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Large automorphism groups of 16-dimensional planes are Lie groups

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Abstract. It is a major problem in topological geometry to describe all compact projective planes $\mathcal{P}$ with an automorphism group Σ of sufficiently large topological dimension. This is greatly facilitated if the group is known to be a Lie group. Slightly improving a result from the first author's dissertation, we show for a 16-dimensional plane $\mathcal{P}$ that the connected component of Σ is a Lie group if its dimension is at least 27.

Compact connected projective planes $\mathcal{P}$ of finite topological dimension exist only in dimensions $d = 2\ell|16$, see [1], 54.11. In the compact-open topology, the automorphism group Σ of such a plane $\mathcal{P}$ is locally compact and has a countable basis [1], 44.3, its topological dimension $\dim\Sigma$ is a suitable measure for the homogeneity of $\mathcal{P}$. The so-called critical dimension c_ℓ is defined as the largest number k such that there exist 2ℓ-dimensional planes with $\dim\Sigma = k$ other than the classical Moufang plane over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ respectively, compare [1], § 65. Analogously, there is a critical dimension $\widetilde{c}_\ell$ for smooth planes, and $\widetilde{c}_\ell \le c_\ell - 2$ by recent work of Bödi [3].

The classification program requires to determine all planes $\mathcal{P}$ admitting a connected subgroup Δ of Σ with $\dim\Delta$ sufficiently close to c_ℓ; most results that have been obtained so far fall into the range $5\ell - 3 \le \dim\Delta \le c_\ell$. Additional assumptions on the structure of Δ or on its geometric action must be made for smaller values of $\dim\Delta$. The cases $\ell \le 4$ are understood fairly well. For $\ell = 8$, however, results are still less complete, and we shall concentrate on 16-dimensional planes from now on. It is known that $c_8 = 40$, and all planes with $\dim\Sigma = 40$ can be coordinatized by a so-called mutation of the octonion algebra $\mathbb{O}$, see [1], 87.7. All *translation planes* with $\dim\Sigma \ge 38$ have been described explicitly by their quasi-fields [1], 82.28. If $\mathcal{P}$ is a proper translation plane, then Σ is an extension of the translation group $\mathsf{T} \cong \mathbb{R}^{16}$ by a linear group, in particular, Σ is then a Lie group.

In her dissertation, the first author proved the following result under the hypothesis $\dim\Sigma \ge 28$. With only minor modifications, her proof yields

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Theorem L. *If* $\dim \Sigma \geq 27$*, then the connected component* Σ^1 *of* Σ *is a Lie group.*

This covers all known examples and all cases in which a classification might be hoped for. A weaker version of Theorem L is given in [1], 87.1, for 8-dimensional planes see also Priwitzer [5]. Here we shall present a proof of Theorem L. The whole structure theory of real Lie groups then becomes available for the classification of sufficiently homogeneous 16-dimensional planes. How such a classification can be achieved has been explained in [1], § 87, second part. Two of the results mentioned there have been improved considerably in the meantime:

Theorem S. *Let* Δ *be a semi-simple group of automorphisms of the* 16*-dimensional plane* $\mathcal{P}$*. If* $\dim \Delta > 28$*, then* $\mathcal{P}$ *is the classical Moufang plane, or* $\Delta \cong \mathrm{Spin}_9(\mathbb{R}, r)$ *and* $r \leq 1$*, or* $\Delta \cong \mathrm{SL}_3\mathbb{H}$ *and* $\mathcal{P}$ *is a Hughes plane as described in* [1], § 86*.*

The proof can be found in Priwitzer [6, 7].

Theorem T. *Assume that* Δ *has a normal torus subgroup* $\Theta \cong \mathbb{T}$*. If* $\dim \Delta > 30$*, then* Θ *fixes a Baer subplane,* $\Delta' \cong \mathrm{SL}_3\mathbb{H}$*, and* $\mathcal{P}$ *is a Hughes plane.*

To prove Theorem L, we will use the approximation theorem as stated in [1], 93.8. The proof distinguishes between semi-simple groups and groups having a minimal connected, commutative normal subgroup Ξ, compare [1], 94.26. A result of Bödi [2] plays an essential role:

Theorem Q. *If the connected group* Λ *fixes a quadrangle, then* Λ *is isomorphic to the compact Lie group* G_2*, or* $\dim \Lambda \leq 11$*. Moreover,* $\dim \Lambda \leq 8$ *if the fixed points of* Λ *form a* 4*-dimensional subplane.*

The last assertion follows from Salzmann [10], § 1, Corollary.

In translation planes, the stabilizer Λ of a quadrangle is compact. Presumably, the same is true for compact, connected planes in general, but for 25 years all efforts have failed to prove compactness of Λ without additional assumptions. This causes some of the difficulties in the following proofs.

Consider any connected subgroup Δ of Σ. If the center Z of Δ is contained in a group of translations with common axis (or with common center), then Δ is a Lie group by Löwen – Salzmann [4] without any further assumption. *Assume now that* Δ *is not a Lie group.* By the approximation theorem, there is a compact, 0-dimensional central subgroup Θ such that Δ/Θ is a Lie group. In particular, $\Theta \leq \mathsf{Z}$ is infinite. The elements of Z can act on the plane in different ways. This leads to several distinct cases. We say that the collineation η is *straight* if each orbit $x^{\langle \eta \rangle}$ is contained in a line, and η is called *planar* if the fixed elements of η form a proper subplane. By a theorem of Baer [1], 23. 15 and 16, a straight collineation is either planar or axial. Hence Theorem L is an immediate consequence of propositions (a—d) which will be proved in this paper.

