 we obtain that either $\mathrm{Spec}\big(\rho(T)\big)\subseteq 2\mathbb{Z}$ or $\mathrm{Spec}\big(\rho(T)\big)\subseteq 2\mathbb{Z}+1$. In the first case we call V *even*, and we call it *odd* in the latter case.

Proposition VII.16. *Let $(\mathfrak{g},\tau)$ be a (CT) symmetric Lie algebra and $\rho\colon\mathfrak{g}\to\mathfrak{gl}(V)$ an irreducible finite dimensional real representation. Then the following cases arise:*

(1) If V is odd, then there exists a one-dimensional space of skew symmetric $\mathfrak{g}$-invariant forms on V.

(2) If V is even and $\mathbb{D}:=\mathrm{End}_{\mathfrak{g}}(V)$, then $\mathbb{D}\cong\mathbb{R},\mathbb{C},\mathbb{H}$ and in all cases the space of invariant symplectic forms is parametrized by $\mathbb{D}_{\mathrm{im}}=\{d\in\mathbb{D}\colon\overline{d}=-d\}$ as follows. Let ψ be a θ-covariant scalar product on V. Then $d^\top=-\overline{d}$ holds for all $d\in\mathbb{D}$ and $(\psi_d)_C$ with $C=e^{\frac{\pi}{2}\rho(U)}$ is the corresponding $\mathfrak{g}$-invariant skew-symmetric form.

Proof. (1) Let V be an odd $\mathfrak{g}$-module and ϕ be a non-trivial τ-covariant form on V (Proposition VII.15(ii)). We extend ϕ to a hermitian form on $V_{\mathbb{C}}$ which is also denoted by ϕ and define $B\colon=e^{i\frac{\pi}{2}\left(\rho(T)-\mathbf{1}\right)}=ie^{i\frac{\pi}{2}\rho(T)}$. Since $\rho(T)$ is ϕ-skew-hermitian, $i\rho(T)$ is ϕ-hermitian, and therefore $e^{i\frac{\pi}{2}\rho(T)}$ is ϕ-hermitian which implies that B is ϕ-skew-hermitian. Moreover $B^2=\mathbf{1}$, B leaves V invariant, and it is a τ-intertwiner, so that Lemma VII.5 shows that $\Omega:=\phi_B$ is a non-degenerate skew-symmetric $\mathfrak{g}$-invariant form on V. Furthermore the uniqueness up to scalar multiple follows from the uniqueness of ϕ because for each $\mathfrak{g}$-invariant skew-symmetric form $\widetilde{\Omega}$ the form $\widetilde{\Omega}_B$ is symmetric and τ-covariant, where on the other side $\Omega_B=\phi_{B^2}=\phi$.

(2) Now we assume that V is even and that ψ is a θ-covariant scalar product on V (cf. Lemma I.9(ii)). For $d\in\mathbb{D}$ and $X\in\mathfrak{g}$ we then obtain

$$d^\top\rho(X)=-\big(\rho(\theta.X)d\big)^\top=-\big(d\rho(\theta.X)\big)^\top=\rho(X)d^\top$$

which shows that $\mathbb{D}$ is invariant under taking transposes. It is clear that $\mathbb{R}\mathbf{1}\subseteq\mathbb{D}$ consists of symmetric elements. Moreover, if $\mathbb{D}\cong\mathbb{C},\mathbb{H}$, then the compactness of the one-parameter subgroups $e^{\mathbb{R}d}$ for $\overline{d}=-d$ implies that d cannot be symmetric. Hence $\mathbb{R}\mathbf{1}=\{d\in\mathbb{D}\colon d^\top=d\}$. Now the classification of the involutions on $\mathbb{C}$ and $\mathbb{H}$ implies that $d^\top=-\overline{d}$ for all $d\in\mathbb{D}$. According to Lemma VII.5, the θ-covariant skew-symmetric forms on V are given by ψ_d, $d\in\mathbb{D}_{\mathrm{im}}$. Furthermore the operator $C=e^{\frac{\pi}{2}\rho(U)}$ is a θ-intertwiner which is ψ-skew-symmetric. Hence the forms $(\psi_d)_C=(\psi_C)_d$ are $\mathfrak{g}$-invariant and skew-symmetric, and reversing the construction, it follows that these are all such forms. ∎

We note that in terms of the classification scheme for symplectic $\mathfrak{g}$-modules described in [14], the preceding result means that whenever an even $\mathfrak{g}$-module carries an invariant symplectic form, then it is of type $\mathbb{C}_{II}$ or $\mathbb{H}_{II}$. These observations will be quite important for a classification of the quasihermitian symmetric Lie algebras with strong cone potential. To see how such Lie algebras may look like, we use the preceding proposition to construct an important class of examples.

Example VII.17. (a) Let $(\mathfrak{s},\tau)$ be a symmetric (CT) Lie algebra and V an irreducible odd real $\mathfrak{g}$-module. Further let Ω be a non-trivial skew symmetric invariant form on V, which exists by Proposition VII.16. We consider the Lie algebra $\mathfrak{h}_V := V \oplus \mathbb{R}$ with the bracket

$$[(v,s),(w,t)] := \big(0, \Omega(v,w)\big)$$

and put

$$\mathfrak{g} := \mathfrak{h}_V \rtimes \mathfrak{s}.$$

We extend τ to an involution on $\mathfrak{g}$ by setting $\tau\mid_{\mathfrak{z}(\mathfrak{g})} = -\mathrm{id}_{\mathfrak{z}(\mathfrak{g})}$ and $\tau\mid_V = ie^{i\frac{\pi}{2}\rho(T)}$, where ρ denotes the representation of $\mathfrak{s}$ on V. This turns $\mathfrak{g}$ into a symmetric Lie algebra $(\mathfrak{g},\tau)$. Since, in view of Proposition VII.15(iii), the τ-covariant form $\phi(v,w) := \Omega(\tau(v),w)$ is definite on the root spaces V^α, we see that $(\mathfrak{g},\tau)$ has cone potential. Furthermore Proposition VII.15(ii) shows that $(\mathfrak{g},\tau)$ has strong cone potential if and only if Spec $\big(\rho(T)\big) = \{1,-1\}$.
(b) (The symmetric Jacobi algebra) We consider the symmetric (CT) Lie algebra $(\mathfrak{sp}(n,\mathbb{R}),\tau)$, where τ is given by conjugation with $I_{n,n}$. Let $V = \mathbb{R}^{2n}$ be the irreducible module for the standard representation of $\mathfrak{sp}(n,\mathbb{R})$. Here the invariant skew symmetric form Ω on V is given by the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Moreover the involution on V is given by $\tau|_V = I_{n,n}$ and $\mathfrak{h}_V$ is the $(2n+1)$-dimensional Heisenberg algebra $\mathfrak{h}_n$. We call the symmetric Lie algebra given by

$$(\mathfrak{hsp}(n,\mathbb{R}),\tau) := (\mathfrak{h}_n \rtimes \mathfrak{sp}(n,\mathbb{R}),\tau)$$

the *symmetric Jacobi algebra.* Note that $\mathrm{Spec}(T) = \{-1,1\}$, so that the Lie algebra $(\mathfrak{hsp}(n,\mathbb{R}),\tau)$ has strong cone potential. ■

Theorem VII.18. (The Short String Theorem) *Let $(\mathfrak{g},\tau)$ be a quasihermitian symmetric Lie algebra with strong cone potential, Δ^+ be a $\mathfrak{p}$-adapted positive system, and $\alpha \in \Delta_r$, $\beta \in \Delta_p$. Then the following assertions hold:*

(i) *The β-string through α has at most length 2 and contains at most one positive and one negative root. If α,β are positive, then $\alpha+\beta \notin \Delta$, $\alpha-\beta \in \Delta_r^-$ whenever it is a root, and $\langle\alpha,\beta\rangle \geq 0$.*

(ii) *If*

$$\mathfrak{p}_{\mathfrak{s}}^{\pm} = \bigoplus_{\alpha\in\Delta_p^{\pm}} \mathfrak{g}^\alpha, \quad \mathfrak{p}_{\mathfrak{r}}^{\pm} = \bigoplus_{\alpha\in\Delta_r^{\pm}} \mathfrak{g}^\alpha, \quad \textit{and} \quad \mathfrak{p}^{\pm} = \mathfrak{p}_{\mathfrak{r}}^{\pm} + \mathfrak{p}_{\mathfrak{s}}^{\pm},$$

then $[\mathfrak{p}^+,\mathfrak{p}^+] = \{0\}$ and $[\mathfrak{p}_{\mathfrak{s}}^{\pm},\mathfrak{p}_{\mathfrak{r}}^{\mp}] \subseteq \mathfrak{p}_{\mathfrak{r}}^{\pm}$.

(iii) *$C_{\min}(\Delta^+) \subseteq C_{\max}(\Delta^+)$.*

Proof. (i) Let $0 \neq Z \in \mathfrak{g}^\beta$ and $\mathfrak{g}(Z)$ the corresponding 3-dimensional symmetric Lie algebra of type (SR). Note that $\mathfrak{g}(Z)$ is (NCC). Since $(\mathfrak{g},\tau)$ has strong cone potential, we find $\gamma \in \mathfrak{a}_{\mathfrak{z}}^*$ such that ψ^γ is positive definite on the positive root spaces and negative definite on the negative root spaces.

We consider the $\mathfrak{g}(Z)$-submodule $V := \sum_{n\in\mathbb{Z}} \mathfrak{r}^{\alpha+n\beta}$ of $\mathfrak{r}$. If $W \subseteq V$ is an irreducible submodule of maximal dimension, then W intersects each root space in V, and the restriction of ψ^γ to W is a non-degenerate τ-covariant symmetric bilinear form (Proposition VII.7(ii)). Hence Proposition VII.15(ii) implies that ψ^γ has alternating signs on the $\check{\beta}$-eigenspaces in W. This proves that V contains at most one positive and at most one negative root space, hence that the length of the β-string through α has at most length 2 and contains at most one positive and one negative root. From that the assertions of (i) are clear.
(ii) First $[\mathfrak{p}_\mathfrak{r}^+, \mathfrak{p}_\mathfrak{r}^+] = \{0\}$ is a consequence of the cone potential of $(\mathfrak{g}, \tau)$ (Proposition VII.2). Further the $\mathfrak{p}$-adaptedness of Δ and the semisimplicity of $\mathfrak{s}$ imply that $[\mathfrak{p}_\mathfrak{s}^+, \mathfrak{p}_\mathfrak{s}^+] = \{0\}$ (cf. Proposition V.9(vii)). Finally $[\mathfrak{p}_\mathfrak{r}^+, \mathfrak{p}_\mathfrak{s}^+] = \{0\}$ and $[\mathfrak{p}_\mathfrak{r}^\mp, \mathfrak{p}_\mathfrak{s}^\pm] \subseteq \mathfrak{p}_\mathfrak{r}^\pm$ follow from (i).
(iii) The asserted statement is equivalent to

$$\alpha([X_\beta, \tau(X_\beta)]) \geq 0 \qquad \text{for all } \alpha, \beta \in \Delta_n^+, X_\beta \in \mathfrak{g}^\beta, \tag{7.2}$$

If $\beta \in \Delta_p^+$, then (7.2) means that $\alpha(\check{\beta}) \geq 0$. If this is false, then Corollary V.3 implies that $\alpha + \beta$ is a root. For $\alpha \in \Delta_r^+$ this contradicts (i), and for $\alpha \in \Delta_p^+$ this contradicts Proposition V.9(vii). If $\alpha \in \Delta_n^+$ and $\beta \in \Delta_r^+$, then $[X_\beta, \tau(X_\beta)] \in \mathfrak{z}(\mathfrak{g})$ implies that $\alpha([X_\beta, \tau(X_\beta)]) = 0$. This completes the proof.■

VIII. Convexity theorems

The canonical extension of a symmetric Lie algebra

Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra. We think of $(\mathfrak{g}, \tau)$ as sitting in the symmetric Lie algebra $(\mathfrak{g}_\mathbb{C}, \hat{\tau})$, where $\hat{\tau}$ denotes the antilinear extension of τ, i.e., complex conjugation with respect to the real form $\mathfrak{g}^c$ of $\mathfrak{g}_\mathbb{C}$. We also write $\overline{X} := \hat{\tau}(X)$ for $X \in \mathfrak{g}_\mathbb{C}$. The inclusion $(\mathfrak{g}, \tau) \to (\mathfrak{g}_\mathbb{C}, \hat{\tau})$ is an embedding of symmetric Lie algebras. We call $(\mathfrak{g}_\mathbb{C}, \hat{\tau})$ the *canonical extension* of $(\mathfrak{g}, \tau)$ and $\hat{\mathfrak{q}} := \mathfrak{q} + i\mathfrak{h}$ the *canonical extension of* $\mathfrak{q}$. The complex linear extension of τ is again denoted by τ. Note that $\hat{\tau}|_\mathfrak{g} = \tau$.

For the remainder of this section we assume that $(\mathfrak{g}, \tau)$ is quasihermitian.

Theorem VIII.1. (The Inheritage Theorem) *Let $(\mathfrak{g}, \tau)$ be a quasihermitian symmetric Lie algebra, $\mathfrak{a} \subseteq \mathfrak{q}$ a maximal hyperbolic abelian subspace, Δ^+ be a $\mathfrak{p}$-adapted positive system, and assume that $\mathfrak{h}^0$ is compactly embedded in $\mathfrak{g}$. Then the following assertions hold:*

(i) *If $\mathfrak{t}_\mathfrak{h} \subseteq \mathfrak{h}_0$ is a Cartan subalgebra, then $\hat{\mathfrak{a}} := \mathfrak{a} + i\mathfrak{t}_\mathfrak{h}$ is a maximal hyperbolic abelian and maximal abelian subspace $\hat{\mathfrak{a}} \subseteq \hat{\mathfrak{q}}$.*

(ii) *If $\mathfrak{p}$ is a maximal hyperbolic Lie triple system in $\mathfrak{q}$ containing $\mathfrak{a}$, then*

$$\hat{\mathfrak{p}} := \mathfrak{p} + i[\mathfrak{p}, \mathfrak{p}] + i\mathfrak{h}^0$$

is a maximal hyperbolic Lie triple system in $\hat{\mathfrak q}$, *we have* $\mathfrak z(\mathfrak p) \subseteq \mathfrak z(\hat{\mathfrak p})$, *and* $\mathfrak z_{\hat{\mathfrak q}}\big(\mathfrak z(\mathfrak p)\big) = \hat{\mathfrak p}$. *In particular* $(\mathfrak g_{\mathbb C}, \hat\tau)$ *is quasihermitian and* $\hat{\mathfrak p}$ *is the unique maximal hyperbolic Lie triple system containing* $\hat{\mathfrak a}$.

(iii) *There exists a* $\hat{\mathfrak p}$*-adapted positive system* $\hat\Delta^+ \subseteq \Delta(\mathfrak g_{\mathbb C}, \hat{\mathfrak a})$ *which is compatible with* Δ^+ *in the sense that*

$$\hat\Delta_n^+|_{\mathfrak a} = \Delta_n^+, \quad -\tau(\Delta_n^+) = \Delta_n^+, \quad \textit{and} \quad \hat\Delta_k = \{\alpha \in \hat\Delta \colon \alpha|_{\mathfrak a} \in \Delta_k \cup \{0\}\}.$$

(iv) *If* $\hat{\mathcal W}$ *is the Weyl group of* $\hat{\mathfrak a}$, *then there exists for each* $\gamma \in \mathcal W$ *a* $\hat\gamma \in \hat{\mathcal W}$ *with* $\hat\gamma \circ \tau|_{\mathfrak a} = \tau|_{\mathfrak a} \circ \gamma$ *and* $\hat\gamma|_{\mathfrak a} = \gamma$.

(v) *If* $p_{\mathfrak a} \colon \hat{\mathfrak a}^* \to \mathfrak a^*$ *denotes the restriction map and* $\hat\Delta_\alpha := p_{\mathfrak a}^{-1}(\alpha)$ *for* $\alpha \in \Delta$, *then* $(\mathfrak g^\alpha)_{\mathbb C} = \bigoplus_{\beta \in \hat\Delta_\alpha} \mathfrak g_{\mathbb C}^\beta$.

