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Normalizers and centralizers of reductive subgroups of almost connected Lie groups

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Abstract. If M is an almost connected Lie subgroup of an almost connected Lie group G such that the adjoint group M is reductive then the product of M and its centralizer is of finite index in the normalizer. The method of proof also gives a different approach to the Weyl groups in the sense of [6].

An *almost connected Lie group* is a Lie group with finitely many connected components. For a subgroup U of any group H denote by $Z(U,H)$ and by $N(U,H)$ the centralizer and the normalizer of U in H, respectively. The product $UZ(U,H)$ is always contained in $N(U,H)$. By Theorem A of [3], if H is an almost connected Lie group and U is compact, the difference is not too "big," i.e., $UZ(U,H)$ is of finite index in $N(U,H)$. While from the point of view of applications, for instance to probability theory of Lie groups, the hypothesis of the compactness of U may be the most interesting one, this short note presents a theorem, which shows that, from the point of view of the proof, it is not the topological property of compactness which is crucial, but an algebraic consequence of it, namely, the reductivity of the adjoint group. Moreover, we provide additional information on the almost connectedness of certain normalizers. Finally, there are comments on the finiteness of certain generalized "Weyl groups" in the sense of [6]. A few words about our notion of Lie (sub)groups are in order. By defintion we shall assume that a Lie group is *paracompact*, i.e., has countably many components. A subgroup M of a Lie group G is called a *Lie subgroup* if M is endowed with the structure of a smooth manifold such that M is a Lie group and that the inclusion $M \to G$ is an immersion. The smooth structure of M is uniquely determined by the Lie group G and the abstract subgroup M. For instance, denoting by $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ the adjoint representation of G, we observe that the subgroup $\mathrm{Ad}(M)$ is a Lie subgroup of $\mathrm{GL}(\mathfrak{g})$ which may, as a Lie group, be identified with the quotient Lie group $M/(Z(G_0,G) \cap M)$; the uniqueness theorem tells us that this is the only way to view $\mathrm{Ad}(M)$ as a Lie subgroup. If M is not closed in G then the Lie group topology is properly finer than the relative topology. In the sequel, when we speak

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of topological properties of M such as almost connectedness, we shall refer to the intrinsic Lie group topology of M.

A connected Lie subgroup H of some general linear group is called *reductive* if it can be written as a product of a semisimple Lie group and a central connected Lie subgroup consisting of semisimple automorphisms. Because linear semisimple Lie groups are closed this condition is equivalent to the statement that $\bar{H}$ is reductive; moreover, the commutator subgroup $[\bar{H}, \bar{H}]$ is contained in H, indeed $[\bar{H}, \bar{H}] = [H, H]$.

Theorem 1. *Let G be an almost connected Lie group with Lie algebra $\mathfrak{g}$, and let M be an almost connected Lie subgroup of G such that $\mathrm{Ad}(M_0) \leq \mathrm{Aut}(\mathfrak{g}) \leq \mathrm{GL}(\mathfrak{g})$ is reductive. Then $MZ(M,G)$ is of finite index in $N(M,G)$.*

Remark 2. The assumption on $\mathrm{Ad}(M_0)$ implies that the Lie algebra $\mathfrak{m}$ of M is itself reductive as an "abstract Lie algebra," i.e., is the direct sum of its center and a semisimple ideal. To see this one observes that the reductive Lie subgroup $\mathrm{Ad}(M_0)$ acts on $\mathfrak{m}$ and yields a direct decomposition of $\mathfrak{m}$ into simple ideals and central ideals. While $Z(M,G)$ is closed and thus is a Lie subgroup of G, the subgroup $MZ(M,G)$ is not necessarily closed. However, it is a Lie subgroup inheriting its differentiable structure from the product $M \times Z(M,G)$ via the obvious surjective group homomorphism $M \times Z(M,G) \to MZ(M,G)$. As a consequence of Theorem 1, also $N(M,G)$ is a Lie subgroup containing $MZ(M,G)$ as an open submanifold. By the known structure of compact connected Lie groups the assumptions are satisfied if $\mathrm{Ad}(M)$ is relatively compact. In particular, this holds if M itself is compact, which is the case treated in [3].

