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Kazhdan Constants Associated with Laplacian on Connected Lie Groups.

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Abstract. Let G be a finite dimensional connected Lie group. Fix a basis $\{X_i\}_{i=1,\cdots,n}$ of the Lie algebra $\mathfrak{g}$ and form the associated Laplace operator $\Delta = -\sum_{1\leq i\leq n} X_i^2$ in the enveloping algebra $U(\mathfrak{g})$. Let π be a strongly continuous unitary representation of G; let $\overline{d\pi(\Delta)}$ be the closure of the essentially self-adjoint operator $d\pi(\Delta)$. We show that π almost has invariant vectors if and only if 0 belongs to the spectrum of $\overline{d\pi(\Delta)}$. From this, we deduce that G has Kazhdan's property (T) if and only if there exists $\epsilon > 0$ such that, for any unitary representation without non zero fixed vectors, one has $\epsilon < \min\{\mathrm{Sp}(\overline{d\pi(\Delta)})\}$. This answers positively a question of Y. Colin de Verdière. It also allows us to define new Kazhdan constants, that we compare to the classical ones.

1. Introduction

In 1967, Kazhdan introduced property (T), a fixed point property of unitary representations for locally compact groups. More precisely,

Definition 1.1. Let G be a locally compact group.

1. Let $\pi : G \to U(H_\pi)$ be a strongly continuous unitary representation, $\epsilon > 0$ and let $K \subset G$ be a compact subset of G; a vector $\xi \in H_\pi^1$, the set of vectors of length 1 in H_π, is (ϵ, K)*-invariant* if

$$\sup\{\| \pi(g)\xi - \xi \| \mid g \in K\} < \epsilon.$$

2. π *has almost invariant vectors* if for every ϵ and K as above, there exists an (ϵ, K)-invariant vector.
3. G *has property (T)*, if for all representations with almost invariant vectors, there exists a nonzero fixed vector.

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This property has strong consequences both from practical and theoretical points of view (see [10]). In graph theory, the property (T) permits the construction of family of expanders via the knowledge of the Kazhdan constant (see [11]). Let (π, H_π) be a representation of G and let K be a compact generating set of G. We define $\kappa(G, K, \pi)$ as follows :

$$\kappa(G,K,\pi) = \inf_{\xi \in H_\pi^1} \max_{s \in K} \| \pi(s)\xi - \xi \| .$$

The Kazhdan constant of the group G relatively to K is defined by:

$$\kappa(G,K) = \inf\{\kappa(G,K,\pi) \mid \pi \in \widetilde{G}^*\}$$

where $\widetilde{G}^*$ is the set of equivalence classes of unitary representations of G on separable Hilbert spaces, without nonzero fixed vectors.

Proposition 1.2. *For G a locally compact group and K a compact generating set of G, the following assertions are equivalent :*

1. *G has property (T)*
2. *$\kappa(G, K) > 0$*

A proof of this result is given in [4].

In general, it is difficult to see whether a group G has property (T). In this paper we give an equivalent definition of property (T) for connected Lie groups. This approach was suggested by Colin de Verdière [1].

2. Definitions and first properties

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and enveloping algebra $U(\mathfrak{g})$. We use on G a right invariant Haar measure dg.

Any unitary representation (π, H_π) of G induces a representation $d\pi$ of $\mathfrak{g}$ on the subspace $\mathcal{C}^\infty(\mathcal{H}_\pi)$ of $\mathcal{C}^\infty$-vectors of π, i.e. the subspace of vectors $\xi \in H_\pi$ for which the function $x \mapsto \pi(x)\xi$ is a $\mathcal{C}^\infty$ function. $d\pi$ extends to a representation of $U(\mathfrak{g})$ on the same subspace $\mathcal{C}^\infty(\mathcal{H}_\pi)$.

Let ξ and η be two vectors in H_π, we denote by $\varphi_{\xi,\eta}$ the function on G defined by $\varphi_{\xi,\eta}(g) = \langle \pi(g)\xi \mid \eta \rangle$. We call $\varphi_{\xi,\eta}$ the coefficient of π associated to ξ and η.

Lemma 2.1. *Using the same notations as before, if ξ is in $\mathcal{C}^\infty(\mathcal{H}_\pi)$, η is in H_π and X is in $\mathfrak{g}$, then $\varphi_{\xi,\eta}$ satisfies :*

1. $X\varphi_{\xi,\eta} = \varphi_{d\pi(X)\xi,\eta}$,
2. $\Delta\varphi_{\xi,\eta} = \varphi_{d\pi(\Delta)\xi,\eta}$,
3. $\Delta^n\varphi_{\xi,\eta} = \varphi_{d\pi(\Delta)^n\xi,\eta}$ *for every* $n \geq 1$.

Proof. By definition, we get for every g in G :

$$\begin{aligned}(X\varphi_{\xi,\eta})(g) &= \lim_{t\to 0}\frac{\varphi_{\xi,\eta}(g\exp(tX))-\varphi_{\xi,\eta}(g)}{t}\\ &= \lim_{t\to 0}\frac{1}{t}\left[\left\langle \pi(\exp(tX))\xi \,|\, \pi(g^{-1})\eta\right\rangle - \left\langle \xi|\pi(g^{-1})\eta\right\rangle\right]\\ &= \left\langle d\pi(X)\xi \,|\, \pi(g^{-1})\eta\right\rangle = \varphi_{d\pi(X)\xi,\eta}(g)\end{aligned}$$

This proves (1). The assertions (2) and (3) follow from the fact that $d\pi(X)\xi$ is a $\mathcal{C}^\infty$ vector if ξ is. ■

Lemma 2.2. *Let X be an element of $\mathfrak{g}$ and ψ be a $\mathcal{C}^\infty$ function on G such that ψ and $X\psi$ lie in $L^1(G)$; then $\int_G (X\psi)(g)dg = 0$.*

Proof. Suppose first that $\psi \in \mathcal{C}_0^\infty(G)$, the space of $\mathcal{C}^\infty$ with compact support on G :

$$\begin{aligned}\int_G (X\psi)(g)dg &= \int_G\left(\lim_{t\to 0}\frac{\psi(g\exp(tX))-\psi(g)}{t}\right)dg\\ &= \lim_{t\to 0}\int_G\frac{\psi(g\exp(tX))-\psi(g)}{t}dg = 0.\end{aligned}$$

The two last equalities hold because $Supp(\psi)$ is compact and because the measure dg is invariant by right multiplication. Now, if $\psi\in\mathcal{C}^\infty(G)$ is such that ψ and $X\psi$ lie in $L^1(G)$, there exists a compact subset K of G such that $\int_{G-K}|\psi(g)|dg<\epsilon/3$ and $\int_{G-K}|(X\psi)(g)|dg<\epsilon/3$. So we have : $|\int_G (X\psi)(g)dg| < |\int_K (X\psi)(g)dg| + \epsilon/3$.