(a) *If* Δ *leaves some proper closed subplane* $\mathcal{F}$ *invariant (in particular, if* Z *contains a planar element), or if* Δ *is semi-simple, then* $\dim \Delta < 26$.

(b) *If* $\zeta \in \mathsf{Z}$ *is not straight, or if* Z *contains axial collineations with different centers, then* $\dim \Delta \leq 26$.

(c) *If* $\dim \Delta > 26$, *then* Z *is contained in a group* $\Delta_{[a,W]}$ *of homologies. Moreover, a minimal connected, commutative normal subgroup* Ξ *of* Δ *is also contained in* $\Delta_{[a,W]}$.

(d) *If* $\Xi\mathsf{Z} \leq \Delta_{[a,W]}$ *as in* (c), *then* $\dim \Delta \leq 26$, *i.e. case* (c) *does not occur.*

The following criteria will be used repeatedly:

Theorem O. *If* Σ *has an open orbit in the point space, or if the stabilizer* Σ_L *of some line* L *acts transitively on* L, *then* Σ *is a Lie group. (An orbit having the same dimension as the point space* P *is open in* P.*)*

For *proofs* see [1], 53.2 and 62.11. The addendum is a consequence of [1], 51.12 and 96.11(a).

From Szenthe's Theorem [1], 96.14 and again [1], 51.12 and 96.11(a) we infer

Lemma O. *If the stabilizer* Δ_L *of a line* L *has an orbit* $X \subseteq L$ *with* $\dim X = \dim L$, *then* X *is open in* L, *and the induced group* $\Delta_L|_X \cong \Delta_L/\Delta_{[X]}$ *is a Lie group.*

The next result holds without restriction on the dimension of the group:

Theorem P. *The full automorphism group of any* 2*- or* 4*-dimensional compact plane is a Lie group of dimension at most* 8 *or* 16 *respectively.*

Proofs are given in [1], 32.21 and 71.2.

In conjunction with Theorem Q we need

Proposition G. *If* Σ *contains a subgroup* $\Gamma \cong \mathrm{G}_2$, *and if* Γ *fixes some element of the plane, then* Σ *is a Lie group.*

Proof. Assume that Σ is not a Lie group and that Γ fixes the line W. Being simple, Γ acts faithfully on W by [1], 61.26. There are commuting involutions α and β in Γ, and all involutions in Γ are conjugate, see [1], 11.31. Each involution is either a reflection or a Baer involution [1], 55.29, and conjugate involutions are of the same kind. In the case of reflections, one of the involutions α, β, and $\alpha\beta$ would have axis W by [1], 55.35, and Γ would not be effective on W. Hence all involutions are planar [1], 55.29. Because of [1], 55.39, the fixed subplanes $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ intersect in a 4-dimensional plane $\mathcal{F}$. By [1], 55.6, Note, the lines are 8-spheres, and repeated application of [1], 96.35 shows that the fixed elements of Γ form a 2-dimensional subplane $\mathcal{E} < \mathcal{F}$. Moreover, each point $z \in W \setminus \mathcal{E}$ has an orbit $z^\Gamma \approx \mathbb{S}_6$. By the approximation theorem [1], 93.8, some open subgroup of Σ contains a compact central subgroup Θ which is not a Lie group. According to Theorem P, the group Θ induces a Lie group on $\mathcal{F}$, and the kernel $\mathsf{K} = \Theta_{[\mathcal{F}]}$ is infinite. Now choose $z \in W$ such that z belongs to $\mathcal{F}$ but not to $\mathcal{E}$. Then $z^{\mathsf{K}} = z$, and K fixes each point of $z^{\Gamma} \approx \mathbb{S}_6$ (note that $\Gamma \circ \Theta = \mathbb{1}$). Since $\mathcal{F}$ and z^{Γ} together generate the whole plane, we get $\mathsf{K} = \mathbb{1}$. This contradiction proves the proposition. ∎

Finally, we mention a result of M. Lüneburg [1], 55.40 which excludes many semi-simple groups as possible subgroups of Δ:

Lemma R. *The group* $\mathrm{SO}_5\mathbb{R}$ *is never contained in* Σ.

A group Λ of automorphisms is called *straight* if each point orbit x^{Λ} is contained in some line. Baer's theorem mentioned above is true in general for groups which are straight and dually straight. In compact planes of finite positive dimension 2ℓ it holds in the following form:

Theorem B. *If* Λ *is straight, then* Λ *is contained in a group* $\Sigma_{[z]}$ *of central collineations with common center* z, *or the fixed elements of* Λ *form a Baer subplane* $\mathcal{F}_{\Lambda}$.