If $\mathrm{Ad}(M)$ is relatively compact one has a stronger result:

Corollary 3. *Let G, $\mathfrak{g}$, and M be as in the Theorem. If $\mathrm{Ad}(M)$ is relatively compact, then $N(M,G)$ and $Z(M,G)$ are almost connected Lie subgroups. For the connected component $N(M,G)_0$ one has*

$$N(M,G)_0 = M_0 Z(M,G)_0.$$

To deduce the Corollary from the Theorem it is sufficient to show that $Z(M,G)$ is almost connected, which follows from the subsequent proposition.

Proposition 4. *For an almost connected Lie group G with Lie algebra $\mathfrak{g}$, let $d\colon \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ denote the standard homomorphism assigning to each automorphism its differential at the origin. Let F be a subgroup of $\mathrm{Aut}(G)$ such that $d(F)$ is relatively compact. Then the closed subgroup G^F of fixed points is almost connected.*

Proof. W.l.o.g. we assume that G is connected. Since G^F is not changed when we replace F by its closure, we may as well assume that F is a compact Lie group. Then we form the almost connected Lie group $H \stackrel{\text{def}}{=} F \ltimes G$. By [4, p. 180] there exists a maximal compact subgroup K of H with $F \subset K$. We claim that then $K \cap G$ is a maximal compact subgroup of G. If L is a compact subgroup of G containing $K \cap G$ then choose a maximal compact subgroup Q of H with $L \subset Q$. Since maximal compact subgroups of the almost connected Lie group H are conjugate, there exists an element $x \in H$ with $K = xQx^{-1}$. Using that G is normal in H we obtain

$$L \supset K \cap G = (xQx^{-1}) \cap G = x(Q \cap G)x^{-1} \supset xLx^{-1}.$$

Hence $L = xLx^{-1} = K \cap G$, and this proves the claim. This argument applies to any closed normal subgroup of an almost connected Lie group H and a maximal compact subgroup of H. However, in the case at hand, the homogeneous spaces $G/(K \cap G)$ and H/K are diffeomorphic and indeed are diffeomorphic to a euclidean space. Actually, we use the more precise information that there exist K–invariant, in particular F–invariant vector subspaces $\mathfrak{v}_1, \ldots, \mathfrak{v}_r$ of $\mathfrak{g}$ such that the map

$$(K \cap G) \times \mathfrak{v}_1 \times \ldots \times \mathfrak{v}_r \to G,$$

given by $(k, X_1, \ldots, X_r) \longmapsto k \ \exp X_1 \cdot \ldots \cdot \exp X_r$, is a diffeomorphism. Since this map is F–invariant, G^F corresponds to

$$(K \cap G)^F \times \mathfrak{v}_1^F \times \ldots \times \mathfrak{v}_r^F,$$

which has finitely many connected components. ■

We now turn to the theorem and consider first the case that M is connected and abelian, which indeed is the most interesting one. Let us briefly stop for looking an example which comes to mind and illustrates well why the theorem and its corollary should hold. Why does the natural semidirect product $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2$ not allow a faithful representation into some $\mathrm{GL}_n(\mathbb{R})$? If H were the image of such a representation we would take the Zariski closure G of H, and G would then be almost connected. The image of $\mathbb{T}^2$ would be Zariski closed as a compact group. Its normalizer in $\mathrm{GL}_n(\mathbb{R})$ would be Zariski closed and would contain H, hence G. In particular G_0 would normalize the image of $\mathbb{T}^2$ and thus would yield a *connected* group of automorphisms of the torus $\mathbb{T}^2$, whose automorphism group is discrete. Thus G_0 would centralize $\mathbb{T}^2$, giving a contradiction to $H \leq G$ and the finiteness of G/G_0. What is essential in this illustration is the fact that algebraic groups are almost connected, and this will also be crucial in the proof of the following proposition.

Proposition 5. *Let G be an almost connected Lie group and V a connected Lie subgroup of G such that $\mathrm{Ad}(V)$ is abelian and consists of semisimple automorphisms. Then the quotient group $N(V,G)/Z(G_0,G)$ is almost connected and $Z(G_0,G)N(V,G)_0$ is of finite index in $N(V,G)$. Moreover, $N(V,G)_0$ centralizes V, whence $Z(V,G)$ is of finite index in $N(V,G)$.*

Proof. Again we may assume that G is connected. As we observed in Remark 2, the subgroup V itself is abelian. By [2, Proposition 10, p. 24] there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing the Lie algebra $\mathfrak{v}$ of V as a central subalgebra. Set $H = \exp \mathfrak{h}$, and let G' denote the commutator subgroup of