For $\chi\in\mathcal{C}_0^\infty(G)$, $\chi\equiv 1$ on K, $0\le\chi\le 1$, $|(X\chi)(g)|\le 1$, we have

$$\begin{aligned}\int_K (X\psi)(g)dg &= \int_K (X\psi)(g)\chi(g)dg\\ &= \int_K (X(\psi\chi))(g)dg \quad (\text{because } X\chi = 0 \text{ on } K)\\ &= -\int_{G-K}(X(\psi\chi))(g)dg \quad \left(\text{because } \int_G X(\psi\chi)(g)dg = 0\right)\\ &= -\int_{G-K}(X\psi)(g)\chi(g)dg - \int_{G-K}(X\chi)(g)\psi(g)dg\end{aligned}$$

But $|\int_{G-K}(X\psi)(g)\chi(g)dg| \le \int_{G-K}|(X\psi)(g)|dg \le \epsilon/3$ and
$|\int_{G-K}(X\chi)(g)\psi(g)dg| \le \int_{G-K}|\psi(g)|dg \le \epsilon/3$. ■

Corollary 2.3. *Let X be an element of $\mathfrak{g}$ and ψ be a $\mathcal{C}^\infty$ function on G such that ψ and $X\psi$ lie in $L^1(G)$. If ξ belongs to $\mathcal{C}^\infty(H_\pi)$ and η belongs to H_π, then :*

$$\int_G \left\langle \xi|\pi(g^{-1})\eta\right\rangle (X\psi)(g)dg = -\int_G \left\langle d\pi(X)\xi|\pi(g^{-1})\eta\right\rangle \psi(g)dg.$$

Proof. The corollary follows from 2.2, 2.1 (1) and from the equality :

$$\int_G \varphi_{\xi,\eta}(g)(X\psi)(g)dg + \int_G (X\varphi_{\xi,\eta})(g)\psi(g)dg = \int_G (X(\varphi_{\xi,\eta}\psi))(g)dg = 0. \quad ■$$

Let $X_1,\ldots,X_n$ be a basis of $\mathfrak{g}$; set $\Delta = -\sum_{i=1}^n X_i^2 \in U(\mathfrak{g})$ The operator $d\pi(\Delta)$ is essentially selfadjoint [8], and we denote by $\overline{d\pi(\Delta)}$ its closure and by $D(\overline{d\pi(\Delta)})$ the domain of $\overline{d\pi(\Delta)}$.

Corollary 2.4. *If ψ is a $\mathcal{C}^\infty$ function on G such that ψ, $X_i\psi$ and $X_i^2\psi$ belong to $L^1(G)$ for every $i = 1, \ldots, n$, then for every ξ in $D(\overline{d\pi(\Delta)})$ and for every η in H_π :*

$$\int_G \varphi_{\xi,\eta}(g)(\Delta\psi)(g)dg = \int_G \varphi_{\overline{d\pi(\Delta)}\xi,\eta}(g)\psi(g)dg.$$

Proof. By lemmas 2.1 and 2.3, the equality is true for $\xi \in \mathcal{C}^\infty(H_\pi)$. If $\xi \in D(\overline{d\pi(\Delta)})$, there exists a sequence $(\xi_k)_{k\geq 1}$ in $\mathcal{C}^\infty(H_\pi)$ such that

$$\| \xi_k - \xi \| + \| \overline{d\pi(\Delta)}\xi_k - \overline{d\pi(\Delta)}\xi \| \underset{k\to\infty}{\longrightarrow} 0.$$

Then :

$$\begin{aligned} & |\int \varphi_{\xi,\eta}(g)(\Delta\psi)(g)dg - \int \left\langle \overline{d\pi(\Delta)}\xi|\pi(g^{-1})\eta\right\rangle \psi(g)dg| \\ \leq\ & |\int \left\langle \xi - \xi_k|\pi(g^{-1})\eta\right\rangle (\Delta\psi)(g)dg| \\ & + |\int \left\langle \xi_k|\pi(g^{-1})\eta\right\rangle (\Delta\psi)(g)dg \\ & - \int \left\langle \overline{d\pi(\Delta)}\xi_k|\pi(g^{-1})\eta\right\rangle \psi(g)dg| \\ & + |\int \left\langle \overline{d\pi(\Delta)}\xi_k - \overline{d\pi(\Delta)}\xi|\pi(g^{-1})\eta\right\rangle \psi(g)dg| \\ \leq\ & (\| \xi - \xi_k \| \| \Delta\psi \|_1 + \| \overline{d\pi(\Delta)}(\xi_k - \xi) \| \| \psi \|_1) \| \eta \| \underset{k\to\infty}{\longrightarrow} 0. \end{aligned}$$

This finishes the proof. ■

Proposition 2.5. *The following conditions are equivalent :*

1. *π has a non zero fixed vector.*
2. *0 is an eigenvalue for $d\pi(\Delta)$.*
3. *0 is an eigenvalue for $\overline{d\pi(\Delta)}$.*

Proof. The implications 1) ⇒ 2) and 2) ⇒ 3) are obvious.

We prove now 3) ⇒ 1) : Let ξ be a non zero vector in $Ker(\overline{d\pi(\Delta)})$. Let $\eta \in H_\pi$ and $\psi \in \mathcal{C}_0^\infty(G)$. Then, using Corollary 2.4, we have in the sense of weak derivatives :

$$(\Delta\varphi_{\xi,\eta}, \psi) = (\varphi_{\xi,\eta}, \Delta\psi) = (\varphi_{\overline{d\pi(\Delta)}\xi,\eta}, \psi) = 0.$$

Therefore, $\Delta\varphi_{\xi,\eta} = 0$ as a distribution, and since Δ is hypo-elliptic (see [8]), $\varphi_{\xi,\eta}$ is a $\mathcal{C}^\infty$ function on G.