Proof. If all fixed points of Λ with at most one exception lie on one line, then the unique fixed line through any other point must pass through the same point z. If, on the other hand, there is a quadrangle of fixed points and $\Lambda \neq \mathbb{1}$, then $\mathcal{F}_{\Lambda} = (F, \mathfrak{F})$ is a closed proper subplane. Suppose that $\mathcal{F}_{\Lambda}$ is not a Baer subplane. By definition, this means that some line H does not meet the (Λ-invariant) fixed point set F. For each $x \in H$ the line L_x containing x^{Λ} is the unique fixed line through x. Choose $p \in H$ and $\lambda \in \Lambda$ with $p^{\lambda} \neq p$. Then $pp^{\lambda} = L_p \in \mathfrak{F}$ and $L_p \neq H \neq H^{\lambda}$ (since $H \cap F = \emptyset$ and $H \notin \mathfrak{F}$). There is a compact neighbourhood V of p in H such that $V \cap V^{\lambda} = \emptyset$. The map $(x \mapsto xx^{\lambda}) : V \to \mathfrak{F}$ is continuous and injective. Hence $\dim \mathfrak{F} = \ell$. This condition, however, characterizes Baer subplanes, see [1], 55.5. ∎

In the following, Δ will always denote a connected locally compact group of automorphisms of a 16-dimensional compact projective plane $\mathcal{P} = (P, \mathfrak{L})$. We assume again that Θ is a compact, 0-dimensional subgroup in the center Z of Δ such that Δ/Θ is a Lie group but Θ is not. Groups of dimension ≥ 35 are known to be Lie groups [1], 87.1. Hence only the cases $25 < h = \dim \Delta < 35$ have to be considered.

Proof of (a) (1) Assume that $\dim \Delta \geq 26$ and that $\mathcal{F}$ is any Δ-invariant closed proper subplane. Δ induces on $\mathcal{F}$ a group $\Delta^* = \Delta/\Phi$ with kernel Φ. If $\dim \mathcal{F} \leq 4$, then Theorems P and Q imply $\dim \Delta \leq 24$. Hence $\dim \mathcal{F} = 8$ and $\mathcal{F}$ is a Baer subplane. Moreover, the kernel Φ is compact and satisfies $\dim \Phi < 8$, see [1], 83.6. Consequently, $\dim \Delta^* \geq 19$, and then $\mathcal{F}$ is isomorphic to the classical quaternion plane $\mathcal{P}_2\mathbb{H}$, cf. Salzmann [11] or [1], 84.28. In particular, Δ^* is a Lie group, and we may assume $\Theta \leq \Phi$. A semi-simple group Δ^* in the given dimension range is, in fact, one of the simple motion groups $\mathrm{PU}_3(\mathbb{H}, r)$. This is proved in Salzmann [9], for almost simple groups cp. also [1], 84.19.

In all other cases, it has been shown in Salzmann [8] (4.8) that Δ fixes some element of $\mathcal{F}$, say a line W. The lines of $\mathcal{P}$ are homeomorphic to $\mathbb{S}_8$ because the point set of $\mathcal{F}$ is a manifold [1], 41.11(b) and 52.3. Any k-dimensional orbit in a k-dimensional manifold M is open in M, see [1], 92.14 or 96.11. Since Δ is not a Lie group, Theorem O implies $\dim p^\Delta < 16$ for each point p. Moreover, we conclude from Lemma O that the stabilizer of a line of $\mathcal{F}$ has only orbits of dimension at most 7 on this line. The points and lines of $\mathcal{F}$ will be called " inner " elements, the others " outer " ones. There are outer points p and q not on the same inner line such that $\dim \Delta/\Delta_{p,q} \le \dim p^\Delta + \dim q^\Delta \le 15+7$. (If Δ fixes the inner line W, choose $q \in W$; if Δ^* is a motion group corresponding to the polarity π of $\mathcal{F} \cong \mathcal{P}_2\mathbb{H}$, and if p is on the inner line $L = a^\pi$, choose q on the line ap.) Hence the connected component Λ of $\Delta_{p,q}$ satisfies $\dim \Lambda > 3$. Because the infinite group Θ acts freely on the set of outer points, $\Lambda \cap \Theta = \Theta_p = \mathbb{1}$, and Λ is a Lie group. The orbits p^Θ and q^Θ consist of fixed points of Λ, and all fixed elements of Λ form a proper subplane $\mathcal{E}$. Since each outer line meets $\mathcal{F}$ in a unique inner point, $\mathcal{E} \cap \mathcal{F}$ is infinite. Any collineation group of $\mathcal{P}_2\mathbb{H}$ with 3 distinct fixed points on a line fixes even a point set homeomorphic to a circle on that line [1], 13.6 and 11.29. Therefore, $\dim \mathcal{E} \in \{2,4,8\}$. In the first two cases, Θ would be a Lie group by Theorem P. In the last case it follows from [1], 83.6 and 55.32(ii) that Λ is a compact Lie group of torus rank 1, and $\dim \Lambda \le 3$. Thus, $\dim \Delta > 25$ has led to a contradiction.

(2) If Δ is even almost simple, i.e. if $\Delta^* = \Delta/\mathsf{Z}$ is simple, then Δ is a projective limit of covering groups of Δ^*, see Stroppel [12] Th. 8.3. In particular, the fundamental group $\pi_1\Delta^*$ must be infinite. In the range $25 < h < 35$ the last condition is satisfied only by $\Delta^* \cong \mathrm{PSO}_8(\mathbb{R},2)$. Let Φ be a maximal compact subgroup of Δ. The commutator subgroup Φ' covers $\mathrm{PSO}_6\mathbb{R}$. Lemma R implies $\Phi' \cong \mathrm{Spin}_6\mathbb{R} \cong \mathrm{SU}_4\mathbb{C}$. In $\mathrm{SU}_4\mathbb{C}$ there are 6 pairwise commuting diagonal involutions conjugate to $\alpha = \mathrm{diag}\,(1,1,-1-1)$. Let β be one of these conjugates. From [1], 55. 34b and 39 together with [1], 55.29 it follows that the common fixed elements of α and β form a 4-dimensional subplane $\mathcal{F}$. By Theorem P, the kernel K of the action of Θ on $\mathcal{F}$ is infinite. The subplane $\mathcal{Q} < \mathcal{P}$ consisting of all fixed elements of K is Δ-invariant (because $\Theta \le \mathsf{Z}$). On the other hand, it has been proved in [1], 84.9 that Φ' cannot act on any proper subplane of $\mathcal{P}$. This contradiction shows that a semi-simple group Δ has at least two almost simple factors, cp. [1], 94.25.