G. Then $\mathfrak{g} = \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]$ and $G = HG'$ since any Cartan subalgebra supplements $[\mathfrak{g}, \mathfrak{g}]$. The Lie algebra of $\mathrm{Ad}(G') = \mathrm{Ad}(G)'$ is $\mathrm{ad}([\mathfrak{g}, \mathfrak{g}]) = [\mathrm{ad}\,\mathfrak{g}, \mathrm{ad}\,\mathfrak{g}]$, which, as the commutator algebra of a linear Lie algebra is an algebraic Lie algebra (see e.g. [1, (7.9) Corollary, p. 195]. Hence $\mathrm{Ad}(G')$ is the connected identity component of an algebraic subgroup A of $\mathrm{Aut}(\mathfrak{g})$. In particular, $\mathrm{Ad}(G')$ is of finite index in A. The group $B \stackrel{\text{def}}{=} \{\alpha \in A \mid \alpha(\mathfrak{v}) = \mathfrak{v}\}$ is an algebraic subgroup of A and thus is almost connected. Therefore, $\mathrm{Ad}(G') \cap B = \mathrm{Ad}(G' \cap N(V, G))$ is almost connected as well. From $H \subset N(V, G)$ and $G = HG'$ it follows that $N(V, G) = HN(V, G')$ and thus $\mathrm{Ad}(N(V, G)) = \mathrm{Ad}(H)\,\mathrm{Ad}(G' \cap N(V, G))$, which is almost connected. Since $\mathrm{Ad}(N(V, G))$ is isomorphic to $N(V, G)/Z(G, G)$ this latter group is almost connected. This implies immediately that $Z(G, G)N(V, G)_0$ has finite index in $N(V, G)$. For each X in the Lie algebra $\mathfrak{n}$ of $N(V, G)$ one has $[X, \mathfrak{v}] \subset \mathfrak{v}$, hence $\mathrm{ad}(Y)(X) \in \mathfrak{v}$ for all $Y \in \mathfrak{v}$. As $\mathfrak{v}$ is abelian it follows that $\mathrm{ad}(Y)^2(X) = 0$. Since all the $\mathrm{ad}(Y)$, $Y \in \mathfrak{v}$, are semisimple, we conclude that $\mathrm{ad}(Y)(X) = 0$, i.e., $[\mathfrak{n}, \mathfrak{v}] = 0$. Therefore, $N(V, G)_0$ centralizes V. ■

Remark 6. If $\overline{\mathrm{Ad}(V)}$ happens to be even a compact torus, the above arguments can be simplified, and one obtains in addition that $N(V, G)$ is almost connected, in accordance with Corollary 3. In these circumstances, choose a maximal torus T in $\overline{\mathrm{Ad}(G)}$ containing $\overline{\mathrm{Ad}(V)}$; we still assume G to be connected for simplicity. The subgroup $U \stackrel{\text{def}}{=} Ad^{-1}(T)$ is abelian and connected, it clearly contains $Z(G, G)$ and there is a Cartan subalgebra $\mathfrak{h}$ containing the Lie algebra $\mathfrak{u}$ of U as a central subalgebra; for all this compare [8, p. 634, 635]. Here, H contains the center which is not true in general. Therefore the homogeneous space $N(V, G)/H$ is an image of $\mathrm{Ad}(G' \cap N(V, G)) = \mathrm{Ad}(G') \cap \mathrm{Ad}(N(V, G))$. Given that the latter group is almost connected, $N(V, G)/H$ has only finitely many connected components, whence $N(V, G)$ is almost connected.

When we now turn to the proof of the theorem in full generality we shall need some results on $\mathrm{Aut}(M)$, when M satisfies the hypotheses of the theorem. The following is implicit, if not explicit, in [7] or in [3], so we may be brief. If the Lie algebra $\mathfrak{m}$ of an almost connected Lie group M is reductive, then the commutator subgroup $(M_0)'$ of M_0 is a (not necessarily closed) semisimple Lie group and

$$M_0 = (M_0)'Z(M_0, M_0) = (M_0)'Z(M_0, M_0)_0.$$

Denote by $I\colon M \to \mathrm{Aut}(M)$ the canonical homomorphism given by $I(x)(m) = xmx^{-1}$. Let
$\mathcal{A} = \{\alpha \in \mathrm{Aut}(M) \mid \alpha|_{Z(M_0, M_0)_0} = \text{identity}\}$,
$\mathcal{A}_1 = \{\alpha \in \mathcal{A} \mid \alpha$ induces the identity on $M/M_0\}$, and
$\mathcal{A}_2 = \{\alpha \in \mathcal{A}_1 \mid \alpha|_{M_0} = \text{identity}\}$.