Since η is arbitrary, this implies that $g \mapsto \pi(g)\xi$ is weakly $\mathcal{C}^\infty$ and so, by a lemma due to Poulsen (see [15]), $g \mapsto \pi(g)\xi$ is strongly $\mathcal{C}^\infty$. Therefore, $\overline{d\pi(\Delta)}\xi = d\pi(\Delta)\xi = 0$. Hence,

$$\sum_{i=1}^n \| d\pi(X_i)\xi \|^2 = \langle d\pi(\Delta)\xi|\xi\rangle = 0.$$

Since $\{X_i\}_{i=1,\ldots,n}$ is a basis of $\mathfrak{g}$, this implies that $d\pi(X)\xi = 0$ for every element of $\mathfrak{g}$. Hence, $\pi(g)\xi = \xi$ for every g in V, a suitable neighbourhood of e in G. As G is connected, V generates G and so ξ is fixed under the action of G. ■

3. Spectral characterisation of almost invariant vectors

The goal of this section is to prove :

Theorem 3.1. *For a unitary representation (π, H_π) of G, the following conditions are equivalent :*

1. *π almost has invariant vectors.*
2. *0 is an approximate eigenvalue of $d\pi(\Delta)$.*
3. *0 is a spectral value of $\overline{d\pi(\Delta)}$.*

Proof. The equivalence between 2) $\Leftrightarrow$ 3) is clear, since $\overline{d\pi(\Delta)}$ is the closure of $d\pi(\Delta)$.

2) $\Rightarrow$ 1)

If 0 is an approximate eigenvalue of $d\pi(\Delta)$, there exists a sequence $\{\xi_m\}_{m\geq 0}$ of unit vectors in $\mathcal{C}^\infty(H_\pi)$ such that $\lim_{m\to+\infty} \| d\pi(\Delta)\xi_m \| = 0$. So,

$$\lim_{m\to+\infty} \| d\pi(X_i)\xi_m \| = 0 \text{ for every } i = 1, \ldots, n.$$

Let V defined by

$$V = \{ \prod_{1\leq i\leq n} \exp(t_i X_i) \mid -1 \leq t_i \leq 1 \}.$$

V is a compact neighbourhood of e, see [5].

Moreover, for every X in $\mathfrak{g}$ and every $t \geq 0$, we have :

$$\pi(\exp(tX))\xi_m - \xi_m = \int_0^t \pi(\exp(sX)) d\pi(X)\xi_m ds\,.$$

Let $\epsilon > 0$. Then, for $0 \leq t \leq 1$, we have :

$$\| \pi(\exp(tX_i))\xi_m - \xi_m \| \leq t \, \| d\pi(X_i)\xi_m \| \leq \epsilon/n$$

for $i = 1, \cdots, n$ and m sufficiently large.

This implies that $\| \pi(g)\xi_m - \xi_m \| \leq \epsilon$ for every g in V as soon as m is large enough.

So we proved that, for every $\epsilon > 0$, there exist (ϵ, V)-invariant vectors. As G is connected, V generates G and one easily deduces that π has (ϵ, K)-invariant vectors for any compact subset K of G.

The remainder of this section is devoted to the proof of the implication 1) $\Rightarrow$ 3) which is much more involved. For this, we need to recall some facts about the heat kernel associated with Δ.

Let h denote the closure of Δ acting on $L^2(G)$. As $-h$ is a selfadjoint and negative definite operator, by the Hille-Yosida theorem (see [16]), $-h$ is the infinitesimal generator of a strongly continous semi-group of contractions $T(t)$.

It is known (see [13]) that, for every $t > 0$, $T(t)$ is given by the convolution from the right with a function p_t, the heat kernel. The function p_t is a smooth function on G with positive values and integral 1. We briefly recall its construction.

Let δ denote the modular function associated with our right Haar measure, thus :

$$\int_G \delta(x)f(xy)dy = \int_G f(y)dy,$$

for every integrable function f over G. The right regular representation over $L^2(G)$ is given by :

$$(\rho(x)\xi)(y) = \xi(yx).$$

The left regular representation is then given by :

$$(\lambda(x)\xi)(y) = \delta(x^{-1})^{1/2}\xi(x^{-1}y).$$

If $\Delta = -\sum_{i=1}^{n} X_i^2$ and if h is the closure of Δ on $L^2(G)$, then h and $\lambda(x)$ commute for every x in G.

For every $t \geq 0$, let $T(t) = \exp(-th)$.

We claim that for every $t > 0$, $T(t)$ is a regularising operator, that is, for every $\xi \in L^2(G)$, for every integer $m \geq 1$, $h^m T(t)\xi \in D(h)$. Indeed, let $h = \int_0^\infty \lambda dE(\lambda)$ be the spectral decomposition of h. As the function $\lambda \mapsto \lambda^{2m}\exp(-2t\lambda)$ is bounded on $\mathbb{R}_+$, we have :

$$\int_0^\infty \lambda^{2m}\exp(-2t\lambda)dE_\xi(\lambda) = \| h^m T(t)\xi \|^2 < \infty,\ \forall \xi \in L^2(G).$$

By Sobolev's lemma, $T(t)\xi \in \mathcal{C}^\infty(G)$, $\forall \xi \in L^2(G)$.