(3) Consider an almost simple factor A of Δ of minimal dimension such that A is not a Lie group, and denote the product of all other factors by B. We will find successively smaller bounds for $\dim \mathsf{B}$. Write A^* for the simple image of A in $\Delta^* = \Delta/\mathsf{Z}$. Let Φ be a maximal compact subgroup of A. The Mal'cev-Iwasawa theorem [1], 93.10 shows that A is homeomorphic to $\Phi \times \mathbb{R}^k$, and Φ is not a Lie group. By Weyl's theorem [1], 94.29, a compact semi-simple Lie group has only finitely many coverings. Hence Φ^* cannot be semi-simple and has a central torus [1], 94.31(c). In fact, this central torus is one-dimensional as can be seen by inspection of the list of simple Lie groups [1], 94.33. Consequently, the connected component Υ of $\mathsf{Z}(\Phi)$ is a 1-dimensional solenoid. In particular, $\mathsf{A} \ne \Phi$ and A is not compact. In the next steps we will apply Theorem B to Υ and to Z.

(4) Assume first that Υ is straight, and let $\mathbb{1} \neq \zeta \in \Upsilon \cap \mathsf{Z}$. If $\mathcal{F}_\Upsilon$ is a Baer subplane, then $\mathcal{F}_\zeta = \mathcal{F}_\Upsilon$ would be a Δ-invariant proper subplane in contradiction to (1). If $\Upsilon \leq \Delta_{[z]}$, then the center z of ζ is Δ-invariant. In particular, $z^{\mathsf{A}} = z$. Because A is almost simple and Υ is contained in the normal subgroup $\mathsf{A}_{[z]}$, we get $\mathsf{A} \leq \Delta_{[z]}$. Homologies and elations with center z or homologies with different axes and the same center do not commute. Hence Υ consists of elations only or of homologies with the same axis. If Υ is an elation group, so is A, and all elements in A have the same axis, because A is not commutative, cp. [1], 23.13. If $\Upsilon \leq \Delta_{[z,L]}$, then L is the axis of ζ, and $L^{\mathsf{A}} = L$. Consequently, $\mathsf{A}_{[z,L]}$ is a normal subgroup of A, and $\mathsf{A} \leq \Delta_{[z,L]}$. For $z \in L$ the connected group A would be a Lie group [1], 61.5, and in the case $z \notin L$ it would follow from [1], 61.2 that A is compact. This contradicts the last statement in (3).

(5) Therefore, Υ is not straight, and there is some point c such that c^Υ generates a connected subplane. We shall write $\langle c^\Upsilon \rangle = \mathcal{F}$ for the smallest closed subplane containing c^Υ. If $\dim \mathcal{F} \leq 4$, then Υ induces a Lie group on $\mathcal{F}$ by Theorem P, and there is an element $\zeta \in \mathsf{Z}$ such that $\mathcal{F} \leq \mathcal{F}_\zeta < \mathcal{P}$, but this contradicts (1). Thus, $\mathcal{F}$ is a Baer subplane or $\mathcal{F} = \mathcal{P}$. Since $\mathsf{B}\Phi$ and Υ commute elementwise, $(\mathsf{B}\Phi)_c$ induces the identity on $\mathcal{F}$, and $\dim(\mathsf{B}\Phi)_c \leq 7$ by [1], 83.6. From Theorem O it follows that $\dim c^\Delta \leq 15$. If $\dim c^\Delta > 8$, then $\langle c^\Delta \rangle = \mathcal{P}$ and $\mathsf{Z}_c = \mathbb{1}$. Hence, $(\mathsf{B}\Phi)_c$ is a Lie group and we have even $\dim(\mathsf{B}\Phi)_c \leq 3$ as at the end of (1). In any case, the dimension formula [1], 96.10 gives $\dim \mathsf{B} + \dim \Phi \leq 18$ and $\dim \mathsf{A} \geq 8$. Now the classification of simple Lie groups [1], 94.33 shows that $\dim \Phi \geq 4$, and $\dim \mathsf{B} \leq 14$. Consequently, $\dim \mathsf{A} \geq 12$. The remarks in (3) and again the classification [1], 94.33 imply $\dim \mathsf{A} \in \{15, 21, 24\}$, and then $\dim \Phi \geq 7$. We conclude that $\dim \mathsf{B} \leq 11$, and B is a Lie group by the minimality assumption on $\dim \mathsf{A}$.

