Journal of Lie Theory
Volume **7** (1997) 101–120

The Quantum Double of a (locally) Compact Group

T. H. Koornwinder and N. M. Muller

Communicated by B. Ørsted

Abstract. We generalise the quantum double construction of Drinfel'd to the case of the (Hopf) algebra of suitable functions on a compact or locally compact group. We will concentrate on the $*$-algebra structure of the quantum double. If the conjugacy classes in the group are countably separated, then we classify the irreducible $*$-representations by using the connection with so–called transformation group algebras. For finite groups, we will compare our description to the result of Dijkgraaf, Pasquier and Roche. Finally we will work out the explicit examples of $SU(2)$ and $SL(2,\mathbb{R})$.

The quantum double of a Hopf algebra (or, double quantum group) was introduced by Drinfel'd in [9]. Quantum doubles are important examples of quasitriangular (quasi) Hopf algebras, and in that sense they are well-studied, see for instance [6], [23], [18].

The existing theory of quantum doubles has beautiful applications in physics: in [7] Dijkgraaf, Pasquier and Roche show that the representation theory covers the main interesting data of particular orbifolds of Rational Conformal Field Theories. Tightly connected are the topological interactions in spontaneously broken gauge theories. In [2], [3] Bais, Van Driel and De Wild Propitius show that the non-trivial fusion and braiding properties of the excited states in broken gauge theories can be fully described by the representation theory of a quantum double which is constructed from a finite group G via a finite dimensional Hopf algebra $\mathcal{A} = C[G]$ (the corresponding group ring). For a detailed treatment, see [26].

From a physical as well as a mathematical point of view it is natural to ask whether the quantum double construction from a finite group can be generalized to the case of a compact, or even locally compact group. In this report we will explicitly construct the quantum double $\mathcal{D}(G)$ corresponding to a (locally) compact group G. It has a natural $*$-structure, and we will find a class of $*$-representations (unitary representations), and prove rigorously that they form a complete set of irreducible $*$-representations. The construction uses the representation theory of *transformation group algebras*, which we will discuss in detail, and its connections with the theory of induced representations of locally

ISSN 0949–5932 / \$2.50

compact groups via the imprimitivity theorem of Mackey [14]. For an overview of the theory of (induced) representations of locally compact groups, see for instance Chapters 9–10–11 in [12]. In fact, the same construction works more generally for classifying the irreducible $*$-representations of transformation group algebras, see Glimm [11].

To be more precise, the construction is done in the following way: For the generalization of the quantum double, we choose the algebra $C_c(G \times G)$ of continuous functions on $G \times G$ with compact support. This allows us to use the representation theory of transformation group algebras $C_c(X \times G)$, where the locally compact group G acts continuously on a locally compact space X. Under the technical (but crucial) assumption that the conjugate action of G on G is *countably separated*, classification of the irreducible $*$-representations of these algebras turns out to be equivalent to classifying the irreducible representations (τ, P) of the pair $(G, \mathcal{O})$, with $\mathcal{O}$ an orbit of G on X. Writing $G/H \simeq \mathcal{O}$, with H a closed subgroup of G, then $(G, G/H, P)$ is a *system of imprimitivity* for the unitary representation τ of G on a Hilbert space V, and P is a projection valued measure on G/H acting on V. From Mackey's imprimitivity theorem it follows that such representations τ are precisely the representations of G which are induced from unitary irreducible representations α of H. The classification of irreducible $*$-representations of the quantum double is a direct consequence.

We will show that the case of finite G is covered by our description, and it leads to representations which are isomorphic to the ones derived in [7].

Finally we will work out the explicit examples of $G = SU(2)$ (compact) and $G = SL(2, \mathbb{R})$ (non–compact). Their interesting applications in physics will be discussed in a forthcoming paper by the second author, where the connection with a quantization of a Chern–Simons theory in (2+1) dimensions will be described.

We are also studying in detail the coalgebra structure of the continuous quantum double, and in a follow–up of this paper we will give the tensor product decomposition ('fusion rules') for the quantum double representations, the universal R-matrix, and its action on a tensor product state.

In fact, questions about fusion rules were our original motivation for this work. However, it soon became apparent that even the definition of the quantum double of $C(G)$, and the classification of the irreducible representations of this quantum double had to be clarified. This led us to a thorough study of quite some older literature on transformation group algebras, which turned out to be well applicable for our case of the quantum double. To our knowledge these references have not been put together in this combination before.

We do not claim originality in the contents of our main results. Notably, the main Theorem 3.9 occurs in Glimm [16], in a somewhat hidden way. Our reformulation of the theorem may be more suitable for applications, for instance in physics, and makes it possible to treat concrete examples. For expository reasons we have added our own version of the proof, with emphasis on the link with the imprimitivity theorem. It gives insight into the way we have derived the characterisation and classification of the irreducible unitary representations of the quantum double. The latter is necessary for the computation of the fusion rules and for the braiding properties (R-matrix) of the model.

1. Construction

Drinfel'd [9] gives the following definition of the *quantum double* $\mathcal{D}(\mathcal{A})$ of a Hopf algebra $\mathcal{A}$. (This definition is only mathematically precise if $\mathcal{A}$ is a finite dimensional Hopf algebra. However, there is a way out by working with a dual pair of bialgebras, see Majid [18], p.296.)

Definition 1.1. Let $\mathcal{A}$ be a Hopf algebra over the field C, and $\mathcal{A}^0$ the dual Hopf algebra to $\mathcal{A}$ with the opposite comultiplication. Then $\mathcal{D}(\mathcal{A})$ is the unique quasi-triangular Hopf algebra with universal R-matrix $R \in \mathcal{D}(\mathcal{A}) \otimes \mathcal{D}(\mathcal{A})$ such that

i. As a vector space, $\mathcal{D}(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}^0$.

ii. $\mathcal{A} = \mathcal{A} \otimes 1$ and $\mathcal{A}^0 = 1 \otimes \mathcal{A}^0$ are Hopf subalgebras of $\mathcal{D}(\mathcal{A})$.

iii. The mapping $x \otimes \xi \mapsto x\xi : \mathcal{A} \otimes \mathcal{A}^0 \to \mathcal{D}(\mathcal{A})$ is an isomorphism of vector spaces. Here $x\xi$ is short notation for the product $(x \otimes 1)(1 \otimes \xi)$.

iv. Let $(e_i)_{i\in I}$ be a basis of $\mathcal{A}$ and $(e^i)_{i\in I}$ the dual basis of $\mathcal{A}^0$. Then
$$R = \sum_{i\in I} (e_i \otimes 1) \otimes (1 \otimes e^i), \tag{1}$$
independent of the choice of the basis.

Tensor products are taken over the field C. Now write
$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$
$$(\Delta \otimes id) \circ \Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \tag{2}$$
for (iterated) comultiplication on $\mathcal{A}$, and similarly for comultiplication on $\mathcal{A}^0$. Then the comultiplication on $\mathcal{D}(\mathcal{A})$ is given by
$$\Delta(x\xi) = \sum_{(x),(\xi)} x_{(1)}\xi_{(1)} \otimes x_{(2)}\xi_{(2)}, \qquad (x \in \mathcal{A}, \xi \in \mathcal{A}^0). \tag{3}$$
and the multiplication by
$$\xi x = \sum_{(x),(\xi)} \xi_{(1)}(Sx_{(1)})\, \xi_{(3)}(x_{(3)})\, x_{(2)}\xi_{(2)}, \qquad (x \in \mathcal{A}, \xi \in \mathcal{A}^0), \tag{4}$$
where S denotes the antipode.