(vi) *If* $\hat\Delta^+$ *is compatible with* Δ^+ *as in* (iii) *and* $\hat C_{\min}$, *resp.* $\hat C_{\max}$, *is the corresponding minimal, resp. maximal, cone, then*

(a) $\hat C_{\min} \cap \mathfrak a = p_{\mathfrak a}(\hat C_{\min}) = C_{\min}$.
(b) $\hat C_{\max} \cap \mathfrak a = p_{\mathfrak a}(\hat C_{\max}) = C_{\max}$.
(c) $C_{\max}^0 \subseteq \hat C_{\max}^0$.
(d) $p_{\mathfrak a}\big(\operatorname{cone}(\check{\hat\Delta}_k^+)\big) = \operatorname{cone}(\check\Delta_k^+)$.

(vii) *If, in addition,* $(\mathfrak g, \tau)$ *is effective with strong cone potential, then*

(a) *the symmetric Lie algebra* $(\mathfrak g_{\mathbb C}, \hat\tau)$ *has strong cone potential,*
(b) $\hat C_{\min} \subseteq \hat C_{\max}$, *and*
(c) $\hat C_{\min,z} = C_{\min,z}$.

Proof. (i) This follows from Proposition VII.10.

(ii) To see that $\hat{\mathfrak p}$ is a maximal hyperbolic Lie triple system in $\hat{\mathfrak q}$, we show that $\mathfrak k^c := i\hat{\mathfrak p} = i\mathfrak p + [\mathfrak p, \mathfrak p] + \mathfrak h^0$ is a maximal compactly embedded subalgebra of $\mathfrak g^c$ (cf. Corollary III.8). To this end we may assume that $\mathfrak p$ is constructed as in Proposition III.5(iv) as $\mathfrak p = \mathfrak a_{\mathfrak r} \oplus \mathfrak p_{\mathfrak s}$, where $\mathfrak a_{\mathfrak r} = \mathfrak a \cap \mathfrak r$, $\mathfrak s$ is a τ-$\mathfrak a$-invariant Levi complement, $\mathfrak s = \mathfrak s_{\mathfrak k} \oplus \mathfrak s_{\mathfrak p}$ is a τ-invariant Cartan decomposition, and $\mathfrak p_{\mathfrak s} = (\mathfrak s_{\mathfrak p})_{\mathfrak q}$.

First we consider the semisimple symmetric Lie algebra $(\mathfrak s, \tau)$. It is a quasihermitian semisimple symmetric Lie algebra for which $\mathfrak s_{\mathfrak h}^0$ is compactly embedded. Now $\mathfrak s_{\mathfrak k}^c := (\mathfrak s_{\mathfrak k})_{\mathfrak h} + i(\mathfrak s_{\mathfrak p})_{\mathfrak q}$ is a maximal compactly embedded subalgebra of $\mathfrak s^c = \mathfrak s_{\mathfrak h} + i\mathfrak s_{\mathfrak q}$. Since the subalgebra $(\mathfrak s_{\mathfrak p})_{\mathfrak q} + i[(\mathfrak s_{\mathfrak p})_{\mathfrak q}, (\mathfrak s_{\mathfrak p})_{\mathfrak q}]$ of $\mathfrak s_{\mathfrak k}^c$ is an ideal (Lemma II.7), we have $(\mathfrak s_{\mathfrak k})_{\mathfrak h} = \mathfrak s_0 \oplus [(\mathfrak s_{\mathfrak p})_{\mathfrak q}, (\mathfrak s_{\mathfrak p})_{\mathfrak q}]$, where $\mathfrak s_0$ is an ideal of $\mathfrak s_{\mathfrak k}^c$, hence contained in $\mathfrak s_{\mathfrak h}^0$. This proves that $\mathfrak s_{\mathfrak k}^c \subseteq i\hat{\mathfrak p}_{\mathfrak s}$, and the maximality of $\mathfrak s_{\mathfrak k}^c$ gives equality.

Since $\mathfrak s$ is τ-$\mathfrak a$-invariant, we have $\mathfrak g^0 = \mathfrak r^0 \rtimes \mathfrak s^0$ and hence $\mathfrak h^0 = \mathfrak r_{\mathfrak h}^0 \rtimes \mathfrak s_{\mathfrak h}^0$. From $[\mathfrak r_{\mathfrak h}^0, \mathfrak s^\alpha] \subseteq \mathfrak s^\alpha \cap \mathfrak r = \{0\}$ and $\mathfrak g^\alpha = \mathfrak s^\alpha$ for all $\alpha \in \Delta_s$ (Corollary VII.9), it follows that $\mathfrak r_{\mathfrak h}^0$ commutes with $\mathfrak s_{\mathfrak q}$ and therefore with $\mathfrak p_{\mathfrak s}$. Hence

$$\hat{\mathfrak p} = (\mathfrak r_{\mathfrak h}^0 \oplus \mathfrak a_{\mathfrak r}) \oplus (\hat{\mathfrak p} \cap \mathfrak s_{\mathbb C})$$

is a direct sum of Lie algebras. We have already seen above that $(i\hat{\mathfrak p}) \cap \mathfrak s^c$ is maximal compactly embedded in $\mathfrak s^c$. Let $\widetilde{\mathfrak k} \supseteq \mathfrak k^c$ be a maximal compactly embedded subalgebra containing $i\hat{\mathfrak p}$. Projecting onto $\mathfrak s^c$ along $\mathfrak r^c$ and using the maximality of $\mathfrak k^c \cap \mathfrak s^c$ proves that $\widetilde{\mathfrak k} \subseteq \mathfrak r^c + (\mathfrak k^c \cap \mathfrak s^c)$, hence that

$$\widetilde{\mathfrak k} = (\widetilde{\mathfrak k} \cap \mathfrak r^c) \oplus (\mathfrak k^c \cap \mathfrak s^c).$$

Now the compactness of the Lie algebra $\widetilde{\mathfrak{k}}$ shows that $\widetilde{\mathfrak{k}} \cap \mathfrak{r}^c$ is central in $\widetilde{\mathfrak{k}}$, hence contained in $\mathfrak{z}_{\mathfrak{r}^c}(i\mathfrak{a}) = \mathfrak{r}_{\mathfrak{h}}^0 \oplus i\mathfrak{a}_{\mathfrak{r}} = \mathfrak{k}^c \cap \mathfrak{r}^c$. We conclude that $\widetilde{\mathfrak{k}} = \mathfrak{k}^c$ and hence that $\mathfrak{k}^c$ is maximal compactly embedded in $\mathfrak{g}^c$.

It remains to show that $\hat{\mathfrak{p}} = \mathfrak{z}_{\hat{\mathfrak{q}}}\big(\mathfrak{z}(\mathfrak{p})\big)$. The construction of $\hat{\mathfrak{p}}$ shows that $\mathfrak{z}(\mathfrak{p}) \subseteq \mathfrak{z}(\hat{\mathfrak{p}})$ because $\mathfrak{z}(\mathfrak{p}) \subseteq \mathfrak{a}$ commutes with $\mathfrak{h}^0$. The converse inclusion will be proved in (iii) below.

(iii) From the construction of $\hat{\mathfrak{p}}$ in (ii) and the definition of the compact roots, it follows that

$$\hat{\mathfrak{p}}_{\mathbb{C}} = \mathfrak{k}^c_{\mathbb{C}} = (\mathfrak{g}^0)_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \hat{\Delta}_k} (\mathfrak{g}^\alpha)_{\mathbb{C}}.$$

Therefore

$$\hat{\Delta}_k = \{\beta \in \hat{\Delta} \colon \beta|_{\mathfrak{a}} \in \Delta_k \cup \{0\}\} \quad \text{and consequently} \quad \hat{\Delta}_n = \{\beta \in \hat{\Delta} \colon \beta|_{\mathfrak{a}} \in \Delta_n\}.$$

This shows in particular that $\hat{\Delta}_k = \{\beta \in \hat{\Delta} \colon \beta \mid_{\mathfrak{z}(\mathfrak{p})} = 0\}$ and hence that $\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}\big(\mathfrak{z}(\mathfrak{p})\big) = \mathfrak{k}^c_{\mathbb{C}}$. From that we conclude in particular that

$$\mathfrak{z}_{\hat{\mathfrak{q}}}\big(\mathfrak{z}(\mathfrak{p})\big) = \hat{\mathfrak{p}}$$

which completes the proof of (ii).

Now it is easy to find a compatible $\hat{\mathfrak{p}}$-adapted positive system in $\hat{\Delta}$. We simply choose $X_0 \in C^0_{\max}$ and note that no non-compact root in $\hat{\Delta}_n$ vanishes on X_0. Then we pick $X_1 \in C^0_{\max}$ near to X_0 such that no root in Δ vanishes on X_1 and, in addition, all non-compact roots which are positive on X_0 are still positive on X_1. Then we pick $X_2 \in \hat{\mathfrak{a}}$ near to X_1 such that no root in $\hat{\Delta}$ vanishes on X_2 and, in addition, all roots which are positive on X_1 are still positive on X_2. We put $\hat{\Delta}^+ := \{\alpha \in \hat{\Delta} \colon \alpha(X_2) > 0\}$ and obtain a positive $\hat{\mathfrak{p}}$-adapted system satisfying all the requirements if we define $\Delta^+ = \{\alpha \in \Delta \colon \alpha(X_1) > 0\}$. Note in particular that $-\tau$ leaves $\hat{\Delta}_n^+$ invariant because $\hat{\Delta}_n^+ = \{\alpha \in \Delta \colon \alpha(X_0) > 0\}$ and $-\tau(X_0) = X_0$.

(iv) Let $\gamma \in \mathcal{W}$ and $\widetilde{\gamma} \in N_{\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{p},\mathfrak{p}])}(\mathfrak{a})$ be an element with $\gamma = \widetilde{\gamma}|_{\mathfrak{a}}$ (Lemma III.6). Then $\widetilde{\gamma}.\mathfrak{h}^0 = \mathfrak{h}^0$, and since $\widetilde{\gamma}.\mathfrak{t}_{\mathfrak{h}}$ is another compactly embedded Cartan subalgebra of $\mathfrak{h}^0$, there exists an inner automorphism $\sigma \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}^0)$ with $\sigma\widetilde{\gamma}.\mathfrak{t}_{\mathfrak{h}} = \mathfrak{t}_{\mathfrak{h}}$. Then $\sigma\widetilde{\gamma}|_{\mathfrak{a}} = \mathfrak{a}$ and $\sigma\widetilde{\gamma}$ normalizes $\hat{\mathfrak{a}} = \mathfrak{a} + i\mathfrak{t}_{\mathfrak{h}}$. Therefore $\hat{\gamma} := \sigma\widetilde{\gamma}|_{\hat{\mathfrak{a}}} \in \hat{\mathcal{W}}$ satisfies $\hat{\gamma}|_{\mathfrak{a}} = \gamma$.

(v) This follows from the $\hat{\mathfrak{a}}$-invariance of $(\mathfrak{g}^\alpha)_{\mathbb{C}}$.

(vi) We start by proving the first equality in (a) and (b). Note that the finite group $L := \{\mathbf{1}, -\tau\}$ operates on $\hat{\mathfrak{a}}$ with fixed point set $\mathfrak{a}$ and leaves the convex sets $\hat{C}_{\min}$ and $\hat{C}_{\max}$ invariant because it preserves $\hat{\Delta}_n^+$. Hence the assertion follows from ([5, Prop. 1.6]). Now we establish the statements concerning $C_{\max}$. The second equality in (b) follows from $\hat{\Delta}_n^+ \mid_{\mathfrak{a}} = \Delta_n^+$. Let $X \in C^0_{\max}$, which means that $\alpha(X) > 0$ for all $\alpha \in \Delta_n^+$. Then $\beta(X) > 0$ for all $\beta \in \hat{\Delta}_n^+$ and thus $X \in \hat{C}^0_{\max}$. This proves (c).

Next we show that $p_{\mathfrak{a}}(\hat{C}_{\min}) \subseteq C_{\min}$. Let $\alpha \in \Delta_n^+$, $\beta \in \hat{\Delta}$ with $\beta|_{\mathfrak{a}} = \alpha$, and $X_\beta \in \mathfrak{g}^\beta_{\mathbb{C}} \subseteq (\mathfrak{g}^\alpha)_{\mathbb{C}}$. Write $X_\beta = Y_\beta + iZ_\beta$, where $Y_\beta, Z_\beta \in \mathfrak{g}^\alpha$. Then

$$\begin{aligned}[X_\beta, \hat{\tau}(X_\beta)] &= [Y_\beta + iZ_\beta, \tau(Y_\beta) - i\tau(Z_\beta)] \\ &= \underbrace{[Y_\beta, \tau(Y_\beta)] + [Z_\beta, \tau(Z_\beta)]}_{\in C_{\min}} + \underbrace{i\big([Z_\beta, \tau(Y_\beta)] + [\tau(Z_\beta), Y_\beta]\big)}_{\in i\mathfrak{t}_{\mathfrak{h}}}. \end{aligned} \tag{8.1}$$

Hence $p_{\mathfrak a}([X_\beta, \hat\tau.X_\beta]) \in C_{\min}$, which yields $p_{\mathfrak a}(\hat C_{\min}) \subseteq C_{\min}$.

Now we prove the converse inclusion $C_{\min} \subseteq \hat C_{\min}$. Let $\alpha \in \Delta_n^+$, $X_\alpha \in \mathfrak g^\alpha$ and write $X_\alpha = \sum_{\beta\in\hat\Delta_\alpha} X_\beta$ with $X_\beta \in \mathfrak g_{\mathbb C}^\beta$ according to (v). We compute

$$[X_\alpha, \tau(X_\alpha)] = [X_\alpha, \hat\tau(X_\alpha)] = \sum_{\beta\in\hat\Delta_\alpha} [X_\beta, \hat\tau(X_\beta)] + \sum_{\beta\neq\gamma\in\hat\Delta_\alpha} [X_\beta, \hat\tau(X_\gamma)].$$

Since the last summand above is contained in $[\hat{\mathfrak a}, \mathfrak g_{\mathbb C}]$, it vanishes, and so the desired inclusion follows.

To see that (d) holds, let $\alpha \in \hat\Delta_k^+$, $\beta = \alpha|_{\mathfrak a}$, and $X_\alpha = Y_\beta + iZ_\beta \in \mathfrak g_{\mathbb C}^\alpha$, where $Y_\beta, Z_\beta \in \mathfrak g^\beta$ and $\check\alpha = [\hat\tau.X_\alpha, X_\alpha]$. As above we obtain

$$\check\alpha = [\hat\tau.X_\alpha, X_\alpha] \in i\mathfrak t_{\mathfrak h} + [\tau.Y_\beta, Y_\beta] + [\tau.Z_\beta, Z_\beta] \subseteq i\mathfrak t_{\mathfrak h} + \mathbb R^+\check\beta.$$

This proves (d).

(vii) First we prove (c). Let $X_\beta \in \mathfrak r_{\mathbb C}^\beta$. Then the second term of (8.1) vanishes since $[\mathfrak n, \mathfrak n] \subseteq \mathfrak z(\mathfrak g)_{\mathfrak q}$ (Proposition VII.2(iii)), and thus

$$[X_\beta, \hat\tau(X_\beta)] = [Y_\beta, \tau(Y_\beta)] + [Z_\beta, \tau(Z_\beta)] \in C_{\min}. \tag{8.2}$$

So $\hat C_{\min,z} = C_{\min,z}$ and (c) is established.