(6) Suppose that Z is straight. $\mathcal{F}_{\mathsf{Z}}$ cannot be a Baer subplane by (1). Hence $\mathsf{Z} \leq \Delta_{[a]}$ for some center a. If each element of Z is an elation, Δ would be a Lie group by the dual of (2.7) in Löwen – Salzmann [4]. Therefore, the center Z is contained in a group $\Delta_{[a,W]}$ of homologies (note that homologies in $\Delta_{[a]}$ with different axes do not commute). We can now show that Δ has torus rank $\operatorname{rk} \Delta < 4$. Else, it would follow from [1], 55.35 and 39 (a) that there are Baer involutions α and β in Δ such that $\mathcal{F}_{\alpha,\beta}$ is a 4-dimensional subplane. As a group of homologies, Z would act faithfully on $\mathcal{F}_{\alpha,\beta}$, but this contradicts Theorem P. At the end of step (5) we have seen that B is a Lie group of dimension at most 11. This implies $\dim \mathsf{A} \geq 15$ and then $\operatorname{rk} \mathsf{A} \geq 2$, see [1], 94. 32(e) or 33. If $\dim \mathsf{B} = 11$, then B is a product $\Psi\Omega$ of two almost simple Lie groups such that $\dim \Psi = 8$ and $\dim \Omega = 3$. It follows that $\operatorname{rk} \Psi = 1$ and $\operatorname{rk} \Omega = 0$. Hence Ω is the universal covering group of $\mathrm{SL}_2\mathbb{R}$, and Ω is not compact [1], 94.37. Since any almost simple subgroup of $\Delta_{[a,W]}$ is compact by [1], 61.2, the group Ω acts non-trivially on W, and there is a point x such that $\langle x^{\Omega\mathsf{Z}} \rangle = \mathcal{B}$ is a connected subplane of $\mathcal{P}$. Because Z consists of homologies, Z acts faithfully on $\mathcal{B}$, and Theorem P shows that $\dim \mathcal{B} \geq 8$, i.e. $\mathcal{B}$ is a Baer subplane, or $\mathcal{B} = \mathcal{P}$. The stabilizer $\Lambda = (\mathsf{A}\Psi)_x$ fixes $\mathcal{B}$ pointwise, moreover, $\Lambda \cap \mathsf{Z} = \mathbb{1}$, and Λ is a Lie group. From [1], 83.6 and 55.32(ii) we conclude again that Λ is compact,

that $\operatorname{rk}\Lambda \le 1$, and hence $\dim\Lambda \le 3$. With $\dim\Psi = 8$ we get $\dim\mathsf{A} < 11$, a contradiction. The only remaining possibility $\dim\mathsf{B} < 11$ and $\dim\mathsf{A} \ge 21$ can be excluded by similar arguments: If B acts non-trivially on W, then $\mathcal{B} = \langle x^{\mathsf{BZ}}\rangle$ is a subplane of dimension at least 8, and $\dim\mathsf{A}_x \le 3$, $\dim\mathsf{A} < 20$. If $\mathsf{B} \le \Delta_{[a,W]}$, however, then B is compact by [1], 61.2. At the end of (5) it has been stated that B is a Lie group, and we know also that $\operatorname{rk}\mathsf{B} \le 1$. Consequently, $\dim\mathsf{B} = 3$, $\dim\mathsf{A} = 24$, and $\operatorname{rk}\mathsf{A} = 3$, but we have proved above that $\operatorname{rk}\Delta < 4$.

(7) Finally, we consider the case that Z is not straight. There is a point c such that the orbit c^{Z} is not contained in a line. In particular, $\langle c^{\Delta}\rangle$ is a Δ-invariant subplane, and $\langle c^{\Delta}\rangle = \mathcal{P}$ by step (1). Hence $\mathsf{Z}_c = \mathbb{1}$, and $\langle c^{\mathsf{Z}}\rangle$ is a non-degenerate subplane. By Theorems Q and G, we have $\dim\Delta_c \le 11$, and we conclude from Theorem O that $\dim c^{\Delta} < 16$. The dimension formula gives $\dim\Delta = 26$. If $\dim\mathsf{A} > 15$, then $\dim\mathsf{A} \in \{21, 24\}$ and $\dim\mathsf{B} \in \{5, 2\}$, and B would not be semi-simple. Consequently, $\dim\mathsf{B} = 11$, and we have again that B is a product of two almost simple factors Ψ and Ω with $\dim\Omega = 3$. Let C be the set of all points x such that x^{Z} is not contained in any line. Then C is an open neighborhood of c, and $\Omega|_C \ne \mathbb{1}$. We may assume that $c^{\Omega} \ne c$. Consider the subplane $\mathcal{B} = \langle c^{\Omega\mathsf{Z}}\rangle$. Because $\mathsf{Z}_c = \mathbb{1}$, it follows as in step (6) that $\dim\mathcal{B} \ge 8$, and then $\dim(\mathsf{A}\Psi) < 20$. This contradiction completes the proof of (a). ∎

Proof of (b) By Theorem B and (a), each assumption implies that Z is not straight. As in step (7) above, some orbit c^{Z} contains a quadrangle, and from Theorems Q and G we get $\dim\Delta_c \le 11$. Theorem O shows that $\dim c^{\Delta} < 16$, and the dimension formula gives $\dim\Delta \le 26$. ∎

Proof of (c) (1) Let $\dim\Delta \ge 27$. Then Δ cannot be semi-simple by (a). This means that Δ has a minimal commutative connected normal subgroup Ξ, and Ξ is either compact, (and then Ξ is contained in the center Z, see [1], 93.19), or Ξ is a vector group $\mathbb{R}^t$, (and Δ induces an irreducible representation on Ξ). The proof of (b) shows that Z is straight. The dual statement is also true. Z is not planar by (a), and Theorem B implies that Z is contained in a group $\Delta_{[a,W]}$. As mentioned in the introduction, Z does not consist of elations, and $a \notin W$. This proves the first assertion of (c) . In particular, $\Xi \le \Delta_{[a,W]}$ if Ξ is compact.