We consider the case where G is a finite group and $\mathcal{A} := C(G)$, the space of all complex valued functions on G, which becomes a Hopf algebra under pointwise multiplication, with comultiplication
$$(\Delta f)(x,y) := f(xy), \qquad (x, y \in G) \tag{5}$$
and with antipode
$$(Sf)(x) := f(x^{-1}), \qquad (x \in G). \tag{6}$$

The dual of $C(G)$ is the group algebra $C[G]$ with comultiplication

$$\Delta(x) = x \otimes x, \qquad (x \in G). \tag{7}$$

The pairing is given by

$$\langle f, x \rangle = f(x), \qquad (f \in C(G), x \in G). \tag{8}$$

For the quantum double of $C(G)$ we now write

$$\mathcal{D}(G) := \mathcal{D}(C(G)) = C(G) \otimes C[G] \simeq C(G, C[G]). \tag{9}$$

Thus $f \otimes x \in C(G) \otimes C[G]$ can be considered as the mapping

$$z \mapsto f(z)x : G \to C[G]. \tag{10}$$

Also $\mathcal{D}(G) \otimes \mathcal{D}(G) \simeq C(G \times G, C[G] \otimes C[G])$. Now formulas (3) and (4) can be written as:

$$(\Delta(f \otimes x))(y, z) = f(yz)x \otimes x, \qquad (x, y, z \in G, f \in C(G)) \tag{11}$$

$$(1 \otimes x)(f \otimes e) = f(x^{-1}.\, x) \otimes x, \qquad (e \text{ unit in } G). \tag{12}$$

Hence

$$(f_1 \otimes x)(f_2 \otimes y) = f_1(.) f_2(x^{-1}.\, x) \otimes xy : z \mapsto f_1(z) f_2(x^{-1}zx)\, xy. \tag{13}$$

For the antipode we find

$$S(f \otimes x) = f(x(.)^{-1}x^{-1}) \otimes x^{-1} : z \mapsto f(xz^{-1}x^{-1})\, x^{-1}. \tag{14}$$

The unit of $\mathcal{D}(G)$ is given by

$$1 \otimes e : z \mapsto e. \tag{15}$$

The counit ϵ of $\mathcal{D}(G)$ is

$$\epsilon(f \otimes x) = f(e). \tag{16}$$

For the R-matrix, which is an element of $\mathcal{D}(G) \otimes \mathcal{D}(G)$, we have

$$R = \sum_{x \in G} (\delta_x \otimes e) \otimes (1 \otimes x) \tag{17}$$

where δ_x is the Kronecker delta on $x \in G$, and thus a (basis) element of $C(G)$. Hence $R(y, z) = e \otimes y$, for $y, z \in G$.

$C(G)$ becomes a Hopf $*$-algebra with

$$f^*(x) := \overline{f(x)}. \tag{18}$$

The corresponding Hopf $*$-algebra structure on $C[G]$ is given by $x^* := x^{-1}$. Now $\mathcal{D}(G)$ has a $*$-algebra structure such that $C(G)$ and $C[G]$ are $*$-subalgebras:

$$(f \otimes x)^* = ((f \otimes e)(1 \otimes x))^* = (1 \otimes x^*)(f^* \otimes e) = \overline{f(x.\, x^{-1})} \otimes x^{-1}. \tag{19}$$

We verify that

$$((f_1 \otimes x)(f_2 \otimes y))^* = (f_2 \otimes y)^*(f_1 \otimes x)^*, \tag{20}$$

so we get a $*$-algebra structure on $\mathcal{D}(G)$.

Remark 1.2. It has been shown by Majid, [18], Proposition 7.1.4, and [17], that the quantum double of any Hopf $*$-algebra naturally becomes a Hopf $*$-algebra. We now give a slight reformulation of our model (9), (13), (19) for $\mathcal{D}(G)$ as a $*$-algebra. The new model will suggest how to generalize the definition of this $*$-algebra to the case where G is a locally compact group. Observe that there is a linear bijection

$$\mathcal{D}(G) = C(G) \otimes C[G] \Longleftrightarrow C(G \times G). \tag{21}$$

For this bijection we can write:

$$\begin{array}{rcl} f \otimes x & \mapsto & ((y,z) \mapsto f(y)\delta_x(z)) \\ \sum_{z\in G} F(\,.\,,z) \otimes z & \leftarrow & F \end{array} \tag{22}$$

Then $C(G \times G)$ is a $*$-algebra with multiplication

$$(F_1 \bullet F_2)(x,y) = \sum_{z\in G} F_1(x,z) F_2(z^{-1}xz, z^{-1}y), \tag{23}$$

and $*$-structure

$$F^*(x,y) = \overline{F(y^{-1}xy, y^{-1})}. \tag{24}$$

Essentially the same algebra would be obtained with some constant nonzero factor added on the right hand side of Eq.(23).

2. The case of a locally compact group G

In the following, compact or locally compact spaces are always supposed to be Hausdorff. If X is a compact space then $C(X)$ will denote the space of complex valued, continuous functions on X. If X is locally compact space then $C_c(X)$ will denote the space of continuous functions with compact support on X and $C_0(X)$ the space of continuous functions f on X for which $\lim_{\xi\to\infty} f(\xi) = 0$. In all cases, $\|.\|_\infty$ will denote the sup-norm.

Let G be a compact group with Haar measure dg. Then the most straightforward generalisation for $\mathcal{D}(G)$ is $\mathcal{D}(G) := C(G \times G)$ with $*$-algebra structure given by

$$(F_1 \bullet F_2)(x,y) := \int_G F_1(x,z)\, F_2(z^{-1}xz, z^{-1}y)\, dz. \tag{25}$$

and (24). One can indeed verify that the axioms of a $*$-algebra are satisfied. Note that this algebra in general has no unit element.

Remark 2.1. It would be attractive to construct the quantum double associated with a non-finite compact group G in the spirit of Majid ([18], p.296). So work with two Hopf algebras in duality (for finite G these were $C(G)$ and $C[G]$), and make their tensor product into a Hopf algebra in the style of the quantum double construction.

However, it is a problem which algebras we should choose, and whether these should be Hopf algebras in the algebraic sense or only in the topological sense. A next problem, for representation theory, is the type of irreducible $*$-representations to be considered: all algebraic representations or only those which are moreover continuous in some sense? Also, will different choices of algebras in the quantum double construction give rise to the same class of representations? We might start with $C(G)$, which is an algebra and which is a coalgebra only in the topological sense, but the Hopf algebra of trigonometric polynomials on G or (if G is a compact Lie group) the algebra $C^\infty(G)$ will also be candidates. For a (topological) Hopf algebra in duality with the first chosen algebra there are also many choices. Whatever we may choose, it has some arbitrariness.

Our decision to follow another approach, namely to generalize the algebra structure of $C(G\times G)$ (obtained from the quantum double construction for finite G) to the case of non–finite G, is motivated because we can then make contact with an existing representation theory of transformation group algebras.

There are several other approaches in literature to the definition of quantum double which look more conceptual, but which will not be followed in the present paper. (We thank Dr. Klaas Landsman for bringing some of these references to our attention.) The first approach of Podlés and Woronowicz [22] defines the double group of a compact matrix quantum group. The second approach of Baaj and Skandalis [1], see also the earlier paper by Skandalis [24], defines the quantum double of a so–called Kac system. They remark that their construction is compatible with both the Drinfel'd [9] and the Podlés–Woronowicz [22] construction of quantum double. A third approach, of Bonneau [5], defines a topological quantum double. Finally, in [19] Müger gives the Von Neumann double of a locally compact group. However, these papers do not give a classification of irreducible $*$-representations of the quantum double.