Next we show that $(\mathfrak g_{\mathbb C}, \hat\tau)$ has cone potential, i.e.,

$$[X_\beta, \hat\tau(X_\beta)] \neq 0 \qquad \text{for} \quad 0 \neq X_\beta \in \mathfrak g_{\mathbb C}^\beta, \beta \in \hat\Delta. \tag{8.3}$$

We have already seen that $(\mathfrak g_{\mathbb C}, \hat\tau)$ is quasihermitian, so that $\hat\Delta$ splits (Proposition V.9(ii)). Hence (8.3) follows from formula (8.1) and the pointedness of $C_{\min}$. As $\hat C_{\min,z} = C_{\min,z}$ is pointed, the cone potential of $(\mathfrak g_{\mathbb C}, \hat\tau)$ implies that it has in fact strong cone potential. Now $\hat C_{\min} \subseteq \hat C_{\max}$ follows from Theorem VII.18(iii).■

Remark VIII.2. The structure of the root decomposition of $\mathfrak g_{\mathbb C}$ with respect to $\hat{\mathfrak a}$ is in general much simpler than that of the root decomposition of $\mathfrak g$ with respect to $\mathfrak a$. This is mainly due to the fact that here we always have that

$$\hat\Delta_r = \{\alpha \in \hat\Delta: \mathfrak g_{\mathbb C}^\alpha \subseteq \mathfrak r_{\mathbb C}\}, \quad \hat\Delta_s = \{\alpha \in \hat\Delta: \mathfrak g_{\mathbb C}^\alpha \subseteq \mathfrak s_{\mathbb C}\},$$

and

$$\hat\Delta_k = \{\alpha \in \hat\Delta: \mathfrak g_{\mathbb C}^\alpha \subseteq \mathfrak k_{\mathbb C}^c\}$$

(cf. [13, Sect. II] or [19, Ch. V]). We also note that in this case there exists a $\hat{\mathfrak p}$-adapted positive system if and only if $\mathfrak z_{\mathfrak g^c}\big(\mathfrak z(\mathfrak k^c)\big) = \mathfrak k^c$ which means that the Lie algebra $\mathfrak g^c$ is quasihermitian. It is instructive to note that here the conditions (1) and (2) of Proposition V.10 are always satisfied so that the condition that $(\mathfrak g_{\mathbb C}, \hat\tau)$, resp. the Lie algebra $\mathfrak g^c$, is quasihermitian reduces to the existence of a $\hat{\mathfrak p}$-adapted, resp. $\mathfrak k^c$-adapted, positive system. ■

Reduced symmetric Lie algebras

For $\omega \in \mathfrak{q}^*$ we write $\mathcal{O}_\omega := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})^*.\omega$ for the coadjoint orbit of ω in $\mathfrak{q}^*$. The next concept will be a crucial tool in the following (cf. [17, Def. III.9] for the group case).

Definition VIII.3. We fix a $\mathfrak{p}$-adapted positive system Δ^+ and consider $C_{\min}^\star$ as a cone in $\mathfrak{a}^* \subset \mathfrak{g}^*$. Then the largest ideal $\mathfrak{b} = \mathfrak{b}(\Delta_n^+)$ in $(C_{\min}^\star)^\perp = \{X \in \mathfrak{g} : \langle X, C_{\min}^\star\rangle = \{0\}\}$ is called the *associated ideal of degeneracy.* We note that $\mathfrak{b}$ is automatically τ-invariant because $-\tau(C_{\min}) = C_{\min}$. ■

Lemma VIII.4. *If $(\mathfrak{g}, \tau)$ is quasihermitian, Δ^+ a $\mathfrak{p}$-adapted positive system, and $\mathfrak{b} \subseteq \mathfrak{g}$ a τ-invariant ideal, then the following assertions hold:*

(i) *If $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{b}$ is the quotient homomorphism, then $\pi(\mathfrak{a})$ is a maximal hyperbolic abelian and maximal abelian subspace of $\pi(\mathfrak{q})$, $\Delta_1^+ := \{\alpha \in \Delta^+ \colon \mathfrak{g}^\alpha \neq \mathfrak{b}^\alpha\}$ is a $\mathfrak{p}$-adapted positive system with respect to $\pi(\mathfrak{a})$, $\Delta_{k,1} = \Delta_k \cap \Delta_1$ is the corresponding system of compact roots, and $\Delta_{n,1} := \Delta_n \cap \Delta_1$ the corresponding system of non-compact roots. The corresponding minimal cone $C_{\min,1}$ coincides with $\overline{\pi(C_{\min})}$.*

If, in addition, $\mathfrak{b} = \mathfrak{b}(\Delta_n^+)$ is the ideal of degeneracy, then

(ii) *$\mathfrak{b} = H(C_{\min}) \oplus \mathfrak{b}_{\mathfrak{h}}^0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{b}^\alpha$, where $\mathfrak{b}^\alpha \subseteq \mathfrak{r}^\alpha$ and $X \in \mathfrak{b}^\alpha$ implies that $[X, \tau(X)] \in H(C_{\min})$,*

(iii) *$\pi(C_{\min})$ is closed and pointed, and the ideal of degeneracy of $\mathfrak{g}/\mathfrak{b}$ with respect to Δ_1^+ is trivial.*

Proof. (i) We write $\mathfrak{c}_1 := \pi(\mathfrak{c})$ for a subspace $\mathfrak{c} \subseteq \mathfrak{g}$ and τ_1 for the involution induced by τ on $\mathfrak{g}_1 = \mathfrak{g}/\mathfrak{b}$. It follows from Lemma III.11 that $\mathfrak{a}_1$ is a maximal hyperbolic abelian subspace and maximal abelian in $\mathfrak{q}_1$, and that $\mathfrak{p}_1$ is a maximal hyperbolic Lie triple system in $\mathfrak{q}_1$. From the decomposition $\mathfrak{p} = \mathfrak{z}(\mathfrak{p}) \oplus \big[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]\big]$ we obtain $\mathfrak{p}_1 = \mathfrak{z}(\mathfrak{p})_1 \oplus \big[\mathfrak{p}_1, [\mathfrak{p}_1, \mathfrak{p}_1]\big]$ so that $\mathfrak{z}(\mathfrak{p})_1 \subseteq \mathfrak{z}(\mathfrak{p}_1)$ implies that $\mathfrak{z}(\mathfrak{p}_1) = \mathfrak{z}(\mathfrak{p})_1$. The fact that $(\mathfrak{g}, \tau)$ is quasihermitian implies that $\mathfrak{q} = \mathfrak{p} \oplus [\mathfrak{z}(\mathfrak{p}), \mathfrak{h}]$ which gives $\mathfrak{q}_1 = \mathfrak{p}_1 \oplus [\mathfrak{z}(\mathfrak{p}_1), \mathfrak{h}_1]$, and therefore $\mathfrak{z}_{\mathfrak{q}_1}\big(\mathfrak{z}(\mathfrak{p}_1)\big) = \mathfrak{p}_1$ follows from the fact that $\mathfrak{g}_1$ is a semisimple $\mathfrak{z}(\mathfrak{g}_1)$-module. This proves that $(\mathfrak{g}_1, \tau_1)$ is quasihermitian. Now the assertions on the root system are consequences of $\mathfrak{p}_1 = \pi(\mathfrak{p})$, and the $\mathfrak{p}_1$-adaptedness of Δ_1^+ follows from the $\mathfrak{p}$-adaptedness of Δ^+. The formula describing the minimal cone follows from $\pi(\mathfrak{g}^\alpha) = \mathfrak{g}_1^\alpha$ for $\alpha \in \Delta_1$ and from $\pi \circ \tau = \tau_1 \circ \pi$.

(ii) Since $\mathfrak{b}$ is an ideal, it is in particular invariant under $\mathfrak{a}$, hence decomposes according to the root space decomposition, i.e., $\mathfrak{b} = \mathfrak{b}^0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{b}^\alpha$.

According to the definition of $\mathfrak{b}$ we have $\mathfrak{b} \cap \mathfrak{a} \subseteq (C_{\min}^\star)^\perp \cap \mathfrak{a} = H(C_{\min})$. To see the converse, it suffices to prove that $H(C_{\min}) \subseteq \mathfrak{z}(\mathfrak{g})$ whenever $(\mathfrak{g}, \tau)$ is quasihermitian. In fact, in Proposition V.9(vii) we have seen that $C_{\min,\mathrm{p}} \subseteq (\Delta_p^+)^\star$. Let $\mathfrak{n} \subseteq \mathfrak{g}$ denote the maximal nilpotent ideal. Then $\mathfrak{n}$ is τ-invariant, $\mathfrak{n} \cap \mathfrak{a} = \mathfrak{z}(\mathfrak{g})_{\mathfrak{q}}$, and $(\mathfrak{g}/\mathfrak{n}, \tau_{\mathfrak{g}/\mathfrak{n}})$ is reductive and quasihermitian (Proposition V.9(v)). Now Proposition V.9(vii) shows that that the corresponding cone

$C_{\min,1} \subseteq \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{n})$ is pointed because only the irreducible components of type (NCC) contribute to $C_{\min}$. We conclude that $H(C_{\min}) \subseteq \mathfrak{a} \cap \mathfrak{n} \subseteq \mathfrak{z}(\mathfrak{g})$ and this proves that $\mathfrak{b} \cap \mathfrak{a} = H(C_{\min})$.

For $X \in \mathfrak{b}^\alpha$ we have

$$[X, \tau.X] \in \mathfrak{b} \cap \mathfrak{a} = H(C_{\min}) \subseteq \mathfrak{z}(\mathfrak{g}).$$

Thus we conclude from Proposition IV.7(v) that the form κ_τ is degenerate of all root spaces $\mathfrak{b}^\alpha$. Since $\mathfrak{b} = (\mathfrak{b} \cap \mathfrak{r}) + (\mathfrak{b} \cap \mathfrak{s})$ holds for any Levi complement $\mathfrak{s}$ ([1, Ch. 1, §6, no. 8, Cor. 4]) and $\mathfrak{b} \cap \mathfrak{s}$ is an ideal of $\mathfrak{s}$, we conclude that $\mathfrak{b}^\alpha \subseteq \mathfrak{r}^\alpha$.
(iii) In view of (i), we have $C_{\min,1} = \overline{\pi(C_{\min})}$ and therefore

$$C^\star_{\min,1} = \overline{\pi(C_{\min})}^\star = \overline{C_{\min} + (\mathfrak{b} \cap \mathfrak{a})}^\star = C^\star_{\min},$$

where we identify $\mathfrak{a}_1^*$ with the subspace $(\mathfrak{b} \cap \mathfrak{a})^\perp$ of $\mathfrak{a}^*$. Similarly we identify $\mathfrak{g}_1^*$ with the subspace $\mathfrak{b}^\perp \subseteq \mathfrak{g}^*$. Then the definition of $\mathfrak{b}$ shows that

$$\mathfrak{b} = \big(\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{g})^*.C^\star_{\min}\big)^\perp$$

which is equivalent to the fact that

$$\mathfrak{g}_1^* \cong \mathfrak{b}^\perp = \mathrm{span}\,\big(\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{g})^*.C^\star_{\min}\big) = \mathrm{span}\,\big(\mathrm{Inn}_{\mathfrak{g}_1}(\mathfrak{g}_1).\widetilde{C}^\star_{\min}\big),$$

and this shows that the ideal of degeneracy in $\mathfrak{g}_1$ is trivial, hence in particular that $C_{\min,1}$ is pointed.

That $\pi(C_{\min})$ is closed follows from the fact that $\mathfrak{b} \cap C_{\min} = H(C_{\min})$ is a vector space which implies that $\mathfrak{b} + C_{\min}$ is a closed convex cone in $\mathfrak{g}$ (cf. [5, Prop. 1.4]). ∎

For every $\alpha \in \Delta$ we define $\mathfrak{r}^\alpha_{(A)} := \{Z \in \mathfrak{r}^\alpha : [Z, \tau(Z)] = 0\}$. Note that in general $\mathfrak{r}^\alpha_{(A)}$ is not a vector space (cf. Example IV.10(b)).

Lemma VIII.5. *If $C_{\min}$ is pointed, then*

(i) *the set $\mathfrak{r}^\alpha_{(A)}$ is a vector space, $[\mathfrak{r}^\alpha_{(A)}, \mathfrak{g}^{-\alpha}] \subseteq \mathfrak{n}^0_{\mathfrak{h}}$, and*

(ii) $[\mathfrak{r}^\alpha_{(A)}, \mathfrak{r}^\beta_{(A)}] \subseteq \mathfrak{r}^{\alpha+\beta}_{(A)}$ *if $\beta \neq -\alpha$.*

(iii) *If, in addition, $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}(\mathfrak{g})$, then $[\mathfrak{r}^\alpha_{(A)}, \mathfrak{g}^\beta] \subseteq \mathfrak{r}^{\alpha+\beta}_{(A)}$ if $\beta \neq -\alpha$.*

Proof. (i) Let $X \in \mathfrak{r}^\alpha_{(A)}$ and $Y \in \mathfrak{g}^\alpha$. We consider the expression

$$[X + Y, \tau(X) + \tau(Y)] = [Y, \tau(Y)] + ([X, \tau(Y)] + [Y, \tau(X)]).$$

We see that the right hand side of this equation is linear in X. Since the left hand side is contained in $C_{\min}$, it follows from the pointedness of $C_{\min}$ that the expression in brackets must vanish. We conclude that $\tau([X, \tau(Y)]) = [X, \tau(Y)]$, which establishes the second assertion because $[\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{n}$. Finally, the first statement drops out if we choose $Y \in \mathfrak{r}^\alpha_{(A)}$.

(ii) For $X \in \mathfrak{r}^\alpha_{(A)}$ and $Y \in \mathfrak{r}^\beta_{(A)}$ we have

$$\begin{aligned}\big[[X,Y],[\tau(X),\tau(Y)]\big] &= \Big[\big[[X,Y],\tau(X)\big],\tau(Y)\Big] + \Big[\tau(X),\big[[X,Y],\tau(Y)\big]\Big] \\ &\in [\mathfrak{g}^\beta, \mathfrak{r}^{-\beta}_{(A)}] + [\mathfrak{r}^{-\alpha}_{(A)}, \mathfrak{g}^\alpha] \subseteq \mathfrak{n}^0_{\mathfrak{h}}.\end{aligned}$$

Therefore the assertion follows from $\big[[X,Y],[\tau(X),\tau(Y)]\big] \in \mathfrak{h} \cap \mathfrak{a} = \{0\}$.
(iii) For $X \in \mathfrak{r}^\alpha_{(A)}$ and $Y \in \mathfrak{g}^\beta$ we now have

$$\begin{aligned}\big[[X,Y],[\tau(X),\tau(Y)]\big] &= \Big[\big[[X,Y],\tau(X)\big],\tau(Y)\Big] + \Big[\tau(X),\big[[X,Y],\tau(Y)\big]\Big] \\ &\in [\mathfrak{z}(\mathfrak{g}), \mathfrak{g}^{-\beta}] + [\mathfrak{r}^{-\alpha}_{(A)}, \mathfrak{g}^\alpha] \subseteq \mathfrak{n}^0_{\mathfrak{h}},\end{aligned}$$

so we can argue as in (ii). ■

We call $(\mathfrak{g},\tau)$ or, more precisely, $(\mathfrak{g},\tau,\Delta_n^+)$ *reduced* if the associated ideal of degeneracy is trivial. For every element in $\omega \in \mathfrak{a}^*$ we denote by $\mathfrak{d}(\omega)$ the largest ideal in $\ker\omega$. Note that $\mathfrak{d}(\omega)$ is τ-invariant. We write π for the quotient homomorphism from $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{d}(\omega)$. We call the Lie algebra $\mathfrak{g}(\omega) = \pi(\mathfrak{g})$ the *strictly reduced Lie algebra associated to* ω. Finally $\omega \in \mathfrak{a}^*$ is called *strictly reduced* if $\mathfrak{d}(\omega) = \{0\}$. Note that the τ-invariance of $\mathfrak{d}(\omega)$ implies that the symmetric structure is inherited by $\mathfrak{g}(\omega)$ and that $\mathfrak{a}(\omega) := \pi(\mathfrak{a})$ is a maximal hyperbolic and maximal abelian subspace in $\mathfrak{q}(\omega) := \pi(\mathfrak{q})$ (Lemma III.11).