(2) Assume now that Ξ is a vector group and that $\Xi|_W \ne \mathbb{1}$. Choose $z \in W$ such that $z^{\Xi} \ne z$, and let $c \in az \setminus \{a, z\}$. The group Ξ induces on the orbit z^{Ξ} a sharply transitive Lie group $\Omega \cong \Xi/\Xi_z$ of dimension at most 8. Consider an element $\omega \in \Omega$ which belongs to a unique one-parameter subgroup Π of Ω. Denote the connected component of $\Delta_c \cap \mathrm{Cs}\,\omega$ by Λ. Then Λ centralizes each element of Π and fixes z^{Π} pointwise. Hence the fixed elements of Λ form a connected subplane $\mathcal{F}_{\Lambda}$. Moreover, Λ is a Lie group since $\Lambda \cap \mathsf{Z} \le \mathsf{Z}_c = \mathbb{1}$, and $\dim\Lambda \ge 27 - \dim c^{\Delta} - \dim\Omega > 3$ by Theorem O. The center Z acts effectively on $\mathcal{F}_{\Lambda}$ because Z consists of homologies. If $\dim\mathcal{F}_{\Lambda} \le 4$, then Z would be a Lie group by Theorem P. Therefore, $\mathcal{F}_{\Lambda}$ is a Baer subplane, and we conclude from [1], 83.6 and 55.32(ii) that Λ is a compact Lie group of torus rank at most 1. Hence $\Lambda \le \mathrm{SU}_2$ and $\dim\Lambda \le 3$. This contradiction proves that $\Xi \le \Delta_{[a,W]}$ as

asserted. If Ξ is not compact, then $\Xi \cong \mathbb{R}$ by [1], 61.2. Together with the first part of (1) this implies that $\dim \Delta/\mathrm{Cs}\,\Xi \leq 1$. ■

Proof of (d) (1) Whenever $a \neq c \notin W$, then Δ_c is a Lie group because $\Delta_c \cap \mathsf{Z} = \mathbb{1}$. If Λ denotes the stabilizer of a quadrangle and $\Phi = \Lambda \cap \mathrm{Cs}\,\Xi$, then $\dim \Lambda/\Phi \leq 1$ by the last remark in (c). Moreover, Φ is a Lie group, and the fixed elements of Φ form a $\Xi\mathsf{Z}$-invariant connected subplane $\mathcal{F}$. Since Z acts effectively on $\mathcal{F}$ and Z is not a Lie group, it follows from Theorem P that $\mathcal{F}$ is a Baer subplane or $\mathcal{F} = \mathcal{P}$. Consequently, Φ is a compact Lie group of torus rank at most 1, and $\dim \Phi \leq 3$. Thus, the existence of Ξ implies $\dim \Lambda \leq 4$. Letting $ac \cap W = z$, we conclude from Lemma O that $\dim c^{\Delta_z} < 8$.

(2) Assuming again that $\dim \Delta \geq 27$, we now study the action of Δ on W. For $v^\Delta \subseteq W$ and $\dim v^\Delta = k > 0$, the dimension formula [1], 96.10 and the last remarks in (1) imply $27 \leq \dim \Delta \leq 3k+7+4$ and $k > 5$. Similarly, if Δ fixes a point $z \in W$, then Δ has only 8-dimensional orbits on $W \setminus z$, and Δ is even doubly transitive on $W \setminus z$. In this case, the action of Δ_v on $v^\Delta \approx \mathbb{R}^8$ is linear [1], 96.16(b), and the stabilizer Λ of a quadrangle has a connected subplane of fixed elements. With the arguments of (c) step (2), we would obtain $\dim \Lambda \leq 3$, but $\dim \Lambda \geq 27-2\cdot 8-7 = 4$. If, on the other hand, $\dim v^\Delta = 8$ for each $v \in W$, then Δ would be transitive on $W \approx \mathbb{S}_8$. Consequently, $\dim \Delta \geq 36$ by [1], 96. 19 and 23, and Δ would be a Lie group, either by [1], 87.1 or by the dual of [1], 62.11. Hence there is some $v \in W$ with $\dim v^\Delta = k \in \{6, 7\}$.

