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Compact Lie Groups with isomorphic Homotopy Groups

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Abstract. Compact connected Lie groups with isomorphic higher homotopy groups are locally isomorphic.

In [6] H. Scheerer has shown that compact, connected Lie groups are locally isomorphic if they are homotopy equivalent (see [4] for a conceptually different proof). Obviously, the converse is not true. If G is a compact, connected Lie group which is not simply connected and $\widetilde{G}$ is its universal covering group, then these two groups do not even have isomorphic fundamental groups: $\pi_1(G)$ is isomorphic to the kernel of the covering $\widetilde{G} \to G$ (see [2], (5.8)–(5.10)), while $\pi_1(\widetilde{G})$ is trivial. But the covering yields for every $n > 1$ an isomorphism $\pi_n(\widetilde{G}) \to \pi_n(G)$ (see [2], (7.12)). So the question arises wether it suffices to replace homotopy equivalence in Scheerer's result by isomorphy of the homotopy groups. As will be shown, the answer to this question is "yes". The main step to this result was done by H. Toda in [9]:

Theorem 1. *Two simply connected, compact (and hence semi-simple) Lie groups are isomorphic to each other if and only if they have isomorphic homotopy groups for each dimension.* ∎

The key to the generalization of H. Toda's theorem to all compact, connected Lie groups is a result due to A. Borel which seems to be not as well known as it would deserve to. In [1] he has shown that a compact, connected Lie group is homeomorphic to the topological direct product of its commutator subgroup and the connected component of its center. This result was generalized to arbitrary compact, connected groups by H. Scheerer (see [7]), and K. H. Hofmann remarked that every compact, connected group is even a semi-direct product of its commutator subgroup and an abelian subgroup (see [8] and [3]). However, we will need only A. Borel's weaker version of the result.

Theorem 2. *Let G_1 and G_2 be two compact, connected Lie groups with isomorphic homotopy groups in each dimension. Then G_1 and G_2 are locally isomorphic.*

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Proof. Let $i \in \{1,2\}$. As G_i is compact, its Lie algebra $\mathfrak{g}_i$ is reductive (see [10], Theorem 4.11.7). Thus $\mathfrak{g}_i$ is the direct sum of its center $\mathfrak{c}_i$ and its derived algebra $\mathfrak{D}\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i]$, which is semi-simple in this situation (see [10], Theorem 3.16.3). The analytic subgroup of G_i corresponding to $\mathfrak{D}\mathfrak{g}_i$ is its commutator subgroup $\mathrm{D}G_i$ (see [10], Theorem 3.18.8), while the connected component C_i of the center of G_i corresponds to $\mathfrak{c}_i$. Now by [1], 3.2 the group G_i is homeomorphic to $C_i \times \mathrm{D}G_i$. For the homotopy groups this means $\pi_n(G_i) \cong \pi_n(C_i) \times \pi_n(\mathrm{D}G_i)$ (see [2], (7.11)). The compact, semi-simple group $\mathrm{D}G_i$ has a compact universal covering group $\widetilde{\mathrm{D}G_i}$ (see [10], Theorem 4.11.6). Therefore, the fundamental group $\pi_1(\mathrm{D}G_i)$ is finite (see [2], (5.8)–(5.10)), whereas the torus group C_i has a fundamental group isomorphic to $\mathbb{Z}^d$, where $d = \dim C_i = \dim \mathfrak{c}_i$ (see [2], (4.4) and (4.8)). So the dimension of the abelian Lie algebra $\mathfrak{c}_i$ is determined by the torsion-free rank of the fundamental group of G_i, and we get $\mathfrak{c}_1 \cong \mathfrak{c}_2$. It remains to show that $\mathfrak{D}\mathfrak{g}_1$ and $\mathfrak{D}\mathfrak{g}_2$ are isomorphic too. Since every torus has trivial higher homotopy groups (see [2], (7.14) and (7.11)), we get $\pi_n(G_i) \cong \pi_n(\mathrm{D}G_i)$ for $n > 1$. The covering of $\mathrm{D}G_i$ by its universal covering group $\widetilde{\mathrm{D}G_i}$ yields an isomorphism $\pi_n(\widetilde{\mathrm{D}G_i}) \to \pi_n(\mathrm{D}G_i)$ in every dimension $n > 1$ (see [2], (7.12)). Hence the compact, simply connected Lie groups $\widetilde{\mathrm{D}G_1}$ and $\widetilde{\mathrm{D}G_2}$ have isomorphic homotopy groups in each dimension, and Theorem 1 says that $\widetilde{\mathrm{D}G_1}$ and $\widetilde{\mathrm{D}G_2}$ are isomorphic. Consequently, their Lie algebras $\mathfrak{D}\mathfrak{g}_1$ and $\mathfrak{D}\mathfrak{g}_2$ are isomorphic as well. ■

The groups $\mathrm{U}_2\,\mathbb{C}$ and $\mathrm{SU}_2\,\mathbb{C} \times \mathrm{SO}_2\,\mathbb{R}$ are homeomorphic but not isomorphic. Therefore, one cannot omit the word "locally" in Theorem 2. A semi-simple example for this situation is given by the groups $\mathrm{SO}_3\,\mathbb{R} \times \mathrm{SU}_2\,\mathbb{C}$ and $\mathrm{SO}_4\,\mathbb{R} \cong \mathrm{SO}_3\,\mathbb{R} \ltimes \mathrm{SU}_2\,\mathbb{C}$. In [9] H. Toda presents an example showing that homology and cohomology do not suffice to determine compact Lie groups up to local isomorphism. A survey on homotopy theory of Lie groups can be found in [5].

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