Proposition VIII.6. *For a quasihermitian symmetric Lie algebra* $(\mathfrak{g},\tau)$ *the following assertions hold:*

(i) *If* $(\mathfrak{g},\tau)$ *is strictly reduced for an element* $\omega \in C^\star_{\min}$, *then* $(\mathfrak{g},\tau)$ *is reduced.*

(ii) *If* $(\mathfrak{g},\tau)$ *is reduced, then it is effective and has strong cone potential. In particular,* $[\mathfrak{n},\mathfrak{n}] \subseteq \mathfrak{z}(\mathfrak{g})$, $\mathfrak{n}^0_{\mathfrak{h}} = \{0\}$ *and* $\mathfrak{a}$ *can be extended to a Cartan subalgebra* $\mathfrak{t}$ *such that* $\mathfrak{t}^c \subseteq \mathfrak{g}^c$ *is a compactly embedded Cartan subalgebra.*

Proof. (i) This is clear because the ideal of degeneracy $\mathfrak{b}$ is contained in $\ker\omega$ for all $\omega \in C^\star_{\min}$.
(ii) Obviously $(\mathfrak{g},\tau)$ is effective. Suppose that $(\mathfrak{g},\tau)$ has cone potential. Then the pointedness of $C_{\min}$ implies strong cone potential and the remaining assertions follow from Proposition VII.12 and Corollary VII.13.

It remains to show that $(\mathfrak{g},\tau)$ has cone potential. First we show that $[\mathfrak{n},\mathfrak{n}] \subseteq \mathfrak{z}(\mathfrak{g})$. Let k be the length of $\mathfrak{n}$ and assume that $k > 2$. According to [5, Lemma 7.13], the ideal $\mathfrak{j} := \mathfrak{n}^{k-1}$ is abelian but not central because $\mathfrak{n}^k \neq \{0\}$. Since $\mathfrak{j}$ is $\mathfrak{a}$-invariant, it decomposes under the root decomposition as

$$\mathfrak{j} = \mathfrak{j}^0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{j}^\alpha.$$

Note that $\mathfrak{j}^\alpha \subseteq \mathfrak{r}^\alpha_{(A)}$ because $\mathfrak{j}$ is abelian. We claim that

$$\mathfrak{j}' = \mathfrak{j}^0_{\mathfrak{h}} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{j}^\alpha$$

is a τ-invariant ideal of $\mathfrak{g}$. In fact, it is clear that $\mathfrak{j}'$ is invariant under $\mathfrak{g}^0 = \mathfrak{a} \oplus \mathfrak{h}^0$, and the invariance under the root spaces $\mathfrak{g}^\alpha$ follows from Lemma VIII.5(i). From $\mathfrak{j}' \subset (C_{\min}^\star)^\perp$ and the reducedness of $(\mathfrak{g}, \tau, \Delta_n^+)$ we now conclude that $\mathfrak{j}' = \{0\}$, i.e., $\mathfrak{j} = \mathfrak{j}_\mathfrak{q}^0 = \mathfrak{j} \cap \mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{g})$. This contradiction shows that $k \leq 2$ and $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}(\mathfrak{n})$. If $\mathfrak{j} := [\mathfrak{n}, \mathfrak{n}]$ is not contained in $\mathfrak{z}(\mathfrak{g})$, we can argue as above and get a contradiction. Thus we have shown that $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}(\mathfrak{g})$.

Let $\gamma \in \operatorname{int} C_{\min}^\star$ and note that such a functional exists because $C_{\min}$ is pointed which in turn follows from the assumption that $(\mathfrak{g}, \tau)$ is reduced (cf. Lemma VIII.4). Now the form

$$\psi^\gamma \colon \mathfrak{n} \times \mathfrak{n} \to \mathbb{R}, \quad (X, Y) \mapsto \gamma([X, \tau.Y])$$

is τ-covariant and semidefinite on each root space (cf. proof of Proposition VII.7). Therefore $\mathfrak{n}^\perp := \mathfrak{n}^{\perp_{\psi^\gamma}}$ is an ideal of $\mathfrak{g}$. Moreover we have $\mathfrak{n}^\perp \cap \mathfrak{n}^\alpha = \mathfrak{n}_{(A)}^\alpha$. On the other hand $\mathfrak{n}^0 = \mathfrak{n}_\mathfrak{h}^0 + \mathfrak{n}_\mathfrak{q}^0 = \mathfrak{n}_\mathfrak{h}^0 + \mathfrak{z}(\mathfrak{g})_\mathfrak{q}$, so that $[\mathfrak{n}^0, \mathfrak{n}^0] \subseteq \mathfrak{n}_\mathfrak{h}^0 \subseteq \ker \gamma$. We conclude that

$$\mathfrak{n}^\perp = \mathfrak{n}^0 + \sum_{\alpha \in \Delta} \mathfrak{n}_{(A)}^\alpha$$

is an ideal of $\mathfrak{g}$. By the same argument as above, we see that $\mathfrak{n}_\mathfrak{h}^0 + \sum_{\alpha \in \Delta} \mathfrak{n}_{(A)}^\alpha$ is also an ideal, which has to vanish because $(\mathfrak{g}, \tau, \Delta_n^+)$ is reduced. Therefore $\mathfrak{n}_{(A)}^\alpha = \{0\}$ for all $\alpha \in \Delta$.

Let $Z \in \mathfrak{g}^\alpha$ be of type (A). Since Δ is split, Proposition IV.7(vi) implies that $Z \in \mathfrak{r}_{(A)}^\alpha = \{0\}$. This proves that $(\mathfrak{g}, \tau)$ has cone potential. ■

In the remainder of this subsection we give some information about the structure of $C_{\min}$ needed later on. We are in particular interested in the case where $C_{\min}$ coincides with the convex hull of the set $\{[X_\alpha, \tau(X_\alpha)] \colon X_\alpha \in \mathfrak{g}^\alpha, \alpha \in \Delta_n^+\}$. As has been shown in [17, Ex. III.8], this is not always the case. But, as the next lemma shows, the situation is rather well behaved if $C_{\min}$ is pointed.

Lemma VIII.7. *If $C_{\min}$ is pointed, then*

$$C_{\min} = \sum_{\mathfrak{g}^\alpha = \mathfrak{s}^\alpha} \mathbb{R}^+ \check{\alpha} + \sum_{\mathfrak{g}^\alpha \neq \mathfrak{s}^\alpha} C_\alpha,$$

where $C_\alpha \colon= \operatorname{conv}(\{[X_\alpha, \tau(X_\alpha)] \colon X_\alpha \in \mathfrak{g}^\alpha\})$. *If $B_\alpha \subseteq \mathfrak{g}^\alpha$ is a 0-neighborhood, then*

$$\operatorname{conv}(\{[X, \tau(X)] \colon X \in B_\alpha\})$$

is a 0-neighborhood in C_α.

Proof. This can be proved in the same way as Lemma III.7 in [17]. One only has to replace the expressions $i[X, \overline{X}]$ by $[X, \tau(X)]$. ■

The convexity theorems

Theorem VIII.8. (Convexity Theorem, coadjoint version) *Let $(\mathfrak{g}, \tau)$ be a quasihermitian symmetric Lie algebra, Δ^+ a $\mathfrak{p}$-adapted positive system and suppose that $\mathfrak{g}$ is strictly reduced for $\omega \in C_{\min}^\star$. Then*

$$p_{\mathfrak{a}^*}(\mathcal{O}_\omega) = \operatorname{conv}(\mathcal{W}.\omega) + \operatorname{cone}(\Delta_\omega^+) \subseteq \operatorname{conv}(\mathcal{W}.\omega) + \operatorname{cone}(\Delta_n^+),$$

where $\Delta_\omega^+ = \{\alpha \in \Delta_n^+ : (\exists \gamma \in \mathcal{W})(\exists X_\alpha \in \mathfrak{g}^\alpha)\ \langle \omega, \gamma.[X_\alpha, \tau(X_\alpha)]\rangle < 0\}$.

Proof. To use [7, Cor. 5.25], we note that the assumption about the pointedness of $C_{\min}^\star$ in this corollary is superfluous. According to Proposition VIII.6(ii), the assumptions of [7, Cor. 5.25] are satisfied and we get the asserted formula. ∎

Corollary VIII.9. *If $(\mathfrak{g}, \tau)$ is quasihermitian and $\omega \in C_{\min}^\star$, then*

$$p_{\mathfrak{a}^*}(\mathcal{O}_\omega) \subseteq \operatorname{conv}(\mathcal{W}.\omega) + \operatorname{cone}(\Delta_n^+).$$

Proof. By the isomorphism $\mathfrak{g}(\omega)^* \cong \mathfrak{d}(\omega)^\perp \subseteq \mathfrak{g}^*$ we realize the coadjoint orbit $\mathcal{O}_\omega$ in $\mathfrak{g}(\omega)^*$. Under this identification we have $\Delta(\omega) \subseteq \Delta$, where $\Delta(\omega)$ denotes the root system w.r.t. $\mathfrak{a}(\omega)$ induced by Δ, $\mathfrak{g}(\omega)$ is quasihermitian, and $\Delta(\omega)^+ := \Delta^+ \cap \Delta(\omega)$ is $\mathfrak{p}$-adapted (Lemma VIII.4(i)). Now the inclusion follows from Theorem VIII.8. ∎

Theorem VIII.10. (Convexity Theorem, adjoint version) *Let Δ^+ be a $\mathfrak{p}$-adapted positive system with $C_{\min} \subseteq C_{\max}$. Then for $X \in C_{\max}^0$ and the adjoint orbit $\mathcal{O}_X = \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}).X$ the following formula holds:*

$$\overline{\operatorname{conv}(\mathcal{O}_X)} \cap \mathfrak{a} = p_{\mathfrak{a}}\big(\overline{\operatorname{conv}(\mathcal{O}_X)}\big) = \operatorname{conv}(\mathcal{W}.X) + C_{\min}.$$

Proof. (cf. [17, Th. 3.12]) Since the cone

$$C := \{X \in C_{\max} : (\forall \alpha \in \Delta_k^+)\ \alpha(X) \geq 0\}$$

is a fundamental domain for the Weyl group action (cf. [19, Prop. III.2.7(i)]), we may assume that $X \in C$.

We will prove the chain of inclusions

$$\overline{\operatorname{conv}(\mathcal{O}_X)} \cap \mathfrak{a} \subseteq p_{\mathfrak{a}}\big(\overline{\operatorname{conv}(\mathcal{O}_X)}\big) \subseteq \operatorname{conv}(\mathcal{W}.X) + C_{\min} \subseteq \overline{\operatorname{conv}(\mathcal{O}_X)} \cap \mathfrak{a},$$

where the first inclusion is trivial. Thus we start by proving the second inclusion.

Let $\omega \in C_{\min}^\star$ be such that $\omega(\check{\alpha}) \leq 0$ for all $\alpha \in \Delta_k^+$. Then Corollary VIII.9 implies that

$$p_{\mathfrak{a}^*}(\mathcal{O}_\omega) \subseteq \operatorname{conv}(\mathcal{W}.\omega) + \operatorname{cone}(\Delta_n^+).$$

Therefore

$$\begin{aligned}\langle \omega, \mathcal{O}_X\rangle = \langle \mathcal{O}_\omega, X\rangle &= \langle p_{\mathfrak{a}^*}(\mathcal{O}_\omega), X\rangle \\ &\subseteq \operatorname{conv}(\langle \mathcal{W}.\omega, X\rangle) + \mathbb{R}^+ = \operatorname{conv}(\langle \omega, \mathcal{W}.X\rangle) + \mathbb{R}^+.\end{aligned}$$

From $\mathcal{W}.X \subseteq X - \mathrm{cone}(\check{\Delta}_k^+)$ ([19, Prop. III.2.7(ii)]) we now get $\langle \omega, \mathcal{O}_X\rangle \subseteq \omega(X) + \mathbb{R}^+$. This means that

$$p_{\mathfrak{a}}(\mathcal{O}_X) - X \subseteq \big(C_{\min}^\star \cap (-\check{\Delta}_k^+)^\star\big)^\star = \overline{C_{\min} - \mathrm{cone}(\check{\Delta}_k^+)} = C_{\min} - \mathrm{cone}(\check{\Delta}_k^+),$$

since $C_{\min} - \mathrm{cone}(\check{\Delta}_k^+)$ is closed (cf. [19, Lemma III.2.15]). Thus $p_{\mathfrak{a}}(\mathcal{O}_X) \subseteq X + C_{\min} - \mathrm{cone}(\check{\Delta}_k^+)$ and therefore the closedness of the right hand side yields

$$p_{\mathfrak{a}}\big(\overline{\mathrm{conv}(\mathcal{O}_X)}\big) \subseteq \bigcap_{\gamma\in\mathcal{W}} \gamma.\big(X + C_{\min} - \mathrm{cone}(\check{\Delta}_k^+)\big) = \mathrm{conv}(\mathcal{W}.X) + C_{\min}$$

([19, Cor. III.2.10]).

To complete the proof, we show that $\mathrm{conv}(\mathcal{W}.X) + C_{\min} \subseteq \overline{\mathrm{conv}(\mathcal{O}_X)} \cap \mathfrak{a}$. Let $\alpha \in \Delta_n^+$ and $Y \in \mathfrak{g}^\alpha$ be a non-zero element. Then Lemma VI.1 shows that

$$p_{\mathfrak{a}}(e^{\mathbb{R}\,\mathrm{ad}(Y+\tau.Y)}.X) = X + \alpha(X)\mathbb{R}^+[Y,\tau.Y] = X + \mathbb{R}^+[Y,\tau.Y]$$

because $\alpha(X) > 0$. Let $F := \mathrm{conv}(\{[X_\alpha, \tau.X_\alpha] \colon \alpha \in \Delta_n^+, X_\alpha \in \mathfrak{g}^\alpha\})$ and note that this is a dense subcone of $C_{\min}$. Now by Proposition VI.3(iii) $X + F \subseteq \mathrm{conv}(\mathcal{O}_X)$ and since $\mathcal{O}_X$ is $\mathcal{W}$-invariant, we conclude that

$$\mathrm{conv}(\mathcal{W}.X) + F = \mathrm{conv}\,\big(\mathcal{W}.(X+F)\big) + F = \mathrm{conv}\,\big(\mathcal{W}.(X+F)\big) \subseteq \mathrm{conv}(\mathcal{O}_X) \cap \mathfrak{a}.$$

As a consequence, we obtain $\mathrm{conv}(\mathcal{W}.X) + C_{\min} \subseteq \overline{\mathrm{conv}(\mathcal{O}_X)} \cap \mathfrak{a}$. This completes the proof. ∎

IX. Existence of hyperbolic invariant convex cones

Up to this section we have only considered consequences of the existence of invariant hyperbolic convex sets in $\mathfrak{q}$. In this section we will use the convexity theorems of Section VIII to prove that hyperbolic invariant convex cones exist in $\mathfrak{q}$ if and only if $(\mathfrak{g}, \tau)$ is quasihermitian and $C_{\min} \subseteq C_{\max}$ holds for a $\mathfrak{p}$-adapted positive system Δ^+. The latter condition is crucial for the applicability of the convexity theorems and we have already seen in Theorem VI.6 that it is necessary. Since there are only finitely many possibilities for positive systems, this condition has the advantage that it can be checked quite easily by computing the cones $C_{\min}$ and $C_{\max}$ which usually is quite easy because it reduces to calculating the brackets $[X, \tau.X]$, $X \in \mathfrak{g}^\alpha$.

Before we turn to the maximal cones, we need some preliminaries on invariant convex sets.

Invariant convex sets and exposed points

In this subsection V denotes a finite dimensional real vector space.

Lemma IX.1. *Let $F \subseteq V$ be a non-empty closed subset and $C := \overline{\mathrm{conv}(F)}$. If $H(C) = \{0\}$, then there exists for each $f \in B(C)^0$ an $x \in F$ with $f(x) = \min f(C)$.*

Proof. If $f \in B(C)^0$, then $f|_C$ is a proper function ([7, Cor. 1.13]). Since $F \subseteq C$ is closed, the function $f|_F$ is also proper. Now the fact that it is bounded from below implies that there exists an $x \in F$ with $f(x) = \min f(F)$. Finally the assertion follows from $f(C) \subseteq [\min f(F), \infty[$. ■

We recall that a point x in a closed convex set $C \subseteq V$ is called *exposed* if there exists a linear functional $f \in B(C)$ with $\{x\} = \{y \in C : f(y) = \min f(C)\}$. We write $\mathrm{Exp}(C)$ for the set of exposed points of C. A convex function ϕ on a convex subset Ω of V is said to be *strictly convex* if it is not affine on any non-trivial line segment, i.e., if $x \neq y$, $x, y \in \Omega$ and $\lambda \in]0,1[$ implies that

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y).$$

Proposition IX.2. *Let $\Omega \subseteq V$ be an open convex set, $\phi : \Omega \to \mathbb{R}$ be a strictly convex function with $\phi(x) \to \infty$ whenever $x \to x_0 \in \partial\Omega$, and $\emptyset \neq F \subseteq \Omega$ a closed subset of V such that $\phi|_F$ is constant. Then*

$$\mathrm{Exp}\left(\overline{\mathrm{conv}(F)}\right) = F.$$

Proof. Since Ω is open, the function ϕ on Ω is continuous and the fact that it tends to infinity at the boundary implies that the sets $\Omega_c := \{x \in C : \phi(x) \leq c\}$ are closed in V. Let $\phi(F) = \{c_0\}$. Then the convexity of ϕ implies that $\mathrm{conv}(F) \subseteq \Omega_{c_0}$, hence that

$$D := \overline{\mathrm{conv}(F)} \subseteq \Omega_{c_0} \subseteq \Omega.$$

If $x \in D$, then $x + H(D) \subseteq D$. Therefore the convex function ϕ is bounded from above on this affine subspace and therefore constant. Now the strict convexity of ϕ implies that $H(D) = \{0\}$.