By Schwartz Kernel theorem, there exists a $\mathcal{C}^\infty$ function $p'_t : G \times G \to \mathbb{R}_+$ such that

$$(T(t)\xi)(x) = \int_G p'_t(x,y)\xi(y)dy,\ \forall \xi \in L^2(G).$$

Hence ,

$$\begin{aligned}(\lambda(x)T(t)\xi)(x') &= \delta(x^{-1})^{1/2}(T(t)\xi)(x^{-1}x') \\ &= \delta(x^{-1})^{1/2}\int_G p'_t(x^{-1}x',y)\xi(y)dy\end{aligned}$$

Moreover

$$\begin{aligned}(T(t)\lambda(x)\xi)(x') &= \int_G p'_t(x',y)(\lambda(x)\xi)(y)dy \\ &= \delta(x^{-1})^{1/2}\int_G p'_t(x',y)\xi(x^{-1}y)dy \\ &= \delta(x^{-1})^{1/2}\int_G \delta(x)p'_t(x',xy)\xi(y)dy\end{aligned}$$

As $T(t)$ and $\lambda(x)$ commute, we deduce that

$$(*)\ p'_t(x^{-1}x',y) = \delta(x)p'_t(x',xy)\ \forall x,x',y \in G.$$

Set $p_t(x) = p'_t(e,x)$.
The above relation $(*)$ shows that, for all $y \in G$,

$$p_t(x^{-1}y)\delta(x^{-1}) = p'_t(e,x^{-1}y)\delta(x^{-1}) = p'_t(x,y).$$

Hence

$$\begin{aligned}(T(t)\xi)(x) &= \int p'_t(x,y)\xi(y)dy = \int p_t(x^{-1}y)\delta(x^{-1})\xi(y)dy \\ &= \int p_t(y)\xi(xy)dy \text{ for } \xi \in L^2(G).\end{aligned}$$

Recall that if π is a representation of G, then $\pi(f)$ is the operator defined for $f \in L^1(G)$ by

$$\pi(f)\eta = \int f(y)\pi(y)\eta dy \text{ for every } \eta \in H_\pi.$$

Taking $\pi = \rho$ and $f = p_t$, we see that :

$$(\rho(p_t)\xi)(x) = \int p_t(y)(\rho(y)\xi)(x)dy = \int p_t(y)\xi(xy)dy = (T(t)\xi)(x).$$

As $T(t) = T(t)^*$, we have $p_t = p_t^*$, and p_t is a solution of the heat equation :

$$\frac{\partial p_t}{\partial t} = -\Delta p_t.$$

Lemma 3.2. *Δp_t belongs to $L^1(G)$ and*

$$\lim_{s\to 0^+} \| \frac{p_{t+s} - p_t}{s} + \Delta p_t \|_1 = 0, \textit{ for every } t > 0.$$

Proof. We denote by ρ_1 the right regular representation of G on $L^1(G)$. Following ([13], theorem 4, p.599), we set for $t > 0$ and for $f \in L^1(G)$:

$$P^t f = \int_G p_t(y)\rho_1(y) f dy.$$

By lemma 7.1 and theorem 4 of [13], $P^t f$ is an analytic vector in the following sense : there exists $s > 0$ such that

$$\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{1\le i_1,\ldots,i_m\le n} \| d\rho_1(X_{i_1})\ldots d\rho_1(X_{i_m})P^t f \|_1 \, s^m < \infty.$$

In particular, $p_t = P^{t/2}p_{t/2}$ is analytic in the preceding sense, and $hp_t \in L^1(G)$. The last assertion is a consequence of mean value theorem and of Lebesgue's dominated convergence theorem. ■

Let now π be a unitary representation of G. Set

$$S(t) = \pi(p_t), \text{ if } t > 0 \text{ and } S(0) = \mathbb{I}_{H_\pi}.$$

$(S(t))_{t\ge 0}$ is a strongly continuous semi-group on H_π with infinitesimal generator A. Moreover $S(t)^* = S(t)$ for every t because $p_t = p_t^*$.

By Corollary 10.6, p.41 of [14], A is selfadjoint. More precisely the following is true.

Lemma 3.3. *With the same notations, one has $A = -\overline{d\pi(\Delta)}$, where $\overline{d\pi(\Delta)}$ denotes the closure of $d\pi(\Delta)$.*

Proof. Let ξ be in $D(\overline{d\pi(\Delta)})$. We claim that, for every $t>0$, $S(t)\xi$ belongs to $D(A)$ and

$$AS(t)\xi = -S(t)\overline{d\pi(\Delta)}\xi.$$

Indeed, let η in H_π. Then, for $0<s\leq 1$:

$$\begin{aligned}\left\langle \frac{S(s)S(t)\xi - S(t)\xi}{s}|\eta\right\rangle &= \frac{1}{s}\langle \pi(p_{t+s})\xi - \pi(p_t)\xi|\eta\rangle \\ &= \frac{1}{s}\int (p_{t+s}(g)-p_t(g))\left\langle \xi|\pi(g^{-1})\eta\right\rangle dg \\ &= \int \frac{p_{t+s}(g)-p_t(g)}{s}\varphi_{\xi,\eta}(g)dg.\end{aligned}$$

By lemma 3.2 and corollary 2.4,

$$\begin{aligned}\lim_{s\to 0^+}\left\langle \frac{S(s)S(t)\xi - S(t)\xi}{s}|\eta\right\rangle &= \int (-\Delta p_t)(g)\varphi_{\xi,\eta}(g)dg \\ &= -\left\langle S(t)\overline{d\pi(\Delta)}\xi|\eta\right\rangle.\end{aligned}$$

By theorem 1.3, p.43 of [14], $S(t)\xi$ belongs to $D(A)$ and $AS(t)\xi = -S(t)\overline{d\pi(\Delta)}\xi$. This proves the claim. But $\| S(t)\xi - \xi \| \underset{t\to 0}{\longrightarrow} 0$ and $\| AS(t)\xi - (-\overline{d\pi(\Delta)}\xi)\| = \| -S(t)\overline{d\pi(\Delta)}\xi + \overline{d\pi(\Delta)}\xi\| \underset{t\to 0}{\longrightarrow} 0$.

Since A is a closed operator, ξ belongs to $D(A)$ and $A\xi = -\overline{d\pi(\Delta)}\xi$.