Proposition 7. *The following quotients are finite:*
(i) $\mathcal{A}/\mathcal{A}_1$, (ii) $\mathcal{A}_1/\mathcal{A}_2 I((M_0)')$, (iii) $\mathcal{A}_2/I(Z(M_0, M_0))$, *and* (iv) $\mathcal{A}/I(M_0)$.

Proof. The finiteness of M/M_0 gives case (i). Case (ii) results from the fact that the automorphism group of a semisimple Lie algebra contains the group of inner automorphisms as a subgroup of finite index. Case (iv) follows from (i)–(iii), thus we are left with (iii). Each $\alpha \in \mathcal{A}_2$ can be written as $\alpha(x) = x\gamma(x)$, $x \in M$, with a map $\gamma\colon M \to M_0$ satisfying

$$\gamma(M_0) = \{1\}, \text{ and } \gamma(xy) = y^{-1}\gamma(x)y\gamma(y) \text{ for all } x,\, y \in M.$$

Choosing $x \in M_0$ one finds that γ may be considered as a map on the finite group M/M_0, choosing $y \in M_0$ we get $\gamma(M/M_0) \subset Z(M_0, M_0)$. Writing the abelian group $Z \overset{\text{def}}{=} Z(M_0, M_0)$ additively for the moment and denoting by $\dot{x} \in M/M_0$ the coset of $x \in M$ we see that $\mathcal{A}_2$ is isomorphic to the (abelian) group $\mathcal{C}$ of all maps $\gamma\colon M/M_0 \to Z$ with

$$(*) \qquad \gamma(\dot{x}\dot{y}) = y^{-1}\gamma(\dot{x})y + \gamma(\dot{y}).$$

The subgroup $I(Z)$ of $\mathcal{A}_2$ corresponds to the subgroup $\mathcal{B} \overset{\text{def}}{=} \{\gamma_z \mid z \in Z\}$ of $\mathcal{C}$ where $\gamma_z(\dot{y}) = y^{-1}zy - z$. As Z is isomorphic to $\mathbb{R}^d \times \mathbb{T}^m \times \mathbb{Z}^k \times F$ with finite F, for each natural number n the subgroup nZ is of finite index in Z, and $\mathrm{Tor}_n(Z) = \{z \in Z \mid nz = 0\}$ is finite. This applies in particular to the order n of M/M_0, from which one easily deduces that $H^1(M/M_0, Z) = \mathcal{C}/\mathcal{B}$ is finite, which concludes the proof. ■

Proof of Theorem 1. The reductivity of $\mathrm{Ad}\, M_0$ implies that with the semi-simple group $(M_0)'$ we have

$$M_0 = (M_0)' Z(M_0, M_0)_0.$$

Set $V \overset{\text{def}}{=} Z(M_0, M_0)_0$; then $N(M, G)$ is contained in $N(V, G)$. From Proposition 5 we know that $Z(V, G) \cap N(M, G)$ is of finite index in $N(M, G)$. With the notations of Proposition 7, considering also $I\colon N(M, G) \to \mathrm{Aut}(M)$, we have

$$\begin{aligned} I(M_0) &\subset I(Z(V, G) \cap N(M, G)) \subset I(N(M, G)), \text{ and} \\ I(M_0) &\subset I(Z(V, G) \cap N(M, G)) \subset \mathcal{A}. \end{aligned}$$

As $\mathcal{A}/I(M_0)$ and $I(N(M, G))/I(Z(V, G) \cap N(M, G))$ are finite, all four groups above are commensurable, in particular, $I(N(M, G))/I(M_0)$ is finite, whence $M_0 Z(M, G)$ is of finite index in $N(M, G)$. ■