Let $x \in \mathrm{Exp}(D)$ and $f \in B(D)$ with $\{x\} = \{y \in D : f(y) = \min f(D)\}$. The closed convex set D contains the cone $x + \lim(D)$ with vertex x. Therefore $f \in \lim(D)^\star$ and 0 is the unique minimum of f in $\lim(D)$. This proves that $f \in \mathrm{int}\lim(D)^\star$, hence, by [7, Lemma 1.9], that $f \in B(D)^0$. Now Lemma IX.1 shows that there exists $y \in F$ with $f(y) = \min f(D)$, hence that $x = y \in F$.

It remains to show that $F \subseteq \mathrm{Exp}(D)$. So let $x \in F$ and $f \in V^*$ a subgradient of ϕ in x, i.e., $\phi(x) + f(y - x) \leq \phi(y)$ for all $y \in \Omega$. For the existence of such functionals we refer to [19, Lemma III.3.16]. For $y \in D \setminus \{x\}$ we then have

$$f(y) \leq f(x) + \phi(y) - \phi(x) = f(x) + \phi(y) - c_0 \leq f(x).$$

If $f(y) = f(x)$, then $\phi(x) = \phi(y)$, so that $\phi(x) = \min \phi(D)$ and the strict convexity of ϕ imply that $x = y$. Hence

$$\{x\} = \{y \in D : -f(y) = \min\big(-f(D)\big)\}$$

shows that $x \in \mathrm{Exp}(D)$. This proves that $\mathrm{Exp}(D) = F$. ■

Remark IX.3. Note that under the assumptions of Proposition IX.2 the set $\mathrm{conv}(F)$ is in general not closed. Let $V = \mathbb{R}^2$, $\Omega =]0,\infty[^2$, $\phi(x,y) = \frac{1}{xy}$, and $F = \{(n, \frac{1}{n}) : n \in \mathbb{N}\}$. Then the points $(1+t, 1)$, $t > 0$ are contained in the closure of $\mathrm{conv}(F)$ but not in $\mathrm{conv}(F)$. ∎

Now we draw a general conclusion from Proposition IX.2.

Theorem IX.4. *Let H be a group and $\pi: H \to \mathrm{Sl}(V)$ a representation with closed image. Furthermore let $\Omega \subseteq V$ be an open convex subset with $H(\overline{\Omega}) = \{0\}$ which is H-invariant. Then for all $x \in \Omega$ the orbit $H.x$ is closed and*

$$\mathrm{Exp}\left(\overline{\mathrm{conv}(H.x)}\right) = H.x.$$

Proof. Using [19, Th. III.5.4], we see that the characteristic function ϕ_Ω of Ω defined by

$$\phi_\Omega(x) := \int_{B(\Omega)} e^{-\alpha(x) + \inf \alpha(\Omega)} \, d\mu_{V^*}(\alpha),$$

where μ_{V^*} denotes Lebesgue measure on V^*, has the following properties:

(1) ϕ_Ω is invariant under H because $\pi(H) \subseteq \mathrm{Sl}(V)$.

(2) If $x_n \to x \in \partial\Omega$, then $\phi_\Omega(x_n) \to \infty$.

(3) ϕ_Ω is strictly convex.

Next we show that the orbit $H.x$ is closed. We consider the vector space $V^\sharp := V \times \mathbb{R}$, the cone $C := \mathrm{cone}(\Omega \times \{1\})$ and the action of H given by $h.(v,t) := (h.v, t)$. Then C is a pointed closed convex cone in $V^\sharp$ which is invariant under the action of H. Therefore the closedness of the orbit $H.x$ follows from [5, Prop. 1.12].

Since the characteristic function ϕ_Ω is H-invariant, it is constant on $H.x$, and the assertion now follows from Proposition IX.2. ∎

Hyperbolicity of the maximal cone

Throughout this section $(\mathfrak{g}, \tau)$ denotes a quasihermitian symmetric Lie algebra, $\mathfrak{p} \subseteq \mathfrak{q}$ a maximal hyperbolic Lie triple system, $\mathfrak{a} \subseteq \mathfrak{p}$ a maximal abelian subspace, and Δ^+ a $\mathfrak{p}$-adapted positive system.

We define

$$W_{\max} := \bigcap_{h \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})} h.p_{\mathfrak{a}}^{-1}(C_{\max}) = \{X \in \mathfrak{q} : p_{\mathfrak{a}}(\mathcal{O}_X) \subseteq C_{\max}\}.$$

Lemma IX.5. *If $C_{\min} \subseteq C_{\max}$, then $W_{\max}$ is a generating closed convex invariant cone in $\mathfrak{q}$ with $p_{\mathfrak{a}}(W_{\max}) = W_{\max} \cap \mathfrak{a} = C_{\max}$, and $\mathfrak{n}_{\mathfrak{q}} \subseteq W_{\max}$.*

Proof. As an intersection of closed convex cones, the cone $W_{\max}$ is closed and convex, and the invariance follows from the definition. Further it is clear that

$$W_{\max} \cap \mathfrak{a} \subseteq p_{\mathfrak{a}}(W_{\max}) \subseteq C_{\max}.$$

For $X \in C_{\max}^0$ the adjoint version of the Convexity Theorem (Theorem VIII.10) shows that

$$p_{\mathfrak{a}}(\mathcal{O}_X) \subseteq \operatorname{conv}(\mathcal{W}.X) + C_{\min} \subseteq C_{\max} + C_{\min} \subseteq C_{\max}$$

because $C_{\min} \subseteq C_{\max}$. Hence $C_{\max}^0 \subseteq W_{\max} \cap \mathfrak{a}$, and $C_{\max} \subseteq W_{\max} \cap \mathfrak{a}$ follows from the closedness of $W_{\max}$. This proves the stated equality. That $W_{\max}$ has interior points follows from Lemma VI.5(ii).

To see that $\mathfrak{n}_{\mathfrak{q}} \subseteq W_{\max}$, let $X \in \mathfrak{n}_{\mathfrak{q}}$. Then $\mathcal{O}_X \subseteq \mathfrak{n}_{\mathfrak{q}}$ and therefore $p_{\mathfrak{a}}(\mathcal{O}_X) \subseteq p_{\mathfrak{a}}(\mathfrak{n}_{\mathfrak{q}}) \subseteq \mathfrak{n}_{\mathfrak{q}} \cap \mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{g})_{\mathfrak{q}} = H(C_{\max})$. This proves that $X \in W_{\max}$, whence $\mathfrak{n}_{\mathfrak{q}} \subseteq W_{\max}$. ∎

Now the main problem is to show that the cone $W_{\max}$ is in fact hyperbolic. Using that $W_{\max}$ contains $\mathfrak{n}_{\mathfrak{q}}$, we will reduce this question to the case where $(\mathfrak{g}, \tau)$ is irreducible. We start with this case.

Proposition IX.6. *If $(\mathfrak{g}, \tau)$ is irreducible and quasihermitian, then the cone $W_{\max}$ is hyperbolic.*

Proof. According to the remark after Definition V.5, we have to consider two cases. If $(\mathfrak{g}, \tau)$ is (NCR), then $\Delta_n = \emptyset$, $C_{\max} = \mathfrak{a}$ and therefore $\mathfrak{p} = \mathfrak{q} = W_{\max}$, showing that $W_{\max}$ is hyperbolic.

Now we assume that $(\mathfrak{g}, \tau)$ is (NCC). First we show that $W_{\max}$ is pointed. We know already that $W_{\max}$ is generating and different from $\mathfrak{q}$ because $C_{\max} \neq \mathfrak{a}$. If $W_{\max}$ is not pointed, then $H(W_{\max})$ is a non-zero $\mathfrak{h}$-submodule of $\mathfrak{q}$ and Lemma II.7(ii)(b) implies that $H(W_{\max})$ is isotropic with respect to the Cartan-Killing form κ. Since κ is positive definite on $\mathfrak{a}$ (Proposition IV.7(ii)), we conclude from $H(C_{\max}) \subseteq H(W_{\max})$ that $C_{\max}$ is pointed. Therefore $p_{\mathfrak{a}}\big(H(W_{\max})\big) \subseteq H(C_{\max}) = \{0\}$ shows that $H(W_{\max})$ is an $\mathfrak{h}$-submodule of $\mathfrak{a}^{\perp_\kappa} = [\mathfrak{a}, \mathfrak{h}]$. Eventually the fact that $(\mathfrak{g}, \tau)$ has cone potential (Proposition V.9(viii)) and Proposition VII.2(iv) entail that $H(W_{\max}) = \{0\}$, i.e., that $W_{\max}$ is pointed.

The fact that $W_{\max}$ is pointed and generating implies that the dual cone $W_{\max}^\star \subseteq \mathfrak{q}^*$ is also pointed and generating. Let θ be a Cartan involution of $\mathfrak{g}$ commuting with τ. Then $\mathfrak{h}$ is θ-invariant and therefore reductive. For $X \in \mathfrak{h}$ we therefore have

$$\operatorname{tr} \operatorname{ad}_{\mathfrak{q}} X = \operatorname{tr} \operatorname{ad} X - \operatorname{tr} \operatorname{ad}_{\mathfrak{h}} X = 0 - 0 = 0$$

and thus $\operatorname{Inn}_{\mathfrak{q}}(\mathfrak{h})^* \subseteq \operatorname{Sl}(\mathfrak{q}^*)$. To see that $\operatorname{Inn}_{\mathfrak{q}}(\mathfrak{h}) \subseteq \operatorname{Gl}(\mathfrak{q})$ is closed, we note that $\mathfrak{g} = \mathfrak{q}_L = \mathfrak{q} + [\mathfrak{q}, \mathfrak{q}]$, so that the non-closedness of $\operatorname{Inn}_{\mathfrak{q}}(\mathfrak{h})$ would imply that $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$ is not closed. But $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{g}) = \operatorname{Aut}(\mathfrak{g})_0$ is closed and therefore $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}) = \operatorname{Aut}(\mathfrak{g})_0^\tau$ is closed.

Now we can apply Theorem IX.4 with $V = \mathfrak{q}^*$, $H = \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$, and $\Omega = \operatorname{int} W_{\max}^\star$, and find that for each $f \in \operatorname{int} W_{\max}^\star$ the coadjoint orbit $\mathcal{O}_f = \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})^*.f$ is closed and satisfies

$$\operatorname{Exp}\left(\overline{\operatorname{conv}(\mathcal{O}_f)}\right) = \mathcal{O}_f.$$

From the definition of $W_{\max}$ we also conclude that $C_{\max}^\star \subseteq W_{\max}^\star \cap \mathfrak{a}^*$ because $X \in W_{\max}$ and $f \in C_{\max}^\star$ implies that $f(X) = \langle f, p_{\mathfrak{a}}(X)\rangle \geq 0$. Using the $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-equivariant isomorphism $\psi\colon \mathfrak{q} \to \mathfrak{q}^*$ given by $\psi(X)(Y) = \kappa(X,Y)$ obtained by the Cartan-Killing form, we see that we can use Proposition III.2 to show that for each element $f \in \operatorname{int} C_{\max}^\star$ with $\mathfrak{z}_{\mathfrak{q}}\big(\psi^{-1}(f)\big) = \mathfrak{a}$ we have $f \in \operatorname{int} W_{\max}^\star$.

Let $f \in \operatorname{int} C_{\max}^\star$ be such an element and $X' \in W_{\max}^0$. Then Lemma IX.1 implies that there exists an element $f' \in \mathcal{O}_f$ with

$$\langle f', X'\rangle = \min\langle\overline{\mathrm{conv}(\mathcal{O}_f)}, X'\rangle.$$

Write $f' = \gamma.f$ with $\gamma \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ and put $X := \gamma^{-1}.X'$. Then

$$\langle f, X\rangle = \langle f', X'\rangle = \min\langle\overline{\mathrm{conv}(\mathcal{O}_f)}, X\rangle = \min\langle\mathcal{O}_f, X\rangle = \min\langle f, \mathcal{O}_X\rangle.$$

Therefore $f([\mathfrak{h}, X]) = \{0\}$ and thus

$$\{0\} = \kappa\big(\psi^{-1}(f), [\mathfrak{h}, X]\big) = \kappa\big([X, \psi^{-1}(f)], \mathfrak{h}\big).$$

Since κ is non-degenerate on $\mathfrak{h}$, this means that $[X, \psi^{-1}(f)] = \{0\}$, i.e., $X \in \mathfrak{a}$ by the choice of f. This proves that $X' \in \mathcal{O}_X$ is hyperbolic and therefore that $W_{\max}^0 \subseteq \mathfrak{q}_{\mathrm{hyp}}$. ∎

Corollary IX.7. *If $(\mathfrak{g}, \tau)$ is reductive and quasihermitian, then the cone $W_{\max}$ is hyperbolic.*

Proof. The reductive symmetric Lie algebra $(\mathfrak{g}, \tau)$ decomposes as a direct sum of quasihermitian irreducible symmetric Lie algebras (Proposition V.9(v)) and so do the cones $C_{\max}$ and $W_{\max}$. Therefore the assertion follows from Proposition IX.6. ∎

To pave the way from the reductive to the general case we will need the following result on solvable symmetric Lie algebras.