If G is a unimodular locally compact group with Haar measure dg and if we put $\mathcal{D}(G) := C_c(G\times G)$, then formulas (25) and (24) still define a $*$-algebra structure on $\mathcal{D}(G)$.

If G is a locally compact group with left Haar measure dg and Haar modulus Δ defined by $d(gh) = \Delta(h)\,dg$, then $\mathcal{D}(G) := C_c(G\times G)$ becomes a $*$-algebra with multiplication (25) and $*$-structure given by

$$F^*(x,y) = \Delta(y^{-1})\,\overline{F(y^{-1}xy,y^{-1})}. \tag{26}$$

In fact, this last case is still a special case of a more general $*$-algebra considered in literature: a *transformation group algebra*.

Definition 2.2. Let G be a locally compact group acting continuously on a locally compact space X. Denote the action by

$$(g,\xi)\mapsto g\xi : G\times X\to X, \qquad (g\in G,\ \xi\in X) \tag{27}$$

Then $C_c(X\times G)$ is called a *transformation group algebra* if it is equipped with a multiplication and $*$-operation given by

$$\begin{aligned}(F_1\bullet F_2)(\xi,y) &= \int_G F_1(\xi,z)F_2(z^{-1}\xi,z^{-1}y)dz\\ F^*(\xi,y) &= \overline{F(y^{-1}\xi,y^{-1})}\Delta(y^{-1})\end{aligned} \tag{28}$$

Straightforward computations show that $C_c(X\times G)$ becomes a $*$-algebra. We recover our earlier case $\mathcal{D}(G)$ when we take $X := G$ and G acting on itself by conjugation.

These algebras first occur in literature in Dixmier [8]. According to Dixmier, the unimodular case was considered earlier in unpublished work by Godement. The study of these algebras was continued by Glimm [11]. Afterwards, they were considered by many authors, often in a more generalized form. See the survey paper by Packer [21] for references.

Remark 2.3. Suppose that G is a finite group acting on a finite set X. Then we can take the set

$$B := \{\delta_\eta \otimes \delta_g \mid \eta \in X, g \in G\} \tag{29}$$

to be a basis of $C_c(X \times G)$ (which is here the space of all complex–valued functions on $X \times G$). In terms of these basis elements, the operations in Eq.(28) become

$$\begin{aligned} (\delta_\eta \otimes \delta_g) \bullet (\delta_{\eta'} \otimes \delta_{g'}) &= \delta_{\eta, g\eta'} \, (\delta_\eta \otimes \delta_{gg'}), \\ (\delta_\eta \otimes \delta_g)^* &= \delta_{g^{-1}\eta} \otimes \delta_{g^{-1}}. \end{aligned} \tag{30}$$

There is also a unit element: $1 = \sum_{\xi \in X} \delta_\xi \otimes \delta_e$. Now it follows easily that the $\mathbb{Z}$–span of B, considered as an associative ring with unit and with involution, is a *based ring*, as defined by Lusztig [13]. For $\mathcal{D}(G)$ (G finite group) this was already observed in [7].

Remark 2.4. Suppose that a unimodular locally compact group G continuously acts on a locally compact space X and that X is equipped with a G–invariant positive Borel measure. Then the algebra $C_c(X \times G)$ can be made into a pre–Hilbert space with inner product

$$\langle F_1, F_2 \rangle := \int_X (F_1 \bullet F_2^*)(\xi, e) d\xi = \int_G \int_X F_1(\xi, z) \overline{F_2(\xi, z)} d\xi dz, \tag{31}$$

and we can easily derive that

$$\langle F F_1, F_2 \rangle = \langle F_1, F^* F_2 \rangle, \qquad (F, F_1, F_2 \in C_c(X \times G)). \tag{32}$$

In particular, if G and X are finite, then $C_c(X \times G)$ becomes a semisimple algebra. (This is true for any based ring with finite $\mathbb{Z}$–basis, see Lusztig [13]).

3. Representation theory of transformation group algebras

In the following a lcsc group (resp. space) will mean a topological group (resp. space) which is locally compact, Hausdorff and second countable. Also, Hilbert spaces will be assumed to be separable. These assumptions are for convenience. We have made no effort to check to which extent the results below remain true without these assumptions.

Let a lcsc group G act continuously on a lcsc space X and consider the transformation group algebra $C_c(X \times G)$ as above. Glimm [11] defines a norm on $C_c(X \times G)$ by

$$\|F\|_1 := \int_G \|F(\,.\,, z)\|_\infty \, dz \tag{33}$$

with $\|.\|_\infty$ the sup-norm on X. Then

$$\|F_1 \bullet F_2\|_1 \le \|F_1\|_1 \|F_2\|_1 \tag{34}$$

We will now classify the irreducible $\|.\|_1$-bounded $*$-representations of $C_c(X \times G)$ (up to equivalence) under a certain assumption about the G-space X, namely that X is *countably separated* (see Definition 3.5). This classification is due to Glimm [11] Theorem 2.2, (see the proof of his implication (2)$\Rightarrow$(3) and take K equal to $\mathcal{K}$). As we will sketch below, the classification will follow from Mackey's [14], [15] imprimitivity theorem, see also Ørsted's [20] short proof of Mackey's theorem. In fact, if X is a transitive G-space, then the classification result is equivalent to the imprimitivity theorem.

In the following we will work with representations π of various kinds of structured sets (a group, a $*$-algebra, Borel algebra, etc.) on a Hilbert space $\mathcal{H}$. In all cases under consideration this will give rise to a $*$-closed subset $\{\pi(A)\}$ of the space $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. Then the commutant $R(\pi)$ and bicommutant $R(\pi)'$ of this set of operators are both a *Von Neumann algebra* (a weakly closed $*$-subalgebra of the algebra $\mathcal{L}(\mathcal{H})$). Note that $R(\pi)'' = R(\pi)$, so $R(\pi)$ and $R(\pi)'$ are the commutant of each other. $R(\pi)$ is called the *Von Neumann algebra associated with the representation* π. Representations can be classified according to their corresponding Von Neumann algebras.

A representation π is called *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ which are invariant under all operators $\pi(A)$. A well known theorem says that π is irreducible iff $R(\pi) = C\,I$, iff $R(\pi)' = \mathcal{L}(\mathcal{H})$, see for instance Theorem 4.7, Chap.VII in [12]

Definition 3.1. A mapping

$$P : E \mapsto P_E : \{\text{Borel subsets of } X\} \to \{\text{projection operators on Hilbert space } \mathcal{H}\} \tag{35}$$

is called a *projection valued measure* if

i. $P_\emptyset = 0,\ P_X = I$

ii. $P_{E\cap F} = P_E P_F$

iii. $P_E = \sum_{i=1}^\infty P_{E_i}$ (strong convergence) if $E = \cup_{i=1}^\infty E_i$ (disjoint union)

If $v, w \in \mathcal{H}$, then $E \mapsto \langle P_E v, w\rangle$ is a complex measure on X, which we write as $dP_{v,w}(\xi)$.