Suppose that Δ is doubly transitive on $V = v^\Delta$. By results of Tits [1], 96. 16 and 17, either V is a sphere, or V is an affine or projective space and the stabilizer of two points fixes a real or complex line. In the latter case, the stabilizer Ω of three "collinear" points of V would have dimension at least $27-15$, but the remarks at the end of (1) show that $\dim \Omega \leq 11$. If $V \approx \mathbb{S}_6$, then Δ has a subgroup $\Gamma \cong \mathrm{G}_2$, see [1], 96. 19 and 23, and Δ would be a Lie group by Theorem G. Therefore, $V \approx \mathbb{S}_7$, and the Tits list [1], 96.17(b) shows that Δ induces on V a group $\mathrm{PSU}_5(\mathbb{C}, 1)$ or $\mathrm{PU}_3(\mathbb{H}, 1)$. In the first case, Δ contains a subgroup $\mathrm{SU}_4\mathbb{C}$. As in the proof of (a) step (2), the element $\mathrm{diag}(1, 1, -1, -1)$ and its conjugates are Baer involutions. Two of these involutions fix a 4-dimensional subplane $\mathcal{F}$. The center Z acts effectively on $\mathcal{F}$ and hence would be a Lie group by Theorem P. In the only remaining case, Δ has a subgroup Ψ which is locally isomorphic to $\mathrm{U}_3(\mathbb{H}, 1)$, compare [1], 94.27. Consider a maximal compact subgroup Φ of Ψ and its 10-dimensional factor Υ. From Lemma R we conclude that $\Upsilon \cong \mathrm{U}_2\mathbb{H} \cong \mathrm{Spin}_5\mathbb{R}$ and that the central involution σ of Υ is not planar. Thus, σ is a reflection [1], 55.29, and σ fixes only the points on the axis and the center. Moreover, the map $\Upsilon \to \mathrm{PU}_3(\mathbb{H}, 1)$ is injective and σ acts freely on V. Therefore, W is not the axis of σ, and σ fixes exactly two points on W. Hence $\mathrm{Cs}\,\sigma = \nabla$ is the stabilizer of a triangle. Let K be the connected component of the kernel $\Delta_{[V]}$. Then $\dim \Delta/\mathsf{K} = \dim \mathrm{U}_3(\mathbb{H}, 1) = 21$, and $\dim \mathsf{K} \geq 6$. On the other hand, K acts effectively on the line av, and $\dim \mathsf{K} \leq 7$ by Lemma O. Any representation of Ψ in dimension < 12 is trivial, see [1], 95.10. Therefore, Ψ induces the identity on the Lie algebra of K, and $\Psi \circ \mathsf{K} = \mathbb{1}$. Consequently, $\mathsf{K}\Phi \leq \nabla$, and $\dim \nabla \geq 6 + 13$, but step (1) implies

$\dim \nabla \le 2\cdot 7+4 = 18$. This contradiction shows that Δ is not doubly transitive on V.

(3) Choose $v \in W$ such that $v^{\Delta} = V$ has dimension < 8, and let $c \in av \setminus \{a, v\}$. The connected component $\mathsf{\Gamma}$ of Δ_c is not transitive on $V \setminus \{v\}$ and hence has an orbit $u^{\mathsf{\Gamma}} = U \subset V$ of dimension ≤ 6. By the last remarks in (1), we have $\dim \mathsf{\Gamma} \ge 13$ and $\dim U \ge 5$. Consequently, $\mathsf{\Gamma}$ acts effectively on U. Assume that $\dim U = 6$ and that $\mathsf{\Gamma}$ is doubly transitive on U. Step (1) implies that $\dim \mathsf{\Gamma} \le 2\cdot 6+4 = 16$, and $\mathsf{\Gamma}$ cannot be simple by [1], 96.17. From [1], 96.16 we conclude that $U \approx \mathbb{R}^6$ and that $\mathsf{\Gamma}_u$ has a subgroup $\mathsf{\Phi} \cong \mathrm{SU}_3\mathbb{C}$. The representation of $\mathsf{\Phi}$ on $U \approx \mathbb{C}^3$ shows that each involution in $\mathsf{\Phi}$ fixes a 2-dimensional subspace of U and so is planar. Two commuting involutions fix a 4-dimensional subplane, and Z would be a Lie group by Theorem P. Therefore, the connected component Ω of $\mathsf{\Gamma}_u$ has an orbit in U of dimension < 6. By step (1), we obtain $\dim \Omega \le 9$ and $\dim \mathsf{\Gamma} \le 15$. If ζ is in the center of $\mathsf{\Gamma}$, and $z^{\zeta} \ne z \in U$, then $\mathsf{\Gamma}_z = \mathsf{\Gamma}_{z^{\zeta}}$ fixes a quadrangle, and $\dim \mathsf{\Gamma}_z \le 4$ by step (1), but $\dim \mathsf{\Gamma}_z \ge 13-6$. Because $\mathsf{\Gamma}$ acts effectively on U, this shows that the center of $\mathsf{\Gamma}$ is trivial. Either $\mathsf{\Gamma}$ has a minimal normal subgroup $\mathsf{X} \cong \mathbb{R}^s$, or $\mathsf{\Gamma}$ is a direct product of simple Lie groups, cp. [1], 94. 26 and 23. We will discuss the two possibilities separately in the next steps.

(4) Let $\mathsf{\Gamma}$ be semi-simple. Any reflection $\alpha \in \mathsf{\Gamma}$ has axis av, and $\alpha^{\mathsf{\Gamma}} \ne \alpha$ since $\mathsf{\Gamma}$ has trivial center. The set $\alpha^{\mathsf{\Gamma}}\alpha$ is contained in the connected component E of the elation group $\mathsf{\Gamma}_{[v,av]}$, and E is a normal subgroup of $\mathsf{\Gamma}$. Hence E is itself a product of simple Lie groups, and E contains a non-trivial torus, but an involution is never an elation [1], 55.29. This contradiction shows that each involution in $\mathsf{\Gamma}$ is planar. Because $\dim \mathsf{\Gamma} > 8$, there exists a pair of commuting involutions. Their common fixed elements form a 4-dimensional subplane [1], 55.39, and Z would be a Lie group by Theorem P. Therefore, $\mathsf{\Gamma}$ cannot be semi-simple.