Proposition IX.8. *Let $(\mathfrak{r}, \tau)$ be a solvable symmetric Lie algebra, $X_1, X_2 \in \mathfrak{r}_{\mathfrak{q}}$ regular elements and $\mathfrak{t}_j := \mathfrak{r}^0(\operatorname{ad} X_j)$ the corresponding Cartan subalgebras. Then $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{r}_{\mathfrak{h}})$.*

Proof. We prove the proposition by induction over the dimension of $\mathfrak{r}$. Let $\mathfrak{z} := \mathfrak{z}(\mathfrak{n})$ denote the center of the nilradical $\mathfrak{n}$ of $\mathfrak{r}$. If $\mathfrak{r} \neq \{0\}$, then $\mathfrak{z}$ is a non-zero abelian ideal which is invariant under τ. Let $\mathfrak{r}_1 := \mathfrak{r}/\mathfrak{z}$ and write $\pi\colon \mathfrak{r} \to \mathfrak{r}_1$ for the quotient homomorphism. Then $\pi(X_j) \in \mathfrak{q}_1 := \pi(\mathfrak{r}_{\mathfrak{q}})$ are regular elements in the Lie algebra $\mathfrak{g}_1$ ([1, Ch. 7, §2, no. 2, Prop. 8]) and $\pi(\mathfrak{t}_j) = \pi\big(\mathfrak{r}^0(\operatorname{ad} X_j)\big) = \mathfrak{r}_1^0\big(\operatorname{ad} \pi(X_j)\big)$. Hence our induction hypothesis implies the existence of $\gamma \in \mathrm{Inn}_{\mathfrak{r}_1}(\mathfrak{r}_{\mathfrak{h},1})$ with $\gamma.\pi(\mathfrak{t}_1) = \pi(\mathfrak{t}_2)$. Let $\widetilde{\gamma} \in \mathrm{Inn}_{\mathfrak{r}}(\mathfrak{r}_{\mathfrak{h}})$ with $\pi \circ \widetilde{\gamma} = \gamma \circ \pi$. Then $\pi(\widetilde{\gamma}.\mathfrak{t}_1) = \pi(\mathfrak{t}_2)$ shows that $\widetilde{\gamma}.\mathfrak{t}_1 \subseteq \mathfrak{t}_2 + \mathfrak{z}$. From now on we may therefore w.l.o.g. assume that $\mathfrak{t}_1 \subseteq \mathfrak{t}_2 + \mathfrak{z}$. Now $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are τ-invariant Cartan subalgebras of the solvable Lie algebra $\mathfrak{t}_2 + \mathfrak{z}$. Hence there exists $X \in \mathfrak{z}$ with $e^{\operatorname{ad} X}.\mathfrak{t}_1 = \mathfrak{t}_2$ ([1, Ch. 7, §3, no. 4, Th. 3]). The fact that the ideal $\mathfrak{z}$ is abelian implies that $(\mathrm{ad}\,X)^2=\{0\}$. Therefore $e^{\mathrm{ad}\,X}=\mathbf{1}+\mathrm{ad}\,X$ and if we write $X=X_{\mathfrak h}+X_{\mathfrak q}$ with $X_{\mathfrak h}\in\mathfrak z_{\mathfrak h}$ and $X_{\mathfrak q}\in\mathfrak z_{\mathfrak q}$, we obtain

$$e^{\mathrm{ad}\,X}.X_1=\underbrace{X_1+[X_{\mathfrak h},X_1]}_{\in\mathfrak r_{\mathfrak q}}+\underbrace{[X_{\mathfrak q},X_1]}_{\in\mathfrak r_{\mathfrak h}}\in e^{\mathrm{ad}\,X}.\mathfrak t_1=\mathfrak t_2=(\mathfrak t_2)_{\mathfrak h}\oplus(\mathfrak t_2)_{\mathfrak q}.$$

We conclude that $e^{\mathrm{ad}\,X_{\mathfrak h}}.X_1\in\mathfrak t_2$ and hence that $\mathfrak t_2=\mathfrak r^0\big(\mathrm{ad}(e^{\mathrm{ad}\,X_{\mathfrak h}}.X_1)\big)=e^{\mathrm{ad}\,X_{\mathfrak h}}.\mathfrak r^0(\mathrm{ad}\,X_1)=e^{\mathrm{ad}\,X_{\mathfrak h}}.\mathfrak t_1$. Thus $\mathfrak t_1$ and $\mathfrak t_2$ are conjugate under $\mathrm{Inn}_{\mathfrak r}(\mathfrak r_{\mathfrak h})$. ■

Theorem IX.9. *If $(\mathfrak g,\tau)$ is a quasihermitian symmetric Lie algebra and Δ^+ a $\mathfrak p$-adapted positive system such that $C_{\min}\subseteq C_{\max}$, then the cone $W_{\max}$ is hyperbolic.*

Proof. Let $\mathfrak n$ denote the nilradical of $\mathfrak g$, $\mathfrak g_1:=\mathfrak g/\mathfrak n$, and $\pi\colon\mathfrak g\to\mathfrak g_1$ the canonical quotient map. Then Lemma VIII.4(i) implies that $(\mathfrak g_1,\tau_1)$ is quasihermitian with root system $\{\alpha\in\Delta\colon\mathfrak g^\alpha\neq\mathfrak n^\alpha\}=\Delta_s$. Moreover $C_{\min,1}=\overline{\pi(C_{\min})}\subseteq C_{\max,1}$ follows from $C_{\min}+(\mathfrak a\cap\mathfrak n)\subseteq C_{\max}\subseteq C_{\max,1}$. From $C_{\max}\subseteq C_{\max,1}$ we also conclude that $\pi(W_{\max})\subseteq W_{\max,1}$ and hence that $\pi(W_{\max}^0)\subseteq W_{\max,1}^0\subseteq\mathfrak q_{\mathrm{hyp},1}$ (Corollary IX.7).

Let $X\in W_{\max}^0$. We have to show that X is hyperbolic, i.e., that it is conjugate to an element in $\mathfrak a$. The preceding observations show that there exists $\gamma\in\mathrm{Inn}_{\mathfrak g_1}(\mathfrak h_1)$ with $\gamma.\pi(X)\in\mathfrak a_1$. Choosing $\widetilde\gamma\in\mathrm{Inn}_{\mathfrak g}(\mathfrak h)$ with $\pi\circ\widetilde\gamma=\gamma\circ\pi$, we obtain $\pi(\widetilde\gamma.X)\in\mathfrak a_1$, hence that $\widetilde\gamma.X\in\pi^{-1}(\mathfrak a)\cap\mathfrak q=\mathfrak a+\mathfrak n_{\mathfrak q}$, and we may w.l.o.g. assume that $X\in\mathfrak a+\mathfrak n_{\mathfrak q}$.

Let $\mathfrak d:=\mathfrak n+\mathfrak a$. Then $\mathfrak d$ is τ-invariant so that $(\mathfrak d,\tau\,|_{\mathfrak d})$ is a solvable symmetric Lie algebra. Since the fact that $\mathfrak g$ is quasihermitian implies that Δ is split (Proposition V.9(ii)), the compact roots do not contribute to $\mathfrak n$, and we obtain the root decomposition

$$\mathfrak d=\mathfrak n_{\mathfrak h}^0\oplus\mathfrak a\oplus\bigoplus_{\alpha\in\Delta_n}\mathfrak n^\alpha.$$

An element $Y\in\mathfrak d$ is regular if and only if the dimension of $\mathfrak d^0(\mathrm{ad}\,Y)=\ker(\mathrm{ad}\,Y)_s$, where $(\mathrm{ad}\,Y)_s$ denote the semisimple part in the Jordan decomposition of $\mathrm{ad}\,Y$, is minimal. For $Y\in\mathfrak a$ and $Z\in\mathfrak n$ the fact that $\mathfrak n$ is a nilpotent ideal implies that the eigenvalues and their multiplicities for $\big(\mathrm{ad}(Y+Z)\big)_s$ are the same as for $(\mathrm{ad}\,Y)_s=\mathrm{ad}\,Y$. Hence $Y+Z$ is regular if and only if Y is regular which in turn is equivalent to $\alpha(Y)\neq 0$ for all $\alpha\in\Delta_n$ with $\mathfrak n^\alpha\neq\{0\}$.

Our element $X\in\mathfrak a+\mathfrak n_{\mathfrak q}$ from above satisfies $p_{\mathfrak a}(X)\in C_{\max}^0$ and $X-p_{\mathfrak a}(X)\in\mathfrak n$. Hence the fact that $\alpha\big(p_{\mathfrak a}(X)\big)>0$ for all $\alpha\in\Delta_n^+$ and the preceding remark show that X and $p_{\mathfrak a}(X)$ are regular elements in $\mathfrak d$. Now Proposition IX.8 implies the existence of $\gamma\in\mathrm{Inn}_{\mathfrak g}(\mathfrak n_{\mathfrak h})$ with

$$\gamma.X\in\mathfrak d^0\big(\mathrm{ad}\,p_{\mathfrak a}(X)\big)\cap\mathfrak q=\mathfrak z_{\mathfrak d_{\mathfrak q}}\big(p_{\mathfrak a}(X)\big)=\mathfrak z_{\mathfrak d_{\mathfrak q}}(\mathfrak a)=\mathfrak a.$$

This completes the proof. ■

Corollary IX.10. *If $(\mathfrak{g},\tau)$ is a symmetric Lie algebra, then $\mathfrak{q}$ contains hyperbolic invariant convex cones if and only if $(\mathfrak{g},\tau)$ is quasihermitian and there exists a $\mathfrak{p}$-adapted positive system Δ^+ such that $C_{\min} \subseteq C_{\max}$.*

If this condition is satisfied, then the cone $W_{\max}$ is hyperbolic and each invariant hyperbolic cone $W \subseteq \mathfrak{q}$ is contained in a unique cone $W_{\max}$, where Δ_n^+ is determined by $W \cap \mathfrak{a} \subseteq C_{\max}$.

Proof. The first part follows by combining Theorem IX.9 with Theorem VI.6(ii).

For the second part it only remains to apply Theorem VI.6(ii) to see that there exists a $\mathfrak{p}$-adapted positive system with $W \cap \mathfrak{a} \subseteq C_{\max}$. Then $W = \overline{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).(W \cap \mathfrak{a})} \subseteq \overline{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).C_{\max}} \subseteq W_{\max}$ and it is clear that the fact that $W \cap \mathfrak{a}$ has interior points determines Δ_n^+ and therefore $W_{\max}$ uniquely. ■

Problems IX. Is it true that for a solvable symmetric Lie algebra $(\mathfrak{g},\tau)$ all τ-invariant Cartan subalgebras $\mathfrak{t}$ are conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$? ■

X. Characterization of invariant convex hyperbolic sets

In the preceding section we have seen whenever $(\mathfrak{g},\tau)$ is a quasihermitian symmetric Lie algebra and Δ^+ is a $\mathfrak{p}$-adapted positive system such that $C_{\min} \subseteq C_{\max}$, the cone $W_{\max}$ is hyperbolic. Throughout this section we will make these assumptions.

The characterization of invariant convex sets

Theorem X.1. *For $X \in W_{\max}^0$ we have*

(i) $\overline{\mathrm{conv}(\mathcal{O}_X)}$ *is contained in* $W_{\max}^0$.

(ii) $\overline{\mathrm{conv}(\mathcal{O}_X)} = \{Y \in W_{\max}^0 : p_{\mathfrak{a}}(\mathcal{O}_Y) \subseteq \mathrm{conv}(\mathcal{W}_{\mathfrak{k}}.X) + C_{\min}\}$.

(iii) *If the cone $C_{\min}$ is pointed or if $C_{\min} = \mathrm{conv}\{[X_\alpha, \tau.X_\alpha] : \alpha \in \Delta_n^+, X_\alpha \in \mathfrak{g}^\alpha\}$, then $\mathrm{conv}(\mathcal{O}_X)$ is closed.*

Proof. (cf. [17, Th. III.12]) (i) Let $\phi: W_{\max}^0 \to \mathbb{R}^+$ denote the characteristic function of $W_{\max}$, i.e.,

$$\phi(X) = \int_{W_{\max}^\star} e^{-\alpha(X)}\, d\mu(\alpha),$$

where μ is the restriction of a Lebesgue measure in the subspace $W_{\max}^\star - W_{\max}^\star$. In view of [19, Th. III.5.4], this function has the following properties:

(1) $\phi(\gamma.x) = \chi(\gamma)\phi(x)$ for $\gamma \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$, where $\chi(\gamma) = \det_{\mathfrak{q}/H(W_{\max})}(\gamma)^{-1}$.

(2) If $x_n \to x \in \partial W_{\max}$, then $\phi(x_n) \to \infty$.

(3) ϕ is a convex function.

To strengthen (1), we show that the character χ of $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ is trivial. Since $\mathfrak{n}_{\mathfrak{q}} \subseteq H(W_{\max})$ (Lemma IX.5), this reduces the problem to the corresponding assertion on the Lie algebra $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{n}$ which is reductive. Then $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$

is a reductive subalgebra and hence $\mathrm{ad}_{\mathfrak{q}}(\mathfrak{h}) \subseteq \mathfrak{sl}(\mathfrak{q})$ follows as in the proof of Proposition IX.6. This proves that χ is trivial, hence that the function ϕ is invariant under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$.

Now the convexity of ϕ implies that

$$\mathrm{conv}(\mathcal{O}_X) \subseteq \{Y \in W_{\max}^0 : \phi(Y) \leq \phi(X)\}.$$

According to the continuity of ϕ and (2), the set $\{Y \in W_{\max}^0 : \phi(Y) \leq \phi(X)\}$ is closed in $\mathfrak{g}$, and therefore contains $\overline{\mathrm{conv}(\mathcal{O}_X)}$.

(ii) According to (i), both sets are invariant and contained in $W_{\max}^0$. Hence it suffices to check that their intersection with $\mathfrak{a}$ is the same. This is a consequence of Theorem VIII.10.

(iii) If $C_{\min}$ is pointed, then Lemma VIII.7 shows that

$$C_{\min} = F := \mathrm{conv}(\{[X, \tau.X] : X \in \mathfrak{g}^{\alpha}, \alpha \in \Delta_n^+\}).$$

Suppose that $F = C_{\min}$. Since $\mathcal{O}_X$ meets $\mathfrak{a}$, we may w.l.o.g. assume that $X \in \mathfrak{a}$. Then we have seen in the proof of Theorem VIII.10 that

$$\overline{\mathrm{conv}(\mathcal{O}_X)} \cap \mathfrak{a} = \mathrm{conv}(\mathcal{W}.X) + C_{\min} = \mathrm{conv}(\mathcal{W}.X) + F \subseteq \mathrm{conv}(\mathcal{O}_X).$$

In view of (i), this proves that $\mathrm{conv}(\mathcal{O}_X)$ is closed. ■

Theorem X.2. (Characterization Theorem) *Let $(\mathfrak{g}, \tau)$ be a quasihermitian symmetric Lie algebra and Δ^+ a $\mathfrak{p}$-adapted positive system with $C_{\min} \subseteq C_{\max}$.*

(i) *If $C \subseteq W_{\max}^0$ is an invariant subset, then* (2)$\Rightarrow$(1) *holds for the following statements:*

(1) *C is convex.*

(2) *$C_{\mathfrak{a}} := C \cap \mathfrak{a}$ is convex and $C_{\mathfrak{a}} + C_{\min} \subseteq C_{\mathfrak{a}}$.*

Furthermore, if is either C is closed or open or if $C_{\min}$ is pointed, then also (1)$\Rightarrow$(2).

(ii) *If $C_{\mathfrak{a}} \subseteq C_{\max}^0$ is a convex $\mathcal{W}$-invariant subset satisfying $C_{\mathfrak{a}} + C_{\min} \subseteq C_{\mathfrak{a}}$, then $C := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).C_{\mathfrak{a}}$ is an invariant convex subset of $W_{\max}^0$ with $C \cap \mathfrak{a} = p_{\mathfrak{a}}(C) = C_{\mathfrak{a}}$.*

Proof. (i) (cf. [17, Prop. III.14]) (2)$\Rightarrow$(1): Since C is invariant, the convex set $C_{\mathfrak{a}}$ is in particular invariant under the Weyl group $\mathcal{W}$. Moreover, (2) and Theorem VIII.10 imply for each $X \in C_{\mathfrak{a}}$ that

$$p_{\mathfrak{a}}(\mathcal{O}_X) \subseteq C_{\mathfrak{a}} + C_{\min} \subseteq C_{\mathfrak{a}}$$

and therefore that

$$p_{\mathfrak{a}}(C) = p_{\mathfrak{a}}\big(\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).C_{\mathfrak{a}}\big) \subseteq C_{\mathfrak{a}}.$$

Since $C_{\mathfrak{a}}$ is convex, it even follows that

$$\mathrm{conv}(C) \cap \mathfrak{a} \subseteq p_{\mathfrak{a}}\big(\mathrm{conv}(C)\big) = \mathrm{conv}\, p_{\mathfrak{a}}(C) \subseteq C_{\mathfrak{a}}.$$

So the invariance of $\operatorname{conv}(C) \subseteq W_{\max}^0$ and the hyperbolicity of $W_{\max}$ (Theorem IX.9) yield

$$\operatorname{conv}(C) = \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}).\big(\operatorname{conv}(C) \cap \mathfrak{a}\big) \subseteq \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}).C_{\mathfrak{a}} = C.$$

This means that C is convex.
(1) $\Rightarrow$ (2): If C is convex, then $C_{\mathfrak{a}}$ is trivially convex. Let $X \in C_{\mathfrak{a}} \subseteq C$. Then $\operatorname{conv}(\mathcal{O}_X) \subseteq C$ and therefore $X + C_{\min} \subseteq \overline{\operatorname{conv}(\mathcal{O}_X)} \cap \mathfrak{a} \subseteq \overline{C_{\mathfrak{a}}}$. Hence $C_{\min} \subseteq \lim(\overline{C_{\mathfrak{a}}})$ follows from Theorem VIII.10. If either C is closed, open, or $C_{\min}$ is pointed, then Lemma VIII.7 and Theorem X.1 show that $C_{\mathfrak{a}} + C_{\min} \subseteq C_{\mathfrak{a}}$, where we have used that if C is closed or open, then $\lim C = \lim \overline{C}$ is closed.
(ii) First we note that $\mathcal{O}_X \cap \mathfrak{a} = \mathcal{W}.X$ (Theorem III.10) and the $\mathcal{W}$-invariance of $C_{\mathfrak{a}}$ show that $C \cap \mathfrak{a} = C_{\mathfrak{a}}$. Thus (i) implies that C is convex, and the proof of (i) implies that $p_{\mathfrak{a}}(C) \subseteq C_{\mathfrak{a}}$. Therefore $C_{\mathfrak{a}} \subseteq C \cap \mathfrak{a} \subseteq p_{\mathfrak{a}}(C) \subseteq C_{\mathfrak{a}}$ and the equality follows. ∎

Theorem X.3. (Characterization and Reconstruction of Invariant Cones) *Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra.*

(i) *There is a pointed generating invariant hyperbolic closed convex cone in $\mathfrak{q}$ if and only if*

(1) $(\mathfrak{g}, \tau)$ has strong cone potential,

(2) is quasihermitian, and

(3) $\mathfrak{z}(\mathfrak{p}) \neq \{0\}$ holds for a maximal hyperbolic Lie triple system $\mathfrak{p} \subseteq \mathfrak{q}$.