So $-\overline{d\pi(\Delta)} \subset A$, and as they are selfadjoint, we have $-\overline{d\pi(\Delta)} = A$. ■

Lemma 3.4. *If (π, H_π) is a unitary representation of a locally compact group G with almost invariant vectors and if μ is a probability measure over G, the spectrum of the operator $\pi(\mu)=\int \pi(g)d\mu(g)$ contains 1.*

Proof. Let $(\xi_k)_{k\geq 1}$ a sequence of unit vectors in H_π such that φ_{ξ_k,ξ_k} tends to 1 uniformly over every compact set of G. We show that

$$\lim_{n\to\infty} \| \pi(\mu)\xi_k - \xi_k \| = 0.$$

Let $\epsilon>0$; there exists a compact set K in G such that $\mu(G-K)\leq\epsilon$. So we have

$$\| \pi(\mu)\xi_k - \xi_k \| \leq \int_K \| \pi(g)\xi_k - \xi_k \| \, d\mu(g) + 2\epsilon < 3\epsilon$$

for k sufficiently large. ■

As we now show, the converse is true under some restrictions on μ. Although we shall not use this result, we give the proof as we think it is of interest.

Lemma 3.5. *Let G be a locally compact group and let μ be a absolutely continuous probability measure with respect to the Haar measure and such that the support generates topologically G. Let π be a unitary representation on a Hilbert space H_π. Then π has almost invariant vectors, if the spectrum of the operator $\pi(\mu)=\int \pi(g)d\mu(g)$ contains 1.*

Proof. By replacing μ by $\frac{1}{2}(\mu+\check{\mu})$, we can assume that μ is symmetric. Then $\pi(\mu)$ is selfadjoint and there exist unit vectors ξ_n such that $\|\pi(\mu)\xi_n - \xi_n\| \to 0$.

So we have :

$$\lim_{n\to\infty} \int_G (1 - \mathrm{Re}\,\langle \pi(x)\xi_n|\xi_n\rangle) d\mu(x) = 0.$$

As $\mu \geq 0$ and $1 - \mathrm{Re}\,\langle \pi(x)\xi_n|\xi_n\rangle \geq 0$, there exist a subsequence of $\{\xi_n\}_{n\geq 0}$ (that we will also denote by $\{\xi_n\}_{n\geq 0}$), such that

$$\lim_{n\to\infty} 1 - \mathrm{Re}\,\langle \pi(x)\xi_n|\xi_n\rangle \geq 0$$

for μ-almost every $x \in G$.

On an other hand, by compactness of the unit ball of L^∞ gifted with the weak* topology, there exists a subsequence of $\{\xi_n\}_{n\geq 0}$ (still denoted by $\{\xi_n\}_{n\geq 0}$) and a positive type function φ on G such that

$$\lim_{n\to\infty} \varphi(x) - \mathrm{Re}\,\langle \pi(x)\xi_n|\xi_n\rangle \geq 0$$

almost everywhere with respect to the Haar measure (this is true first for the weak* topology $\sigma(L^\infty, L^1)$ and the claim follows by the same arguments as before).

As μ is absolutely continuous, we have $\varphi = 1$ μ-almost everywhere. As φ is a measurable, positive definite function, φ is continuous. Hence $\varphi = 1$ on the support of μ.

Therefore, $\varphi = 1$ on the closed subgroup generated by $Supp(\mu)$ which is G, by assumption.

So $\lim_{n\to\infty} \mathrm{Re}\,\langle \pi(x)\xi_n|\xi_n\rangle = 1$ almost everywhere on G and by Lebesgue's dominated convergence, this is also true in the $\sigma(L^\infty, L^1)$ topology. By [2], Theorem 13.5.2, it follows that $\lim_{n\to\infty} \mathrm{Re}\,\langle \pi(x)\xi_n|\xi_n\rangle = 1$ uniformly on every compact subset of G and, hence, $\lim_{n\to\infty} \|\pi(x)\xi_n - \xi_n\| = 0$ uniformly on compact subsets of G. This shows that π almost has invariant vectors. ■

Now we are able to finish the proof of Theorem 3.1.

If π almost has invariant vectors, by lemma 3.4, the spectrum of $\pi(p_t)$ contains 1, for every $t > 0$.

Now $-\overline{d\pi(\Delta)}$ is the infinitesimal generator of the semi-group $(\pi(p_t))_{t\geq 0}$ so that 0 is in the spectrum of $\overline{d\pi(\Delta)}$ by functional calculus and lemma 3.3. ■

One may wonder whether it is necessary to use, as we did, arguments involving the heat kernel to prove 1) $\Rightarrow$ 3). For instance, one might think that if, for a compact neighbourhood V of e in G, $\{\xi_n\}_{n\geq 0}$ is a family of $(1/n, V)$-invariant $\mathcal{C}^\infty$ vectors, then $\|d\pi(\Delta)\xi_n\|$ should tend to 0.
The following example shows that this is not always true.

Example 3.6. Let $G = \mathbb{R}$ be the real line. We define first, the following family of unitary representations of degree 2 :

$$s \mapsto \pi_n(s) = \begin{pmatrix} \exp(is/n) & 0 \\ 0 & \exp(isn) \end{pmatrix}.$$

We consider the unitary representation π defined by $s \mapsto \bigoplus_{n\geq 0} \pi_n(s)$.

We also define $\xi_n = (0,\dots,0,\sqrt{\frac{n-1}{n}},\sqrt{\frac{1}{n}},0,\dots)$ where the nonzero components of ξ_n are in $2n^{th}$ and $(2n+1)^{th}$ places.

By construction the ξ_n have norm 1. Let I be a compact subset of $\mathbb{R}$. Then, for every s in I, we have :

$$\begin{aligned} \| \pi(s)\xi_n - \xi_n \|^2 &= |(\exp(is/n)-1)\sqrt{\frac{n-1}{n}}\,|^2 + |(\exp(isn)-1)\sqrt{\frac{1}{n}}\,|^2 \\ &= |(\exp(is/n)-1)|^2 \frac{n-1}{n} + |\exp(isn)-1|^2 \frac{1}{n}\,. \end{aligned}$$

As $|\exp(is/n)-1|^2 \underset{n\to\infty}{\longrightarrow} 0$ uniformly for s in I and as $|\exp(isn)-1|^2 \leq 4$, the ξ_n are (ϵ_n, I)-invariant vectors for a sequence $\{\epsilon_n\}_{n\geq 0}$ with $\epsilon_n > 0$ and $\lim \epsilon_n = 0$.

