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The Euler-Poincaré characteristic of a Lie algebra

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Abstract. We will show that the Euler-Poincaré characteristic of a finite dimensional Lie algebra is zero if the ground field has characteristic zero or the Lie algebra is not perfect.

Let $V^* = \bigoplus_{i\geq 0} V^i$ be a graded vector space over a field k. We assume that each V^i has finite dimension. We recall that *Euler-Poincaré characteristic* of V^* is given by

$$\chi(V^*) := \sum_{i\geq 0}(-1)^i \dim V^i.$$

Let $\mathbf{g}$ be a Lie algebra over k and M be a $\mathbf{g}$-module. Assume $\dim \mathbf{g} < \infty$ and $\dim M < \infty$. We let $\chi(\mathbf{g}, M)$ denote the Euler-Poincaré characteristic of $H^*(\mathbf{g}, M)$. When $M = k$ with trivial action, we write $\chi(\mathbf{g})$ instead of $\chi(\mathbf{g}, k)$. The goal of this paper is to prove the following result.

Theorem 1. *Let $\mathbf{g}$ be a finite dimensional Lie algebra and M be a finite dimensional $\mathbf{g}$-module. Assume that one of the following conditions hold*

i) $H_1(\mathbf{g}) \neq 0$.

ii) char $k = 0$ and $\mathbf{g} \neq 0$.

Then $\chi(\mathbf{g}, M) = 0$.

For solvable Lie algebras this was proved before in [1]. In order to prove the theorem we need to make some simple observations.

Lemma 2. *Let V^* be a finite dimensional graded vector space, whose nonzero components are concentrated in odd degrees. Then*

$$\chi(\Lambda^*(V^*)) = 0,$$

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where Λ^* denotes the exterior algebra.

Proof. Since $\Lambda^*(V^* \oplus W^*) \cong \Lambda^*(V^*) \otimes \Lambda^*(W^*)$ and the Euler-Poincaré characteristic has multiplicative property, one can reduce the statement to the case where V^* is concentrated in degree $2k+1$ and is one-dimensional. Then one has $\chi(\Lambda^*(V^*)) = 1 + (-1)^{2k+1} = 0$. ■

Lemma 3. *If $\mathbf{g}$ is one dimensional Lie algebra and M is a finite dimensional $\mathbf{g}$-module, then $\chi(\mathbf{g}, M) = 0$.*

Proof. By definition of Lie algebra cohomology, one has an exact sequence

$$0 \to H^0(\mathbf{g}, M) \to M \to M \to H^1(\mathbf{g}, M) \to 0$$

and $H^i(\mathbf{g}, M) = 0$ for $i > 1$. Thus $\chi(\mathbf{g}, M) = 0$. ■

Lemma 4. *If k has characteristic zero, $\mathbf{g}$ is a finite dimensional semi-simple Lie algebra and M is a finite dimensional $\mathbf{g}$-module, then $\chi(\mathbf{g}, M) = 0$.*

Proof. It is well known that $H^*(\mathbf{g})$ is an exterior algebra on odd degree generators, thus Lemma 2 gives that $\chi(\mathbf{g}) = 0$. Moreover, assume that M is a finite dimensional $\mathbf{g}$-module. Then one has an isomorphism $H^*(\mathbf{g}, M) \cong H^*(\mathbf{g}) \otimes H^0(\mathbf{g}, M)$. Therefore

$$\chi(\mathbf{g}, M) = \chi(\mathbf{g})\ (\dim H^0(\mathbf{g}, M)) = 0. \quad ■$$

Lemma 5. *Let*

$$0 \to \mathbf{a} \to \mathbf{g} \to \mathbf{h} \to 0$$

be a short exact sequence of finite dimensional Lie algebras and M be a finite dimensional $\mathbf{g}$-module. Then

$$\chi(\mathbf{g}, M) = \sum_{i \geq 0} (-1)^i \chi(\mathbf{h}, H^i(\mathbf{a}, M)).$$

Moreover, if the action of $\mathbf{h}$ on the cohomology $H^*(\mathbf{a}, M)$ is trivial, then we have

$$\chi(\mathbf{g}, M) = \chi(\mathbf{a}, M)\chi(\mathbf{h}).$$

Proof. Since the Euler-Poincaré characteristic does not change after taking homology, the Hochschild-Serre spectral sequence gives:

$$\chi(\mathbf{g}, M) = \chi(E_2^{**}) = \sum_{p,q \geq 0} (-1)^{p+q} \dim E_2^{pq}.$$

To finish the proof one remarks that

$$\sum_{q \geq 0} (-1)^q \sum_{p \geq 0} (-1)^p \dim H^p(\mathbf{h}, H^q(\mathbf{a}, M)) = \sum_{i \geq 0} (-1)^i \chi(\mathbf{h}, H^i(\mathbf{a}, M)). \quad ■$$

Proof of the Theorem. i) By assumption there exists an epimorphism from $\mathbf{g}$ into a nonzero abelian Lie algebra. On the other hand, from any nonzero abelian Lie algebra there exists an epimorphism onto a one dimensional Lie algebra. Hence there exists an exact sequence of Lie algebras

$$0 \to \mathbf{a} \to \mathbf{g} \to \mathbf{h} \to 0$$

with $\dim \mathbf{h} = 1$. Now we can use Lemma 5 and Lemma 3 to finish the proof.

ii) By i) one can assume that $\mathbf{g}$ is not solvable. Let $\mathbf{r}$ be the radical of $\mathbf{g}$ and $\mathbf{s}$ be the factor-algebra $\mathbf{g}/\mathbf{r}$. By Lemma 5 one has $\chi(\mathbf{g}, M) = \sum_{i\geq 0}(-1)^i \chi(\mathbf{s}, H^i(\mathbf{r}, M))$. Since $\mathbf{s} \neq \mathbf{0}$ is a semi-simple Lie algebra, it has zero Euler-Poincaré characteristic (by Lemma 4) and the result follows. ■

References

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