(5) We use the notation of (3) and determine the action of Ω on X. Note that $u^{\mathsf{X}} \ne u$ because $\mathsf{\Gamma}$ acts effectively on U. If $u \ne z \in u^{\mathsf{X}}$, then $z^{\Omega} \subset u^{\mathsf{X}}$. By step (1), we have $\dim \Omega_z \le 4$. From $\dim \mathsf{\Gamma} \ge 13$ it follows that $\dim \Omega \ge 7$ and hence $3 \le \dim z^{\Omega} \le \dim u^{\mathsf{X}}$. The stabilizer X_u fixes each point of the connected subplane $\langle a, c, u^{\mathsf{X}}\rangle$, and this subplane has dimension at least 8, since u^{X} is contained in a line and $\dim u^{\mathsf{X}} > 2$. From [1], 83.6 we infer that X_u is compact, and then $\mathsf{X}_u = \mathbb{1}$ since X is a vector group. Because Ω acts linearly on X, the fixed elements of the connected component $\mathsf{\Lambda}$ of Ω_z form a connected subplane $\mathcal{F}$. As a group of homologies, the non-Lie group Z acts effectively on $\mathcal{F}$, and Theorem P implies that $\mathcal{F}$ is a Baer subplane. As at the end of (c) step (2) it follows that $\mathsf{\Lambda}$ is isomorphic to a subgroup of SU_2. We know that $\dim \Omega \ge 7$, and we conclude from (3) that there is a point z with $\dim z^{\Omega} < 6$. This gives $\dim \mathsf{\Lambda} \ge 2$ and then $\mathsf{\Lambda} \cong \mathrm{SU}_2 \cong \mathrm{Spin}_3$. In particular, $\dim \mathsf{\Lambda} = 3$ and $\dim z^{\Omega} \ge 4$. Therefore, any minimal Ω-invariant subgroup of X has dimension at least 4. Calling such a subgroup X from now on, we may assume that Ω acts irreducibly on $\mathsf{X} \cong \mathbb{R}^s$, where $4 \le s \le 6$. These three possibilities will be discussed in the last steps. Each case will lead to a contradiction.

(6) The connected component $\mathsf{\Lambda}$ of Ω_z acts reducibly on X by its very

definition. If $s = 4$, then Λ induces on X either the identity or a group SO_3. Each non-trivial orbit of Ω on X is 4-dimensional, and Ω is transitive on $\mathsf{X}\setminus\{\mathbb{1}\}$. In particular, Ω is not compact, and a maximal compact (connected) subgroup Φ of Ω has dimension at most 6. A theorem of Montgomery [1], 96.19 shows that Φ is transitive on the 3-sphere consisting of the rays in $\mathsf{X} \cong \mathbb{R}^4$. Let r denote the ray determined by z. Then $\Lambda \leq \Phi_r$ and $\Phi/\Phi_r \approx \mathbb{S}_3$. This implies $\dim \Phi = 6$ and $\dim \Phi_r = 3$. Moreover, Φ_r is connected by [1], 94.4(a), and hence $\Phi_r = \Lambda$ is simply connected. The exact homotopy sequence [1], 96.12 shows that Φ is also simply connected. Consequently, $\Phi \cong \mathrm{Spin}_4 \cong (\mathrm{Spin}_3)^2$, compare [1], 94.31(c), and Φ contains exactly 3 involutions. If $\dim w^\Phi = 6$ for some $w \in U$, then w^Φ is open in U by [1], 96.11. Since w^Φ is also compact and U is connected, Φ would be transitive on U, but $u^\Omega = u$. Hence, each stabilizer Φ_w has positive dimension and contains a (planar) involution γ. Let $F_\gamma = \{x \in W \mid x^\gamma = x\}$. Then U is covered by the 3 sets F_γ, and these are homeomorphic to $\mathbb{S}_4$. The sum theorem [1], 92.9 implies $\dim U \leq 4$, but we have seen at the beginning of (3) that $\dim U \geq 5$. This contradiction excludes the case $s = 4$.

(7) If $s = 5$, then Ω acts effectively on X, and Ω' is irreducible and simple, see [1], 95. 5 and 6(b). A table of irreducible representations [1], 95.10 shows that $\dim \Omega' \in \{3, 10\}$, but we know from step (5) that $6 \leq \dim \Omega' \leq 8$.

(8) Finally, let $s = 6$. Then Ω' is semi-simple by [1], 95.6(b), and $\dim \Omega' \in \{6, 8\}$. Note that $\mathrm{SU}_2 \cong \Lambda < \Omega'$. Either Ω' is even almost simple, or Ω' has a factor $\Phi \cong \mathrm{SU}_2$. By [1], 95.5, any Φ-invariant subspace of X has a dimension d dividing 6, but effective irreducible representations of SU_2 exist only in dimensions $4k$, compare [1], 95.10. Therefore, Ω' is almost simple. The table [1], 95.10 shows that Ω' must be one of the groups $\mathrm{SO}_3\mathbb{C}$, $\mathrm{SL}_3\mathbb{R}$, or $\mathrm{SU}_3(\mathbb{C}, r)$. The first two have no subgroup SU_2 and can be discarded. The two unitary groups contain 3 diagonal involutions. Each one of these has an eigenvalue 1 and thus is planar. By [1], 55.39 their common fixed elements form a 4-dimensional subplane $\mathcal{F}$. The center Z acts effectively on $\mathcal{F}$, and Z would be a Lie group by Theorem P. This completes the proof of (d) and hence of Theorem L. ∎

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