However this family ξ_n does not satisfy $\| d\pi(\Delta)\xi_n \| \underset{n\to\infty}{\longrightarrow} 0.$

In fact,

$$d\pi(\Delta)\xi_n = (0,\dots,0,\frac{1}{n^2}\sqrt{\frac{n-1}{n}},n^2\sqrt{\frac{1}{n}},0,\dots).$$

Thus, $\| d\pi(\Delta)\xi_n \| \approx n^{3/2}$; so it does not tend to 0.

Lemma 3.7. *Let h be a selfadjoint operator on a Hilbert space H, with domain $D(h)$, and such that its spectrum is bounded from below. Then*

$$\min(\mathrm{Sp}\, h) = \inf_{\xi\in D(H)^1} \langle h\xi|\xi\rangle \ , \textit{ where } D(H)^1 = \{\xi \in D(H) | \, \|\xi\| = 1\}.$$

Proof. As h is selfadjoint, its residual spectrum is empty. So every spectral value is an approximate eigenvalue. For every spectral value λ, there exists a sequence $\{\xi_n\}_{n\geq 0}$ in $D(h)^1$ such that $\langle h\xi_n|\xi_n\rangle \to \lambda$.

Hence,

$$\inf_{\xi\in D(h)^1} \langle h\xi|\xi\rangle \ \leq \ \lambda\,.$$

Let $\lambda_0 = \min(\mathrm{Sp}(h))$ (the minimum exists since the spectrum of h is real, closed and bounded below). If we apply the last inequality to λ_0, we get :

$$\inf_{\xi\in D(h)^1} \langle h\xi|\xi\rangle \ \leq \ \lambda_0\,.$$

As for the other inequality, let $h = \int \lambda dE(\lambda)$ be the spectral decomposition of h; we have $\langle h\xi|\xi\rangle = \int \lambda \langle dE(\lambda)\xi|\xi\rangle$ and $\langle \xi|\xi\rangle = \int \langle dE(\lambda)\xi|\xi\rangle$ for every fixed ξ in $D(h)$. Then, for ξ in $D(h)^1$,

$$\langle h\xi|\xi\rangle = \int_{\mathrm{spec}(h)} \lambda \langle dE(\lambda)\xi|\xi\rangle \ \geq \lambda_0 \int_{\mathrm{spec}(h)} \langle dE(\lambda)\xi|\xi\rangle \ = \lambda_0\,.$$

This finishes the proof. ∎

Definition 3.8. For a unitary representation π of G, we define the constant $k(\overline{d\pi(\Delta)}, G)$ by

$$k(\overline{d\pi(\Delta)}, G) = \inf_{\xi \in D(\overline{d\pi(\Delta)})^1} \left\langle \overline{d\pi(\Delta)}\xi | \xi \right\rangle .$$

Corollary 3.9. *The following holds :*

$$k(\overline{d\pi(\Delta)}, G) = \min \mathrm{Sp}(\overline{d\pi(\Delta)}) = \inf_{\xi \in [\mathcal{C}^\infty(H_\pi)]^1} \sum_{i=1}^{n} \| d\pi(X_i)\xi \|^2$$

Proof. The first equality comes from the preceding lemma. On the other hand, it is clear that

$$k(\overline{d\pi(\Delta)}, G) \leq \inf_{\xi \in [\mathcal{C}^\infty(H_\pi)]^1} \left\langle \overline{d\pi(\Delta)}\xi | \xi \right\rangle = \inf_{\xi \in [\mathcal{C}^\infty(H_\pi)]^1} \sum_{i=1}^{n} \| d\pi(X_i)\xi \|^2 .$$

To obtain the reverse inequality, it suffices to show that, for $\epsilon > 0$, there exists a $\mathcal{C}^\infty$ vector η of norm 1 such that

$$|k(\overline{d\pi(\Delta)}, G) - \langle d\pi(\Delta)\eta | \eta \rangle \,| < \epsilon .$$

By definition, there exists ξ in $D(\overline{d\pi(\Delta)})$ of norm 1 such that

$$0 \leq \left\langle \overline{d\pi(\Delta)}\xi | \xi \right\rangle - k(\overline{d\pi(\Delta)}, G) < \epsilon/3 .$$

As $\overline{d\pi(\Delta)}$ is the closure of $d\pi(\Delta)$, there exists a $\mathcal{C}^\infty$ vector η of norm 1 which is arbitrarily close to ξ with respect to the graph norm. As,

$$\begin{aligned} |k(\overline{d\pi(\Delta)}, G) - \langle d\pi(\Delta)\eta | \eta \rangle \,| \;\; \leq \;\; & |k(\overline{d\pi(\Delta)}, G) - \langle d\pi(\Delta)\xi | \xi \rangle \,| + \\ & | \left\langle \overline{d\pi(\Delta)}\xi | \xi \right\rangle - \left\langle \overline{d\pi(\Delta)}\xi | \eta \right\rangle | + \\ & | \left\langle \overline{d\pi(\Delta)}\xi | \eta \right\rangle - \langle d\pi(\Delta)\eta | \eta \rangle \,| \end{aligned}$$

this proves the claim. ■

Theorem 3.10. *G has property (T) if and only if there exists an $\epsilon > 0$ such that $k(\overline{d\pi(\Delta)}, G) \geq \epsilon$ for every unitary representation π of G without non zero fixed vectors.*

Proof. $\Leftarrow$) If π has almost invariant vectors, by thm 3.1, $k(\overline{d\pi(\Delta)}, G) = \min \mathrm{Sp}(\overline{d\pi(\Delta)}) = 0$. The assumption then implies that π has non zero fixed vectors, i.e. G has property (T).
$\Rightarrow$) Assume by contradiction that there exists a sequence of unitary representations $\{\pi_n\}_{n\geq 0}$ without non zero fixed vector such that $k(\overline{d\pi_n(\Delta)}, G) \to 0$. We claim that the representation $\sigma = \bigoplus_{n\geq 0} \pi_n$ satisfies $k(\overline{d\sigma(\Delta)}, G) = 0$. By the assumption, there exists, for every n, a vector $\xi_n \in \mathcal{C}^\infty(H_{\pi_n})^1$ such that $\langle d\pi_n(h)\xi_n \,|\, \xi_n \rangle \; < \; 1/n + k(\overline{d\pi_n(\Delta)}, G)$.