The projection valued measures P on X are in one–to–one correspondence with the non–degenerate $*$-representations π of $C_0(X)$:

$$\pi(f) = \int_X f(\xi)\, dP(\xi), \qquad (f \in C_0(X)), \tag{36}$$

with the following interpretation:

$$\langle \pi(f)v, w\rangle = \int_X f(\xi) dP_{v,w}(\xi), \qquad (v, w \in \mathcal{H}). \tag{37}$$

The operators P_E and the operators $\pi(f)$ generate the same Von Neumann algebra.

Definition 3.2. Let (G, X) be as before, τ be a unitary representation of G on $\mathcal{H}$, then a *system of imprimitivity* (s.o.i.) for τ based on X is given by (G, X, P), where P is a projection valued measure on X acting on $\mathcal{H}$ such that

$$\tau(y) P_E \tau(y)^{-1} \;=\; P_{yE}, \qquad \forall y \in G,\ \forall \text{ Borel subsets } E \subset X. \tag{38}$$

We now say that (τ, P) furnishes a representation of (G, X).

Theorem 3.3. (Glimm)[1] *There is a one–to–one correspondence between $*$-representations τ_0 of $C_c(X \times G)$ which are bounded in the norm* (33) *and representations (τ, P) of (G, X) (both on the same Hilbert space $\mathcal{H}$), in the following way*

$$\tau_0(F) := \int_G \int_X F(\xi, z)\, dP(\xi)\, \tau(z)\, dz \tag{39}$$

which has to be interpreted as

$$\langle \tau_0(F) v, w \rangle = \int_G \int_X \ F(\xi, z)\, dP_{\tau(z)v,w}(\xi)\, dz, \quad \forall v, w \in \mathcal{H}. \tag{40}$$

All such representations τ_0 are norm-decreasing, i.e. $\|\tau_0(F)\| \le \|F\|_1$. The operators $\tau_0(F)$ and the operators $\tau(z)$ and P_E generate the same Von Neumann algebra. In particular, (τ, P) is irreducible iff τ_0 is irreducible. An isometry of Hilbert spaces implements an equivalence between representations (τ, P) and (τ', P') of (X, G) iff it implements an equivalence of the corresponding representations τ_0 and τ_0' of $C_c(X \times G)$.

Remark 3.4. In view of the one–to–one correspondence $\pi \leftrightarrow P$ given by Eq.(36), we can describe representations (τ, P) of (G, X) equivalently as so–called *covariant representations* (τ, π) of $(G, C_0(X))$, where τ is a unitary representation of G on $\mathcal{H}$ and π is a nondegenerate $*$-representation of $C_0(X)$ on $\mathcal{H}$ such that

$$\tau(y)\pi(f)\tau(y)^{-1} = \pi(f(y^{-1}.)), \qquad \forall y \in G, \forall f \in C_0(X). \tag{41}$$

More generally, Takesaki [25], Def.3.1, defined covariant representations of (G, A) where A is a (possibly noncommutative) C^*–algebra, and G is a locally compact automorphism group of A.

Combining Eqs.(37) and (40) shows how a representation τ_0 of $C_c(X \times G)$ can be obtained from the corresponding covariant representation (τ, π) of $(G, C_0(X))$:

$$\tau_0(F) = \int_G \pi(F(., z))\, \tau(z)\, dz \tag{42}$$

which can also be interpreted weakly, like in Eq.(40).

Note that if (τ, P) is an irreducible representation of (G, X), and if E, E' are G-invariant Borel sets, then P_E commutes with $P_{E'}$ and with all $\tau(z)$, and we conclude that $P_E = \text{const.}I$, i.e. $P_E = I$ or $P_E = 0$ (P_E is a projection operator). A projection valued measure P such that, for all G-invariant Borel sets E, $P_E = 0$ or I, is called *ergodic*.

[1] See [11] Theorem 1.51.

Definition 3.5. (G, X) as above, is called *countably separated*, if there are countably many Borel sets $B_1, B_2, \ldots$ in X which are G-invariant, such that for every G-orbit $\mathcal{O}$ in X we have that

$$\mathcal{O} = \cap_{\mathcal{O} \subset B_i} B_i. \tag{43}$$

This holds for instance, if G and X are compact and second countable.

With (G, X) as above, the following conditions are equivalent (see Glimm [10], Theorem 1):

i. (G, X) is countably separated;

ii. The orbit space $G \backslash X$ is T_0 in the quotient topology. (A topological space is T_0 if for any two distinct points at least one of the points has a neighbourhood to which the other point does not belong.)

iii. Each orbit in X is relatively open in its closure.

iv. For each $\xi \in X$ the map $zG_\xi \mapsto z\xi : G/G_\xi \to G\xi$ is a homeomorphism, where $G\xi$ has the relative topology of a subspace of X. (Here G_ξ denotes the stabilizer of ξ in G.)

Hence, if G is compact then (G, X) is countably separated. Also note that each G-orbit in X is necessarily a Borel set and that property (iv) above implies that the mapping $zG_\xi \mapsto z\xi$ is a Borel isomorphism.

Lemma 3.6.[2] *If (G, X) is countably separated and P is an ergodic system of imprimitivity on $\mathcal{H}$, then there is a unique G-orbit $\mathcal{O}$ in X such that $P_{\mathcal{O}} = I$ and $P_E = 0$ if E is the complement of $\mathcal{O}$. (Then we say that P is concentrated on $\mathcal{O}$.) In particular, the conclusion holds if (τ, P) is an irreducible representation of (G, X).*

From now on assume that (G, X) is countably separated. Let $\mathcal{O}$ be a G-orbit in X. Take some $\xi \in \mathcal{O}$ and let G_ξ be the stabilizer of ξ in G. There is a one–to–one correspondence between representations (τ, P) of (G, X) concentrated on $\mathcal{O}$, representations (τ, P') of $(G, \mathcal{O})$, and representations (τ, P'') of $(G, G/G_\xi)$. Here P is related to P' by $P_E = P'_{\mathcal{O} \cap E}$ (E Borel set of X), and P' is related to P'' via the homeomorphism, hence Borel isomorphism $zG_\xi \mapsto z\xi$. The three representations (τ, P), (τ, P') and (τ, P'') are associated with the same Von Neumann algebra.

Under this correspondence, equivalent representations of $(G, G/G_\xi)$ will give rise to equivalent representations of (G, X) which are concentrated on $\mathcal{O}$. Conversely, if (τ, P) and (σ, Q) are equivalent representation of (G, X) and if P is concentrated on $\mathcal{O}$, then Q will also be concentrated on $\mathcal{O}$ and, under the above correspondence, the two equivalent representations of (G, X) will correspond to two equivalent representations of $(G, G/G_\xi)$.

[2] See for instance Van der Meer in [12], Ch. XI, Lemma 3.3

Thus, the classification of irreducible $\|.\|_1$-bounded $*$-reps of $C_c(X \times G)$ (up to equivalence) is reduced in several steps to the problem of classifying the irreducible representations of $(G, G/H)$, where H is a closed subgroup of G. Now we make contact with the notion of induced representation of a locally compact group and with the imprimitivity theorem.