(ii) *Suppose that $(\mathfrak{g}, \tau)$ is quasihermitian and Δ^+ is a $\mathfrak{p}$-adapted positive system with $C_{\min} \subseteq C_{\max}$. A closed convex subset $C_{\mathfrak{a}} \subseteq C_{\max}$ with interior points arises as the trace $C \cap \mathfrak{a}$ of a closed convex hyperbolic set $C \subseteq \mathfrak{q}$ if and only if $C_{\mathfrak{a}}$ is $\mathcal{W}$-invariant and $C_{\min} \subseteq \lim(C_{\mathfrak{a}})$.*

Suppose that (1)-(3) *in* (i) *are satisfied.*

(iii) *A pointed generating closed convex cone $C_{\mathfrak{a}} \subseteq \mathfrak{a}$ arises as the trace $C \cap \mathfrak{a}$ of a pointed generating invariant closed convex hyperbolic cone $C \subseteq \mathfrak{q}$ if and only if $C_{\mathfrak{a}}$ is $\mathcal{W}$-invariant and there exists a $\mathfrak{p}$-adapted positive system Δ^+ such that $C_{\min} \subseteq C \subseteq C_{\max}$.*

Proof. (i) The necessity of (1)-(3) follows from Theorem VI.6. To prove the sufficiency, let us assume that (1)-(3) are satisfied. We are going to construct a cone with the desired properties. Let Δ^+ be a $\mathfrak{p}$-adapted positive system. According to Theorem VII.18(iii), we have $C_{\min} \subseteq C_{\max}$.

We claim that there exists a pointed generating $\mathcal{W}$-invariant closed convex cone $C_{\mathfrak{a}} \subseteq \mathfrak{a}$ satisfying $C_{\min} \subseteq C_{\mathfrak{a}} \subseteq C_{\max}$.

Let $0 \neq X \in \mathfrak{z}(\mathfrak{p}) \cap C_{\max}^0$ (Proposition V.4(3)) and $K \subseteq C_{\max}^0$ a $\mathcal{W}$-invariant compact convex neighborhood of X not containing 0. Then $\mathbb{R}^+ K$ is a pointed generating $\mathcal{W}$-invariant closed convex cone in $\mathfrak{a}$. We put $C_{\mathfrak{a}} := C_{\min} + \mathbb{R}^+ K$ and show that $C_{\mathfrak{a}}$ satisfies all our requirements:

First we note that the cone

$$C_{\min} \cap -\mathbb{R}^+ K \subseteq H(C_{\max}) \cap -\mathbb{R}^+ K$$

is $\{0\}$. In fact, if $C_{\max} \neq \mathfrak{a}$ this follows from $H(C_{\max}) \cap -\mathbb{R}^+K = \{0\}$, and if $C_{\max} = \mathfrak{a}$, then $\Delta_n = \emptyset$, hence $C_{\min} = \{0\}$ and it is also clear. Now [5, Prop. 1.4] implies that C is closed with $H(C_{\mathfrak{a}}) = H(C_{\min}) + H(\mathbb{R}^+K) = \{0\} + \{0\} = \{0\}$. Since K has interior points, the same holds for $C_{\mathfrak{a}}$, and the $\mathcal{W}$-invariance of $\mathbb{R}^+K$ implies the $\mathcal{W}$-invariance of $C_{\mathfrak{a}}$. Moreover, $C_{\min} \subseteq C_{\mathfrak{a}} \subseteq C_{\max}$ holds by construction. Now the assertion follows from (iii).

(ii) If $C \subseteq \mathfrak{q}$ is an invariant hyperbolic convex set, then clearly $C_{\mathfrak{a}} = C \cap \mathfrak{a}$ is $\mathcal{W}$-invariant, and $C_{\min} \subseteq \lim(C_{\mathfrak{a}})$ follows from Theorem VI.6 because the fact that $C_{\mathfrak{a}}$ has interior points determines the cone $C_{\max}$ containing $C_{\mathfrak{a}}$ uniquely.

Suppose that these conditions are satisfied by $C_{\mathfrak{a}} \subseteq C_{\max}$ and put $C := \overline{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).C_{\mathfrak{a}}}$. That C is generating follows from Lemma VI.5(ii), and since closures of convex sets are convex, we obtain from Theorem X.2 and $C = \overline{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).C_{\mathfrak{a}}^0}$ that C is hyperbolic and convex. Moreover,

$$C_{\mathfrak{a}} \subseteq C \cap \mathfrak{a} \subseteq p_{\mathfrak{a}}(C) \subseteq \overline{p_{\mathfrak{a}}\big(\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).C_{\mathfrak{a}}^0\big)} \subseteq \overline{C_{\mathfrak{a}}}^0 = C_{\mathfrak{a}}$$

follows from Theorem X.2(ii). This proves that $C_{\mathfrak{a}} = C \cap \mathfrak{a}$.

(iii) As noted in (i), the necessity of the condition follows from Theorem VI.6(ii). If, conversely, they are satisfied, then (ii) implies the existence of a hyperbolic closed convex invariant subset $C \subseteq \mathfrak{q}$ with $C \cap \mathfrak{a} = C_{\mathfrak{a}}$. The hyperbolicity of C implies that $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}).C_{\mathfrak{a}}$ is dense in C, hence that C is a cone. So it remains to show that C is pointed. As $p_{\mathfrak{a}}\big(H(C)\big) \subseteq H(C) \cap C_{\mathfrak{a}} = \{0\}$, we conclude that $H(C) \subseteq [\mathfrak{a}, \mathfrak{h}]$. Since, according to Proposition VII.2(iv), $[\mathfrak{a}, \mathfrak{h}]$ contains no non-zero $\mathfrak{h}$-submodule, $H(C) = \{0\}$ follows. ∎

Extension of convex invariant hyperbolic sets

In this section we deal with the problem of extending hyperbolic invariant convex sets in $\mathfrak{q}$ to hyperbolic invariant convex sets in $\hat{\mathfrak{q}} = i\mathfrak{g}^c$. First we have a look at some crucial examples.

Example X.4. (a) We consider the five dimensional Heisenberg algebra

$$\mathfrak{h}_2 = \mathbb{R}Z \oplus \mathbb{R}Q_1 \oplus \mathbb{R}P_1 \oplus \mathbb{R}Q_2 \oplus \mathbb{R}P_2$$

with non-zero brackets $[Q_1, P_1] = [Q_2, P_2] = Z$. Let $\mathrm{mot}(2) = V \rtimes \mathbb{R}U$, where $V \cong \mathbb{R}^2$ and U as in Example IV.11, be the motion algebra of the euclidean plane. We define a homomorphism of $\mathfrak{l} = \mathrm{mot}(2) \oplus \mathbb{R}T$ into $\mathrm{der}(\mathfrak{h}_2)$ by

$$V.\mathfrak{h}_2 = \{0\}, \quad U.Q_1 = Q_2, \quad U.Q_2 = -Q_1, \quad U.P_1 = P_2, \quad U.P_2 = -P_1$$

and $T.Q_i = P_i$, $T.P_i = Q_i$ for $i \in \{1,2\}$. This turns $\mathfrak{g} = \mathfrak{h}_2 \rtimes \mathfrak{l}$ into a Lie algebra and

$$\mathfrak{h} = \mathrm{span}\{P_1, P_2\} \rtimes \mathrm{mot}(2) \quad \text{and} \quad \mathfrak{q} = \mathrm{span}\{Z, Q_1, Q_2, T\}$$

defines an involutive automorphism τ on $\mathfrak{g}$ with $\tau|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$ and $\tau|_{\mathfrak{q}} = -\mathrm{id}_{\mathfrak{q}}$. A maximal hyperbolic abelian subspace which is also maximal abelian in $\mathfrak{q}$ is given by

$$\mathfrak{a} = \mathbb{R}Z \oplus \mathbb{R}T.$$

We have $\Delta = \Delta_r = \{\pm\alpha\}$ with $\alpha(T) = 1$ and $\mathfrak{g}^\alpha = \mathbb{R}(P_1 + Q_1) + \mathbb{R}(P_2 + Q_2)$. Therefore $(\mathfrak{g}, \tau)$ has strong cone potential, $\mathfrak{p} = \mathfrak{a}$ is a maximal hyperbolic Lie triple system, so that the fact that $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$ implies that $(\mathfrak{g}, \tau)$ is quasihermitian. Therefore $\mathfrak{q}$ admits invariant hyperbolic cones (Theorem X.3) and $\mathfrak{a}$ extends to a maximal hyperbolic abelian and maximal abelian subspace $\hat{\mathfrak{a}} \subseteq \hat{\mathfrak{q}}$ (Theorem VIII.1), namely,

$$\hat{\mathfrak{a}} = \mathbb{R}Z \oplus \mathbb{R}T \oplus \mathbb{R}iU.$$

Also $(\mathfrak{g}_{\mathbb{C}}, \hat{\tau})$ is quasihermitian, but does not contain any pointed generating invariant hyperbolic cone since it has no cone potential (Theorem X.3). In particular, it is impossible to extend an invariant pointed generating hyperbolic convex cone $C \subseteq \mathfrak{q}$ to an invariant pointed generating hyperbolic convex cone in $\hat{\mathfrak{q}}$. We also note that $\mathfrak{h}^0 \cong \mathfrak{mot}(2)$ is not compactly embedded, but it has a compactly embedded Cartan subalgebra.

This situation changes drastically if we factor out the ideal $V \subseteq \mathfrak{g}$ and obtain the effective Lie algebra $(\mathfrak{g}_0, \tau_0) := (\mathfrak{g}/V, \widetilde{\tau})$, where $\widetilde{\tau}$ is the canonical involution induced on the quotient. Now any invariant pointed generating hyperbolic convex cone in $\mathfrak{q}_0$ can be extended to an invariant pointed generating hyperbolic convex cone in $\hat{\mathfrak{q}}$ (cf. Theorem X.7 below).

(b) (cf. [17, Ex. II.9(c)]) Let

$$\mathfrak{s}_1 := \mathfrak{sl}(2,\mathbb{R}) := \mathrm{span}\{H, T, U\} \quad \text{and} \quad \mathfrak{s}_2 := \mathfrak{sl}(2,\mathbb{R}) := \mathrm{span}\{H', T', U'\}$$

as in the notation of Example IV.10. We denote by V_j, $j \in \{1, 2\}$ the 2-dimensional real $\mathfrak{s}_j$-module and let $P = (1, 1)$, $Q = (1, -1) \in V_1$. Now we define

$$\mathfrak{g} = (V_1 \otimes V_2) \rtimes (\mathfrak{s}_1 \oplus \mathfrak{s}_2).$$

and equip $\mathfrak{g}$ with an involution τ on $\mathfrak{g}$ via

$$\mathfrak{h} = (\mathbb{R}P \otimes V_2) \rtimes (\mathbb{R}T \oplus \mathfrak{s}_2) \quad \text{and} \quad \mathfrak{q} = (\mathbb{R}Q \otimes V_2) \oplus \mathrm{span}\{H, U\}.$$

A maximal hyperbolic abelian and maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{q}$ is given by $\mathfrak{a} = \mathbb{R}H$. Note that $\mathfrak{a}$ is a maximal hyperbolic Lie triple system and thus $(\mathfrak{g}, \tau)$ is quasihermitian. We can extend $\mathfrak{a}$ to a maximal hyperbolic abelian subspace $\hat{\mathfrak{a}} \subseteq \hat{\mathfrak{q}}$, which is also maximal abelian in $\mathfrak{q}$, by

$$\hat{\mathfrak{a}} = \mathbb{R}H \oplus \mathbb{R}iU'.$$

Since $\hat{\mathfrak{a}}$ is a maximal hyperbolic Lie triple system in $\hat{\mathfrak{q}}$ we see that $(\mathfrak{g}_{\mathbb{C}}, \hat{\tau})$ is quasihermitian. Note also that $(\mathfrak{g}, \tau)$ and $(\mathfrak{g}_{\mathbb{C}}, \hat{\tau})$ are effective. Let $\alpha \in \mathfrak{a}^*$ be given by $\alpha(H) = 1$ and extend α to an element of $\hat{\mathfrak{a}}^*$ by setting $\alpha(iU) = 0$.

Further we define $\beta \in \hat{\mathfrak{a}}^*$ by $\beta(iU') = 1$ and $\beta(H) = 0$. Then the root systems are given by

$$\Delta = \{\underbrace{\pm\alpha}_{\Delta_r}, \underbrace{\pm 2\alpha}_{\Delta_s}\} \qquad \text{and} \qquad \hat{\Delta} = \{\underbrace{\pm\alpha \pm \beta}_{\hat{\Delta}_r}, \underbrace{\pm 2\alpha, \pm 2\beta}_{\hat{\Delta}_s}\}.$$

Hence for any positive system Δ^+ we always have $C_{\min} \subseteq C_{\max}$ but, as it has already been shown in [17, Ex.II.9(c)], there is no positive system $\hat{\Delta}^+$ such that $\hat{C}_{\min} \subseteq \hat{C}_{\max}$. By Theorem VI.6 this means that there are no generating invariant hyperbolic convex sets in $\hat{\mathfrak{q}}$. In particular, we cannot extend any invariant generating hyperbolic convex set in $\mathfrak{q}$ to an invariant hyperbolic convex set in $\hat{\mathfrak{q}}$. ■

The first example above shows that for the sake of extension theorems it is reasonable to assume that $(\mathfrak{g}, \tau)$ is effective. Note that passing to the effective quotient algebra touches in no way all the properties of hyperbolic sets nor changes the root structure and the minimal and maximal cone.

From now on we assume that the subalgebra $\mathfrak{h}^0$ is compactly embedded so that Theorem VIII.1 applies to $(\mathfrak{g}_{\mathbb{C}}, \hat{\tau})$.

Proposition X.5. *We consider the subspaces $\mathfrak{a} \subseteq \hat{\mathfrak{a}}$ and the corresponding Weyl groups $\mathcal{W}$ and $\hat{\mathcal{W}}$. Let further Δ_k^+ and $\hat{\Delta}_k^+$ be positive systems with $\hat{\Delta}_k^+|_{\mathfrak{a}} \subseteq \Delta_k^+ \cup \{0\}$, and $\mathfrak{a}^+ = (\Delta_k^+)^\star$, resp. $\hat{\mathfrak{a}}^+ = (\hat{\Delta}_k^+)^\star$, denote the corresponding fundamental domains for $\mathcal{W}$, resp. $\hat{\mathcal{W}}$. Then the following assertions hold:*

(i) $p_{\mathfrak{a}}(\hat{\mathfrak{a}}^+) = \hat{\mathfrak{a}}^+ \cap \mathfrak{a} = \mathfrak{a}^+$.