The vector η_n defined by $\eta_n = (0, \cdots, 0, \xi_n, 0 \cdots)$, with ξ_n at the nth place, is a $\mathcal{C}^\infty$ vector in H^1_σ and

$$\left\langle \overline{d\sigma(\Delta)}\eta_n \,|\, \eta_n \right\rangle \; = \; \left\langle \overline{d\pi_n(\Delta)}\xi_n \,|\, \xi_n \right\rangle \; < \; 1/n + k(\overline{d\pi_n(\Delta)}, G).$$

Hence, $k(\overline{d\sigma(\Delta)}, G) = 0$ and 0 is in the spectrum of $\overline{d\sigma(\Delta)}$. By Theorem 3.1, this implies that σ has almost invariant vectors. Since G has property T, σ has a nonzero fixed vector $\{\beta_n\}_{n\geq 0}$. Choose n so that $\beta_n \neq 0$. Then β_n is a non zero fixed vector for π_n, contradicting the assumption. ■

We can define $K(\Delta, G) = \inf_{\pi\in\widetilde{G}^*} k(\overline{d\pi(\Delta)}, G)$, the Laplacian Kazhdan constant.

Corollary 3.11. *G has property (T) if and only if $K(\Delta, G) > 0$.*

This corollary is a direct consequence of the preceding proposition.

Remark 3.12. The above results (Theorem 3.1, Theorem 3.10 and Corollary 3.11) remain true if the vectors $X_1, \ldots, X_n$ are only assumed to generate $\mathfrak{g}$ as a Lie algebra. The relevant facts about the heat kernel associated with the sub-Laplacian $\Delta = -\sum_{i=1}^n X_i^2$ hold in this more general situation.

4. Comparison with the classical constants

Proposition 4.1. *Let G be a connected Lie group and let $\{X_i\}_{i=1,\ldots,n}$ be a basis of its Lie algebra. Fix $\epsilon > 0$ and set $S = \{\exp(tX_i) \mid t \in [0,\epsilon], i \in \{1,\ldots,n\}\}$. Then, for any unitary representation π of G, one has :*

$$\kappa(G, S, \pi) \leq \epsilon\sqrt{k(\overline{d\pi(\Delta)}, G)}.$$

In particular $\kappa(G, S) \leq \epsilon\sqrt{K(\Delta, G)}$.

Proof. Let π be a unitary representation of G and let $\xi \in \mathcal{C}^\infty(H_\pi)$.

$$\begin{aligned}
\text{Then } \sup_{s\in S} \|\pi(s)\xi - \xi\| &= \sup_{t\in[0,\epsilon]} \max_{i\in\{1\ldots n\}} \|\pi(e^{tX_i})\xi - \xi\| \\
&= \sup_{t\in[0,\epsilon]} \max_{i\in\{1\ldots n\}} \left\| \int_0^t \pi(e^{sX_i}) d\pi(X_i)\xi ds \right\| \\
&\leq \sup_{t\in[0,\epsilon]} \max_{i\in\{1\ldots n\}} \int_0^t \|\pi(e^{sX_i}) d\pi(X_i)\xi\| \, ds \\
&\leq \epsilon \max_{i\in\{1\ldots n\}} \|d\pi(X_i)\xi\| \\
&\leq \epsilon\sqrt{\sum_{i=1}^n \|d\pi(X_i)\xi\|^2}\,.
\end{aligned}$$

So

$$\begin{aligned}
\kappa(G, S, \pi) &= \inf_{\xi\in H_\pi^1} \sup_{s\in S} \|\pi(s)\xi - \xi\| \\
&\leq \inf_{\xi\in(\mathcal{C}^\infty(H_\pi))^1} \sup_{s\in S} \|\pi(s)\xi - \xi\| \\
&\leq \epsilon \inf_{\xi\in(\mathcal{C}^\infty(H_\pi))^1} \sqrt{\sum_{i=1}^n \|d\pi(X_i)\xi\|^2} \\
&= \epsilon\sqrt{k(\overline{d\pi(\Delta)}, G)}
\end{aligned}$$

This ends the proof. ∎

Remark 4.2. (i) 4.1 gives an alternative proof of implication $(\Rightarrow)$ in 3.10.

(ii) Proposition 4.1 states that :

$$\kappa(G,S,\pi) \leq \epsilon\sqrt{k(\overline{d\pi(\Delta)},G)},$$

for every unitary representation π of G. However the converse fails very strongly : there exists no continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, such that $f(0)=0$ and

$$k(\overline{d\pi(\Delta)},G) \leq f(\kappa(G,S,\pi)),$$

for every connected Lie group G and for every unitary representation π of G. Indeed, $\kappa(G,S,\pi) \leq 2$ for any representation π, and we shall exhibit a sequence $(\pi_k)_{k\geq 1}$ of unitary representations of $\mathbb{R}$ such that $k(\overline{d\pi_k(\Delta)},G) = k^2$ for every $k \geq 1$.

Let $H_k = L^2([k,+\infty))$ et T_k be the selfadjoint operator on H_k defined by :

$$D(T_k) = \{\xi \in H_k \mid \int_k^\infty \lambda^2|\xi(\lambda)|^2 d\lambda < \infty\} \text{ and } (T_k\xi)(\lambda) = \lambda\,\xi(\lambda).$$

If $\pi_k(t) = \exp(itT_k)$, we have :

$$k(\overline{d\pi_k(\Delta)},\mathbb{R}) = \inf_{\xi\in D(T_k)^1} \left\langle T_k^2\xi \mid \xi\right\rangle = k^2.$$

Example 4.3. 1. Let $G = SL(2,\mathbb{R})$. Take, as basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, the matrices

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; V = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; W = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $\Delta = -(H^2+V^2+W^2)$ is equal to $\frac{1}{4}(\omega + 2W^2)$ with ω the Casimir operator in $U(\mathfrak{sl}_2(\mathbb{R}))$.

Let us use the coordinates (x,y,θ) on G, where a group element is expressed as

$$g = \begin{pmatrix} \sqrt{y} & x \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \; (x,\theta\in\mathbb{R}, y>0).$$

Then, in these coordinates, we find, viewing H,V,W as left invariant vector fields on G :

$$\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y\frac{\partial^2}{\partial x\partial\theta} + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}$$

(see S. Lang, [9], Chap. X, §1 and §2).