3.1. Connection with induced representations

Let α be a unitary representation of H on the Hilbert space V_α. Choose a non-zero quasi-invariant measure $d\mu$ on G/H. Then $d\mu(z\xi) = R(\xi, z)\, d\mu(\xi)$ $(z \in G)$, with R a strictly positive continuous function on $G/H \times G$ satisfying

$$R(\xi, yz) = R(z\xi, y)\, R(\xi, z). \tag{44}$$

If there is an invariant measure μ on G/H, which is certainly the case if H is compact, we can take $R = 1$. Introduce the following space of functions:

$$\begin{aligned}\mathcal{L}^2_\alpha(G, V_\alpha) \;\; := \;\; & \Big\{ f : G \to V_\alpha \Big| f(gh) = \alpha(h^{-1}) f(g),\ \forall h \in H, \text{ for almost all } g \in G, \\ & \quad \text{and} \quad \|f\|^2 := \int_{G/H} \|f(z)\|^2_{V_\alpha}\, d\mu(zH) < \infty \Big\}\end{aligned} \tag{45}$$

It has a positive semi-definite inner product

$$\langle f_1, f_2 \rangle := \int_{G/H} \langle f_1(z), f_2(z) \rangle_{V_\alpha}\, d\mu(zH). \tag{46}$$

We obtain a Hilbert space by taking the quotient space with respect to the subspace of functions with norm zero:

$$L^2_\alpha(G, V_\alpha) = \mathcal{L}^2_\alpha(G, V_\alpha) \,/\, \{ f \in \mathcal{L}^2_\alpha(G, V_\alpha) \mid \|f\| = 0 \} \tag{47}$$

Now we can write down the following unitary representation of G on $L^2_\alpha(G, V_\alpha)$:

$$(\tau_\alpha(y) f)(x) := (R(xH, y^{-1}))^{1/2}\, f(y^{-1}x), \tag{48}$$

and the following projection valued measure:

$$(P^\alpha_E f)(x) = \chi_E(xH) f(x), \qquad E \text{ Borel subset in } G/H \tag{49}$$

where χ_E is the characteristic function of E (namely, $\chi_E(x) = 1$ if $x \in E$, and zero elsewhere).

Proposition 3.7. *The representation (τ_α, P^α) is a unitary representation of $(G, G/H)$ on $L^2_\alpha(G, V_\alpha)$. The Von Neumann algebras $R(\alpha)$ and $R(\tau_\alpha, P^\alpha)$ are isomorphic. In particular, α is irreducible iff (τ_α, P^α) is irreducible. Also, the equivalence class of α corresponds to the equivalence class of (τ_α, P^α).*

Now we can use Mackey's imprimitivity theorem, adapted to our specific situation:

Theorem 3.8. (Mackey) *If (τ, P) is a representation of $(G, G/H)$, then (τ, P) is equivalent to an (induced) representation (τ_α, P^α), with α a unitary representation of H. (τ, P) is irreducible iff α is irreducible.*

Summarizing we have the following equivalences:
Irreducible representation (irrep) τ_0 of $C_c(X \times G) \iff$
irrep (τ, P) of $(G, X) \iff$ (if (G, X) is countably separated)
irrep (τ, P) of $(G, \mathcal{O})$, with $\mathcal{O} \simeq G/H$, and H the stabilizer of point $\xi_0 \in X \iff$ (imprimitivity)
irrep (τ_α, P^α) of $(G, \mathcal{O})$, with α unitary irrep of H.

3.2. Induced representations of transformation group algebras

Let $\tau_{\alpha,0}$ be the representation of $C_c(X \times G)$ obtained by extending the representation (τ_α, P^α) of $(G, \mathcal{O})$ to (G, X) (putting $P^\alpha = 0$ on the complement of $\mathcal{O}$) and next lifting it to a representation of $C_c(X \times G)$. Then it follows from (40) that

$$\langle \tau_{\alpha,0}(F)\phi, \psi\rangle = \int_G \int_X F(\xi, z)\, dP^\alpha_{\tau_\alpha(z)\phi,\psi}(\xi)\, dz, \tag{50}$$

where $F \in C_c(X \times G)$, $\phi, \psi \in L^2_\alpha(G, V_\alpha)$. Now use that

$$\int_X f(\xi)\, dP^\alpha_{\phi,\psi}(\xi) = \int_{G/H} f(x\xi_0)\, \langle \phi(x), \psi(x)\rangle\, d\mu(xH), \tag{51}$$

where $f \in C_c(X)$, $\phi, \psi \in L^2_\alpha(G, V_\alpha)$, $\xi_0 \in X$. This follows since, for a Borel set E in X,

$$\begin{aligned} P^\alpha_{\phi,\psi}(E) &= \langle P^\alpha_E \phi, \psi\rangle \\ &= \begin{cases} 0, & \text{if } E \text{ is in the complement of } \mathcal{O}, \\ \int_{G/H} \chi_E(xH)\, \langle \phi(x), \psi(x)\rangle\, d\mu(xH) = \int_E \langle \phi(x), \psi(x)\rangle\, d\mu(xH) & \\ \qquad \text{if } E \text{ is a Borel set of } \mathcal{O} \text{ and is transfered to a Borel set of } G/H. & \end{cases} \end{aligned}$$

Here we used formula (49). From (50) and (51), we obtain:

$$\langle \tau_{\alpha,0}(F)\phi, \psi\rangle = \int_G \int_{G/H} F(x\xi_0, z)\, \langle (\tau_\alpha(z)\phi)(x), \psi(x)\rangle\, d\mu(xH)\, dz. \tag{52}$$

So finally we arrive at

$$(\tau_{\alpha,0}(F)\phi)(x) := \int_G F(x\xi_0, z)(\tau_\alpha(z)\phi)(x)\, dz. \tag{53}$$

Theorem 3.9. *Let (G, X) be countably separated. Let $\{\mathcal{O}_A\}_{A\in\mathcal{A}}$ be the collection of G-orbits in X ($\mathcal{A}$ an index set). For each $A \in \mathcal{A}$ choose some $\xi_A \in \mathcal{O}_A$, let N_A be the stablizer of ξ_A in G, choose some quasi-invariant measure μ_A on $G/N_A \simeq \mathcal{O}_A$ and let R_A be the corresponding R-function given by (44). For each $\alpha \in \widehat{N_A}$ choose a representative, also denoted by α, which is an irreducible unitary representation of N_A on some Hilbert space V_α. Then, for $A \in \mathcal{A}$ and $\alpha \in \widehat{N_A}$ we have mutually inequivalent irreducible $\|\cdot\|_1$-bounded $*$-representations τ^A_α of $C_c(X \times G)$ on $L^2_\alpha(G, V_\alpha)$ given by*

$$(\tau^A_\alpha(F)\phi)(x) := \int_G F(x\xi_A, z)\, (R_A(x\xi_A, z^{-1}))^{1/2}\, \phi(z^{-1}x)\, dz, \tag{54}$$

and all irreducible $\|.\|_1$*-bounded* $*$*-representations of* $C_c(X \times G)$ *are equivalent to some* τ_α^A*.*

Proof: follows from the statements before.

For purposes of later reference we formulate the specialization of Theorem 3.9 to the case $\mathcal{D}(G) := C(G \times G)$ (G compact group) with $*$-algebra operations given by (25) and (24), and with norm $\|.\|_1$ defined by (33). Since G is compact, the requirement of countable separability is certainly fulfilled. The orbits $\mathcal{O}_A$ are identified as the conjugacy classes C_A.