(ii) *If $\hat{C} \subseteq \hat{\mathfrak{a}}$ is a $\hat{\mathcal{W}}$-invariant set and $\hat{C}^+ := \hat{C} \cap \mathfrak{a}^+$ is convex, then*

$$p_{\mathfrak{a}}(\operatorname{conv} \hat{C}) = \operatorname{conv}\big(\mathcal{W}.p_{\mathfrak{a}}(\hat{C}^+)\big).$$

(iii) *For $X \in \hat{\mathfrak{a}}^+$ we have $p_{\mathfrak{a}}(\operatorname{conv} \hat{\mathcal{W}}.X) = \operatorname{conv}\big(\mathcal{W}.p_{\mathfrak{a}}(X)\big)$.*

(iv) *Let $C \subseteq \mathfrak{a}$ be a $\mathcal{W}$-invariant convex set. Then*

$$\hat{C} := \{X \in \hat{\mathfrak{a}} \colon p_{\mathfrak{a}}(\hat{\mathcal{W}}.X) \subseteq C\}$$

is a $\hat{\mathcal{W}}$-invariant convex subset of $\hat{\mathfrak{a}}$ with $\hat{C} \cap \mathfrak{a} = p_{\mathfrak{a}}(\hat{C}) = C$. If C has interior points, then the same holds for $\hat{C}$.

Proof. (i) From the compatibility of the positive systems Δ_k^+ and $\hat{\Delta}_k^+$ it follows that $\mathfrak{a}^+ = \hat{\mathfrak{a}}^+ \cap \mathfrak{a} \subseteq p_{\mathfrak{a}}(\hat{\mathfrak{a}}^+)$. Let $X \in \hat{\mathfrak{a}}^+$. Then $p_{\mathfrak{a}}(X) = \frac{1}{2}(X - \tau.X)$. If $\alpha \in \Delta_k^+$, then there exists $\widetilde{\alpha} \in \hat{\Delta}_k$ with $\widetilde{\alpha}|_{\mathfrak{a}} = \alpha$. Moreover $\widetilde{\alpha}$ must be positive because otherwise $\widetilde{\alpha}|_{\mathfrak{a}} \in \Delta_k^- \cup \{0\}$, and since $-\tau.\alpha$ has the same restriction to $\mathfrak{a}$, this root must also be positive. We conclude that

$$2\langle \alpha, p_{\mathfrak{a}}(X)\rangle = \langle \alpha, X - \tau.X\rangle = \langle \alpha - \tau.\alpha, X\rangle \geq 0.$$

Hence $p_{\mathfrak{a}}(\hat{\mathfrak{a}}^+) \subseteq \mathfrak{a}^+$, and (i) follows.

(ii) Let $D := \operatorname{conv} \hat{C}$. Then D is a $\hat{\mathcal{W}}$-invariant convex set. Since for each element $w \in \mathcal{W}$ there exists $\hat{w} \in \hat{\mathcal{W}}$ with $\hat{w}|_{\mathfrak{a}} = w$ and $\hat{w}$ commutes with $\tau|_{\hat{\mathfrak{a}}}$

(Theorem VIII.1(iv)), the set $p_{\mathfrak a}(D)$ is convex and invariant under $\mathcal W$. It follows in particular that "$\supseteq$" holds.

On the other hand [19, Prop. III.2.9] implies that $D \subseteq \hat C^+ - \mathrm{cone}(\check{\hat\Delta}_k^+)$ so that $p_{\mathfrak a}\big(\mathrm{cone}(\check{\hat\Delta}_k^+)\big) = \mathrm{cone}(\check\Delta_k^+)$ (Theorem VIII.1(vi)(d)) implies that

$$p_{\mathfrak a}(D) \subseteq p_{\mathfrak a}(\hat C^+) - p_{\mathfrak a}\big(\mathrm{cone}(\check{\hat\Delta}_k^+)\big) = p_{\mathfrak a}(\hat C^+) - \mathrm{cone}(\check\Delta_k^+),$$

where (i) implies that $p_{\mathfrak a}(\hat C^+) \subseteq \mathfrak a^+$. Now the $\mathcal W$-invariance of $p_{\mathfrak a}(D)$ shows that

$$p_{\mathfrak a}(D) \subseteq \bigcap_{w\in\mathcal W} w.\big(p_{\mathfrak a}(\hat C^+) - \mathrm{cone}(\check\Delta_k^+)\big) = \mathrm{conv}\,\big(\mathcal W.p_{\mathfrak a}(\hat C^+)\big)$$

(cf. [19, Prop. III.2.9]). This completes the proof of (ii).
(iii) This is a special case of (ii).
(iv) The inclusions $\hat C \cap \mathfrak a \subseteq p_{\mathfrak a}(\hat C) \subseteq C$ are trivial consequences of the definition of $\hat C$. Let $X \in C$. Then there exists $X' \in \mathfrak a^+$ with $X \in \mathcal W.X'$. Now (iii) implies that

$$p_{\mathfrak a}(\hat{\mathcal W}.X) = p_{\mathfrak a}(\hat{\mathcal W}.X') \subseteq \mathrm{conv}(\mathcal W.X') \subseteq C.$$

Therefore $C \subseteq \hat C \cap \mathfrak a$, and thus $C = \hat C \cap \mathfrak a = p_{\mathfrak a}(\hat C)$.

If, in addition, C has interior points, then C^0 contains a fixed point X_0 for $\mathcal W$, i.e., $\alpha(X) = 0$ holds for all $\alpha \in \Delta_k$. Then we also have $\alpha(X) = 0$ for all $\alpha \in \hat\Delta_k$, i.e. X is fixed by $\hat{\mathcal W}$. Now it is clear that $\hat C$ contains a sufficiently small neighborhood of X, so that $\hat C$ has interior points. ■

Let G^c be a connected Lie group with Lie algebra $\mathfrak g^c$ and suppose that τ integrates to an involution of G^c, which is also denoted by τ. Further we denote by H^τ the group of τ-fixed points and by H_0^τ its identity component. Let $H \subseteq G^c$ be any subgroup of G such that

$$H_0^\tau \subseteq H \subseteq H^\tau.$$

For an element $X \in \mathfrak q$ we define $\mathcal O_X^H = \mathrm{Ad}(H).X$ and $\mathcal O_X^{G^c} = \mathrm{Ad}(G^c).X$.

Theorem X.6. *Suppose that $\mathfrak h^0$ is compactly embedded, $(\mathfrak g,\tau)$ is quasihermitian, Δ^+ is a $\mathfrak p$-adapted positive system, and $\hat\Delta^+$ a compatible $\hat{\mathfrak p}$-adapted positive system satisfying $\hat C_{\min} \subseteq \hat C_{\max}$. Then for $X \in C_{\max}^0$ the following assertions hold:*

$$\overline{\mathrm{conv}(\mathcal O_X^{G^c})} \cap \hat{\mathfrak a} = p_{\hat{\mathfrak a}}\big(\overline{\mathrm{conv}(\mathcal O_X^{G^c})}\big) = \mathrm{conv}(\hat{\mathcal W}.X) + \hat C_{\min} \tag{10.1}$$

$$\overline{\mathrm{conv}(\mathcal O_X^{H})} \cap \mathfrak a = p_{\mathfrak a}\big(\overline{\mathrm{conv}(\mathcal O_X^{H})}\big) = \mathrm{conv}(\mathcal W.X) + C_{\min}. \tag{10.2}$$

In particular, $p_{\mathfrak a}(\overline{\mathrm{conv}(\mathcal O_X^H)}) = p_{\mathfrak a}(\overline{\mathrm{conv}(\mathcal O_X)})$ is independent of the choice of H. Moreover

$$\begin{aligned} p_{\mathfrak a}\big(\overline{\mathrm{conv}(\mathcal O_X^{G^c})}\big) &= \overline{\mathrm{conv}(\mathcal O_X^{G^c})} \cap \mathfrak a = \overline{\mathrm{conv}(\mathcal O_X^{H})} \cap \mathfrak a \\ &= p_{\mathfrak a}\big(\overline{\mathrm{conv}(\mathcal O_X^{H})}\big) = \mathrm{conv}(\mathcal W.X) + C_{\min}. \end{aligned} \tag{10.3}$$

Proof. It follows from Theorem VIII.1(vi)(c) that $X \in \hat{C}^0_{\max}$. Equation (10.2) for $H = H_0$ is obtained from Theorem VIII.10. Since $\hat{C}_{\min} \subseteq \hat{C}_{\max}$ by assumption, we can apply Theorem VIII.10 also to the canonical extension $(\mathfrak{g}_{\mathbb{C}}, \hat{\tau})$ and obtain (10.1). Using Theorem VIII.1(vi)(a) and Proposition X.5(iii), we get

$$\begin{aligned}\operatorname{conv}(\mathcal{W}.X) + C_{\min} &= p_{\mathfrak{a}}\big(\overline{\operatorname{conv}(\mathcal{O}_X^{H_0})}\big) \subseteq p_{\mathfrak{a}}\big(\overline{\operatorname{conv}(\mathcal{O}_X^{H})}\big)\\ &\subseteq p_{\mathfrak{a}}\big(\overline{\operatorname{conv}(\mathcal{O}_X^{G^c})}\big) \subseteq p_{\mathfrak{a}}\big(\operatorname{conv}(\hat{\mathcal{W}}.X) + \hat{C}_{\min}\big)\\ &= p_{\mathfrak{a}}\big(\operatorname{conv}(\hat{\mathcal{W}}.X)\big) + p_{\mathfrak{a}}(\hat{C}_{\min})\\ &= \operatorname{conv}(\mathcal{W}.X) + C_{\min}.\end{aligned}$$

This proves (10.2) for arbitrary H and also (10.3). ∎

Theorem X.7. (The Extension Theorem) *We keep the assumptions from* Theorem X.6 *and write* $\hat{W}_{\max} := \overline{\operatorname{Ad}(G^c).\hat{C}_{\max}}$ *for the maximal cone in* $\hat{\mathfrak{q}} = i\mathfrak{g}^c$.

(i) *Every* $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$*-invariant closed hyperbolic convex set* $C \subseteq W_{\max}$ *is* H*-invariant and can be extended by*

$$\hat{C} := \{X \in \hat{W}_{\max} : p_{\mathfrak{a}}(\mathcal{O}_X^{G^c}) \subseteq C \cap \mathfrak{a}\}$$

to a $\{G^c, -\tau\}$*-invariant hyperbolic convex subset of* $\hat{W}_{\max}$. *The extension* $\hat{C}$ *is maximal with respect to*

$$\hat{C} \cap \mathfrak{q} = p_{\mathfrak{q}}(\hat{C}) = C. \tag{10.4}$$

(ii) *If, in addition,* C *is a pointed generating invariant hyperbolic cone and* $(\mathfrak{g}, \tau)$ *is effective, then* $\hat{C}$ *is a pointed generating invariant hyperbolic cone.*

Proof. (i) From $\hat{C} = \hat{W}_{\max} \cap \bigcap_{g \in G^c} g.p_{\mathfrak{a}}^{-1}(C \cap \mathfrak{a})$ it follows that $\hat{C}$ is a closed convex $\{G^c, -\tau\}$-invariant subset of $\hat{W}_{\max}$. Moreover (10.3) in Theorem X.6 implies that for $X \in C^0 \cap \mathfrak{a}$ we have

$$p_{\mathfrak{a}}(\mathcal{O}_X^{G^c}) \subseteq \overline{\operatorname{conv}(\mathcal{O}_X^{H_0})} \cap \mathfrak{a} \subseteq C \cap \mathfrak{a},$$

hence that $C^0 \cap \mathfrak{a} \subseteq \hat{C}$, and the hyperbolicity of C entails that $C \subseteq \hat{C} \cap \mathfrak{q}$. On the other hand the $-\tau$-invariance entails $\hat{C} \cap \mathfrak{q} = p_{\mathfrak{q}}(\hat{C})$, so it remains to show that $\hat{C}$ is hyperbolic and that $\hat{C} \cap \mathfrak{q} \subseteq C$.

We put $C_{\mathfrak{a}} := C \cap \mathfrak{a}$ and $\hat{C}_{\mathfrak{a}} := \{X \in \hat{C}_{\max} : p_{\mathfrak{a}}(\hat{\mathcal{W}}.X) \subseteq C_{\mathfrak{a}}\}$. Then Proposition X.5(iv) shows that $p_{\mathfrak{a}}(\hat{C}_{\mathfrak{a}}) = \hat{C}_{\mathfrak{a}} \cap \mathfrak{a} = C_{\mathfrak{a}}$ and that $\hat{C}_{\mathfrak{a}}$ has interior points.

For $X \in \hat{C}^0_{\mathfrak{a}}$ we now conclude with Theorem X.6 that

$$p_{\mathfrak{a}}(\mathcal{O}_X^{G^c}) = \operatorname{conv}(\mathcal{W}.X) + C_{\min} \subseteq C_{\mathfrak{a}} + C_{\min} \subseteq C_{\mathfrak{a}}.$$

This proves that $\hat{C}^0_{\mathfrak{a}}$ and therefore that $\hat{C}_{\mathfrak{a}}$ is contained in $\hat{C} \cap \hat{\mathfrak{a}}$. The converse inclusion follows from $\hat{\mathcal{W}}.X \subseteq \mathcal{O}_X^{G^c}$, and thus we obtain $\hat{C} \cap \hat{\mathfrak{a}} = \hat{C}_{\mathfrak{a}}$. This proves that $\hat{C}$ has interior points (Lemma VI.5(ii)), and so $\hat{C}$ is hyperbolic.

To see that $\hat{C} \cap \mathfrak{q} \subseteq C$, we note that the set on the left hand side is hyperbolic because $(\hat{C} \cap \mathfrak{q})^0 = \hat{C}^0 \cap \mathfrak{q} \subseteq \hat{W}^0_{\max}$. On the other hand it is $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant, so that it suffices to show that $\hat{C} \cap \mathfrak{a} \subseteq C_{\mathfrak{a}}$ which trivially follows from the definition of $\hat{C}$. This proves (10.4). Finally, as $\hat{C}$ is H-invariant and $p_{\mathfrak{q}}$ commutes with the action of H, we conclude that C is also H-invariant.
(ii) Let C be a pointed generating invariant hyperbolic cone in $\mathfrak{q}$. Then $\hat{C}$ is a generating invariant hyperbolic cone in $\hat{\mathfrak{q}}$, and it remains to show that $\hat{C}$ is pointed. Since C is pointed, $p_{\mathfrak{q}}\big(H(\hat{C})\big) = \{0\}$, so that $H(\hat{C})$ is a $\mathfrak{g}^c$-invariant subspace of $i\mathfrak{h} \subseteq \hat{\mathfrak{q}}$. This means that $iH(\hat{C})$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$. Now the assumption that $(\mathfrak{g}, \tau)$ is effective shows that $H(\hat{C}) = \{0\}$, i.e., that $\hat{C}$ is pointed. ∎

Remark X.8. In the preceding two theorems we have needed the assumptions that $\mathfrak{h}^0$ is compactly embedded to apply Theorem VIII.1 and that $\hat{C}_{\min} \subseteq \hat{C}_{\max}$ to apply the convexity theorems to $\hat{\mathfrak{q}}$ itself. Moreover, to extend pointed cones in $\mathfrak{q}$ to pointed cones in $\hat{\mathfrak{q}}$, the effectiveness was a crucial condition.

Suppose that $(\mathfrak{g}, \tau)$ is semisimple and quasihermitian. Then $(\mathfrak{g}_{\mathbb{C}}, \hat{\tau})$ also is semisimple and quasihermitian (Theorem VIII.1), and Proposition V.9(vii) shows that $\hat{C}_{\min} \subseteq \hat{C}_{\max}$ is automatically satisfied. In this case the effectiveness is also not a severe restriction because each ideal is a direct summand. ∎

Problems X. (1) Does the requirement that $X \in \hat{\mathfrak{q}}$ satisfies $p_{\mathfrak{a}}(\mathcal{O}_X^{G^c}) \subseteq C_{\max}$ imply that $X \in \hat{W}_{\max}$? In this case it would not be necessary in the Extension Theorem to take the intersection with $\hat{W}_{\max}$.
(2) Does the assumption that $\mathfrak{h}^0$ is compactly embedded (cf. Theorem VIII.1) and $C_{\min} \subseteq C_{\max}$ imply that $\hat{C}_{\min} \subseteq \hat{C}_{\max}$. Note that in Example X.4(b) the subalgebra $\mathfrak{h}^0 \cong \mathfrak{sl}(2, \mathbb{R})$ is not compactly embedded. ∎

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