Let $\mathbb{H} \cong G/K$, $K = SO(2)$, be the Poincaré upper half space

$$\mathbb{H} = \{z = x+iy \mid x,y\in\mathbb{R}, y>0\},$$

with invariant measure $y^{-2}dxdy$.

Let π be the left regular representation of G on $L^2(\mathbb{H})$. As the functions on $\mathbb{H}$ are independent of θ on the right, we have

$$d\pi(\Delta) = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$

which is the Laplace-Beltrami operator on $\mathbb{H}$.

It is well known that $\mathrm{Sp}(d\pi(\Delta)) \subseteq [1/4; +\infty[$. Here is a very elementary argument for this due to Mac Kean [12]: Let f be a real smooth function on $\mathbb{H}$ with compact support. Then

$$\begin{aligned}\frac{1}{4}\left(\int_0^\infty f(x,y)^2 y^{-2} dy\right)^2 &= \left(\int_0^\infty f(x,y)\frac{\partial f}{\partial y}(x,y) y^{-1} dy\right)^2 \\ &\leq \int_0^\infty f(x,y)^2 y^{-2} dy \int_0^\infty \left(\frac{\partial f}{\partial y}(x,y)\right)^2 dy\end{aligned}$$

So

$$\frac{1}{4}\int_0^\infty f(x,y)^2 y^{-2} dy \leq \int_0^\infty \left(\frac{\partial f}{\partial y}(x,y)\right)^2 dy.$$

Hence

$$\begin{aligned}\frac{1}{4}\,\| f \|^2_{L^2(\mathbb{H})} &\leq \int_{-\infty}^{+\infty} dx \int_0^\infty \left(\frac{\partial f}{\partial y}(x,y)\right)^2 dy \\ &\leq \int_{-\infty}^{+\infty} dx \int_0^\infty \left[\left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2\right] dy \\ &= -\int_{-\infty}^{+\infty} dx \int_0^\infty f(x,y)\left(\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)\right) dy \\ &= \langle d\pi(\Delta) f | f\rangle .\end{aligned}$$

This implies that $\lambda \geq 1/4$ for any $\lambda \in \mathrm{Sp}(\pi(\Delta))$.

It is actually known that $\inf \mathrm{Sp}(\overline{d\pi(\Delta)}) = 1/4$, that is, $k(\overline{d\pi(\Delta)}, G) = 1/4$ (see [9]).

2. Let G be the (three-dimensional) Heisenberg group. Thus $G = \mathbb{R}^3$ with group law

$$(p,q,t)(p',q',t') = (p+p', q+q', t+t' + \frac{1}{2}(pq' - qp')).$$

The left invariant vectors fields on G corresponding to the coordinates (p,q,t) are :

$$P = \frac{\partial}{\partial p} - \frac{1}{2}q\frac{\partial}{\partial t},\ Q = \frac{\partial}{\partial q} + \frac{1}{2}p\frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t}.$$

As is well known, G has for each $h \in \mathbb{R}$, $h \neq 0$, an infinite dimensional unitary representation ρ_h on $L^2(\mathbb{R})$ so that

$$d\rho_h(P) = hD, d\rho_h(Q) = M, d\rho_h(T) = \frac{h}{2\pi i}\mathbb{I},$$

where $D = \frac{1}{2\pi i}\frac{\partial}{\partial x}$ and M is the multiplication operator by x on $L^2(\mathbb{R})$.

A version of the Heisenberg Uncertainty Principle states that :

$$\| Mu \|_2^2 + \| Du \|_2^2 \geq \frac{1}{2\pi} \| u \|_2^2, \ \forall u \in L^2(\mathbb{R})$$

with equality if and only if u is a multiple of the Gaussian $x \mapsto \exp(-\pi x^2)$ (see [3], Corollary (1.37)). Replacing in this inequality u by the function $x \mapsto |h|^{1/4} u(h^{1/2}x)$ yields

$$\| Mu \|_2^2 + \| hDu \|_2^2 \geq \frac{|h|}{2\pi} \| u \|_2^2$$

with equality if $u(x) = |h|^{1/4} \exp(-\pi |h| x^2)$.

Thus, with $\Delta = -(P^2 + Q^2 + T^2)$, we see that

$$\langle d\rho_h(\Delta)u|u\rangle = \| Mu \|_2^2 + \| hDu \|_2^2 + \frac{|h|^2}{4\pi^2} \| u \|_2^2 \geq \left(\frac{|h|}{2\pi} + \frac{|h|^2}{4\pi^2} \right) \| u \|_2^2$$

with equality for the above Gaussian function.

Hence, we obtain the exact value for the Kazhdan constant

$$\begin{aligned} k(\overline{d\rho_h(\Delta)}, G) &= \inf\{\lambda \mid \lambda \in \mathrm{Sp}(d\rho_h(\Delta))\} \\ &= \left(\frac{|h|}{2\pi} + \frac{|h|^2}{4\pi^2} \right) \end{aligned}$$

3. The following well-known example was pointed out to us by A. Valette. Let G be a connected compact semi-simple Lie group. Let $\{X_i\}_{i=1,\cdots,n}$ be a basis of its Lie algebra $\mathfrak{g}$, which is orthogonal relatively to the Killing form. Then $\Delta = -\sum_{i=1}^{n} X_i^2$ is the Casimir operator of G. As $\Delta \in Z(U(\mathfrak{g}))$, the Schur lemma insures that, for $\pi \in \hat{G}$, $d\pi(\Delta) = c\,\mathbb{I}$. This constant c can be determined in the following way : if λ is the highest weight of π and ρ is the half sum of the positive roots, as we can see in [7], p. 247, we have

$$k(d\pi(\Delta), G) = \langle \lambda + 2\rho \mid \lambda \rangle = \| \lambda + \rho \|^2 - \| \rho \|^2 .$$

Remark 4.4. Let G be a simply connected, connected nilpotent Lie group, and let π be a unitary irreducible representation of G. In [6], a bound is given for $k(\overline{d\pi(\Delta)}, G)$ in terms of the distance from 0 of the Kirillov orbit associated to π.

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