Corollary 3.10. *Let* $\{C_A\}_{A\in\mathcal{A}}$ *be the collection of conjugacy classes of the compact group* G *(*$\mathcal{A}$ *an index set). For each* $A \in \mathcal{A}$ *choose some* $g_A \in C_A$ *and let* N_A *be the centralizer of* g_A *in* G*. For each* $\alpha \in \widehat{N_A}$ *choose a representative, also denoted by* α*, which is an irreducible unitary representation of* N_A *on some Hilbert space* V_α*. Then, for* $A \in \mathcal{A}$ *and* $\alpha \in \widehat{N_A}$ *we have mutually inequivalent irreducible* $\|.\|_1$*-bounded* $*$*-representations* τ_α^A *of* $C(G \times G)$ *on* $L^2_\alpha(G, V_\alpha)$ *given by*

$$(\tau_\alpha^A(F)\phi)(x) := \int_G F(xg_Ax^{-1}, z)\, \phi(z^{-1}x)\, dz, \tag{55}$$

and all irreducible $\|.\|_1$*-bounded* $*$*-representations of* $C(G \times G)$ *are equivalent to some* τ_α^A*.*

Remark 3.11. Because of Eq.(42), we can describe the representation τ_α^A in Eq.(54) equivalently as the covariant representation $(\tau_\alpha, \pi_\alpha^A)$ of $(G, C_0(X))$, where τ_α is the unitary representation of G which is induced by the representation α of N_A, and where

$$(\pi_\alpha^A(f)\phi)(x) := f(x\xi_A)\phi(x), \qquad (\phi \in L^2_\alpha(G, V_\alpha),\ f \in C_0(X)). \tag{56}$$

By the definition of induced covariant representations in [25], the covariant representation $(\tau_\alpha, \pi_\alpha^A)$ of $(G, C_0(X))$ is induced by the covariant representation (α, ξ_A) of $(N_A, C_0(X))$ on V_α, where

$$\xi_A(f)v := f(\xi_A)v, \qquad (v \in V_\alpha,\ f \in C_0(X)). \tag{57}$$

Remark 3.12. In the situation of Theorem 3.9, a *Borel cross–section* for $G/N_A \simeq \mathcal{O}_A$ means a Borel mapping $s_A : \mathcal{O}_A \to G$ such that $s_A(\xi)\xi_A = \xi$, for all $\xi \in \mathcal{O}_A$. By a theorem of Mackey (see [16]), such a Borel cross section always exists. In terms of s_A the representation τ_α^A in (54) can be equivalently described as a $*$-representation acting on $L^2(\mathcal{O}_A, V_\alpha; \mu_A)$ (where μ_A is an invariant measure on the orbit (or conjugacy class) $\mathcal{O}_A$) by

$$(\tau_\alpha^A(F)\phi)(\xi) = \int_G F(\xi, z) R_A(\xi, z^{-1}) \alpha(s_A(\xi)^{-1} z\, s_A(z^{-1}\xi))\, \phi(z^{-1}\xi)\, dz$$
$$\xi \in \mathcal{O}_A, \phi \in L^2(\mathcal{O}_A, V_\alpha; \mu_A). \tag{58}$$

Consider in particular the case of $A \in \mathcal{A}$ such that $\mathcal{O}_A$ has only one element ξ_A (equivalently, $N_A = G$). Then R_A is identically 1, we can take $s_A(\xi_A) = e$, and we see from Eq.(58) that τ_α^A is a $*$-representation of $C_c(G \times G)$ on V_α given by

$$\tau_\alpha^A(F) = \int_G F(\xi_A, z)\alpha(z)\, dz, \quad N_A = G,\ \alpha \in \widehat{G}. \tag{59}$$

4. The case of finite G

As an example we specialise to the case of a finite group G, where G acts on itself via conjugation. We derive the unitary irreducible representations in the way outlined in the last section, and will show that our result is isomorphic to the representations derived in [7].

For a finite group the space $X = G$ is countably separated. The Hilbert space is

$$\mathcal{H}_\alpha^A := \{v \in L^2(G; V_\alpha) \,|\, \forall n \in N_A : v(xn) = \alpha(n^{-1})v(x) \text{ for almost all } x\}. \quad (60)$$

Corollary 3.10 holds, and Eq.(55) can now be rewritten as

$$(\pi_\alpha^A(F)v)(x) = \sum_{y \in G} F(xg_A x^{-1}, y)v(y^{-1}x), \qquad F \in C(G \times G), \quad (61)$$

where v is in the Hilbert space (60). In [7] these representations were derived by means of induction of algebra representations. For completeness we will show here how that works in this particular example. Let

$$\mathcal{B}_A := C(G) \otimes C[N_A], \quad (62)$$

considered as a subalgebra of $\mathcal{D}(G)$. Denote a general element of a spanning set for this subalgebra by $f \otimes n$, with $f \in C(G)$ and $n \in N_A$. Then define the representation of $\mathcal{B}_A$ on V_α:

$$\Pi_\alpha(f \otimes n)v := f(g_A)\alpha(n)v, \quad v \in V_\alpha. \quad (63)$$

This is indeed a representation, since

$$\begin{aligned}\Pi_\alpha(f_1 \otimes n_1)\Pi_\alpha(f_2 \otimes n_2) = f_1(g_A)f_2(g_A)\alpha(n_1 n_2) &= \Pi_\alpha(f_1(.)f_2(n_1^{-1}.\,n_1) \otimes n_1 n_2) \\ &= \Pi_\alpha((f_1 \otimes n_1)(f_2 \otimes n_2)). \quad (64)\end{aligned}$$

Now we induce this representation of $\mathcal{B}_A$ on V_α to a representation Π_α^A of $\mathcal{D}(G)$ on the representation space

$$V_\alpha^A := \mathcal{D}(G) \otimes_{\mathcal{B}_A} V_\alpha, \quad (65)$$

that is, V_α^A is a left module of $\mathcal{D}(G)$ [3]. We note that a general element of (a spanning set of) this representation space can be written in the following way:

$$(f \otimes x) \otimes_{\mathcal{B}_A} v = (1 \otimes x)(f(x.\,x^{-1}) \otimes e) \otimes_{\mathcal{B}_A} v = f(xg_A x^{-1})(1 \otimes x) \otimes_{\mathcal{B}_A} v. \quad (66)$$

This is effectively an element of $(1 \otimes C[G]) \otimes_{\mathcal{B}_A} V_\alpha$, which equals $C[G] \otimes_\alpha V_\alpha$, where $\otimes_\alpha$ denotes the tensor product with equivalence relation

$$xh \otimes v \equiv x \otimes \alpha(h)v, \quad h \in N_A. \quad (67)$$

[3] For the case of G a finite group this tensor product is well defined. This procedure is also valid for the case where G is a locally compact group. Then by this representation space we mean a certain well–chosen completion of the tensor product space.

Therefore, there is a bijection

$$\begin{array}{rcl} \mathcal{D}(G) \otimes_{\mathcal{B}_A} V_\alpha & \Longleftrightarrow & C[G] \otimes_\alpha V_\alpha \\ (f \otimes x) \otimes_{\mathcal{B}_A} v & \mapsto & f(xg_A x^{-1}) x \otimes_\alpha v \\ (1 \otimes x) \otimes_{\mathcal{B}_A} v & \leftarrow & x \otimes_\alpha v \end{array} \tag{68}$$

Transfer the representation Π^A_α from V^A_α to $C[G] \otimes_\alpha V_\alpha$ under this bijection, and call the resulting representation again Π^A_α. Because

$$\Pi^A_\alpha(1 \otimes y)(x \otimes_\alpha v) = yx \otimes_\alpha v \tag{69}$$

$\Pi^A_\alpha(1 \otimes \,\cdot\,)$ is the representation of G induced by the representation α of N_A on V_α. For $C(G)$ we have

$$\Pi^A_\alpha(f \otimes e)(x \otimes_\alpha v) = f(xg_A x^{-1}) x \otimes_\alpha v. \tag{70}$$

and so the representation Π^A_α can be completely described by inducing the representation α of N_A to G and by specifying the representation $\Pi^A_\alpha(. \otimes e)$ of $C(G)$ on the representation space of the representation of G obtained by inducing α.