Our approach differs from that one given in [3] essentially in the treatment of the torus–case, Proposition 5. While we give an (almost) self-contained proof the authors of [3] refer to [6] where, among others, the finiteness of certain "generalized Weyl groups" is studied in more generality. Actually, as we shall explain in the last part of this article, our proof of Proposition 5 can be used to reprove some of the results in the beginning of [6]. Both methods rest on the theory of algebraic groups, but while we use that $\mathrm{Ad}(G)'$ is almost algebraic, the authors in [6] apply Ado's theorem and take then the "algebraic hull," which is a less canonical procedure. Recall the situation of [6]. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\mathfrak{a}$ be an ideal in $\mathfrak{h}$. Then define

$$\begin{aligned} N(\mathfrak{a}, G) &= \{x \in G \mid \mathrm{Ad}(x)(\mathfrak{a}) = \mathfrak{a}\}, \\ C(\mathfrak{a}, G) &= \{x \in N(\mathfrak{a}, G) \mid \mathrm{Ad}(x) \text{ induces the identity on } \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}]\}, \\ Z(\mathfrak{a}, G) &= \{x \in N(\mathfrak{a}, G) \mid \mathrm{Ad}(x)|_{\mathfrak{a}} = \text{identity on } \mathfrak{a}\}, \end{aligned}$$

and the corresponding Lie subalgebras $\mathfrak{n}(\mathfrak{a}, \mathfrak{g})$, $\mathfrak{c}(\mathfrak{a}, \mathfrak{g})$, and $\mathfrak{z}(\mathfrak{a}, \mathfrak{g})$ of $\mathfrak{g}$. Clearly, one has $Z(\mathfrak{a}, G) \subset C(\mathfrak{a}, G) \subset N(\mathfrak{a}, G)$.

Proposition 8. *With the above assumptions and notations the following assertions hold true.*

(a) $N(\mathfrak{a}, G)/Z(G, G)$ *is almost connected,*

(b) $N(\mathfrak{a}, G)_0 Z(G, G)$ *is of finite index in* $N(\mathfrak{a}, G)$*,*

(c) $\mathfrak{n}(\mathfrak{a}, \mathfrak{g}) = \mathfrak{h} + \mathfrak{z}(\mathfrak{a}, \mathfrak{g})$*,*

(d) $HZ(\mathfrak{a}, G)$ *is of finite index in* $N(\mathfrak{a}, G)$*, where* $H = \exp \mathfrak{h}$*.*

Proof. Concerning (a) and (b), the arguments in the proof of Proposition 5 apply here as well without any alteration worth mentioning; still H normalizes $\mathfrak{a}$ and $\exp \mathfrak{a}$. Concerning the centralizing property the situation is different, as the "semisimplicity argument" is no longer available. To see (c) consider the Fitting decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_+$ relative to $\mathfrak{h}$ (or a regular element in $\mathfrak{h}$). To prove the nontrivial inclusion $\mathfrak{n}(\mathfrak{a}, \mathfrak{g}) \subset \mathfrak{h} + \mathfrak{z}(\mathfrak{a}, \mathfrak{g})$ decompose an $X \in \mathfrak{n}(\mathfrak{a}, \mathfrak{g})$ accordingly, $X = X_h + X_+$. Since $X_h \in \mathfrak{h} \subset \mathfrak{n}(\mathfrak{a}, \mathfrak{g})$, also X_+ has to be in $\mathfrak{n}(\mathfrak{a}, \mathfrak{g})$, hence $[X_+, \mathfrak{a}] \subset \mathfrak{a} \subset \mathfrak{h}$. On the other hand, as $\mathfrak{a} \subset \mathfrak{h}$, one has $[\mathfrak{a}, X_+] \subset \mathfrak{g}_+$, whence $[\mathfrak{a}, X_+] = 0$, i.e., $X_+ \in \mathfrak{z}(\mathfrak{a}, \mathfrak{g})$. Claim (d) is an immediate consequence of (b) and (c). ■

So far the group $C(\mathfrak{a}, G)$ hasn't appeared yet. In order to introduce Weyl groups, the authors of [6] impose a stronger condition on $\mathfrak{a}$ than just being an ideal, namely that $[\mathfrak{a}, \mathfrak{h}] \subset [\mathfrak{a}, \mathfrak{a}]$ or, in other words, that $\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}]$ is central in $\mathfrak{h}/[\mathfrak{a}, \mathfrak{a}]$ or that $\mathfrak{h}$ is contained in $\mathfrak{c}(\mathfrak{a}, \mathfrak{g})$. Then clearly $H\,Z(\mathfrak{a}, G)$ is contained in $C(\mathfrak{a}, G)$, so that from (d) we deduce the following consequence.

Corollary 9. *If* $\mathfrak{a} \leq \mathfrak{h}$ *satisfies* $[\mathfrak{a}, \mathfrak{h}] \subset [\mathfrak{a}, \mathfrak{a}]$ *then* $N(\mathfrak{a}, G)/C(\mathfrak{a}, G)$ *is a finite group.*

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