In order to show that the representation in Eqs.(69), (70) is equivalent to the representation in Eq.(61), we give the linear intertwining bijections between the Hilbert spaces from Eqs.(60) and (65). For $x \otimes_\alpha v \in C[G] \otimes_\alpha V_\alpha$, and $w \in \mathcal{H}^A_\alpha$ as in Eq.(60) we have:

$$\begin{array}{rcl} x \otimes_\alpha v & \mapsto & \displaystyle\frac{1}{N_A} \sum_{n \in N_A} \delta_{xn^{-1}}(\,\cdot\,) \alpha(n) v \\ \displaystyle\sum_{x \in G} x \otimes_\alpha w(x) & \leftarrow & w(.) \end{array} \tag{71}$$

The last mapping shows that if $\bar{x}$ is taken to be the representative of the right coset xN_A, and $\{e_k\}_{k=1}^{dim V_\alpha}$ a basis of V_α with $v = v^k e_k$, then $\bar{x} \otimes e_k$ is a basis for V^A_α, and we can expand v on this basis:

$$v = \sum_{\bar{x} \in G/N_A} \sum_k v^k(\bar{x}) \, \bar{x} \otimes e_k \tag{72}$$

For completeness we mention that there are also two other vector spaces which are isomorphic to V^A_α. We define a mapping $s : G/N_A \to G$ such that $s(gN_A)N_A = gN_A$ for all $gN_A \in G/N_A$. Then

i. $V^A_{\alpha,s} := C[G/N_A] \otimes V_\alpha$

ii. $\widetilde{V^A_{\alpha,s}} := \{\Phi : G/N_A \to V_\alpha\}$

and we can find compatible linear isomorphisms between them and the representation spaces V^A_α and $\mathcal{H}^A_\alpha$. This concludes our discussion of the connection with [7].

5. Examples

5.1. $SU(2)$

To illustrate the construction of the quantum double for a compact group G and its irreducible representations, we now consider the case of $G = SU(2)$.

Let

$$g_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in SU(2) \tag{73}$$

be the representative of the conjugacy class $C_\theta := \{g g_\theta g^{-1} \,|\, g \in SU(2)\}$, with $0 \leq \theta \leq \pi$. For $0 < \theta < \pi$, the centralizer N_θ equals $U(1)$, which is embedded in $SU(2)$ in the following way: $U(1) := \{g_\theta \,|\, -\pi < \theta \leq \pi\}$. For $\theta = 0, \pi$ the centralizers N_0, N_π are equal to $SU(2)$. The irreducible unitary representations of $U(1)$ are given by $\alpha_n(g_\theta) = e^{in\theta}, (n \in \mathbb{Z})$. Let

$$L^2_n(SU(2)) := \{\phi \in L^2(SU(2)) \mid \phi(g g_\theta) = \alpha_n(g_\theta^{-1})\phi(g), \quad \forall\theta \in [0, 2\pi]\}. \tag{74}$$

Then, by Corollary 3.10, we have for $0 < \theta < \pi$ and $n \in \mathbb{Z}$ irreducible $*$-representations τ^θ_n of $\mathcal{D}(SU(2))$ on $L^2_n(SU(2))$ given by

$$(\tau^\theta_n(F)\phi)(x) = \int_{SU(2)} F(x g_\theta x^{-1}, z)\phi(z^{-1}x)\, dz, \qquad x \in SU(2), \phi \in L^2_n(SU(2)). \tag{75}$$

For $\theta = 0, \pi$ we find by Remark 3.12 the irreducible $*$-representations

$$\tau^\theta_l(F) = \int_{SU(2)} F(g_\theta, z)\pi_l(z)\, dz, \qquad l = 0, \frac{1}{2}, 1, \ldots \tag{76}$$

on C^{2l+1}, where $\pi_l(z)$ denotes the $(2l+1)$-dimensional unitary irreducible representation of $SU(2)$.

Up to equivalence, the representations τ^θ_n, with $(0 < \theta < \pi,\ n \in \mathbb{Z})$ and $\tau^\theta_l, (\theta = 0, \pi,\ l = 0, \frac{1}{2}, 1, \ldots)$ give all irreducible $\|.\|_1$-bounded $*$-representations of $\mathcal{D}(SU(2))$.

5.2. $SL(2, \mathbb{R})$

As an example of a non–compact group we take $SL(2, \mathbb{R})$. The (generalized) eigenvalues λ_1, λ_2 of elements of this group are either real, or complex conjugated, and $\lambda_1\lambda_2 = 1$. We see that the space of conjugacy classes can be split in the following way, according to the eigenvalues of elements of the classes:

i.& ii. For $\lambda_1 = e^{i\theta},\ \lambda_2 = e^{-i\theta}\quad (0 < \theta < \pi)$, we find two disjoint conjugacy classes:
$C_\theta := \{g\, u_\theta\, g^{-1} \mid g \in SL(2, \mathbb{R})\}$, and $C_{-\theta} := \{g\, u_{-\theta}\, g^{-1} \mid g \in SL(2, \mathbb{R})\}$, with

$$u_\psi = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \in SL(2, \mathbb{R}) \tag{77}$$

They have the same centralizer $N_\theta = N_{-\theta} = U(1) := \{u_\psi \,|\, -\pi < \psi \leq \pi\}$, embedded in $SL(2, \mathbb{R})$. This centralizer has irreducible unitary representations labeled by $n \in \mathbb{Z}$, like in the example of $SU(2)$.

iii. For $\lambda_1 = e^t, \lambda_2 = e^{-t}, t > 0$, there is one conjugacy class:
$C_t := \{g\, a_t\, g^{-1} \mid g \in SL(2,\mathbb{R})\}$, with

$$a_s = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \in SL(2,\mathbb{R}). \tag{78}$$

This has centralizer $N_t := \{\pm a_s \mid s \in \mathbb{R}\}$, which means that $N_t \simeq \mathbb{R} \times \mathbb{Z}_2$. The irreducible unitary representations of N_t are labeled by the pairs (b, ϵ), with $b \in \mathbb{R},\ \epsilon = \pm 1$.

iv. $\lambda_1 = -e^t, \lambda_2 = -e^{-t}$ is associated to the class
$\bar{C}_t := \{g\, \bar{a}_t\, g^{-1} \mid g \in SL(2,\mathbb{R})\}$ and

$$\bar{a}_s = \begin{pmatrix} -e^s & 0 \\ 0 & -e^{-s} \end{pmatrix} \in SL(2,\mathbb{R}) \tag{79}$$

It has the same centralizer as C_t.

v. & vi. In case the eigenvalues are $\lambda_1 = \lambda_2 = 1$ we distinguish two conjugacy classes C_e and C_1: $C_e := \{I\}$, which has centralizer $N_e := SL(2,\mathbb{R})$. The unitary irreducible representations of $SL(2,\mathbb{R})$ have been classified by Bargmann [4], see for instance Van Dijk in [12].
$C_1 := \{g\, n_1\, g^{-1} \mid g \in SL(2,\mathbb{R})\}$,

$$n_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{R}). \tag{80}$$

This class has centralizer N_1 consisting of matrices

$$\pm \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{R} \tag{81}$$

and thus $N_1 \simeq \mathbb{R} \times \mathbb{Z}_2$. The irreducible unitary representations of N_1 are labeled by (d, ϵ), with $d \in \mathbb{R}, \epsilon = \pm 1$.

vii. & viii. Finally for $\lambda_1 = \lambda_2 = -1$ we find the classes $C_{-e} := \{-I\}$, again with centralizer $SL(2,\mathbb{R})$, and $\bar{C}_1 := \{g\, \bar{n}_1\, g^{-1} \mid g \in SL(2,\mathbb{R})\}$,

$$\bar{n}_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in SL(2,\mathbb{R}). \tag{82}$$

with centralizer $\bar{N}_1 = N_1$.

A straightforward argument from linear algebra shows that the various classes given above are indeed distinct conjugacy classes, and that they are all conjugacy classes.

Next we prove that $SL(2,\mathbb{R})$ is countably separated, by using the second equivalence under Definition 3.5. To that aim we note that taking the trace tr is a continuous function on the space of conjugacy classes, so conjugacy classes with different values of tr are lying in disjoint open subsets. There are two possible types of obstructions:

The trace equals $(2\cos\theta)$ on C_θ and $C_{-\theta}$, $(0<\theta<\pi)$. However, then the preimage for the continuous function

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \tag{83}$$

on $SL(2,\mathbb{R})$ takes $b>0$ to an open subset containing C_θ, and $b<0$ to an open subset containing $C_{-\theta}$. So by considering inverse images C_θ and $C_{-\theta}$ are in disjoint open subsets.

The trace equals 2 on C_e and C_1. However, C_1 is included in the open subset $SL(2,\mathbb{R})\backslash\{I\}$ of $SL(2,\mathbb{R})$. This means that in the quotient topology it lies in an open subset which does not include $C_e=\{I\}$. Similarly if $tr=-2$. This means that the space of conjugacy classes is T_0, and thus that $SL(2,\mathbb{R})$ is countably separated. □

This classification of conjugacy classes and the representations of their centralizers enables us to classify the irreducible $*$-representations of $\mathcal{D}(SL(2,\mathbb{R}))$. Using Corollary 3.10 for the cases (i) and (ii) listed above, we find for example

$$(\tau_n^\theta(F)\phi)(x)=\int_{SL(2,\mathbb{R})} F(xu_\theta x^{-1},z)\phi(z^{-1}x)\,dz, \quad x\in SL(2,\mathbb{R}),\phi\in L_n^2(SL(2,\mathbb{R})), \tag{84}$$

for $-\pi<\theta<\pi, \theta\neq 0$. On the various (nontrivial) conjugacy classes there is an invariant measure, and therefore the R-function is equal to 1.

For C_e and C_{-e} it follows from Remark 3.12 that

$$\tau_r^{\pm e}(F)=\int_{SL(2,\mathbb{R})} F(\pm I,z)r(z)\,dz \tag{85}$$

where $r(z)$ denotes a unitary irreducible representation of $SL(2,\mathbb{R})$.

Acknowledgements

The second author was supported by the Dutch Science Foundation FOM/NWO. We would like to thank professor Sander Bais for stimulating discussions and putting us on the track of the subject of this paper

References

[1] Baaj, S., and G. Skandalis, *Unitaires multiplcatifs et dualité pour les produits croisés de C^*-algèbres*, Ann. Sci. Ecole Norm. Sup. (4) **26** (1993), 425–488.

[2] Bais, F.A., M. de Wild Propitius, and P. van Driel, *Quantum symmetries in discrete gauge theories*, Phys. Lett. **B280** (1992), 63–70.

[3] Bais, F. A., M. de Wild Propitius, and P. van Driel, *Anyons in discrete gauge theories with Chern–Simons terms*, Nucl. Phys. **B393** (1993), 547–570.

[4] Bargmann, V., *Irreducible unitary representations of the Lorentz group*, Ann. of Math. **48** (1947), 568–640.

[5] Bonneau, Ph., *Topological quantum double*, Rev. in Math. Phys. **6** No.2 (1994), 305–318.

[6] Chari, V., and A. Pressley, “A Guide to Quantum Groups,” Cambridge University Press, 1994.

[7] Dijkgraaf, R. H., V. Pasquier, and P. Roche, *Quasi Hopf algebras, group cohomology and orbifold models*, Nucl. Phys. B (Proc. Suppl.) **18B** (1990), 60–72.

[8] Dixmier, J., *Algèbres quasi-unitaires*, Comment. Math. Helvetici **26** (1952), 275–322.

[9] Drinfel'd, V. G., *Quantum groups,* in: “Proceedings of the I.C.M., Berkeley,” (1986), American Math. Soc. (1987), 798–820.

[10] Glimm, J., *Locally compact transformation groups*, Trans. Amer. Math. Soc. **101** (1961), 124–138.

[11] —, *Families of induced representations*, Pacific J. Math. **12** (1962), 825–911.

[12] Koornwinder, T. H., Ed., “Representations of Locally Compact Groups with Applications, Part II,” MC Syllabus **38.2**, Math. Centrum Amsterdam (1979).

[13] Lusztig, G., *Leading coefficients of character values of Hecke Algebras*, Proc. of Symposia in Pure Math. **47** Part 2 (1987), 235–262.

[14] Mackey, G. W., *Imprimitivity for representations of locally compact groups I*, Proc. Nat. Acad. Sci. **35** (1949), 537–545.

[15] —, *Unitary representations of group extentions I*, Acta Math. **99** (1958), 265–311.

[16] —, “The Theory of Unitary Group Representations,” Univ. of Chicago Press, 1976.

[17] Majid, S., *The quantum double as quantum mechanics*, J. Geom. Phys. **13** (1994), 169–202.

[18] —, “ Foundations of Quantum Group Theory,” Cambridge University Press, 1995.

[19] Müger, M., *Quantum double actions on operator algebras and orbifold quantum field theories*, preprint DESY 96-117, hep-th/9606175.

[20] Ørsted, B., *Induced representations and a new proof of the imprimitivity theorem*, J. Funct. Anal. **31** (1979), 355–359.

[21] Packer, J., *Transformation group C^*-algebras: A selective survey,* in “C^*-algebras: 1943–1993. A Fifty Year Celebration,” American Mathematical Society, R. S. Doran, Ed., Contemporary Math. **167** (1994), 183–217.

[22] Podlés, P, and S. L. Woronowicz, *Quantum deformation of Lorentz group*, Comm. Math. Phys. **130** (1990), 381–431.

[23] Shnider, S., and S. Sternberg, “Quantum Groups,” International Press, 1993.

[24] Skandalis, G., *Operator algebras and duality*, in “Proceedings of the I.C.M, Kyoto 1990,” The Japanese Mathematical Society, Springer–Verlag (1991), 997–1009.

[25] Takesaki, M., *Covariant representations of C^*-algebras*, Acta Math. **119** (1967), 273–303.

[26] Wild Propitius, M. D. F. de, and F. A. Bais, *Discrete gauge theories*, Proc. CRM–CAP Summer School "Particles and Fields '94," Banff, Springer Verlag (1996) (to be published), and
Wild Propitius, M. D. F de, *Topological Interactions in Broken Gauge Theories*, PhD thesis University of Amsterdam, 1995.

Department of Mathematics
University of Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
The Netherlands
thk@wins.uva.nl

Institute for Theoretical Physics
University of Amsterdam
Valckenierstraat 65
1018 XE Amsterdam
The Netherlands
nmuller@phys.uva.nl

Received June 26, 1996